Probability and Stochastic Processes

A Friendly Introduction for Electrical and Computer Engineers

Third Edition

INSTRUCTOR'S SOLUTION MANUAL

Roy D. Yates, David J. Goodman, David Famolari

September 8, 2014

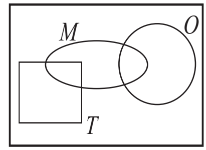
Comments on this Solutions Manual

- This solution manual is mostly complete. Please send error reports, suggestions, and comments to ***ryates@winlab.rutgers.edu***.
- To make solution sets for your class, use the *Solution Set Constructor* at the instructors site ***www.winlab.rutgers.edu/probsolns***.
- Send email to ***ryates@winlab.rutgers.edu*** for access to the instructors site.
- MATLAB functions written as solutions to homework problems can be found in the archive `matsoln3e.zip` (available to instructors). Other MATLAB functions used in the text or in these homework solutions can be found in the archive `matcode3e.zip`. The .m files in `matcode3e` are available for download from the Wiley website. Two other documents of interest are also available for download:
  - A manual `probmatlab3e.pdf` describing the `matcode3e .m` functions
  - The quiz solutions manual `quizsol.pdf`.
- This manual uses a page size matched to the screen of an iPad tablet. If you do print on paper and you have good eyesight, you may wish to print two pages per sheet in landscape mode. On the other hand, a "Fit to Paper" printing option will create "Large Print" output.

Problem Solutions – Chapter 1

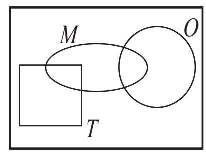
Problem 1.1.1 Solution

Based on the Venn diagram on the right, the complete Gerlandas pizza menu is



- Regular without toppings
- Regular with mushrooms
- Regular with onions
- Regular with mushrooms and onions
- Tuscan without toppings
- Tuscan with mushrooms

Problem 1.1.2 Solution



Based on the Venn diagram on the right, the answers are mostly fairly straightforward. The only trickiness is that a pizza is either Tuscan (T) or Neapolitan (N) so $\{N, T\}$ is a partition but they are not depicted as a partition. Specifically, the event N is the region of the Venn diagram outside of the "square block" of event T. If this is clear, the questions are easy.

(a) Since $N = T^c$, $N \cap M \neq \phi$. Thus N and M are not mutually exclusive.

(b) Every pizza is either Neapolitan (N), or Tuscan (T). Hence $N \cup T = S$ so that N and T are collectively exhaustive. Thus its also (trivially) true that $N \cup T \cup M = S$. That is, R, T and M are also collectively exhaustive.

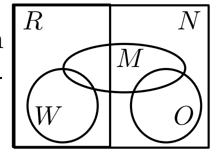
(c) From the Venn diagram, T and O are mutually exclusive. In words, this means that Tuscan pizzas never have onions or pizzas with onions are never Tuscan. As an aside, "Tuscan" is a fake pizza designation; one shouldn't conclude that people from Tuscany actually dislike onions.

(d) From the Venn diagram, $M \cap T$ and O are mutually exclusive. Thus Gerlanda's doesn't make Tuscan pizza with mushrooms and onions.

(e) Yes. In terms of the Venn diagram, these pizzas are in the set $(T \cup M \cup O)^c$.

Problem 1.1.3 Solution

At Ricardo's, the pizza crust is either Roman (R) or Neapolitan (N). To draw the Venn diagram on the right, we make the following observations:



- The set $\{R, N\}$ is a partition so we can draw the Venn diagram with this partition.
- Only Roman pizzas can be white. Hence $W \subset R$.
- Only a Neapolitan pizza can have onions. Hence $O \subset N$.
- Both Neapolitan and Roman pizzas can have mushrooms so that event M straddles the $\{R, N\}$ partition.
- The Neapolitan pizza can have both mushrooms and onions so $M \cap O$ cannot be empty.
- The problem statement does not preclude putting mushrooms on a white Roman pizza. Hence the intersection $W \cap M$ should not be empty.

Problem 1.2.1 Solution

(a) An outcome specifies whether the connection speed is high (h), medium (m), or low (l) speed, and whether the signal is a mouse click (c) or a tweet (t). The sample space is

$$S = \{ht, hc, mt, mc, lt, lc\}. \tag{1}$$

(b) The event that the wi-fi connection is medium speed is $A_1 = \{mt, mc\}$.

(c) The event that a signal is a mouse click is $A_2 = \{hc, mc, lc\}$.

(d) The event that a connection is either high speed or low speed is $A_3 = \{ht, hc, lt, lc\}$.

(e) Since $A_1 \cap A_2 = \{mc\}$ and is not empty, A_1, A_2, and A_3 are not mutually exclusive.

(f) Since

$$A_1 \cup A_2 \cup A_3 = \{ht, hc, mt, mc, lt, lc\} = S, \tag{2}$$

the collection A_1, A_2, A_3 is collectively exhaustive.

Problem 1.2.2 Solution

(a) The sample space of the experiment is

$$S = \{aaa, aaf, afa, faa, ffa, faf, aff, fff\}. \tag{1}$$

(b) The event that the circuit from Z fails is

$$Z_F = \{aaf, aff, faf, fff\}. \tag{2}$$

The event that the circuit from X is acceptable is

$$X_A = \{aaa, aaf, afa, aff\}. \tag{3}$$

(c) Since $Z_F \cap X_A = \{aaf, aff\} \neq \phi$, Z_F and X_A are not mutually exclusive.

(d) Since $Z_F \cup X_A = \{aaa, aaf, afa, aff, faf, fff\} \neq S$, Z_F and X_A are not collectively exhaustive.

(e) The event that more than one circuit is acceptable is

$$C = \{aaa, aaf, afa, faa\}. \tag{4}$$

The event that at least two circuits fail is

$$D = \{ffa, faf, aff, fff\}. \tag{5}$$

(f) Inspection shows that $C \cap D = \phi$ so C and D are mutually exclusive.

(g) Since $C \cup D = S$, C and D are collectively exhaustive.

Problem 1.2.3 Solution

The sample space is

$$S = \{A\clubsuit, \ldots, K\clubsuit, A\diamondsuit, \ldots, K\diamondsuit, A\heartsuit, \ldots, K\heartsuit, A\spadesuit, \ldots, K\spadesuit\}. \tag{1}$$

The event H that the first card is a heart is the set

$$H = \{A\heartsuit, \ldots, K\heartsuit\}. \tag{2}$$

The event H has 13 outcomes, corresponding to the 13 hearts in a deck.

Problem 1.2.4 Solution

The sample space is

$$S = \left\{ \begin{array}{l} 1/1 \ldots 1/31, 2/1 \ldots 2/29, 3/1 \ldots 3/31, 4/1 \ldots 4/30, \\ 5/1 \ldots 5/31, 6/1 \ldots 6/30, 7/1 \ldots 7/31, 8/1 \ldots 8/31, \\ 9/1 \ldots 9/31, 10/1 \ldots 10/31, 11/1 \ldots 11/30, 12/1 \ldots 12/31 \end{array} \right\}. \tag{1}$$

The event H defined by the event of a July birthday is given by the following et with 31 sample outcomes:

$$H = \{7/1, 7/2, \ldots, 7/31\}. \tag{2}$$

Problem 1.2.5 Solution

Of course, there are many answers to this problem. Here are four partitions.

