

# Instructor's Solutions Manual

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## for Numerical Analysis

9th EDITION

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# Preface

This Instructor's Manual for the Ninth Edition of Numerical Analysis by Burden and Faires contains solutions to all the exercises in the book. Although the answers to the odd exercises are also in the back of the text, we have found that users of the book appreciate having all the solutions in one source. In addition, the results listed in this Instructor's Manual often go beyond those given in the back of the book. For example, we do not place the long solutions to theoretical and applied exercises in the book. You will find them here.

A Student Study Guide for the Ninth Edition of Numerical Analysis is also available and the solutions given in the Guide are generally more detailed than those in the Instructor's Manual. In order to make it convenient for instructors, we have placed an asterisk (\*) in this Manual in front of each exercise whose solution is given in the Student Study Guide. Hopefully this will help with your homework assignments and test problems.

We have added a number of exercises to the text that involve the use of a Computer Algebra System. We have chosen Maple as our standard, because their *Numerical-Analysis* package parallels the algorithms in this book, but any of these systems can be used. In our recent teaching of the course we found that students understood the concepts better when they worked through the algorithms step-by-step, but let the computer algebra system do the tedious computation.

It has been our practice to include structured algorithms in our Numerical Analysis book for all the techniques discussed in the text. The algorithms are given in a form that can be coded in any appropriate programming language, by students with even a minimal amount of programming expertise.

At our website for the book,

<http://www.math.yzu.edu/~fares/Numerical-Analysis/>

you will find all the algorithms written in the programming languages FORTRAN, Pascal, C, Java, as well as in the Computer Algebra Systems, Maple, MATLAB, and Mathematica. For the Ninth Edition, we have added new Maple programs to reflect the changes in their system and to include portions of their *NumericalAnalysis* package.

The website also contains additional information about the book and will be updated regularly to reflect any modifications that might be made. For example, we will place there any responses to questions from users of the book concerning interpretations of the exercises and appropriate applications of the techniques.

We will have a set of presentation material ready soon for many of the methods in the book. These are being constructed by Professor John Carroll of Dublin City University using the Beamer package of L<sup>A</sup>T<sub>E</sub>X, and will be placed on the website. The Beamer package creates PDF files that are similar to Power Point presentations but incorporates mathematical elements easily and correctly. We are quite excited about this material and expect to have many of the presentations ready before they would normally be covered in the Fall term of 2010. If you send us an e-mail we will keep you posted on our progress.

We hope our supplement package provides flexibility for instructors teaching Numerical Analysis. If you have any suggestions for improvements that can be incorporated into future editions of the book or the supplements, we would be most grateful to receive your comments. We can be most easily contacted by electronic mail at the addresses listed below.

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August 20, 2010

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# Mathematical Preliminaries

**Note: An asterisk (\*) before an exercise indicates that there is a solution in the Student Study Guide.**

## Exercise Set 1.1, page 14

- \*1. For each part,  $f \in C[a, b]$  on the given interval. Since  $f(a)$  and  $f(b)$  are of opposite sign, the Intermediate Value Theorem implies that a number  $c$  exists with  $f(c) = 0$ .
2. (a)  $[0, 1]$   
(b)  $[0, 1], [4, 5], [-1, 0]$   
\*(c)  $[-2, -2/3], [0, 1], [2, 4]$   
(d)  $[-3, -2], [-1, -0.5],$  and  $[-0.5, 0]$
3. For each part,  $f \in C[a, b]$ ,  $f'$  exists on  $(a, b)$  and  $f(a) = f(b) = 0$ . Rolle's Theorem implies that a number  $c$  exists in  $(a, b)$  with  $f'(c) = 0$ . For part (d), we can use  $[a, b] = [-1, 0]$  or  $[a, b] = [0, 2]$ .
4. The maximum value for  $|f(x)|$  is given below.  
\*(a)  $(2 \ln 2)/3 \approx 0.4620981$   
(b) 0.8  
(c) 5.164000  
(d) 1.582572
- \*5. For  $x < 0$ ,  $f(x) < 2x + k < 0$ , provided that  $x < -\frac{1}{2}k$ . Similarly, for  $x > 0$ ,  $f(x) > 2x + k > 0$ , provided that  $x > -\frac{1}{2}k$ . By Theorem 1.11, there exists a number  $c$  with  $f(c) = 0$ . If  $f(c) = 0$  and  $f(c') = 0$  for some  $c' \neq c$ , then by Theorem 1.7, there exists a number  $p$  between  $c$  and  $c'$  with  $f'(p) = 0$ . However,  $f'(x) = 3x^2 + 2 > 0$  for all  $x$ .
6. Suppose  $p$  and  $q$  are in  $[a, b]$  with  $p \neq q$  and  $f(p) = f(q) = 0$ . By the Mean Value Theorem, there exists  $\xi \in (a, b)$  with
$$f(p) - f(q) = f'(\xi)(p - q).$$
But,  $f(p) - f(q) = 0$  and  $p \neq q$ . So  $f'(\xi) = 0$ , contradicting the hypothesis.
7. (a)  $P_2(x) = 0$   
(b)  $R_2(0.5) = 0.125$ ; actual error = 0.125

(c)  $P_2(x) = 1 + 3(x - 1) + 3(x - 1)^2$

(d)  $R_2(0.5) = -0.125$ ; actual error =  $-0.125$

8.  $P_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

$x$	0.5	0.75	1.25	1.5
$P_3(x)$	1.2265625	1.3310547	1.5517578	1.6796875
$\frac{P_3(x)}{\sqrt{x+1}}$	1.2247449	1.3228757	1.5	1.5811388
$ \sqrt{x+1} - P_3(x) $	0.0018176	0.0081790	0.0517578	0.0985487

\*9. Since

$$P_2(x) = 1 + x \quad \text{and} \quad R_2(x) = \frac{-2e^\xi(\sin \xi + \cos \xi)}{6}x^3$$

for some  $\xi$  between  $x$  and 0, we have the following:

(a)  $P_2(0.5) = 1.5$  and  $|f(0.5) - P_2(0.5)| \leq 0.0932$ ;

(b)  $|f(x) - P_2(x)| \leq 1.252$ ;

(c)  $\int_0^1 f(x) dx \approx 1.5$ ;

(d)  $|\int_0^1 f(x) dx - \int_0^1 P_2(x) dx| \leq \int_0^1 |R_2(x)| dx \leq 0.313$ , and the actual error is 0.122.

10.  $P_2(x) = 1.461930 + 0.617884(x - \frac{\pi}{6}) - 0.844046(x - \frac{\pi}{6})^2$  and  $R_2(x) = -\frac{1}{3}e^\xi(\sin \xi + \cos \xi)(x - \frac{\pi}{6})^3$  for some  $\xi$  between  $x$  and  $\frac{\pi}{6}$ .

(a)  $P_2(0.5) = 1.446879$  and  $f(0.5) = 1.446889$ . An error bound is  $1.01 \times 10^{-5}$ , and the actual error is  $1.0 \times 10^{-5}$ .

(b)  $|f(x) - P_2(x)| \leq 0.135372$  on  $[0, 1]$

(c)  $\int_0^1 P_2(x) dx = 1.376542$  and  $\int_0^1 f(x) dx = 1.378025$

(d) An error bound is  $7.403 \times 10^{-3}$ , and the actual error is  $1.483 \times 10^{-3}$ .

11.  $P_3(x) = (x - 1)^2 - \frac{1}{2}(x - 1)^3$

(a)  $P_3(0.5) = 0.312500$ ,  $f(0.5) = 0.346574$ . An error bound is  $0.291\bar{6}$ , and the actual error is 0.034074.

(b)  $|f(x) - P_3(x)| \leq 0.291\bar{6}$  on  $[0.5, 1.5]$

(c)  $\int_{0.5}^{1.5} P_3(x) dx = 0.08\bar{3}$ ,  $\int_{0.5}^{1.5} (x - 1) \ln x dx = 0.088020$

(d) An error bound is  $0.058\bar{3}$ , and the actual error is  $4.687 \times 10^{-3}$ .

