Chapter 1: Logic and Proof

Exercise 1.1.2 Refer to Figure 1.1.1. The coordinates of the midpoints, M_1 and M_2, are $\left(\frac{c}{2}, \frac{d}{2}\right)$ and $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$, respectively. Thus the slope of $\overline{M_1M_2}$ is

$$\frac{\dfrac{b+d}{2} - \dfrac{d}{2}}{\dfrac{a+c}{2} - \dfrac{c}{2}} = \frac{a}{b},$$

the same as the slope of the segment joining $(0,0)$ with (a,b).

Exercise 1.1.6 The first occurrence is 90, 91, 92, 93, 94, 95, 96.

Exercise 1.2.2 (a) True (b) False (c) True (d) False

Exercise 1.3.3 (a) $3^2 + 4^2 = 5^2$ and $7^2 + 12^2 = 15^2$. (False)
(b) $3^2 + 4^2 = 5^2$ or $7^2 + 12^2 = 15^2$. (True)

Exercise 1.3.5(a) $7^2 + 24^2 \neq 25^2$.
(b) $\dfrac{1}{2} + \dfrac{2}{3} + \dfrac{3}{4} < \dfrac{1+2+3}{2+3+4}$.
(c) In 1983, the ice went out of Lake Minnetonka on or after April 18.
(d) Doug cannot solve the equation $x^2 - 7x - 18 = 0$.
(e) Ambrose scored at least 90 on the last exam.

Exercise 1.3.7 The last column in each truth table is shown.

(a)	(b)	(c)	(d)	(e)
F	T	T	T	T
T	T	F	T	T
F	T	F	T	T
F	T	T	T	T

Exercise 1.3.10 (a)

P	$\neg P$	$\neg(\neg P)$
T	F	T
F	T	F

(b)

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

(c)

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Exercise 1.3.11 (a) The function f is either not decreasing on $(-\infty, 0]$ or not increasing on $[0, \infty)$.
(b) The number 0 is in the domain of f and $\lim_{x \to 0} f(x) = f(0)$.

Exercise 1.3.12 (a) $\neg C\left(\tan, \frac{\pi}{2}\right) \wedge \neg C\left(\sec, \frac{\pi}{2}\right)$.
(b) $(a > 0) \vee \neg D(\ln, a)$.
(c) $C(|x|, 0) \wedge \neg D(|x|, 0)$.

Exercise 1.3.13

P	Q	$P \oplus Q$	$P \vee Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$(P \vee Q) \wedge \neg(P \wedge Q)$
T	T	F	T	T	F	F
T	F	T	T	F	T	T
F	T	T	T	F	T	T
F	F	F	F	F	T	F

Exercise 1.3.14

P	Q	R	$Q \oplus R$	$P \oplus (Q \oplus R)$	$P \oplus Q$	$(P \oplus Q) \oplus R$
T	T	T	F	T	F	T
T	T	F	T	F	F	F
T	F	T	T	F	T	F
T	F	F	F	T	T	T
F	T	T	F	F	T	F
F	T	F	T	T	T	T
F	F	T	T	T	F	T
F	F	F	F	F	F	F

Note that the fifth and last columns are identical.

Exercise 1.4.3

P	Q	$\neg Q$	$P \Rightarrow Q$	$\neg(P \Rightarrow Q)$	$P \wedge \neg Q$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

Exercise 1.4.7 (a) Converse: If you live in Minnesota, then you live in Minneapolis.
Contrapositive: If you don't live in Minnesota, then you don't live in Minneapolis.
(b) Converse: If n^2 is an even integer, then n is an even integer.
Contrapositive: If n^2 is not an even integer, then n is not an even integer.
(c) Converse: If f is continuous at $x = a$, then f is differentiable at $x = a$.
Contrapositive: If f is not continuous at $x = a$, then f is not differentiable at $x = a$.
(d) Converse: If $x^p + 1$ is not factorable, then p is not prime.
Contrapositive: If $x^p + 1$ is factorable, then p is prime.
(e) Converse: If you don't live in Syracuse, then you live in Minnesota.
Contrapositive: If you live in Syracuse, then you don't live in Minnesota.