1. We can divide students into engineers or non-engineers. Let A_1 equal the set of engineering students and A_2 the non-engineers. The pair $\{A_1, A_2\}$ is a partition.

2. To separate students by GPA, let B_i denote the subset of students with GPAs G satisfying $i - 1 \leq G < i$. At Rutgers, $\{B_1, B_2, \ldots, B_5\}$ is a partition. Note that B_5 is the set of all students with perfect 4.0 GPAs. Of course, other schools use different scales for GPA.

3. We can also divide the students by age. Let C_i denote the subset of students of age i in years. At most universities, $\{C_{10}, C_{11}, \ldots, C_{100}\}$ would be an event space. Since a university may have prodigies either under 10 or over 100, we note that $\{C_0, C_1, \ldots\}$ is always a partition.

4. Lastly, we can categorize students by attendance. Let D_0 denote the number of students who have missed zero lectures and let D_1 denote all other students. Although it is likely that D_0 is an empty set, $\{D_0, D_1\}$ is a well defined partition.

Problem 1.2.6 Solution

Let R_1 and R_2 denote the measured resistances. The pair (R_1, R_2) is an outcome of the experiment. Some partitions include

1. If we need to check that neither resistance is too high, a partition is

$$A_1 = \{R_1 < 100, R_2 < 100\}, \qquad A_2 = \{R_1 \geq 100\} \cup \{R_2 \geq 100\}. \tag{1}$$

2. If we need to check whether the first resistance exceeds the second resistance, a partition is

$$B_1 = \{R_1 > R_2\} \qquad B_2 = \{R_1 \leq R_2\}. \tag{2}$$

3. If we need to check whether each resistance doesn't fall below a minimum value (in this case 50 ohms for R_1 and 100 ohms for R_2), a partition is C_1, C_2, C_3, C_4 where

$$C_1 = \{R_1 < 50, R_2 < 100\}, \qquad C_2 = \{R_1 < 50, R_2 \geq 100\}, \tag{3}$$

$$C_3 = \{R_1 \geq 50, R_2 < 100\}, \qquad C_4 = \{R_1 \geq 50, R_2 \geq 100\}. \tag{4}$$

4. If we want to check whether the resistors in parallel are within an acceptable range of 90 to 110 ohms, a partition is

$$D_1 = \left\{(1/R_1 + 1/R_2)^{-1} < 90\right\}, \tag{5}$$

$$D_2 = \left\{90 \leq (1/R_1 + 1/R_2)^{-1} \leq 110\right\}, \tag{6}$$

$$D_2 = \left\{110 < (1/R_1 + 1/R_2)^{-1}\right\}. \tag{7}$$

Problem 1.3.1 Solution

(a) A and B mutually exclusive and collectively exhaustive imply $P[A]+P[B] = 1$. Since $P[A] = 3\,P[B]$, we have $P[B] = 1/4$.

(b) Since $P[A \cup B] = P[A]$, we see that $B \subseteq A$. This implies $P[A \cap B] = P[B]$. Since $P[A \cap B] = 0$, then $P[B] = 0$.

(c) Since it's always true that $P[A \cup B] = P[A] + P[B] - P[AB]$, we have that

$$P[A] + P[B] - P[AB] = P[A] - P[B]. \tag{1}$$

This implies $2\,P[B] = P[AB]$. However, since $AB \subset B$, we can conclude that $2\,P[B] = P[AB] \leq P[B]$. This implies $P[B] = 0$.

Problem 1.3.2 Solution

The roll of the red and white dice can be assumed to be independent. For each die, all rolls in $\{1, 2, \ldots, 6\}$ have probability $1/6$.

(a) Thus

$$P[R_3W_2] = P[R_3]\,P[W_2] = \frac{1}{36}.$$

(b) In fact, each pair of possible rolls R_iW_j has probability $1/36$. To find $P[S_5]$, we add up the probabilities of all pairs that sum to 5:

$$P[S_5] = P[R_1W_4] + P[R_2W_3] + P[R_3W_2] + P[R_4W_1] = 4/36 = 1/9.$$

Problem 1.3.3 Solution

An outcome is a pair (i, j) where i is the value of the first die and j is the value of the second die. The sample space is the set

$$S = \{(1, 1), (1, 2), \ldots, (6, 5), (6, 6)\}. \tag{1}$$

with 36 outcomes, each with probability $1/36$ Note that the event that the absolute value of the difference of the two rolls equals 3 is

$$D_3 = \{(1, 4), (2, 5), (3, 6), (4, 1), (5, 2), (6, 3)\}. \tag{2}$$

Since there are 6 outcomes in D_3, $P[D_3] = 6/36 = 1/6$.

Problem 1.3.4 Solution

(a) FALSE. Since $P[A] = 1 - P[A^c] = 2\,P[A^c]$ implies $P[A^c] = 1/3$.

(b) FALSE. Suppose $A = B$ and $P[A] = 1/2$. In that case,

$$P[AB] = P[A] = 1/2 > 1/4 = P[A]\,P[B]. \tag{1}$$

(c) TRUE. Since $AB \subseteq A$, $P[AB] \leq P[A]$, This implies

$$P[AB] \leq P[A] < P[B]. \tag{2}$$

(d) FALSE: For a counterexample, let $A = \phi$ and $P[B] > 0$ so that $A = A \cap B = \phi$ and $P[A] = P[A \cap B] = 0$ but $0 = P[A] < P[B]$.

Problem 1.3.5 Solution

The sample space of the experiment is

$$S = \{LF, BF, LW, BW\}. \tag{1}$$

From the problem statement, we know that $P[LF] = 0.5$, $P[BF] = 0.2$ and $P[BW] = 0.2$. This implies $P[LW] = 1 - 0.5 - 0.2 - 0.2 = 0.1$. The questions can be answered using Theorem 1.5.

(a) The probability that a program is slow is

$$P[W] = P[LW] + P[BW] = 0.1 + 0.2 = 0.3. \tag{2}$$

(b) The probability that a program is big is

$$P[B] = P[BF] + P[BW] = 0.2 + 0.2 = 0.4. \tag{3}$$

(c) The probability that a program is slow or big is

$$P[W \cup B] = P[W] + P[B] - P[BW] = 0.3 + 0.4 - 0.2 = 0.5. \tag{4}$$

Problem 1.3.6 Solution

A sample outcome indicates whether the cell phone is handheld (H) or mobile (M) and whether the speed is fast (F) or slow (W). The sample space is

$$S = \{HF, HW, MF, MW\}. \tag{1}$$

The problem statement tells us that $P[HF] = 0.2$, $P[MW] = 0.1$ and $P[F] = 0.5$. We can use these facts to find the probabilities of the other outcomes. In particular,

$$P[F] = P[HF] + P[MF]. \tag{2}$$

This implies

$$P[MF] = P[F] - P[HF] = 0.5 - 0.2 = 0.3. \tag{3}$$

Also, since the probabilities must sum to 1,

$$\begin{aligned} P[HW] &= 1 - P[HF] - P[MF] - P[MW] \\ &= 1 - 0.2 - 0.3 - 0.1 = 0.4. \end{aligned} \tag{4}$$

Now that we have found the probabilities of the outcomes, finding any other probability is easy.