12. (a)  $P_3(x) = -4 + 6x - x^2 - 4x^3$ ;  $P_3(0.4) = -2.016$

(b)  $|R_3(0.4)| \leq 0.05849$ ;  $|f(0.4) - P_3(0.4)| = 0.013365367$

(c)  $P_4(x) = -4 + 6x - x^2 - 4x^3$ ;  $P_4(0.4) = -2.016$

(d)  $|R_4(0.4)| \leq 0.01366$ ;  $|f(0.4) - P_4(0.4)| = 0.013365367$

13.  $P_4(x) = x + x^3$

(a)  $|f(x) - P_4(x)| \leq 0.012405$

(b)  $\int_0^{0.4} P_4(x) dx = 0.0864, \int_0^{0.4} xe^{x^2} dx = 0.086755$

(c)  $8.27 \times 10^{-4}$

(d)  $P_4'(0.2) = 1.12, f'(0.2) = 1.124076$ . The actual error is  $4.076 \times 10^{-3}$ .

\*14. First we need to convert the degree measure for the sine function to radians. We have  $180^\circ = \pi$  radians, so  $1^\circ = \frac{\pi}{180}$  radians. Since,

$$f(x) = \sin x, \quad f'(x) = \cos x, \quad f''(x) = -\sin x, \quad \text{and} \quad f'''(x) = -\cos x,$$

we have  $f(0) = 0, f'(0) = 1,$  and  $f''(0) = 0$ .

The approximation  $\sin x \approx x$  is given by

$$f(x) \approx P_2(x) = x, \quad \text{and} \quad R_2(x) = -\frac{\cos \xi}{3!}x^3.$$

If we use the bound  $|\cos \xi| \leq 1,$  then

$$\left| \sin \frac{\pi}{180} - \frac{\pi}{180} \right| = \left| R_2 \left( \frac{\pi}{180} \right) \right| = \left| \frac{-\cos \xi}{3!} \left( \frac{\pi}{180} \right)^3 \right| \leq 8.86 \times 10^{-7}.$$

15. Since  $42^\circ = 7\pi/30$  radians, use  $x_0 = \pi/4$ . Then

$$\left| R_n \left( \frac{7\pi}{30} \right) \right| \leq \frac{\left( \frac{\pi}{4} - \frac{7\pi}{30} \right)^{n+1}}{(n+1)!} < \frac{(0.053)^{n+1}}{(n+1)!}.$$

For  $|R_n(\frac{7\pi}{30})| < 10^{-6},$  it suffices to take  $n = 3$ . To 7 digits,

$$\cos 42^\circ = 0.7431448 \quad \text{and} \quad P_3(42^\circ) = P_3\left(\frac{7\pi}{30}\right) = 0.7431446,$$

so the actual error is  $2 \times 10^{-7}$ .

\*16. (a)  $P_3(x) = \frac{1}{3}x + \frac{1}{6}x^2 + \frac{23}{648}x^3$

(b) We have

$$f^{(4)}(x) = \frac{-119}{1296}e^{x/2} \sin \frac{x}{3} + \frac{5}{54}e^{x/2} \cos \frac{x}{3},$$

so

$$\left| f^{(4)}(x) \right| \leq \left| f^{(4)}(0.60473891) \right| \leq 0.09787176, \quad \text{for } 0 \leq x \leq 1,$$

and

$$|f(x) - P_3(x)| \leq \frac{|f^{(4)}(\xi)|}{4!} |x|^4 \leq \frac{0.09787176}{24} (1)^4 = 0.004077990.$$

17. (a)  $P_3(x) = \ln(3) + \frac{2}{3}(x-1) + \frac{1}{9}(x-1)^2 - \frac{10}{81}(x-1)^3$   
 (b)  $\max_{0 \leq x \leq 1} |f(x) - P_3(x)| = |f(0) - P_3(0)| = 0.02663366$   
 (c)  $\tilde{P}_3(x) = \ln(2) + \frac{1}{2}x^2$   
 (d)  $\max_{0 \leq x \leq 1} |f(x) - \tilde{P}_3(x)| = |f(1) - \tilde{P}_3(1)| = 0.09453489$   
 (e)  $P_3(0)$  approximates  $f(0)$  better than  $\tilde{P}_3(1)$  approximates  $f(1)$ .
18.  $P_n(x) = \sum_{k=0}^n x^k, n \geq 19$
19.  $P_n(x) = \sum_{k=0}^n \frac{1}{k!} x^k, n \geq 7$
20. For  $n$  odd,  $P_n(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + \frac{1}{n}(-1)^{(n-1)/2}x^n$ . For  $n$  even,  $P_n(x) = P_{n-1}(x)$ .
21. A bound for the maximum error is 0.0026.
22. (a)  $P_n^{(k)}(x_0) = f^{(k)}(x_0)$  for  $k = 0, 1, \dots, n$ . The shapes of  $P_n$  and  $f$  are the same at  $x_0$ .  
 (b)  $P_2(x) = 3 + 4(x-1) + 3(x-1)^2$ .
23. (a) The assumption is that  $f(x_i) = 0$  for each  $i = 0, 1, \dots, n$ . Applying Rolle's Theorem on each on the intervals  $[x_i, x_{i+1}]$  implies that for each  $i = 0, 1, \dots, n-1$  there exists a number  $z_i$  with  $f'(z_i) = 0$ . In addition, we have

$$a \leq x_0 < z_0 < x_1 < z_1 < \dots < z_{n-1} < x_n \leq b.$$

- (b) Apply the logic in part (a) to the function  $g(x) = f'(x)$  with the number of zeros of  $g$  in  $[a, b]$  reduced by 1. This implies that numbers  $w_i$ , for  $i = 0, 1, \dots, n-2$  exist with
- $$g'(w_i) = f''(w_i) = 0, \quad \text{and} \quad a < z_0 < w_0 < z_1 < w_1 < \dots < w_{n-2} < z_{n-1} < b.$$
- (c) Continuing by induction following the logic in parts (a) and (b) provides  $n+1-j$  distinct zeros of  $f^{(j)}$  in  $[a, b]$ .
- (d) The conclusion of the theorem follows from part (c) when  $j = n$ , for in this case there will be (at least)  $(n+1) - n = 1$  zero in  $[a, b]$ .
- \*24. First observe that for  $f(x) = x - \sin x$  we have  $f'(x) = 1 - \cos x \geq 0$ , because  $-1 \leq \cos x \leq 1$  for all values of  $x$ . Also, the statement clearly holds when  $|x| \geq \pi$ , because  $|\sin x| \leq 1$ .
- (a) The observation implies that  $f(x)$  is non-decreasing for all values of  $x$ , and in particular that  $f(x) > f(0) = 0$  when  $x > 0$ . Hence for  $x \geq 0$ , we have  $x \geq \sin x$ , and when  $0 \leq x \leq \pi$ ,  $|\sin x| = \sin x \leq x = |x|$ .
- (b) When  $-\pi < x < 0$ , we have  $\pi \geq -x > 0$ . Since  $\sin x$  is an odd function, the fact (from part (a)) that  $\sin(-x) \leq (-x)$  implies that  $|\sin x| = -\sin x \leq -x = |x|$ .  
 As a consequence, for all real numbers  $x$  we have  $|\sin x| \leq |x|$ .

25. Since  $R_2(1) = \frac{1}{6}e^\xi$ , for some  $\xi$  in  $(0, 1)$ , we have  $|E - R_2(1)| = \frac{1}{6}|1 - e^\xi| \leq \frac{1}{6}(e - 1)$ .

26. (a) Use the series

$$e^{-t^2} = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!} \quad \text{to integrate} \quad \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

and obtain the result.

(b) We have

$$\begin{aligned} \frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{1 \cdot 3 \cdots (2k+1)} &= \frac{2}{\sqrt{\pi}} \left[ 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 + \cdots \right] \\ &\quad \cdot \left[ x + \frac{2}{3}x^3 + \frac{4}{15}x^5 + \frac{8}{105}x^7 + \frac{16}{945}x^9 + \cdots \right] \\ &= \frac{2}{\sqrt{\pi}} \left[ x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \frac{1}{216}x^9 + \cdots \right] = \operatorname{erf}(x) \end{aligned}$$

(c) 0.8427008

(d) 0.8427069

(e) The series in part (a) is alternating, so for any positive integer  $n$  and positive  $x$  we have the bound

$$\left| \operatorname{erf}(x) - \frac{2}{\sqrt{\pi}} \sum_{k=0}^n \frac{(-1)^k x^{2k+1}}{(2k+1)k!} \right| < \frac{x^{2n+3}}{(2n+3)(n+1)!}.$$

We have no such bound for the positive term series in part (b).