Exercise 1.4.9

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$	$P \Leftrightarrow Q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Exercise 1.4.10

P	Q	R	$Q \Rightarrow R$	U	$P \Rightarrow Q$	V	$V \Rightarrow U$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	T
T	F	T	T	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	T	T	T
F	T	F	F	T	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	F	T

Note that U is not equivalent to V, so $\Rightarrow$ is not associative. Also note that $V \Rightarrow U$ is true, but the sixth line shows that $U \Rightarrow V$ is false.

Exercise 1.4.11

P	Q	R	$P \Leftrightarrow Q$	$Q \Leftrightarrow P$	$Q \Leftrightarrow R$	$(P \Leftrightarrow Q) \Leftrightarrow R$	$P \Leftrightarrow (Q \Leftrightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	F	F
T	F	T	F	F	F	F	F
T	F	F	F	F	T	T	T
F	T	T	F	F	T	F	F
F	T	F	F	F	F	T	T
F	F	T	T	T	F	T	T
F	F	F	T	T	T	F	F

Exercise 1.4.12 (a) Let $P \Rightarrow Q$ be a conditional statement. Then $\neg Q \Rightarrow \neg P$ is its contrapositive. The contrapositive of $\neg Q \Rightarrow \neg P$ is $\neg(\neg P) \Rightarrow \neg(\neg Q)$ which is equivalent to $P \Rightarrow Q$ by Theorem 1.3.9(i).
(b) The converse of $Q \Rightarrow P$ is $P \Rightarrow Q$, the original conditional statement.
(c) Let $P \Rightarrow Q$ be a conditional statement. The contrapositive of $Q \Rightarrow P$ is $\neg P \Rightarrow \neg Q$. The converse of $\neg Q \Rightarrow \neg P$ is $\neg P \Rightarrow \neg Q$.
(d) The conditional $P \Rightarrow Q$ is false only when P is true and Q is false. In this case, $Q \Rightarrow P$ is true.

Exercise 1.4.14 (a)

P	Q	R	$P \Leftrightarrow Q$	$R \wedge P$	$R \wedge Q$	$(R \wedge P) \Leftrightarrow (R \wedge Q)$	$(P \Leftrightarrow Q) \Rightarrow ((R \wedge P) \Leftrightarrow (R \wedge Q))$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	T
T	F	T	F	T	F	F	T
T	F	F	F	F	F	T	T
F	T	T	F	F	T	F	T
F	T	F	F	F	F	T	T
F	F	T	T	F	F	T	T
F	F	F	T	F	F	T	T

(b)

P	Q	R	$P \Leftrightarrow Q$	$R \vee P$	$R \vee Q$	$(R \vee P) \Leftrightarrow (R \vee Q)$	$(P \Leftrightarrow Q) \Rightarrow ((R \vee P) \Leftrightarrow (R \vee Q))$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	F	F	T
F	T	T	F	T	T	T	T
F	T	F	F	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	F	F	T	T

Exercise 1.4.15 (a) Proposition 1.3.9(i); (b) Proposition 1.3.9(ii); (c) Proposition 1.3.9(iii); (d) Proposition 1.3.9(iv); (e) Proposition 1.3.9(v); (f) Proposition 1.4.2; (g) Theorem 1.4.6.

Exercise 1.4.16 (a) Neither; (b) Tautology; (c) Contradiction (see Proposition 1.4.2); (d) Neither; (e) Tautology; (f) Tautology; (g) Tautology; (h) Neither.