(a) The probability a cell phone is slow is

$$P[W] = P[HW] + P[MW] = 0.4 + 0.1 = 0.5. \tag{5}$$

(b) The probability that a cell hpone is mobile and fast is $P[MF] = 0.3$.

(c) The probability that a cell phone is handheld is

$$P[H] = P[HF] + P[HW] = 0.2 + 0.4 = 0.6. \tag{6}$$

Problem 1.3.7 Solution

A reasonable probability model that is consistent with the notion of a shuffled deck is that each card in the deck is equally likely to be the first card. Let H_i denote the event that the first card drawn is the ith heart where the first heart is the ace, the second heart is the deuce and so on. In that case, $P[H_i] = 1/52$ for $1 \le i \le 13$. The event H that the first card is a heart can be written as the mutually exclusive union

$$H = H_1 \cup H_2 \cup \cdots \cup H_{13}. \quad (1)$$

Using Theorem 1.1, we have

$$P[H] = \sum_{i=1}^{13} P[H_i] = 13/52. \quad (2)$$

This is the answer you would expect since 13 out of 52 cards are hearts. The point to keep in mind is that this is not just the common sense answer but is the result of a probability model for a shuffled deck and the axioms of probability.

Problem 1.3.8 Solution

Let s_i denote the outcome that the down face has i dots. The sample space is $S = \{s_1, \ldots, s_6\}$. The probability of each sample outcome is $P[s_i] = 1/6$. From Theorem 1.1, the probability of the event E that the roll is even is

$$P[E] = P[s_2] + P[s_4] + P[s_6] = 3/6. \quad (1)$$

Problem 1.3.9 Solution

Let s_i equal the outcome of the student's quiz. The sample space is then composed of all the possible grades that she can receive.

$$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}. \quad (1)$$

Since each of the 11 possible outcomes is equally likely, the probability of receiving a grade of i, for each $i = 0, 1, \ldots, 10$ is $P[s_i] = 1/11$. The probability that the student gets an A is the probability that she gets a score of 9 or higher. That is

$$P[\text{Grade of A}] = P[9] + P[10] = 1/11 + 1/11 = 2/11. \quad (2)$$

The probability of failing requires the student to get a grade less than 4.

$$\begin{aligned} P[\text{Failing}] &= P[3] + P[2] + P[1] + P[0] \\ &= 1/11 + 1/11 + 1/11 + 1/11 = 4/11. \end{aligned} \quad (3)$$

Problem 1.3.10 Solution

Each statement is a consequence of part 4 of Theorem 1.4.

(a) Since $A \subset A \cup B$, $P[A] \leq P[A \cup B]$.

(b) Since $B \subset A \cup B$, $P[B] \leq P[A \cup B]$.

(c) Since $A \cap B \subset A$, $P[A \cap B] \leq P[A]$.

(d) Since $A \cap B \subset B$, $P[A \cap B] \leq P[B]$.

Problem 1.3.11 Solution

Specifically, we will use Theorem 1.4(c) which states that for any events A and B,

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]. \tag{1}$$

To prove the union bound by induction, we first prove the theorem for the case of $n = 2$ events. In this case, by Theorem 1.4(c),

$$P[A_1 \cup A_2] = P[A_1] + P[A_2] - P[A_1 \cap A_2]. \tag{2}$$

By the first axiom of probability, $P[A_1 \cap A_2] \geq 0$. Thus,

$$P[A_1 \cup A_2] \leq P[A_1] + P[A_2]. \tag{3}$$

which proves the union bound for the case $n = 2$.Now we make our induction hypothesis that the union-bound holds for any collection of $n-1$ subsets. In this case, given subsets $A_1, \ldots, A_n$, we define

$$A = A_1 \cup A_2 \cup \cdots \cup A_{n-1}, \qquad B = A_n. \tag{4}$$

By our induction hypothesis,

$$P[A] = P[A_1 \cup A_2 \cup \cdots \cup A_{n-1}] \leq P[A_1] + \cdots + P[A_{n-1}]. \tag{5}$$

This permits us to write

$$\begin{aligned} P[A_1 \cup \cdots \cup A_n] &= P[A \cup B] \\ &\leq P[A] + P[B] \qquad \text{(by the union bound for } n = 2) \\ &= P[A_1 \cup \cdots \cup A_{n-1}] + P[A_n] \\ &\leq P[A_1] + \cdots P[A_{n-1}] + P[A_n] \end{aligned} \tag{6}$$

which completes the inductive proof.

Problem 1.3.12 Solution

It is tempting to use the following proof:

> Since S and ϕ are mutually exclusive, and since $S = S \cup \phi$,
>
> $$1 = P[S \cup \phi] = P[S] + P[\phi].$$
>
> Since $P[S] = 1$, we must have $P[\phi] = 0$.

The above "proof" used the property that for mutually exclusive sets A_1 and A_2,

$$P[A_1 \cup A_2] = P[A_1] + P[A_2]. \tag{1}$$

The problem is that this property is a consequence of the three axioms, and thus must be proven. For a proof that uses just the three axioms, let A_1 be an arbitrary set and for $n = 2, 3, \ldots$, let $A_n = \phi$. Since $A_1 = \cup_{i=1}^{\infty} A_i$, we can use Axiom 3 to write

$$P[A_1] = P[\cup_{i=1}^{\infty} A_i] = P[A_1] + P[A_2] + \sum_{i=3}^{\infty} P[A_i]. \tag{2}$$

By subtracting $P[A_1]$ from both sides, the fact that $A_2 = \phi$ permits us to write

$$P[\phi] + \sum_{n=3}^{\infty} P[A_i] = 0. \tag{3}$$

By Axiom 1, $P[A_i] \geq 0$ for all i. Thus, $\sum_{n=3}^{\infty} P[A_i] \geq 0$. This implies $P[\phi] \leq 0$. Since Axiom 1 requires $P[\phi] \geq 0$, we must have $P[\phi] = 0$.

Problem 1.3.13 Solution

Following the hint, we define the set of events $\{A_i | i = 1, 2, \ldots\}$ such that $i = 1, \ldots, m$, $A_i = B_i$ and for $i > m$, $A_i = \phi$. By construction, $\cup_{i=1}^{m} B_i = \cup_{i=1}^{\infty} A_i$. Axiom 3 then implies

$$P[\cup_{i=1}^{m} B_i] = P[\cup_{i=1}^{\infty} A_i] = \sum_{i=1}^{\infty} P[A_i]. \tag{1}$$

For $i > m$, $\mathrm{P}[A_i] = \mathrm{P}[\phi] = 0$, yielding the claim $\mathrm{P}[\cup_{i=1}^m B_i] = \sum_{i=1}^m \mathrm{P}[A_i] = \sum_{i=1}^m \mathrm{P}[B_i]$.