27. (a) Let  $x_0$  be any number in  $[a, b]$ . Given  $\epsilon > 0$ , let  $\delta = \epsilon/L$ . If  $|x - x_0| < \delta$  and  $a \leq x \leq b$ , then  $|f(x) - f(x_0)| \leq L|x - x_0| < \epsilon$ .

(b) Using the Mean Value Theorem, we have

$$|f(x_2) - f(x_1)| = |f'(\xi)||x_2 - x_1|,$$

for some  $\xi$  between  $x_1$  and  $x_2$ , so

$$|f(x_2) - f(x_1)| \leq L|x_2 - x_1|.$$

(c) One example is  $f(x) = x^{1/3}$  on  $[0, 1]$ .

\*28. (a) The number  $\frac{1}{2}(f(x_1) + f(x_2))$  is the average of  $f(x_1)$  and  $f(x_2)$ , so it lies between these two values of  $f$ . By the Intermediate Value Theorem 1.11 there exist a number  $\xi$  between  $x_1$  and  $x_2$  with

$$f(\xi) = \frac{1}{2}(f(x_1) + f(x_2)) = \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2).$$

(b) Let  $m = \min\{f(x_1), f(x_2)\}$  and  $M = \max\{f(x_1), f(x_2)\}$ . Then  $m \leq f(x_1) \leq M$  and  $m \leq f(x_2) \leq M$ , so

$$c_1 m \leq c_1 f(x_1) \leq c_1 M \quad \text{and} \quad c_2 m \leq c_2 f(x_2) \leq c_2 M.$$

Thus

$$(c_1 + c_2)m \leq c_1 f(x_1) + c_2 f(x_2) \leq (c_1 + c_2)M$$

and

$$m \leq \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2} \leq M.$$

By the Intermediate Value Theorem 1.11 applied to the interval with endpoints  $x_1$  and  $x_2$ , there exists a number  $\xi$  between  $x_1$  and  $x_2$  for which

$$f(\xi) = \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2}.$$

(c) Let  $f(x) = x^2 + 1$ ,  $x_1 = 0$ ,  $x_2 = 1$ ,  $c_1 = 2$ , and  $c_2 = -1$ . Then for all values of  $x$ ,

$$f(x) > 0 \quad \text{but} \quad \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2} = \frac{2(1) - 1(2)}{2 - 1} = 0.$$

29. (a) Since  $f$  is continuous at  $p$  and  $f(p) \neq 0$ , there exists a  $\delta > 0$  with

$$|f(x) - f(p)| < \frac{|f(p)|}{2},$$

for  $|x - p| < \delta$  and  $a < x < b$ . We restrict  $\delta$  so that  $[p - \delta, p + \delta]$  is a subset of  $[a, b]$ . Thus, for  $x \in [p - \delta, p + \delta]$ , we have  $x \in [a, b]$ . So

$$-\frac{|f(p)|}{2} < f(x) - f(p) < \frac{|f(p)|}{2} \quad \text{and} \quad f(p) - \frac{|f(p)|}{2} < f(x) < f(p) + \frac{|f(p)|}{2}.$$

If  $f(p) > 0$ , then

$$f(p) - \frac{|f(p)|}{2} = \frac{f(p)}{2} > 0, \quad \text{so} \quad f(x) > f(p) - \frac{|f(p)|}{2} > 0.$$

If  $f(p) < 0$ , then  $|f(p)| = -f(p)$ , and

$$f(x) < f(p) + \frac{|f(p)|}{2} = f(p) - \frac{f(p)}{2} = \frac{f(p)}{2} < 0.$$

In either case,  $f(x) \neq 0$ , for  $x \in [p - \delta, p + \delta]$ .

(b) Since  $f$  is continuous at  $p$  and  $f(p) = 0$ , there exists a  $\delta > 0$  with

$$|f(x) - f(p)| < k, \quad \text{for} \quad |x - p| < \delta \quad \text{and} \quad a < x < b.$$

We restrict  $\delta$  so that  $[p - \delta, p + \delta]$  is a subset of  $[a, b]$ . Thus, for  $x \in [p - \delta, p + \delta]$ , we have

$$|f(x)| = |f(x) - f(p)| < k.$$

## Exercise Set 1.2, page 28

1. We have

	Absolute error	Relative error
(a)	0.001264	$4.025 \times 10^{-4}$
(b)	$7.346 \times 10^{-6}$	$2.338 \times 10^{-6}$
(c)	$2.818 \times 10^{-4}$	$1.037 \times 10^{-4}$
(d)	$2.136 \times 10^{-4}$	$1.510 \times 10^{-4}$
(e)	$2.647 \times 10^1$	$1.202 \times 10^{-3}$
(f)	$1.454 \times 10^1$	$1.050 \times 10^{-2}$
(g)	420	$1.042 \times 10^{-2}$
(h)	$3.343 \times 10^3$	$9.213 \times 10^{-3}$

2. The largest intervals are:

- (a) (3.1412784, 3.1419068)
- (b) (2.7180100, 2.7185536)
- \* (c) (1.4140721, 1.4143549)
- (d) (1.9127398, 1.9131224)

3. The largest intervals are

- (a) (149.85, 150.15)
- (b) (899.1, 900.9)
- (c) (1498.5, 1501.5)
- (d) (89.91, 90.09)

4. The calculations and their errors are:

- (a) (i) 17/15 (ii) 1.13 (iii) 1.13 (iv) both  $3 \times 10^{-3}$
- (b) (i) 4/15 (ii) 0.266 (iii) 0.266 (iv) both  $2.5 \times 10^{-3}$
- (c) (i) 139/660 (ii) 0.211 (iii) 0.210 (iv)  $2 \times 10^{-3}$ ,  $3 \times 10^{-3}$
- (d) (i) 301/660 (ii) 0.455 (iii) 0.456 (iv)  $2 \times 10^{-3}$ ,  $1 \times 10^{-4}$

5. We have

	Approximation	Absolute error	Relative error
(a)	134	0.079	$5.90 \times 10^{-4}$
(b)	133	0.499	$3.77 \times 10^{-3}$
(c)	2.00	0.327	0.195
(d)	1.67	0.003	$1.79 \times 10^{-3}$
* (e)	1.80	0.154	0.0786
(f)	-15.1	0.0546	$3.60 \times 10^{-3}$
(g)	0.286	$2.86 \times 10^{-4}$	$10^{-3}$
(h)	0.00	0.0215	1.00

6. We have

	Approximation	Absolute error	Relative error
(a)	133.9	0.021	$1.568 \times 10^{-4}$
(b)	132.5	0.001	$7.55 \times 10^{-6}$
(c)	1.700	0.027	0.01614
(d)	1.673	0	0
(e)	1.986	0.03246	0.01662
(f)	-15.16	0.005377	$3.548 \times 10^{-4}$
(g)	0.2857	$1.429 \times 10^{-5}$	$5 \times 10^{-5}$
(h)	-0.01700	0.0045	0.2092

7. We have

	Approximation	Absolute error	Relative error
(a)	133	0.921	$6.88 \times 10^{-3}$
(b)	132	0.501	$3.78 \times 10^{-3}$
(c)	1.00	0.673	0.402
(d)	1.67	0.003	$1.79 \times 10^{-3}$
*(e)	3.55	1.60	0.817
(f)	-15.2	0.0454	0.00299
(g)	0.284	0.00171	0.00600
(h)	0	0.02150	1

8. We have

	Approximation	Absolute error	Relative error
(a)	133.9	0.021	$1.568 \times 10^{-4}$
(b)	132.5	0.001	$7.55 \times 10^{-6}$
(c)	1.600	0.073	0.04363
(d)	1.673	0	0
(e)	1.983	0.02945	0.01508
(f)	-15.15	0.004622	$3.050 \times 10^{-4}$
(g)	0.2855	$2.143 \times 10^{-4}$	$7.5 \times 10^{-4}$
(h)	-0.01700	0.0045	0.2092