Exercise 1.4.17

P	Q	$P \Rightarrow Q$	$(P \Rightarrow Q) \wedge P$	$((P \Rightarrow Q) \wedge P) \Rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Exercise 1.5.4 ($\Rightarrow$) Assume $|a| \leq b$. If $a \geq 0$, then we have $|a| = a$ and so $a \leq b$. Since $b \geq 0$, we also have that $-b \leq 0 \leq a$. Thus $-b \leq a \leq b$. If $a < 0$, then $|a| = -a$ and so $-a \leq b$, or $a \geq -b$. We also have, because $a < 0$, that $a \leq b$. And, in this case, too, we have $-b \leq a \leq b$.
($\Leftarrow$) Assume $-b \leq a \leq b$. If $a \geq 0$, then we have $|a| = a$ and thus $|a| \leq b$. If $a < 0$, then $|a| = -a$. Since $-b \leq a$, this implies that $b \geq -a$. Thus $b \geq |a|$.

Exercise 1.5.7 The contrapositive of the statement is

$$x + y \leq \frac{4xy}{x+y} \quad \Rightarrow \quad x = y.$$

Assume that $x + y \leq \dfrac{4xy}{x+y}$. This means, since $x + y > 0$, that $(x+y)^2 \leq 4xy$. This implies that $x^2 - 2xy + y^2 \leq 0$. That is, $(x-y)^2 \leq 0$. Then, since the square of a real number is at least zero, we must have that $(x-y)^2 = 0$, which implies that $x - y = 0$, or that $x = y$.

Exercise 1.5.9

E	R	$\neg E$	$\neg R$	$R \wedge \neg R$	$\neg E \Rightarrow (R \wedge \neg R)$	$[\neg E \Rightarrow (R \wedge \neg R)] \Rightarrow E$
T	T	F	F	F	T	T
T	F	F	T	F	T	T
F	T	T	F	F	F	T
F	F	T	T	F	F	T

Exercise 1.5.11 Let m be a positive integer with an odd divisor greater than 1. Suppose that $\log_2 m$ is an integer. That is, $\log_2 m = n$ for some integer n. By the definition of logarithm, $2^n = m$. This means that m is a power of 2, and thus m has no odd divisors greater than 1. This contradicts our hypothesis. Thus $\log_2 m$ is not an integer.

Exercise 1.6.7 (a) $(\exists x)\,[P(x) \wedge (\neg Q(x) \vee \neg R(x))]$. (b) $(\forall y)\,[\neg P(y) \wedge (\exists x)\,[Q(x) \wedge R(x)]]$.
(c) $(\exists x)(\forall y)(\exists z)\,[(P(x,y) \wedge \neg Q(y,z)) \vee (Q(y,z) \wedge \neg P(x,y))]$.

Exercise 1.6.11 (a) T; (b) F; (c) F; (d) F.

Exercise 1.6.12 (a) There exists an even number ≥ 4 that cannot be written as the sum of two prime numbers. *Also,* there exists an even number ≥ 4 that if written as a sum of two natural numbers, then at least one of them is not prime.
(b) All differentiable functions are unbounded.

Exercise 1.6.13 (a) $(\forall \ell)\neg P(\ell, \ell)$.
(b) $(\forall \ell_1, \ell_2)\,[(\ell_1 \neq \ell_2 \wedge P(\ell_1, \ell_2)) \Rightarrow (\forall p)\,(\neg J(p, \ell_1) \vee \neg J(p, \ell_2))]$.
(c) $(\forall \ell)(\exists p_1, p_2)\,[p_1 \neq p_2 \wedge J(p_1, \ell) \wedge J(p_2, \ell)]$.

Exercise 1.7.1 (a)(i) Either n is not divisible by 2 or n is not divisible by 3. (ii) If n is divisible by 2 or 3 and not divisible by both 2 and 3, then n is not divisible by 6.
(b) $R \Leftrightarrow (P \wedge Q)$.
(c) $R \Rightarrow [(\neg P \wedge \neg Q) \vee (P \wedge Q)]$.