Note that the fact that $\mathrm{P}[\phi] = 0$ follows from Axioms 1 and 2. This problem is more challenging if you just use Axiom 3. We start by observing

$$\mathrm{P}\left[\cup_{i=1}^m B_i\right] = \sum_{i=1}^{m-1} \mathrm{P}\left[B_i\right] + \sum_{i=m}^{\infty} \mathrm{P}\left[A_i\right]. \tag{2}$$

Now, we use Axiom 3 again on the countably infinite sequence $A_m, A_{m+1}, \ldots$ to write

$$\sum_{i=m}^{\infty} \mathrm{P}\left[A_i\right] = \mathrm{P}\left[A_m \cup A_{m+1} \cup \cdots\right] = \mathrm{P}\left[B_m\right]. \tag{3}$$

Thus, we have used just Axiom 3 to prove Theorem 1.3:

$$\mathrm{P}\left[\cup_{i=1}^m B_i\right] = \sum_{i=1}^{m} \mathrm{P}\left[B_i\right]. \tag{4}$$

Problem 1.3.14 Solution

Each claim in Theorem 1.4 requires a proof from which we can check which axioms are used. However, the problem is somewhat hard because there may still be a simpler proof that uses fewer axioms. Still, the proof of each part will need Theorem 1.3 which we now prove.

For the mutually exclusive events $B_1, \ldots, B_m$, let $A_i = B_i$ for $i = 1, \ldots, m$ and let $A_i = \phi$ for $i > m$. In that case, by Axiom 3,

$$\begin{aligned}
\mathrm{P}\left[B_1 \cup B_2 \cup \cdots \cup B_m\right] &= \mathrm{P}\left[A_1 \cup A_2 \cup \cdots\right] \\
&= \sum_{i=1}^{m-1} \mathrm{P}\left[A_i\right] + \sum_{i=m}^{\infty} \mathrm{P}\left[A_i\right] \\
&= \sum_{i=1}^{m-1} \mathrm{P}\left[B_i\right] + \sum_{i=m}^{\infty} \mathrm{P}\left[A_i\right]. \qquad (1)
\end{aligned}$$

Now, we use Axiom 3 again on $A_m, A_{m+1}, \ldots$ to write

$$\sum_{i=m}^{\infty} \mathrm{P}\left[A_i\right] = \mathrm{P}\left[A_m \cup A_{m+1} \cup \cdots\right] = \mathrm{P}\left[B_m\right]. \tag{2}$$

Thus, we have used just Axiom 3 to prove Theorem 1.3:

$$\mathrm{P}\left[B_1 \cup B_2 \cup \cdots \cup B_m\right] = \sum_{i=1}^{m} \mathrm{P}\left[B_i\right]. \tag{3}$$

(a) To show $\mathrm{P}[\phi] = 0$, let $B_1 = S$ and let $B_2 = \phi$. Thus by Theorem 1.3,

$$\mathrm{P}\left[S\right] = \mathrm{P}\left[B_1 \cup B_2\right] = \mathrm{P}\left[B_1\right] + \mathrm{P}\left[B_2\right] = \mathrm{P}\left[S\right] + \mathrm{P}\left[\phi\right]. \tag{4}$$

Thus, $\mathrm{P}[\phi] = 0$. Note that this proof uses only Theorem 1.3 which uses only Axiom 3.

(b) Using Theorem 1.3 with $B_1 = A$ and $B_2 = A^c$, we have

$$\mathrm{P}\left[S\right] = \mathrm{P}\left[A \cup A^c\right] = \mathrm{P}\left[A\right] + \mathrm{P}\left[A^c\right]. \tag{5}$$

Since, Axiom 2 says $\mathrm{P}[S] = 1$, $\mathrm{P}[A^c] = 1 - \mathrm{P}[A]$. This proof uses Axioms 2 and 3.

(c) By Theorem 1.8, we can write both A and B as unions of mutually exclusive events:

$$A = (AB) \cup (AB^c), \qquad B = (AB) \cup (A^cB). \tag{6}$$

Now we apply Theorem 1.3 to write

$$\mathrm{P}\left[A\right] = \mathrm{P}\left[AB\right] + \mathrm{P}\left[AB^c\right], \qquad \mathrm{P}\left[B\right] = \mathrm{P}\left[AB\right] + \mathrm{P}\left[A^cB\right]. \tag{7}$$

We can rewrite these facts as

$$\mathrm{P}[AB^c] = \mathrm{P}[A] - \mathrm{P}[AB], \qquad \mathrm{P}[A^cB] = \mathrm{P}[B] - \mathrm{P}[AB]. \tag{8}$$

Note that so far we have used only Axiom 3. Finally, we observe that $A \cup B$ can be written as the union of mutually exclusive events

$$A \cup B = (AB) \cup (AB^c) \cup (A^cB). \tag{9}$$

Once again, using Theorem 1.3, we have

$$P[A \cup B] = P[AB] + P[AB^c] + P[A^c B] \tag{10}$$

Substituting the results of Equation (8) into Equation (10) yields

$$P[A \cup B] = P[AB] + P[A] - P[AB] + P[B] - P[AB], \tag{11}$$

which completes the proof. Note that this claim required only Axiom 3.

(d) Observe that since $A \subset B$, we can write B as the mutually exclusive union $B = A \cup (A^c B)$. By Theorem 1.3 (which uses Axiom 3),

$$P[B] = P[A] + P[A^c B]. \tag{12}$$

By Axiom 1, $P[A^c B] \geq 0$, hich implies $P[A] \leq P[B]$. This proof uses Axioms 1 and 3.

Problem 1.4.1 Solution

Each question requests a conditional probability.

(a) Note that the probability a call is brief is

$$P[B] = P[H_0 B] + P[H_1 B] + P[H_2 B] = 0.6. \tag{1}$$

The probability a brief call will have no handoffs is

$$P[H_0|B] = \frac{P[H_0 B]}{P[B]} = \frac{0.4}{0.6} = \frac{2}{3}. \tag{2}$$

(b) The probability of one handoff is $P[H_1] = P[H_1 B] + P[H_1 L] = 0.2$. The probability that a call with one handoff will be long is

$$P[L|H_1] = \frac{P[H_1 L]}{P[H_1]} = \frac{0.1}{0.2} = \frac{1}{2}. \tag{3}$$

(c) The probability a call is long is $P[L] = 1 - P[B] = 0.4$. The probability that a long call will have one or more handoffs is

$$\begin{aligned} P[H_1 \cup H_2|L] &= \frac{P[H_1 L \cup H_2 L]}{P[L]} \\ &= \frac{P[H_1 L] + P[H_2 L]}{P[L]} = \frac{0.1 + 0.2}{0.4} = \frac{3}{4}. \end{aligned} \tag{4}$$

Problem 1.4.2 Solution

Let s_i denote the outcome that the roll is i. So, for $1 \leq i \leq 6$, $R_i = \{s_i\}$. Similarly, $G_j = \{s_{j+1}, \ldots, s_6\}$.