9. We have

	Approximation	Absolute error	Relative error
*(a)	3.14557613	$3.983 \times 10^{-3}$	$1.268 \times 10^{-3}$
(b)	3.14162103	$2.838 \times 10^{-5}$	$9.032 \times 10^{-6}$

10. We have

	Approximation	Absolute error	Relative error
(a)	2.7166667	0.0016152	$5.9418 \times 10^{-4}$
(b)	2.718281801	$2.73 \times 10^{-8}$	$1.00 \times 10^{-8}$



11. (a) We have

$$\lim_{x \rightarrow 0} \frac{x \cos x - \sin x}{x - \sin x} = \lim_{x \rightarrow 0} \frac{-x \sin x}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{-\sin x - x \cos x}{\sin x} = \lim_{x \rightarrow 0} \frac{-2 \cos x + x \sin x}{\cos x} = -2$$

(b)  $f(0.1) \approx -1.941$

(c)  $\frac{x(1 - \frac{1}{2}x^2) - (x - \frac{1}{6}x^3)}{x - (x - \frac{1}{6}x^3)} = -2$

(d) The relative error in part (b) is 0.029. The relative error in part (c) is 0.00050.

12. (a)  $\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{x} = \lim_{x \rightarrow 0} \frac{e^x + e^{-x}}{1} = 2$

(b)  $f(0.1) \approx 2.05$

(c)  $\frac{1}{x} \left( \left( 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \right) - \left( 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 \right) \right) = \frac{1}{x} \left( 2x + \frac{1}{3}x^3 \right) = 2 + \frac{1}{3}x^2;$   
 using three-digit rounding arithmetic and  $x = 0.1$ , we obtain 2.00.

(d) The relative error in part (b) is = 0.0233. The relative error in part (c) is = 0.00166.

13. We have

	$x_1$	Absolute error	Relative error	$x_2$	Absolute error	Relative error
(a)	92.26	0.01542	$1.672 \times 10^{-4}$	0.005419	$6.273 \times 10^{-7}$	$1.157 \times 10^{-4}$
(b)	0.005421	$1.264 \times 10^{-6}$	$2.333 \times 10^{-4}$	-92.26	$4.580 \times 10^{-3}$	$4.965 \times 10^{-5}$
(c)	10.98	$6.875 \times 10^{-3}$	$6.257 \times 10^{-4}$	0.001149	$7.566 \times 10^{-8}$	$6.584 \times 10^{-5}$
(d)	-0.001149	$7.566 \times 10^{-8}$	$6.584 \times 10^{-5}$	-10.98	$6.875 \times 10^{-3}$	$6.257 \times 10^{-4}$

14. We have

	Approximation for $x_1$	Absolute error	Relative error
(a)	92.24	0.004580	$4.965 \times 10^{-5}$
(b)	0.005417	$2.736 \times 10^{-6}$	$5.048 \times 10^{-4}$
(c)	10.98	$6.875 \times 10^{-3}$	$6.257 \times 10^{-4}$
(d)	-0.001149	$7.566 \times 10^{-8}$	$6.584 \times 10^{-5}$

	Approximation for $x_2$	Absolute error	Relative error
(a)	0.005418	$2.373 \times 10^{-6}$	$4.377 \times 10^{-4}$
(b)	-92.25	$5.420 \times 10^{-3}$	$5.875 \times 10^{-5}$
(c)	0.001149	$7.566 \times 10^{-8}$	$6.584 \times 10^{-5}$
(d)	-10.98	$6.875 \times 10^{-3}$	$6.257 \times 10^{-4}$

15. The machine numbers are equivalent to
- (a) 3224
  - (b)  $-3224$
  - \*(c) 1.32421875
  - (d) 1.3242187500000002220446049250313080847263336181640625
16. (a) Next Largest: 3224.00000000000045474735088646411895751953125;  
Next Smallest: 3223.9999999999954525264911353588104248046875
- (b) Next Largest:  $-3224.00000000000045474735088646411895751953125$ ;  
Next Smallest:  $-3223.9999999999954525264911353588104248046875$
- \*(c) Next Largest: 1.3242187500000002220446049250313080847263336181640625;  
Next Smallest: 1.3242187499999997779553950749686919152736663818359375
- (d) Next Largest: 1.324218750000000444089209850062616169452667236328125;  
Next Smallest: 1.32421875
17. (b) The first formula gives  $-0.00658$ , and the second formula gives  $-0.0100$ . The true three-digit value is  $-0.0116$ .
18. (a)  $-1.82$
- (b)  $7.09 \times 10^{-3}$
- (c) The formula in (b) is more accurate since subtraction is not involved.
19. The approximate solutions to the systems are
- (a)  $x = 2.451, y = -1.635$
  - (b)  $x = 507.7, y = 82.00$
20. (a)  $x = 2.460 \quad y = -1.634$
- (b)  $x = 477.0 \quad y = 76.93$
- \*21. (a) In nested form, we have  $f(x) = (((1.01e^x - 4.62)e^x - 3.11)e^x + 12.2)e^x - 1.99$ .
- (b)  $-6.79$
  - (c)  $-7.07$
  - (d) The absolute errors are

$$|-7.61 - (-6.71)| = 0.82 \quad \text{and} \quad |-7.61 - (-7.07)| = 0.54.$$

Nesting is significantly better since the relative errors are

$$\left| \frac{0.82}{-7.61} \right| = 0.108 \quad \text{and} \quad \left| \frac{0.54}{-7.61} \right| = 0.071,$$

22. We have  $39.375 \leq \text{Volume} \leq 86.625$  and  $71.5 \leq \text{Surface Area} \leq 119.5$ .

23. (a)  $n = 77$

(b)  $n = 35$

\*24. When  $d_{k+1} < 5$ ,

$$\left| \frac{y - fl(y)}{y} \right| = \frac{0.d_{k+1} \dots \times 10^{n-k}}{0.d_1 \dots \times 10^n} \leq \frac{0.5 \times 10^{-k}}{0.1} = 0.5 \times 10^{-k+1}.$$

When  $d_{k+1} > 5$ ,

$$\left| \frac{y - fl(y)}{y} \right| = \frac{(1 - 0.d_{k+1} \dots) \times 10^{n-k}}{0.d_1 \dots \times 10^n} < \frac{(1 - 0.5) \times 10^{-k}}{0.1} = 0.5 \times 10^{-k+1}.$$

25. (a)  $m = 17$

(b) We have

$$\binom{m}{k} = \frac{m!}{k!(m-k)!} = \frac{m(m-1) \dots (m-k-1)(m-k)!}{k!(m-k)!} = \binom{m}{k} \binom{m-1}{k-1} \dots \binom{m-k-1}{1}$$

(c)  $m = 181707$

(d) 2,597,000; actual error 1960; relative error  $7.541 \times 10^{-4}$

26. (a) The actual error is  $|f'(\xi)\epsilon|$ , and the relative error is  $|f'(\xi)\epsilon| \cdot |f(x_0)|^{-1}$ , where the number  $\xi$  is between  $x_0$  and  $x_0 + \epsilon$ .

(b) (i)  $1.4 \times 10^{-5}$ ;  $5.1 \times 10^{-6}$  (ii)  $2.7 \times 10^{-6}$ ;  $3.2 \times 10^{-6}$

(c) (i) 1.2;  $5.1 \times 10^{-5}$  (ii)  $4.2 \times 10^{-5}$ ;  $7.8 \times 10^{-5}$

27. (a) 124.03

(b) 124.03

(c) -124.03

(d) -124.03

(e) 0.0065

(f) 0.0065

(g) -0.0065

(h) -0.0065

\*28. Since  $0.995 \leq P \leq 1.005$ ,  $0.0995 \leq V \leq 0.1005$ ,  $0.082055 \leq R \leq 0.082065$ , and  $0.004195 \leq N \leq 0.004205$ , we have  $287.61^\circ \leq T \leq 293.42^\circ$ . Note that  $15^\circ\text{C} = 288.16\text{K}$ .