Exercise 1.7.2 (a) If two sets have something in common, then they are equal.
(b) That a given number is a square does not imply that the number is at least 1000.

Exercise 1.7.3 (a) If $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} \neq \lim_{x\to a} \frac{f'(x)}{g'(x)}$, then either f or g does not have a continuous derivative, or $g'(x) = 0$ for some x, or $\lim_{x\to a} f(x) \neq 0$, or $\lim_{x\to a} g(x) \neq 0$.
(b) If $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, then f and g have continuous derivatives in an open interval containing a and $g'(x) \neq 0$ in this interval and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$.
(c) Functions f and g have continuous derivatives in an open interval containing a and $g'(x) \neq 0$ in this interval and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ and $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} \neq \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

Exercise 1.7.4 Denote the $n+1$ couples, including the host and hostess, by (A_i, B_i) for $i = 0, 1, \ldots, n$. The numbers told to the hostess must be the $2n+1$ integers $0, 1, 2, \cdots, 2n$. Let $f(A_i)$ or $f(B_i)$ be the number of handshakes done by A_i or B_i, respectively.

Somebody, say A_0, shakes $2n$ hands, that is, $f(A_0) = 2n$. So A_0 shakes hands with everybody (else) except B_0. Since $f(X) = 0$ for some person X, then X doesn't shake hands even with A_0. Therefore $X = B_0$, that is, $f(B_0) = 0$. Somebody else, say A_1, shakes hands with $2n-1$ people: everybody but his/her spouse B_1 and the timid B_0. Now $f(Y) = 1$ for some person Y. Hence Y shakes hands only with the gregarious A_0 (and not with A_1). Hence $Y = B_1$, and $f(B_1) = 1$. Next suppose that $f(A_2) = 2n-2$, and so A_2 does not shake hands with B_0, B_1, and B_2. If $f(Z) = 2$, then Z must have shaken hands only with A_0 and A_1 (and not with A_2). Hence $Z = B_2$.

Continuing in this way, we see that $f(A_i) + f(B_i) = 2n$ for each $i = 0, 1, \cdots, 2n$. Thus, the handshake numbers for the n visiting couples are paired off, leaving only the number n. It follows that the host himself must have shaken n hands. By Theorem 1.1.4, the hostess must have shaken hands with the same number of people as *somebody* else, and the somebody can only be her own spouse. $\square$

If one knows the formula $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$ (which we establish later as Proposition 4.2.2), there is another way to conclude the proof once it is established that $f(A_i) + f(B_i) = 2n$ for each $i = 0, 1, \cdots, 2n$. Since there are $n+1$ couples, the sum the numbers of handshakes done is $2n(n+1)$. But this sum is also equal to $h + \sum_{i=0}^{2n} i$, where h is the number of hands shaken by the hostess. Setting these two quantities equal and using the formula mentioned gives

$$2n(n+1) = h + \frac{2n(2n+1)}{2},$$

which yields $h = n$.

Exercise 1.7.5 (a)

P	Q	R	$\neg R$	$Q \vee R$	$P \Rightarrow (Q \vee R)$	$P \wedge \neg R$	$(P \wedge \neg R) \Rightarrow Q$
T	T	T	F	T	T	F	T
T	T	F	T	T	T	T	T
T	F	T	F	T	T	F	T
T	F	F	T	F	F	T	F
F	T	T	F	T	T	F	T
F	T	F	T	T	T	F	T
F	F	T	F	T	T	F	T
F	F	F	T	F	T	F	T

(b)

P	Q	$P \oplus Q$	$P \Leftrightarrow Q$	$\neg(P \Leftrightarrow Q)$
T	T	F	T	F
T	F	T	F	T
F	T	T	F	T
F	F	F	T	F

(c)

P	Q	R	$Q \vee R$	$P \Rightarrow (Q \vee R)$	$P \Rightarrow Q$	$P \Rightarrow R$	$(P \Rightarrow Q) \vee (P \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T

(d)