(a) Since $G_1 = \{s_2, s_3, s_4, s_5, s_6\}$ and all outcomes have probability 1/6, $P[G_1] = 5/6$. The event $R_3G_1 = \{s_3\}$ and $P[R_3G_1] = 1/6$ so that

$$P[R_3|G_1] = \frac{P[R_3G_1]}{P[G_1]} = \frac{1}{5}. \tag{1}$$

(b) The conditional probability that 6 is rolled given that the roll is greater than 3 is

$$P[R_6|G_3] = \frac{P[R_6G_3]}{P[G_3]} = \frac{P[s_6]}{P[s_4, s_5, s_6]} = \frac{1/6}{3/6}. \tag{2}$$

(c) The event E that the roll is even is $E = \{s_2, s_4, s_6\}$ and has probability 3/6. The joint probability of G_3 and E is

$$P[G_3E] = P[s_4, s_6] = 1/3. \tag{3}$$

The conditional probabilities of G_3 given E is

$$P[G_3|E] = \frac{P[G_3E]}{P[E]} = \frac{1/3}{1/2} = \frac{2}{3}. \tag{4}$$

(d) The conditional probability that the roll is even given that it's greater than 3 is

$$P[E|G_3] = \frac{P[EG_3]}{P[G_3]} = \frac{1/3}{1/2} = \frac{2}{3}. \tag{5}$$

Problem 1.4.3 Solution

Since the 2 of clubs is an even numbered card, $C_2 \subset E$ so that $P[C_2E] = P[C_2] = 1/3$. Since $P[E] = 2/3$,

$$P[C_2|E] = \frac{P[C_2E]}{P[E]} = \frac{1/3}{2/3} = 1/2. \tag{1}$$

The probability that an even numbered card is picked given that the 2 is picked is

$$P[E|C_2] = \frac{P[C_2E]}{P[C_2]} = \frac{1/3}{1/3} = 1. \tag{2}$$

Problem 1.4.4 Solution

Let A_i and B_i denote the events that the ith phone sold is an Apricot or a Banana respectively. Our goal is to find $P[B_1B_2]$, but since it is not clear where to start, we should plan on filling in the table

	A_2	B_2
A_1		
B_1		

This table has four unknowns: $P[A_1A_2]$, $P[A_1B_2]$, $P[B_1A_2]$, and $P[B_1B_2]$. We start knowing that

$$P[A_1A_2] + P[A_1B_2] + P[B_1A_2] + P[B_1B_2] = 1. \tag{1}$$

We still need three more equations to solve for the four unknowns. From "sales of Apricots and Bananas are equally likely," we know that $P[A_i] = P[B_i] = 1/2$ for $i = 1, 2$. This implies

$$P[A_1] = P[A_1A_2] + P[A_1B_2] = 1/2, \tag{2}$$

$$P[A_2] = P[A_1A_2] + P[B_1A_2] = 1/2. \tag{3}$$

The final equation comes from "given that the first phone sold is a Banana, the second phone is twice as likely to be a Banana," which implies $P[B_2|B_1] = 2\,P[A_2|B_1]$. Using Bayes' theorem, we have

$$\frac{P[B_1B_2]}{P[B_1]} = 2\frac{P[B_1A_2]}{P[B_1]} \qquad \Longrightarrow \qquad P[B_1A_2] = \frac{1}{2}P[B_1B_2]. \tag{4}$$

Replacing $P[B_1A_2]$ with $P[B_1B_2]/2$ in the the first three equations yields

$$P[A_1A_2] + P[A_1B_2] + \frac{3}{2}P[B_1B_2] = 1, \tag{5}$$

$$P[A_1A_2] + P[A_1B_2] = 1/2, \tag{6}$$

$$P[A_1A_2] + \frac{1}{2}P[B_1B_2] = 1/2. \tag{7}$$

Subtracting (6) from (5) yields $(3/2)\,P[B_1B_2] = 1/2$, or $P[B_1B_2] = 1/3$, which is the answer we are looking for.

At this point, if you are curious, we can solve for the rest of the probability table. From (4), we have $\text{P}[B_1A_2] = 1/6$ and from (7) we obtain $\text{P}[A_1A_2] = 1/3$. It then follows from (6) that $\text{P}[A_1B_2] = 1/6$. The probability table is

	A_2	B_2
A_1	1/3	1/6
B_1	1/6	1/3

.

Problem 1.4.5 Solution

The first generation consists of two plants each with genotype yg or gy. They are crossed to produce the following second generation genotypes, $S = \{yy, yg, gy, gg\}$. Each genotype is just as likely as any other so the probability of each genotype is consequently 1/4. A pea plant has yellow seeds if it possesses at least one dominant y gene. The set of pea plants with yellow seeds is

$$Y = \{yy, yg, gy\}. \tag{1}$$

So the probability of a pea plant with yellow seeds is

$$\text{P}[Y] = \text{P}[yy] + \text{P}[yg] + \text{P}[gy] = 3/4. \tag{2}$$

Problem 1.4.6 Solution

Define D as the event that a pea plant has two dominant y genes. To find the conditional probability of D given the event Y, corresponding to a plant having yellow seeds, we look to evaluate

$$\text{P}[D|Y] = \frac{\text{P}[DY]}{\text{P}[Y]}. \tag{1}$$

Note that $\text{P}[DY]$ is just the probability of the genotype yy. From Problem 1.4.5, we found that with respect to the color of the peas, the genotypes yy, yg, gy, and gg were all equally likely. This implies

$$\text{P}[DY] = \text{P}[yy] = 1/4 \qquad \text{P}[Y] = \text{P}[yy, gy, yg] = 3/4. \tag{2}$$

Thus, the conditional probability can be expressed as

$$\text{P}[D|Y] = \frac{\text{P}[DY]}{\text{P}[Y]} = \frac{1/4}{3/4} = 1/3. \tag{3}$$

Problem 1.4.7 Solution

The sample outcomes can be written ijk where the first card drawn is i, the second is j and the third is k. The sample space is

$$S = \{234, 243, 324, 342, 423, 432\}. \tag{1}$$

and each of the six outcomes has probability 1/6. The events E_1, E_2, E_3, O_1, O_2, O_3 are

$$E_1 = \{234, 243, 423, 432\}, \qquad O_1 = \{324, 342\}, \tag{2}$$
$$E_2 = \{243, 324, 342, 423\}, \qquad O_2 = \{234, 432\}, \tag{3}$$
$$E_3 = \{234, 324, 342, 432\}, \qquad O_3 = \{243, 423\}. \tag{4}$$

(a) The conditional probability the second card is even given that the first card is even is

$$P[E_2|E_1] = \frac{P[E_2E_1]}{P[E_1]} = \frac{P[243, 423]}{P[234, 243, 423, 432]} = \frac{2/6}{4/6} = 1/2. \tag{5}$$