When  $P$  is doubled and  $V$  is halved,  $1.99 \leq P \leq 2.01$  and  $0.0497 \leq V \leq 0.0503$  so that  $286.61^\circ \leq T \leq 293.72^\circ$ . Note that  $19^\circ\text{C} = 292.16\text{K}$ . The laboratory figures are within an acceptable range.

### Exercise Set 1.3, page 39

1. (a)  $\frac{1}{1} + \frac{1}{4} \dots + \frac{1}{100} = 1.53$ ;  $\frac{1}{100} + \frac{1}{81} + \dots + \frac{1}{1} = 1.54$ .

The actual value is 1.549. Significant round-off error occurs much earlier in the first method.

(b) The following algorithm will sum the series  $\sum_{i=1}^N x_i$  in the reverse order.

INPUT  $N; x_1, x_2, \dots, x_N$

OUTPUT  $SUM$

STEP 1 Set  $SUM = 0$

STEP 2 For  $j = 1, \dots, N$  set  $i = N - j + 1$   
 $SUM = SUM + x_i$

STEP 3 OUTPUT( $SUM$ );  
 STOP.

2. We have

	Approximation	Absolute Error	Relative Error
(a)	2.715	$3.282 \times 10^{-3}$	$1.207 \times 10^{-3}$
(b)	2.716	$2.282 \times 10^{-3}$	$8.394 \times 10^{-4}$
(c)	2.716	$2.282 \times 10^{-3}$	$8.394 \times 10^{-4}$
(d)	2.718	$2.818 \times 10^{-4}$	$1.037 \times 10^{-4}$

\*3. (a) 2000 terms

(b) 20,000,000,000 terms

4. 4 terms

\*5. 3 terms

6. (a)  $O\left(\frac{1}{n}\right)$

(b)  $O\left(\frac{1}{n^2}\right)$

(c)  $O\left(\frac{1}{n^2}\right)$

(d)  $O\left(\frac{1}{n}\right)$

7. The rates of convergence are:

(a)  $O(h^2)$

(b)  $O(h)$

(c)  $O(h^2)$

(d)  $O(h)$

\*8. (a)  $n(n+1)/2$  multiplications;  $(n+2)(n-1)/2$  additions.

(b)  $\sum_{i=1}^n a_i \left( \sum_{j=1}^i b_j \right)$  requires  $n$  multiplications;  $(n+2)(n-1)/2$  additions.

9. The following algorithm computes  $P(x_0)$  using nested arithmetic.

INPUT  $n, a_0, a_1, \dots, a_n, x_0$   
 OUTPUT  $y = P(x_0)$

STEP 1 Set  $y = a_n$ .

STEP 2 For  $i = n-1, n-2, \dots, 0$  set  $y = x_0 y + a_i$ .

STEP 3 OUTPUT ( $y$ );  
 STOP.

\*10. The following algorithm uses the most effective formula for computing the roots of a quadratic equation.

INPUT  $A, B, C$ .  
 OUTPUT  $x_1, x_2$ .

STEP 1 If  $A = 0$  then

if  $B = 0$  then OUTPUT ('NO SOLUTIONS');  
 STOP.

else set  $x_1 = -C/B$ ;  
 OUTPUT ('ONE SOLUTION',  $x_1$ );  
 STOP.

STEP 2 Set  $D = B^2 - 4AC$ .

STEP 3 If  $D = 0$  then set  $x_1 = -B/(2A)$ ;  
 OUTPUT ('MULTIPLE ROOTS',  $x_1$ );  
 STOP.

STEP 4 If  $D < 0$  then set

$b = \sqrt{-D}/(2A)$ ;  
 $a = -B/(2A)$ ;  
 OUTPUT ('COMPLEX CONJUGATE ROOTS');  
 $x_1 = a + bi$ ;  
 $x_2 = a - bi$ ;  
 OUTPUT ( $x_1, x_2$ );  
 STOP.

STEP 5 If  $B \geq 0$  then set

$d = B + \sqrt{D}$ ;  
 $x_1 = -2C/d$ ;  
 $x_2 = -d/(2A)$

else set

$d = -B + \sqrt{D}$ ;  
 $x_1 = d/(2A)$ ;  
 $x_2 = 2C/d$ .

STEP 6 OUTPUT ( $x_1, x_2$ );  
 STOP.

11. The following algorithm produces the product  $P = (x - x_0), \dots, (x - x_n)$ .

INPUT  $n, x_0, x_1, \dots, x_n, x$   
 OUTPUT  $P$ .

STEP 1 Set  $P = x - x_0$ ;  
 $i = 1$ .

STEP 2 While  $P \neq 0$  and  $i \leq n$  set  
 $P = P \cdot (x - x_i)$ ;  
 $i = i + 1$

STEP 3 OUTPUT ( $P$ );  
 STOP.

12. The following algorithm determines the number of terms needed to satisfy a given tolerance.

INPUT number  $x$ , tolerance  $TOL$ , maximum number of iterations  $M$ .  
 OUTPUT number  $N$  of terms or a message of failure.

STEP 1 Set  $SUM = (1 - 2x)/(1 - x + x^2)$ ;  
 $S = (1 + 2x)/(1 + x + x^2)$ ;  
 $N = 2$ .

STEP 2 While  $N \leq M$  do Steps 3–5.

STEP 3 Set  $j = 2^{N-1}$ ;  
 $y = x^j$   
 $t_1 = \frac{xy}{x}(1 - 2y)$ ;  
 $t_2 = y(y - 1) + 1$ ;  
 $SUM = SUM + \frac{t_1}{t_2}$ .

STEP 4 If  $|SUM - S| < TOL$  then  
 OUTPUT ( $N$ );  
 STOP.

STEP 5 Set  $N = N + 1$ .

STEP 6 OUTPUT('Method failed');  
 STOP.

When  $TOL = 10^{-6}$ , we need to have  $N \geq 4$ .

13. (a) If  $|\alpha_n - \alpha|/(1/n^p) \leq K$ , then

$$|\alpha_n - \alpha| \leq K(1/n^p) \leq K(1/n^q) \quad \text{since } 0 < q < p.$$

Thus

$$|\alpha_n - \alpha|/(1/n^p) \leq K \quad \text{and} \quad \{\alpha_n\}_{n=1}^{\infty} \rightarrow \alpha$$

with rate of convergence  $O(1/n^p)$ .

(b)

$n$	$1/n$	$1/n^2$	$1/n^3$	$1/n^5$
5	0.2	0.04	0.008	0.0016
10	0.1	0.01	0.001	0.0001
50	0.02	0.0004	$8 \times 10^{-6}$	$1.6 \times 10^{-7}$
100	0.01	$10^{-4}$	$10^{-6}$	$10^{-8}$

The most rapid convergence rate is  $O(1/n^4)$ .

14. (a) If  $F(h) = L + O(h^p)$ , there is a constant  $k > 0$  such that

$$|F(h) - L| \leq kh^p,$$

for sufficiently small  $h > 0$ . If  $0 < q < p$  and  $0 < h < 1$ , then  $h^q > h^p$ . Thus,  $kh^p < kh^q$ , so

$$|F(h) - L| \leq kh^q \quad \text{and} \quad F(h) = L + O(h^q).$$

(b) For various powers of  $h$  we have the entries in the following table.

$h$	$h^2$	$h^3$	$h^4$
0.5	0.25	0.125	0.0625
0.1	0.01	0.001	0.0001
0.01	0.0001	0.00001	$10^{-8}$
0.001	$10^{-6}$	$10^{-9}$	$10^{-12}$

The most rapid convergence rate is  $O(h^4)$ .

- \*15. Suppose that for sufficiently small  $|x|$  we have positive constants  $k_1$  and  $k_2$  independent of  $x$ , for which

$$|F_1(x) - L_1| \leq K_1|x|^\alpha \quad \text{and} \quad |F_2(x) - L_2| \leq K_2|x|^\beta.$$

Let  $c = \max(|c_1|, |c_2|, 1)$ ,  $K = \max(K_1, K_2)$ , and  $\delta = \max(\alpha, \beta)$ .