P	Q	R	$Q \wedge R$	$P \Rightarrow (Q \wedge R)$	$P \Rightarrow Q$	$P \Rightarrow R$	$(P \Rightarrow Q) \wedge (P \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	F	F	T	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

(e)

P	Q	R	$P \vee Q$	$(P \vee Q) \Rightarrow R$	$P \Rightarrow R$	$Q \Rightarrow R$	$(P \Rightarrow R) \wedge (Q \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

(f)

P	Q	R	$P \wedge Q$	$(P \wedge Q) \Rightarrow R$	$P \Rightarrow R$	$Q \Rightarrow R$	$(P \Rightarrow R) \vee (Q \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	F	T	T
F	T	T	F	T	T	T	T
F	T	F	F	T	T	F	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Exercise 1.7.6 (a)

P	Q	$\neg Q$	$Q \wedge P$	$\neg Q \vee (P \wedge Q)$	$P \Diamond Q$
T	T	F	T	T	T
T	F	T	F	T	T
F	T	F	F	F	F
F	F	T	F	T	T

(b) $\neg Q \vee (P \wedge Q) \iff (\neg Q \vee P) \wedge (\neg Q \vee Q) \iff \neg Q \vee P$.
(c) No.

P	Q	$P \Diamond Q$	$Q \Diamond P$
T	T	T	T
T	F	T	F
F	T	F	T
F	F	T	T

(d) No.

P	Q	R	$Q \Diamond R$	$P \Diamond (Q \Diamond R)$	$P \Diamond Q$	$(P \Diamond Q) \Diamond R$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	F	T	T	T
T	F	F	T	T	T	T
F	T	T	T	F	F	F
F	T	F	T	F	F	T
F	F	T	F	T	T	T
F	F	F	T	F	T	T

Exercise 1.7.7

P	Q	$\neg P$	$\neg Q$	$P \Rightarrow Q$	$(P \Rightarrow Q) \wedge \neg Q$	$((P \Rightarrow Q) \wedge \neg Q) \Rightarrow \neg P$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

The statement is a tautology.

Exercise 1.7.8 (a)

P	Q	R	$P \Rightarrow Q$	$Q \Rightarrow R$	$(P \Rightarrow Q) \wedge (Q \Rightarrow R)$	$P \Rightarrow R$	$[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

(b)

P	Q	R	$P \Leftrightarrow Q$	$Q \Leftrightarrow R$	$(P \Leftrightarrow Q) \wedge (Q \Leftrightarrow R)$	$P \Leftrightarrow R$	$[(P \Leftrightarrow Q) \wedge (Q \Leftrightarrow R)] \Rightarrow (P \Leftrightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	F	F	T	T
T	F	F	F	T	F	F	T
F	T	T	F	T	F	F	T
F	T	F	F	F	F	T	T
F	F	T	T	F	F	F	T
F	F	F	T	T	T	T	T

Exercise 1.7.9 Consider that should Professor Y's result be false, the implication"if Y's result, then anything" is true and if Professor X's result is true and Y's result is false then the implication "if X's result, then Y's result" is false. Ouch.

Exercise 1.7.10 Suppose $z = \overline{z}$, that is, $a + bi = a - bi$. Then $bi = -bi$ and consequently $2bi = 0$ and thus $b = 0$. Therefore b is real.

Exercise 1.7.11 Write $a = a - b + b$. By the Triangle Inequality, we have $|a| = |a - b + b| \leq |a - b| + |b|$ so that

$$|a| - |b| \leq |a - b|. \qquad (\star)$$

Now write $b = b - a + a$. Similarly we have $|b| - |a| \leq |b - a|$. This implies that

$$|a| - |b| \geq -|b - a|. \qquad (\star\star)$$

Note that $|a - b| = |b - a|$. Combine the inequalities $(\star)$ and $(\star\star)$ to get that

$$-|a - b| \leq |a| - |b| \leq |a - b|.$$

By Exercise 1.5.4, $\big||a| - |b|\big| \leq |a - b|$.