(b) The conditional probability the first card is even given that the second card is even is

$$P[E_1|E_2] = \frac{P[E_1E_2]}{P[E_2]} = \frac{P[243, 423]}{P[243, 324, 342, 423]} = \frac{2/6}{4/6} = 1/2. \tag{6}$$

(c) The probability the first two cards are even given the third card is even is

$$P[E_1E_2|E_3] = \frac{P[E_1E_2E_3]}{P[E_3]} = 0. \tag{7}$$

(d) The conditional probabilities the second card is even given that the first card is odd is

$$P[E_2|O_1] = \frac{P[O_1E_2]}{P[O_1]} = \frac{P[O_1]}{P[O_1]} = 1. \tag{8}$$

(e) The conditional probability the second card is odd given that the first card is odd is

$$P[O_2|O_1] = \frac{P[O_1O_2]}{P[O_1]} = 0. \tag{9}$$

Problem 1.4.8 Solution

The problem statement yields the obvious facts that P[L] = 0.16 and P[H] = 0.10. The words "10% of the ticks that had either Lyme disease or HGE carried both diseases" can be written as

$$\mathrm{P}[LH|L \cup H] = 0.10. \tag{1}$$

(a) Since $LH \subset L \cup H$,

$$\mathrm{P}[LH|L \cup H] = \frac{\mathrm{P}[LH \cap (L \cup H)]}{\mathrm{P}[L \cup H]} = \frac{\mathrm{P}[LH]}{\mathrm{P}[L \cup H]} = 0.10. \tag{2}$$

Thus,

$$\mathrm{P}[LH] = 0.10\,\mathrm{P}[L \cup H] = 0.10\,(\mathrm{P}[L] + \mathrm{P}[H] - \mathrm{P}[LH]). \tag{3}$$

Since P[L] = 0.16 and P[H] = 0.10,

$$\mathrm{P}[LH] = \frac{0.10\,(0.16 + 0.10)}{1.1} = 0.0236. \tag{4}$$

(b) The conditional probability that a tick has HGE given that it has Lyme disease is

$$\mathrm{P}[H|L] = \frac{\mathrm{P}[LH]}{\mathrm{P}[L]} = \frac{0.0236}{0.16} = 0.1475. \tag{5}$$

Problem 1.5.1 Solution

From the table we look to add all the mutually exclusive events to find each probability.

(a) The probability that a caller makes no hand-offs is

$$\mathrm{P}[H_0] = \mathrm{P}[LH_0] + \mathrm{P}[BH_0] = 0.1 + 0.4 = 0.5. \tag{1}$$

(b) The probability that a call is brief is

$$\mathrm{P}[B] = \mathrm{P}[BH_0] + \mathrm{P}[BH_1] + \mathrm{P}[BH_2] = 0.4 + 0.1 + 0.1 = 0.6. \tag{2}$$

(c) The probability that a call is long or makes at least two hand-offs is

$$\begin{aligned}\mathrm{P}[L \cup H_2] &= \mathrm{P}[LH_0] + \mathrm{P}[LH_1] + \mathrm{P}[LH_2] + \mathrm{P}[BH_2] \\ &= 0.1 + 0.1 + 0.2 + 0.1 = 0.5.\end{aligned} \tag{3}$$

Problem 1.5.2 Solution

(a) From the given probability distribution of billed minutes, M, the probability that a call is billed for more than 3 minutes is

$$\begin{aligned} P[L] &= 1 - P[3 \text{ or fewer billed minutes}] \\ &= 1 - P[B_1] - P[B_2] - P[B_3] \\ &= 1 - \alpha - \alpha(1-\alpha) - \alpha(1-\alpha)^2 \\ &= (1-\alpha)^3 = 0.57. \end{aligned} \quad (1)$$

(b) The probability that a call will billed for 9 minutes or less is

$$P[9 \text{ minutes or less}] = \sum_{i=1}^{9} \alpha(1-\alpha)^{i-1} = 1 - (0.57)^3. \quad (2)$$

Problem 1.5.3 Solution

(a) For convenience, let $p_i = P[FH_i]$ and $q_i = P[VH_i]$. Using this shorthand, the six unknowns $p_0, p_1, p_2, q_0, q_1, q_2$ fill the table as

	H_0	H_1	H_2
F	p_0	p_1	p_2
V	q_0	q_1	q_2

. (1)

However, we are given a number of facts:

$$p_0 + q_0 = 1/3, \qquad p_1 + q_1 = 1/3, \quad (2)$$

$$p_2 + q_2 = 1/3, \qquad p_0 + p_1 + p_2 = 5/12. \quad (3)$$

Other facts, such as $q_0+q_1+q_2 = 7/12$, can be derived from these facts. Thus, we have four equations and six unknowns, choosing p_0 and p_1 will specify the other unknowns. Unfortunately, arbitrary choices for either p_0 or p_1 will lead to negative values for the other probabilities. In terms of p_0 and p_1, the other unknowns are

$$q_0 = 1/3 - p_0, \qquad p_2 = 5/12 - (p_0 + p_1), \tag{4}$$

$$q_1 = 1/3 - p_1, \qquad q_2 = p_0 + p_1 - 1/12. \tag{5}$$

Because the probabilities must be nonnegative, we see that

$$0 \le p_0 \le 1/3, \tag{6}$$

$$0 \le p_1 \le 1/3, \tag{7}$$

$$1/12 \le p_0 + p_1 \le 5/12. \tag{8}$$

Although there are an infinite number of solutions, three possible solutions are:

$$p_0 = 1/3, \qquad p_1 = 1/12, \qquad p_2 = 0, \tag{9}$$

$$q_0 = 0, \qquad q_1 = 1/4, \qquad q_2 = 1/3. \tag{10}$$

and

$$p_0 = 1/4, \qquad p_1 = 1/12, \qquad p_2 = 1/12, \tag{11}$$

$$q_0 = 1/12, \qquad q_1 = 3/12, \qquad q_2 = 3/12. \tag{12}$$

and

$$p_0 = 0, \qquad p_1 = 1/12, \qquad p_2 = 1/3, \tag{13}$$

$$q_0 = 1/3, \qquad q_1 = 3/12, \qquad q_2 = 0. \tag{14}$$

(b) In terms of the p_i, q_i notation, the new facts are $p_0 = 1/4$ and $q_1 = 1/6$. These extra facts uniquely specify the probabilities. In this case,

$$p_0 = 1/4, \qquad p_1 = 1/6, \qquad p_2 = 0, \tag{15}$$

$$q_0 = 1/12, \qquad q_1 = 1/6, \qquad q_2 = 1/3. \tag{16}$$

Problem 1.6.1 Solution

This problems asks whether A and B can be independent events yet satisfy $A = B$? By definition, events A and B are independent if and only if $\mathrm{P}[AB] = \mathrm{P}[A]\,\mathrm{P}[B]$. We can see that if $A = B$, that is they are the same set, then

$$\mathrm{P}[AB] = \mathrm{P}[AA] = \mathrm{P}[A] = \mathrm{P}[B]. \tag{1}$$

Thus, for A and B to be the same set and also independent,

$$\mathrm{P}[A] = \mathrm{P}[AB] = \mathrm{P}[A]\,\mathrm{P}[B] = (\mathrm{P}[A])^2. \tag{2}$$

There are two ways that this requirement can be satisfied:

- $\mathrm{P}[A] = 1$ implying $A = B = S$.
- $\mathrm{P}[A] = 0$ implying $A = B = \phi$.