(a) We have

$$\begin{aligned} |F(x) - c_1L_1 - c_2L_2| &= |c_1(F_1(x) - L_1) + c_2(F_2(x) - L_2)| \\ &\leq |c_1|K_1|x|^\alpha + |c_2|K_2|x|^\beta \leq cK[|x|^\alpha + |x|^\beta] \\ &\leq cK|x|^\gamma[1 + |x|^{\delta-\gamma}] \leq \tilde{K}|x|^\gamma, \end{aligned}$$

for sufficiently small  $|x|$  and some constant  $\tilde{K}$ . Thus,  $F(x) = c_1L_1 + c_2L_2 + O(x^\gamma)$ .

(b) We have

$$\begin{aligned} |G(x) - L_1 - L_2| &= |F_1(c_1x) + F_2(c_2x) - L_1 - L_2| \\ &\leq K_1|c_1x|^\alpha + K_2|c_2x|^\beta \leq Kc^\delta[|x|^\alpha + |x|^\beta] \\ &\leq Kc^\delta|x|^\gamma[1 + |x|^{\delta-\gamma}] \leq \tilde{K}|x|^\gamma, \end{aligned}$$

for sufficiently small  $|x|$  and some constant  $\tilde{K}$ . Thus,  $G(x) = L_1 + L_2 + O(x^\gamma)$ .

\*16. Since

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} x_{n+1} = x \quad \text{and} \quad x_{n+1} = 1 + \frac{1}{x_n},$$

we have

$$x = 1 + \frac{1}{x}, \quad \text{so} \quad x^2 - x - 1 = 0.$$

The quadratic formula implies that

$$x = \frac{1}{2} \left( 1 + \sqrt{5} \right).$$

This number is called the *golden ratio*. It appears frequently in mathematics and the sciences.

\*17. (a) To save space we will show the Maple output for each step in one line. Maple would produce this output on separate lines.

```
n := 98; f := 1; s := 1
```

```
n := 98  f := 1  s := 1
```

```
for i from 1 to n do
```

```
  l := f + s; f := s; s := l; od :
```

```
l :=2  f := 1  s := 2
```

```
l :=3  f := 2  s := 3
```

```
⋮
```

```
l :=218922995834555169026  f := 135301852344706746049  s := 218922995834555169026
```

```
l :=354224848179261915075
```

$$(b) \quad F_{100} := \frac{1}{\text{sqrt}(5)} \left( \left( \frac{(1 + \text{sqrt}(5))}{2} \right)^{100} - \left( \frac{(1 - \text{sqrt}(5))}{2} \right)^{100} \right)$$

$$F_{100} := \frac{1}{\sqrt{5}} \left( \left( \frac{1}{2} + \frac{1}{2}\sqrt{5} \right)^{100} - \left( \frac{1}{2} - \frac{1}{2}\sqrt{5} \right)^{100} \right)$$

```
evalf(F100)
```

```
0.3542248538 × 1021
```



- (c) The result in part (a) is computed using exact integer arithmetic, and the result in part (b) is computed using ten-digit rounding arithmetic.
  - (d) The result in part (a) required traversing a loop 98 times.
  - (e) The result is the same as the result in part (a).
18. (a)  $n = 50$   
(b)  $n = 500$



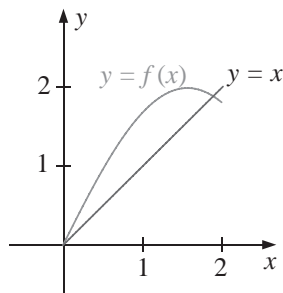
# Solutions of Equations of One Variable

## Exercise Set 2.1, page 54

- \*1.  $p_3 = 0.625$
2. \*(a)  $p_3 = -0.6875$   
(b)  $p_3 = 1.09375$
3. The Bisection method gives:
  - (a)  $p_7 = 0.5859$
  - (b)  $p_8 = 3.002$
  - (c)  $p_7 = 3.419$
4. The Bisection method gives:
  - (a)  $p_7 = -1.414$
  - (b)  $p_8 = 1.414$
  - (c)  $p_7 = 2.727$
  - (d)  $p_7 = -0.7265$
5. The Bisection method gives:
  - (a)  $p_{17} = 0.641182$
  - (b)  $p_{17} = 0.257530$
  - (c) For the interval  $[-3, -2]$ , we have  $p_{17} = -2.191307$ , and for the interval  $[-1, 0]$ , we have  $p_{17} = -0.798164$ .
  - (d) For the interval  $[0.2, 0.3]$ , we have  $p_{14} = 0.297528$ , and for the interval  $[1.2, 1.3]$ , we have  $p_{14} = 1.256622$ .
6.
  - (a)  $p_{17} = 1.51213837$
  - (b)  $p_{17} = 0.97676849$
  - (c) For the interval  $[1, 2]$ , we have  $p_{17} = 1.41239166$ , and for the interval  $[2, 4]$ , we have  $p_{18} = 3.05710602$ .

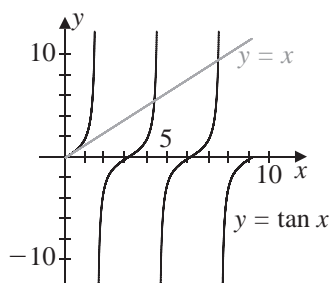
- (d) For the interval  $[0, 0.5]$ , we have  $p_{16} = 0.20603180$ , and for the interval  $[0.5, 1]$ , we have  $p_{16} = 0.68196869$ .

7. (a)



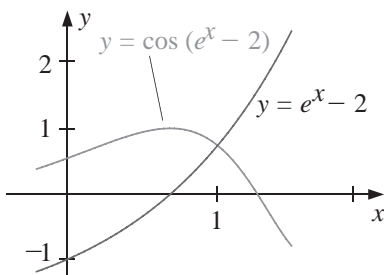
- (b) Using  $[1.5, 2]$  from part (a) gives  $p_{16} = 1.89550018$ .

\*8. (a)



- (b) Using  $[4.2, 4.6]$  from part (a) gives  $p_{16} = 4.4934143$ .

9. (a)



- (b)  $p_{17} = 1.00762177$

10. (a) 0  
 (b) 0  
 (c) 2  
 (d) -2

11. \*(a) 2  
 (b) -2  
 \*(c) -1  
 (d) 1

\*12. We have  $\sqrt{3} \approx p_{14} = 1.7320$ , using [1, 2].

13. The third root of 25 is approximately  $p_{14} = 2.92401$ , using [2, 3].

\*14. A bound for the number of iterations is  $n \geq 12$  and  $p_{12} = 1.3787$ .

15. A bound is  $n \geq 14$ , and  $p_{14} = 1.32477$ .

16. For  $n > 1$ ,

$$|f(p_n)| = \left(\frac{1}{n}\right)^{10} \leq \left(\frac{1}{2}\right)^{10} = \frac{1}{1024} < 10^{-3},$$

so

$$|p - p_n| = \frac{1}{n} < 10^{-3} \Leftrightarrow 1000 < n.$$

\*17. Since  $\lim_{n \rightarrow \infty} (p_n - p_{n-1}) = \lim_{n \rightarrow \infty} 1/n = 0$ , the difference in the terms goes to zero. However,  $p_n$  is the  $n$ th term of the divergent harmonic series, so  $\lim_{n \rightarrow \infty} p_n = \infty$ .

18. Since  $-1 < a < 0$  and  $2 < b < 3$ , we have  $1 < a + b < 3$  or  $1/2 < 1/2(a + b) < 3/2$  in all cases. Further,

$$\begin{aligned} f(x) &< 0, & \text{for } -1 < x < 0 & \text{ and } 1 < x < 2; \\ f(x) &> 0, & \text{for } 0 < x < 1 & \text{ and } 2 < x < 3. \end{aligned}$$

Thus,  $a_1 = a$ ,  $f(a_1) < 0$ ,  $b_1 = b$ , and  $f(b_1) > 0$ .