Exercise 1.7.12 $(\forall x)\,[\neg P(x) \vee (\exists y)\,[P(y) \wedge (x \neq y)]]$.

Exercise 1.7.13 (a) $(\forall p)(\exists \ell_1, \ell_2)\,[(\ell_1 \neq \ell_2) \wedge J(p, \ell_1) \wedge J(p, \ell_2)]$.
(b) $(\forall p_1, p_2)\,[(p_1 \neq p_2) \Rightarrow (\exists! \ell)\,(J(p_1, \ell) \wedge J(p_2, \ell))]$.
(c) $(\forall p)(\forall \ell)\,[\neg J(p, \ell) \Rightarrow (\exists m)\,(J(p, m) \wedge P(\ell, m))]$.
(d) $(\forall \ell_1, \ell_2)\,\big[(\ell_1 \neq \ell_2) \Rightarrow \big((\exists! p)\,[J(p, \ell_1) \wedge J(p, \ell_2)] \vee P(\ell_1, \ell_2)\big)\big]$.
(e) $(\forall \ell_1, \ell_2, \ell_3)\,\big[(\ell_1 \neq \ell_2) \Rightarrow \big((P(\ell_1, \ell_3) \wedge P(\ell_2, \ell_3)) \Rightarrow P(\ell_1, \ell_2)\big)\big]$.

Exercise 1.7.14 (a) $(\forall a, b)\,\big[\big(a \neq b \wedge E(a) \wedge E(b)\big) \Rightarrow \neg R(a, b)\big]$.
(b) $(\forall a, b)\,\big[\big(a \neq b \wedge \neg E(a) \wedge \neg E(b)\big) \Rightarrow R(a, b)\big]$.
(c) $(\forall a)\,[E(a) \Leftrightarrow \neg R(a, 2)]$.
(d) $(\forall a, b, c)\,\big[\big(D(a, b) \wedge D(b, c)\big) \Rightarrow D(a, c)\big]$.
(e) $(\forall a, b, c)\,\big[D(a, bc) \Rightarrow \big(D(a, b) \vee D(a, c)\big)\big]$.
(f) $(\forall a, b, c)\,\big[\big(R(a, b) \wedge D(a, bc)\big) \Rightarrow D(a, c)\big]$.
(g) $(\forall z)\,[R(z, 2) \Rightarrow D(8, z^2 - 1)]$.

Exercise 1.7.15 (a) $(\exists n)\,[L(n, 3)]$.
(b) $(\forall n)\,[L(n, 4)]$.
(c) $(\exists n)(\forall k)\,[L(n, k)]$.
(d) $(\forall p)\,\big[(\forall n)\big(\big((n \neq p) \wedge (n \neq 1) \wedge (n > 0)\big) \Rightarrow \neg D(n, p)\big) \Rightarrow \big(D(4, p - 1) \Rightarrow L(p, 2)\big)\big]$.
(e) $(\forall k)(\exists n)\,[L(n, k)]$.
(f) $(\forall m, n)\,\big[\big(R(m, n) \wedge L(m, 2) \wedge L(n, 2)\big) \Rightarrow L(mn, 4)\big]$.