Problem 1.6.2 Solution

From the problem statement, we learn three facts:

$$\begin{aligned} \mathrm{P}[AB] &= 0 && \text{(since A and B are mutually exclusive)} && (1)\\ \mathrm{P}[AB] &= \mathrm{P}[A]\,\mathrm{P}[B] && \text{(since A and B are independent)} && (2)\\ \mathrm{P}[A] &= \mathrm{P}[B] && \text{(since A and B are equiprobable)} && (3) \end{aligned}$$

Applying these facts in the given order, we see that

$$0 = \mathrm{P}[AB] = \mathrm{P}[A]\,\mathrm{P}[B] = (\mathrm{P}[A])^2. \tag{4}$$

It follows that $\mathrm{P}[A] = 0$.

Problem 1.6.3 Solution

Let A_i and B_i denote the events that the ith phone sold is an Apricot or a Banana respectively. The works "each phone sold is twice as likely to be an Apricot than a Banana" tells us that

$$\mathrm{P}[A_i] = 2\,\mathrm{P}[B_i]. \tag{1}$$

However, since each phone sold is either an Apricot or a Banana, A_i and B_i are a partition and

$$\mathrm{P}[A_i] + \mathrm{P}[B_i] = 1. \tag{2}$$

Combining these equations, we have $\mathrm{P}[A_i] = 2/3$ and $\mathrm{P}[B_i] = 1/3$. The probability that two phones sold are the same is

$$\mathrm{P}[A_1A_2 \cup B_1B_2] = \mathrm{P}[A_1A_2] + \mathrm{P}[B_1B_2]. \tag{3}$$

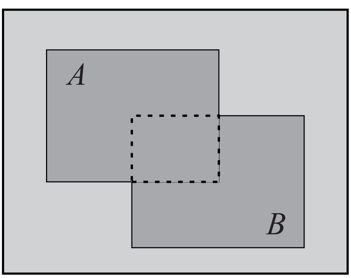
Since "each phone sale is independent,"

$$\mathrm{P}[A_1A_2] = \mathrm{P}[A_1]\,\mathrm{P}[A_2] = \frac{4}{9}, \qquad \mathrm{P}[B_1B_2] = \mathrm{P}[B_1]\,\mathrm{P}[B_2] = \frac{1}{9}. \tag{4}$$

Thus the probability that two phones sold are the same is

$$\mathrm{P}[A_1A_2 \cup B_1B_2] = \mathrm{P}[A_1A_2] + \mathrm{P}[B_1B_2] = \frac{4}{9} + \frac{1}{9} = \frac{5}{9}. \tag{5}$$

Problem 1.6.4 Solution



In the Venn diagram, assume the sample space has area 1 corresponding to probability 1. As drawn, both A and B have area 1/4 so that $\mathrm{P}[A] = \mathrm{P}[B] = 1/4$. Moreover, the intersection AB has area 1/16 and covers 1/4 of A and 1/4 of B. That is, A and B are independent since

$$\mathrm{P}[AB] = \mathrm{P}[A]\,\mathrm{P}[B]. \tag{1}$$

Problem 1.6.5 Solution

(a) Since A and B are mutually exclusive, $\mathrm{P}[A \cap B] = 0$. Since $\mathrm{P}[A \cap B] = 0$,

$$\mathrm{P}[A \cup B] = \mathrm{P}[A] + \mathrm{P}[B] - \mathrm{P}[A \cap B] = 3/8. \tag{1}$$

A Venn diagram should convince you that $A \subset B^c$ so that $A \cap B^c = A$. This implies

$$\mathrm{P}[A \cap B^c] = \mathrm{P}[A] = 1/4. \tag{2}$$

It also follows that $\mathrm{P}[A \cup B^c] = \mathrm{P}[B^c] = 1 - 1/8 = 7/8$.

(b) Events A and B are dependent since $\mathrm{P}[AB] \neq \mathrm{P}[A]\,\mathrm{P}[B]$.

Problem 1.6.6 Solution

(a) Since C and D are independent,

$$\mathrm{P}[C \cap D] = \mathrm{P}[C]\,\mathrm{P}[D] = 15/64. \tag{1}$$

The next few items are a little trickier. From Venn diagrams, we see

$$\mathrm{P}[C \cap D^c] = \mathrm{P}[C] - \mathrm{P}[C \cap D] = 5/8 - 15/64 = 25/64. \tag{2}$$

It follows that

$$\mathrm{P}[C \cup D^c] = \mathrm{P}[C] + \mathrm{P}[D^c] - \mathrm{P}[C \cap D^c] \tag{3}$$

$$= 5/8 + (1 - 3/8) - 25/64 = 55/64. \tag{4}$$

Using DeMorgan's law, we have

$$\mathrm{P}[C^c \cap D^c] = \mathrm{P}[(C \cup D)^c] = 1 - \mathrm{P}[C \cup D] = 15/64. \tag{5}$$

(b) Since $\mathrm{P}[C^cD^c] = \mathrm{P}[C^c]\,\mathrm{P}[D^c]$, C^c and D^c are independent.

Problem 1.6.7 Solution

(a) Since $A \cap B = \emptyset$, $\mathrm{P}[A \cap B] = 0$. To find $\mathrm{P}[B]$, we can write

$$\mathrm{P}[A \cup B] = \mathrm{P}[A] + \mathrm{P}[B] - \mathrm{P}[A \cap B] \tag{1}$$

or

$$5/8 = 3/8 + \mathrm{P}[B] - 0. \tag{2}$$

Thus, $\mathrm{P}[B] = 1/4$. Since A is a subset of B^c, $\mathrm{P}[A \cap B^c] = \mathrm{P}[A] = 3/8$. Furthermore, since A is a subset of B^c, $\mathrm{P}[A \cup B^c] = \mathrm{P}[B^c] = 3/4$.