- (a) Since  $a + b < 2$ , we have  $p_1 = \frac{a+b}{2}$  and  $1/2 < p_1 < 1$ . Thus,  $f(p_1) > 0$ . Hence,  $a_2 = a_1 = a$  and  $b_2 = p_1$ . The only zero of  $f$  in  $[a_2, b_2]$  is  $p = 0$ , so the convergence will be to 0.
- (b) Since  $a + b > 2$ , we have  $p_1 = \frac{a+b}{2}$  and  $1 < p_1 < 3/2$ . Thus,  $f(p_1) < 0$ . Hence,  $a_2 = p_1$  and  $b_2 = b_1 = b$ . The only zero of  $f$  in  $[a_2, b_2]$  is  $p = 2$ , so the convergence will be to 2.
- (c) Since  $a + b = 2$ , we have  $p_1 = \frac{a+b}{2} = 1$  and  $f(p_1) = 0$ . Thus, a zero of  $f$  has been found on the first iteration. The convergence is to  $p = 1$ .

\*19. The depth of the water is 0.838 ft.

20. The angle  $\theta$  changes at the approximate rate  $w = -0.317059$ .

## Exercise Set 2.2, page 64

1. For the value of  $x$  under consideration we have

$$(a) \quad x = (3 + x - 2x^2)^{1/4} \Leftrightarrow x^4 = 3 + x - 2x^2 \Leftrightarrow f(x) = 0$$

$$(b) \quad x = \left( \frac{x + 3 - x^4}{2} \right)^{1/2} \Leftrightarrow 2x^2 = x + 3 - x^4 \Leftrightarrow f(x) = 0$$

$$(c) \quad x = \left( \frac{x + 3}{x^2 + 2} \right)^{1/2} \Leftrightarrow x^2(x^2 + 2) = x + 3 \Leftrightarrow f(x) = 0$$

$$(d) \quad x = \frac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1} \Leftrightarrow 4x^4 + 4x^2 - x = 3x^4 + 2x^2 + 3 \Leftrightarrow f(x) = 0$$

2. (a)  $p_4 = 1.10782$ ; (b)  $p_4 = 0.987506$ ; (c)  $p_4 = 1.12364$ ; (d)  $p_4 = 1.12412$ ;

(b) Part (d) gives the best answer since  $|p_4 - p_3|$  is the smallest for (d).

- \*3. The order in descending speed of convergence is (b), (d), and (a). The sequence in (c) does not converge.

4. The sequence in (c) converges faster than in (d). The sequences in (a) and (b) diverge.

5. With  $g(x) = (3x^2 + 3)^{1/4}$  and  $p_0 = 1$ ,  $p_6 = 1.94332$  is accurate to within 0.01.

6. With  $g(x) = \sqrt{1 + \frac{1}{x}}$  and  $p_0 = 1$ , we have  $p_4 = 1.324$ .

7. Since  $g'(x) = \frac{1}{4} \cos \frac{x}{2}$ ,  $g$  is continuous and  $g'$  exists on  $[0, 2\pi]$ . Further,  $g'(x) = 0$  only when  $x = \pi$ , so that  $g(0) = g(2\pi) = \pi \leq g(x) \leq g(\pi) = \pi + \frac{1}{2}$  and  $|g'(x)| \leq \frac{1}{4}$ , for  $0 \leq x \leq 2\pi$ . Theorem 2.3 implies that a unique fixed point  $p$  exists in  $[0, 2\pi]$ . With  $k = \frac{1}{4}$  and  $p_0 = \pi$ , we have  $p_1 = \pi + \frac{1}{2}$ . Corollary 2.5 implies that

$$|p_n - p| \leq \frac{k^n}{1 - k} |p_1 - p_0| = \frac{2}{3} \left( \frac{1}{4} \right)^n.$$

For the bound to be less than 0.1, we need  $n \geq 4$ . However,  $p_3 = 3.626996$  is accurate to within 0.01.

8. Using  $p_0 = 1$  gives  $p_{12} = 0.6412053$ . Since  $|g'(x)| = 2^{-x} \ln 2 \leq 0.551$  on  $[\frac{1}{3}, 1]$  with  $k = 0.551$ , Corollary 2.5 gives a bound of 16 iterations.

\*9. For  $p_0 = 1.0$  and  $g(x) = 0.5(x + \frac{3}{x})$ , we have  $\sqrt{3} \approx p_4 = 1.73205$ .

10. For  $g(x) = 5/\sqrt{x}$  and  $p_0 = 2.5$ , we have  $p_{14} = 2.92399$ .

11. (a) With  $[0, 1]$  and  $p_0 = 0$ , we have  $p_9 = 0.257531$ .

(b) With  $[2.5, 3.0]$  and  $p_0 = 2.5$ , we have  $p_{17} = 2.690650$ .

(c) With  $[0.25, 1]$  and  $p_0 = 0.25$ , we have  $p_{14} = 0.909999$ .

(d) With  $[0.3, 0.7]$  and  $p_0 = 0.3$ , we have  $p_{39} = 0.469625$ .

(e) With  $[0.3, 0.6]$  and  $p_0 = 0.3$ , we have  $p_{48} = 0.448059$ .

(f) With  $[0, 1]$  and  $p_0 = 0$ , we have  $p_6 = 0.704812$ .

12. The inequalities in Corollary 2.4 give  $|p_n - p| < k^n \max(p_0 - a, b - p_0)$ . We want

$$k^n \max(p_0 - a, b - p_0) < 10^{-5} \quad \text{so we need} \quad n > \frac{\ln(10^{-5}) - \ln(\max(p_0 - a, b - p_0))}{\ln k}.$$

(a) Using  $g(x) = 2 + \sin x$  we have  $k = 0.9899924966$  so that with  $p_0 = 2$  we have  $n > \ln(0.00001)/\ln k = 1144.663221$ . However, our tolerance is met with  $p_{63} = 2.5541998$ .

(b) Using  $g(x) = \sqrt[3]{2x+5}$  we have  $k = 0.1540802832$  so that with  $p_0 = 2$  we have  $n > \ln(0.00001)/\ln k = 6.155718005$ . However, our tolerance is met with  $p_6 = 2.0945503$ .

\* (c) Using  $g(x) = \sqrt{e^x/3}$  and the interval  $[0, 1]$  we have  $k = 0.4759448347$  so that with  $p_0 = 1$  we have  $n > \ln(0.00001)/\ln k = 15.50659829$ . However, our tolerance is met with  $p_{12} = 0.91001496$ .

(d) Using  $g(x) = \cos x$  and the interval  $[0, 1]$  we have  $k = 0.8414709848$  so that with  $p_0 = 0$  we have  $n > \ln(0.00001)/\ln k > 66.70148074$ . However, our tolerance is met with  $p_{30} = 0.73908230$ .

13. For  $g(x) = (2x^2 - 10 \cos x)/(3x)$ , we have the following:

$$p_0 = 3 \Rightarrow p_8 = 3.16193; \quad p_0 = -3 \Rightarrow p_8 = -3.16193.$$

For  $g(x) = \arccos(-0.1x^2)$ , we have the following:

$$p_0 = 1 \Rightarrow p_{11} = 1.96882; \quad p_0 = -1 \Rightarrow p_{11} = -1.96882.$$

\*14. For  $g(x) = \frac{1}{\tan x} - \frac{1}{x} + x$  and  $p_0 = 4$ , we have  $p_4 = 4.493409$ .

15. With  $g(x) = \frac{1}{\pi} \arcsin\left(-\frac{x}{2}\right) + 2$ , we have  $p_5 = 1.683855$ .

16. (a) If fixed-point iteration converges to the limit  $p$ , then

$$p = \lim_{n \rightarrow \infty} p_n = \lim_{n \rightarrow \infty} 2p_{n-1} - Ap_{n-1}^2 = 2p - Ap^2.$$

Solving for  $p$  gives  $p = \frac{1}{A}$ .

(b) Any subinterval  $[c, d]$  of  $\left(\frac{1}{2A}, \frac{3}{2A}\right)$  containing  $\frac{1}{A}$  suffices.

Since

$$g(x) = 2x - Ax^2, \quad g'(x) = 2 - 2Ax,$$

so  $g(x)$  is continuous, and  $g'(x)$  exists. Further,  $g'(x) = 0$  only if  $x = \frac{1}{A}$ .