Exercise 1.7.16 From Exercise 1.7.14:
(a) $(\exists a)(\exists b)\left[\left(a \neq b \wedge E(a) \wedge E(b)\right) \wedge R(a,b)\right]$;
(b) $(\exists a)(\exists b)\left[\left(a \neq b \wedge \neg E(a) \wedge \neg E(b)\right) \wedge \neg R(a,b)\right]$;
(c) $(\exists a)\,[E(a) \Leftrightarrow R(a,2)]$ (Other answers possible);
(d) $(\exists a)(\exists b)(\exists c)\,[D(a,b) \wedge D(b,c) \wedge \neg D(a,c)]$;
(e) $(\exists a)(\exists b)(\exists c)\,[D(a,bc) \wedge \neg D(a,b) \wedge \neg D(a,c)]$;
(f) $(\exists a)(\exists b)(\exists c)\,[R(a,b) \wedge D(a,bc) \wedge \neg D(a,c)]$;
(g) $(\exists z)\,[R(z,2) \wedge \neg D(8, z^2-1)]$.
From Exercise 1.7.15:
(a) $(\forall n)\,[\neg L(n,3)]$;
(b) $(\exists n)\,[\neg L(n,4)]$;
(c) $(\forall n)(\exists k)\,[\neg L(n,k)]$;
(d) $(\exists p)\,\left[(\forall n)\Big(\big((n \neq p) \wedge (n \neq 1) \wedge (n > 0)\big) \Rightarrow \neg D(n,p)\Big) \wedge \big(D(4,p-1) \wedge \neg L(p,2)\big)\right]$;
(e) $(\exists k)(\forall n)\,[\neg L(n,k)]$;
(f) $(\exists m)(\exists n)\left[\left(R(m,n) \wedge L(m,2) \wedge L(n,2)\right) \wedge \neg L(mn,4)\right]$.

Exercise 1.7.17 (a) Equivalent; (b) Not equivalent; (c) Not equivalent; (d) Equivalent; (e) Equivalent; (f) Equivalent.

Exercise 1.7.18 (a) Some cloud doesn't have a silver lining.
(b) There is someone who doesn't like Sara Lee.
(c) There is someone who doesn't love Raymond.
(d) There is some friend that is like an old friend.

Exercise 1.7.19 (a) $(\forall x)(\exists y)(\exists t)L(x,y,t)$.
(b) $(\exists x)(\exists y)(\exists t)H(x,y,t)$.

Exercise 1.7.20 Let G be the set of all things that glitter. In symbols, "All that glitters is not gold" is written $(\forall g \in G)[g$ is not gold$]$. "Not all that glitters is gold" is written $\neg(\forall g \in G)[g$ is gold$]$. Do we agree that gold glitters? If so, then the second statement is true. The statements are not negations of each other. The negations are, respectively, $(\exists g \in G)[g$ is gold$]$ and $(\forall g \in G)[g$ is gold$]$.

Exercise 1.7.21 (a) 3; (b) 1; (c) 2; (d) 1; (e) 1; (f) 3.

Exercise 1.7.22 The area $A(x)$ of the inscribed rectangle at x, is given by

$$A(x) = 4xb\sqrt{1 - \frac{x^2}{a^2}}.$$

Then

$$A'(x) = \frac{4b - \dfrac{8bx^2}{a^2}}{\sqrt{1 - \dfrac{x^2}{a^2}}}.$$

Now $A'(x) = 0$ when $x = \pm\frac{a}{\sqrt{2}}$. Then evaluate $A\left(\frac{a}{\sqrt{2}}\right)$ to see that the maximum area is $2ab$.

Exercise 1.7.23 (a) Note that $d_1 = \sqrt{x^2 + (y-c)^2}$ and $d_2 = y + c$. Set $d_1 = d_2$ and solve for y.
(b) Note that point P has coordinates $\left(a, \frac{a^2}{4c}\right)$ and D has coordinates $(a, -c)$. The slope of the tangent line is

$$\left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{1}{2c}x\right|_{x=a} = \frac{a}{2c}.$$

This means that the equation of the tangent line is $y = \frac{a}{2c}x - \frac{a^2}{4c}$. The equation of the line joining point D to $(0, c)$ is $y = -\frac{2c}{a}x + c$. These two lines are perpendicular since the product of their slopes is -1. The midpoint of $\overline{DF}$ is $\left(\frac{a}{2}, 0\right)$. Verify that $\left(\frac{a}{2}, 0\right)$ lies on the tangent line. Thus the tangent line is the perpendicular bisector of $\overline{DF}$.