(b) The events A and B are dependent because

$$\mathrm{P}[AB] = 0 \neq 3/32 = \mathrm{P}[A]\,\mathrm{P}[B]. \tag{3}$$

Problem 1.6.8 Solution

(a) Since C and D are independent $\mathrm{P}[CD] = \mathrm{P}[C]\,\mathrm{P}[D]$. So

$$\mathrm{P}[D] = \frac{\mathrm{P}[CD]}{\mathrm{P}[C]} = \frac{1/3}{1/2} = 2/3. \tag{1}$$

In addition, $\mathrm{P}[C \cap D^c] = \mathrm{P}[C] - \mathrm{P}[C \cap D] = 1/2 - 1/3 = 1/6$. To find $\mathrm{P}[C^c \cap D^c]$, we first observe that

$$\mathrm{P}[C \cup D] = \mathrm{P}[C] + \mathrm{P}[D] - \mathrm{P}[C \cap D] = 1/2 + 2/3 - 1/3 = 5/6. \tag{2}$$

By De Morgan's Law, $C^c \cap D^c = (C \cup D)^c$. This implies

$$\mathrm{P}[C^c \cap D^c] = \mathrm{P}[(C \cup D)^c] = 1 - \mathrm{P}[C \cup D] = 1/6. \tag{3}$$

Note that a second way to find $\mathrm{P}[C^c \cap D^c]$ is to use the fact that if C and D are independent, then C^c and D^c are independent. Thus

$$\mathrm{P}[C^c \cap D^c] = \mathrm{P}[C^c]\,\mathrm{P}[D^c] = (1 - \mathrm{P}[C])(1 - \mathrm{P}[D]) = 1/6. \tag{4}$$

Finally, since C and D are independent events, $\mathrm{P}[C|D] = \mathrm{P}[C] = 1/2$.

(b) Note that we found $\mathrm{P}[C \cup D] = 5/6$. We can also use the earlier results to show

$$\mathrm{P}[C \cup D^c] = \mathrm{P}[C] + \mathrm{P}[D] - \mathrm{P}[C \cap D^c] \tag{5}$$

$$= 1/2 + (1 - 2/3) - 1/6 = 2/3. \tag{6}$$

(c) By Definition 1.6, events C and D^c are independent because

$$\mathrm{P}[C \cap D^c] = 1/6 = (1/2)(1/3) = \mathrm{P}[C]\,\mathrm{P}[D^c]. \tag{7}$$

Problem 1.6.9 Solution

For a sample space $S = \{1, 2, 3, 4\}$ with equiprobable outcomes, consider the events

$$A_1 = \{1, 2\} \qquad A_2 = \{2, 3\} \qquad A_3 = \{3, 1\}. \tag{1}$$

Each event A_i has probability 1/2. Moreover, each pair of events is independent since

$$P[A_1A_2] = P[A_2A_3] = P[A_3A_1] = 1/4. \tag{2}$$

However, the three events A_1, A_2, A_3 are not independent since

$$P[A_1A_2A_3] = 0 \neq P[A_1] P[A_2] P[A_3]. \tag{3}$$

Problem 1.6.10 Solution

There are 16 distinct equally likely outcomes for the second generation of pea plants based on a first generation of $\{rwyg, rwgy, wryg, wrgy\}$. These are:

$$\begin{array}{cccc} rryy & rryg & rrgy & rrgg \\ rwyy & rwyg & rwgy & rwgg \\ wryy & wryg & wrgy & wrgg \\ wwyy & wwyg & wwgy & wwgg \end{array}$$

A plant has yellow seeds, that is event Y occurs, if a plant has at least one dominant y gene. Except for the four outcomes with a pair of recessive g genes, the remaining 12 outcomes have yellow seeds. From the above, we see that

$$P[Y] = 12/16 = 3/4 \tag{1}$$

and

$$P[R] = 12/16 = 3/4. \tag{2}$$

To find the conditional probabilities $P[R|Y]$ and $P[Y|R]$, we first must find $P[RY]$. Note that RY, the event that a plant has rounded yellow seeds, is the set of outcomes

$$RY = \{rryy, rryg, rrgy, rwyy, rwyg, rwgy, wryy, wryg, wrgy\}. \tag{3}$$

Since $P[RY] = 9/16$,

$$P[Y|R] = \frac{P[RY]}{P[R]} = \frac{9/16}{3/4} = 3/4 \tag{4}$$

and

$$P[R|Y] = \frac{P[RY]}{P[Y]} = \frac{9/16}{3/4} = 3/4. \tag{5}$$

Thus $P[R|Y] = P[R]$ and $P[Y|R] = P[Y]$ and R and Y are independent events. There are four visibly different pea plants, corresponding to whether the peas are round (R) or not (R^c), or yellow (Y) or not (Y^c). These four visible events have probabilities

$$P[RY] = 9/16 \qquad P[RY^c] = 3/16, \tag{6}$$

$$P[R^cY] = 3/16 \qquad P[R^cY^c] = 1/16. \tag{7}$$

Problem 1.6.11 Solution

(a) For any events A and B, we can write the law of total probability in the form of

$$P[A] = P[AB] + P[AB^c]. \tag{1}$$

Since A and B are independent, $P[AB] = P[A] P[B]$. This implies

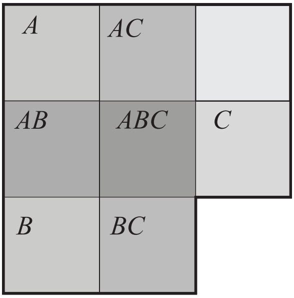
$$P[AB^c] = P[A] - P[A] P[B] = P[A] (1 - P[B]) = P[A] P[B^c]. \tag{2}$$

Thus A and B^c are independent.

(b) Proving that A^c and B are independent is not really necessary. Since A and B are arbitrary labels, it is really the same claim as in part (a). That is, simply reversing the labels of A and B proves the claim. Alternatively, one can construct exactly the same proof as in part (a) with the labels A and B reversed.

(c) To prove that A^c and B^c are independent, we apply the result of part (a) to the sets A and B^c. Since we know from part (a) that A and B^c are independent, part (b) says that A^c and B^c are independent.

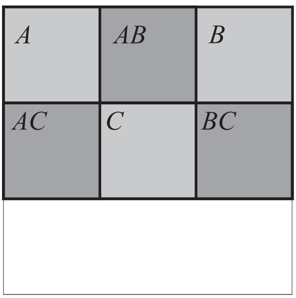
Problem 1.6.12 Solution



In the Venn diagram at right, assume the sample space has area 1 corresponding to probability 1. As drawn, A, B, and C each have area 1/2 and thus probability 1/2. Moreover, the three way intersection ABC has probability 1/8. Thus A, B, and C are mutually independent since

$$P[ABC] = P[A] P[B] P[C]. \qquad (1)$$

Problem 1.6.13 Solution



In the Venn diagram at right, assume the sample space has area 1 corresponding to probability 1. As drawn, A, B, and C each have area 1/3 and thus probability 1/3. The three way intersection ABC has zero probability, implying A, B, and C are not mutually independent since

$$P[ABC] = 0 \neq P[A] P[B] P[C]. \qquad (1)$$

However, AB, BC, and AC each has area 1/9. As a result, each pair of events is independent since

$$P[AB] = P[A] P[B], \quad P[BC] = P[B] P[C], \quad P[AC] = P[A] P[C]. \qquad (2)$$

Problem 1.7.1 Solution

We can generate the 200×1 vector $\mathbf{T}$, denoted `T` in MATLAB, via the command

```
T=50+ceil(50*rand(200,1))
```

Keep in mind that `50*rand(200,1)` produces a 200×1 vector of random numbers, each in the interval $(0, 50)$. Applying the ceiling function converts these random numbers to rndom integers in the set $\{1, 2, \ldots, 50\}$. Finally, we add 50 to produce random numbers between 51 and 100.