Since

$$g\left(\frac{1}{A}\right) = \frac{1}{A}, \quad g\left(\frac{1}{2A}\right) = g\left(\frac{3}{2A}\right) = \frac{3}{4A}, \quad \text{and we have} \quad \frac{3}{4A} \leq g(x) \leq \frac{1}{A}.$$

For  $x$  in  $\left(\frac{1}{2A}, \frac{3}{2A}\right)$ , we have

$$\left|x - \frac{1}{A}\right| < \frac{1}{2A} \quad \text{so} \quad |g'(x)| = 2A \left|x - \frac{1}{A}\right| < 2A \left(\frac{1}{2A}\right) = 1.$$

17. One of many examples is  $g(x) = \sqrt{2x-1}$  on  $[\frac{1}{2}, 1]$ .

\*18. (a) The proof of existence is unchanged. For uniqueness, suppose  $p$  and  $q$  are fixed points in  $[a, b]$  with  $p \neq q$ . By the Mean Value Theorem, a number  $\xi$  in  $(a, b)$  exists with

$$p - q = g(p) - g(q) = g'(\xi)(p - q) \leq k(p - q) < p - q,$$

giving the same contradiction as in Theorem 2.3.

(b) Consider  $g(x) = 1 - x^2$  on  $[0, 1]$ . The function  $g$  has the unique fixed point

$$p = \frac{1}{2}(-1 + \sqrt{5}).$$

With  $p_0 = 0.7$ , the sequence eventually alternates between 0 and 1.

\*19. (a) Suppose that  $x_0 > \sqrt{2}$ . Then

$$x_1 - \sqrt{2} = g(x_0) - g(\sqrt{2}) = g'(\xi)(x_0 - \sqrt{2}),$$

where  $\sqrt{2} < \xi < x_0$ . Thus,  $x_1 - \sqrt{2} > 0$  and  $x_1 > \sqrt{2}$ . Further,

$$x_1 = \frac{x_0}{2} + \frac{1}{x_0} < \frac{x_0}{2} + \frac{1}{\sqrt{2}} = \frac{x_0 + \sqrt{2}}{2}$$

and  $\sqrt{2} < x_1 < x_0$ . By an inductive argument,

$$\sqrt{2} < x_{m+1} < x_m < \dots < x_0.$$

Thus,  $\{x_m\}$  is a decreasing sequence which has a lower bound and must converge.

Suppose  $p = \lim_{m \rightarrow \infty} x_m$ . Then

$$p = \lim_{m \rightarrow \infty} \left( \frac{x_{m-1}}{2} + \frac{1}{x_{m-1}} \right) = \frac{p}{2} + \frac{1}{p}. \quad \text{Thus } p = \frac{p}{2} + \frac{1}{p},$$

which implies that  $p = \pm\sqrt{2}$ . Since  $x_m > \sqrt{2}$  for all  $m$ , we have  $\lim_{m \rightarrow \infty} x_m = \sqrt{2}$ .

(b) We have

$$0 < (x_0 - \sqrt{2})^2 = x_0^2 - 2x_0\sqrt{2} + 2,$$

so  $2x_0\sqrt{2} < x_0^2 + 2$  and  $\sqrt{2} < \frac{x_0}{2} + \frac{1}{x_0} = x_1$ .

(c) Case 1:  $0 < x_0 < \sqrt{2}$ , which implies that  $\sqrt{2} < x_1$  by part (b). Thus,

$$0 < x_0 < \sqrt{2} < x_{m+1} < x_m < \dots < x_1 \quad \text{and} \quad \lim_{m \rightarrow \infty} x_m = \sqrt{2}.$$

Case 2:  $x_0 = \sqrt{2}$ , which implies that  $x_m = \sqrt{2}$  for all  $m$  and  $\lim_{m \rightarrow \infty} x_m = \sqrt{2}$ .

Case 3:  $x_0 > \sqrt{2}$ , which by part (a) implies that  $\lim_{m \rightarrow \infty} x_m = \sqrt{2}$ .



20. (a) Let

$$g(x) = \frac{x}{2} + \frac{A}{2x}.$$

Note that  $g(\sqrt{A}) = \sqrt{A}$ . Also,

$$g'(x) = 1/2 - A/(2x^2) \text{ if } x \neq 0 \text{ and } g'(x) > 0 \text{ if } x > \sqrt{A}.$$

If  $x_0 = \sqrt{A}$ , then  $x_m = \sqrt{A}$  for all  $m$  and  $\lim_{m \rightarrow \infty} x_m = \sqrt{A}$ .

If  $x_0 > A$ , then

$$x_1 - \sqrt{A} = g(x_0) - g(\sqrt{A}) = g'(\xi)(x_0 - \sqrt{A}) > 0.$$

Further,

$$x_1 = \frac{x_0}{2} + \frac{A}{2x_0} < \frac{x_0}{2} + \frac{A}{2\sqrt{A}} = \frac{1}{2}(x_0 + \sqrt{A}).$$

Thus,  $\sqrt{A} < x_1 < x_0$ . Inductively,

$$\sqrt{A} < x_{m+1} < x_m < \dots < x_0$$

and  $\lim_{m \rightarrow \infty} x_m = \sqrt{A}$  by an argument similar to that in Exercise 19(a).

If  $0 < x_0 < \sqrt{A}$ , then

$$0 < (x_0 - \sqrt{A})^2 = x_0^2 - 2x_0\sqrt{A} + A \text{ and } 2x_0\sqrt{A} < x_0^2 + A,$$

which leads to

$$\sqrt{A} < \frac{x_0}{2} + \frac{A}{2x_0} = x_1.$$

Thus

$$0 < x_0 < \sqrt{A} < x_{m+1} < x_m < \dots < x_1,$$

and by the preceding argument,  $\lim_{m \rightarrow \infty} x_m = \sqrt{A}$ .

(b) If  $x_0 < 0$ , then  $\lim_{m \rightarrow \infty} x_m = -\sqrt{A}$ .

21. Replace the second sentence in the proof with: "Since  $g$  satisfies a Lipschitz condition on  $[a, b]$  with a Lipschitz constant  $L < 1$ , we have, for each  $n$ ,

$$|p_n - p| = |g(p_{n-1}) - g(p)| \leq L|p_{n-1} - p|."$$

The rest of the proof is the same, with  $k$  replaced by  $L$ .

22. Let  $\varepsilon = (1 - |g'(p)|)/2$ . Since  $g'$  is continuous at  $p$ , there exists a number  $\delta > 0$  such that for  $x \in [p - \delta, p + \delta]$ , we have  $|g'(x) - g'(p)| < \varepsilon$ . Thus,  $|g'(x)| < |g'(p)| + \varepsilon < 1$  for  $x \in [p - \delta, p + \delta]$ .

By the Mean Value Theorem

$$|g(x) - g(p)| = |g'(c)||x - p| < |x - p|,$$

for  $x \in [p - \delta, p + \delta]$ . Applying the Fixed-Point Theorem completes the problem.

23. With  $g(t) = 501.0625 - 201.0625e^{-0.4t}$  and  $p_0 = 5.0$ ,  $p_3 = 6.0028$  is within 0.01 s of the actual time.
- \*24. Since  $g'$  is continuous at  $p$  and  $|g'(p)| > 1$ , by letting  $\epsilon = |g'(p)| - 1$  there exists a number  $\delta > 0$  such that  $|g'(x) - g'(p)| < |g'(p)| - 1$  whenever  $0 < |x - p| < \delta$ . Hence, for any  $x$  satisfying  $0 < |x - p| < \delta$ , we have

$$|g'(x)| \geq |g'(p)| - |g'(x) - g'(p)| > |g'(p)| - (|g'(p)| - 1) = 1.$$

If  $p_0$  is chosen so that  $0 < |p - p_0| < \delta$ , we have by the Mean Value Theorem that

$$|p_1 - p| = |g(p_0) - g(p)| = |g'(\xi)||p_0 - p|,$$

for some  $\xi$  between  $p_0$  and  $p$ . Thus,  $0 < |p - \xi| < \delta$  so  $|p_1 - p| = |g'(\xi)||p_0 - p| > |p_0 - p|$ .

## Exercise Set 2.3, page 75

- \*1.  $p_2 = 2.60714$
2.  $p_2 = -0.865684$ ; If  $p_0 = 0$ ,  $f'(p_0) = 0$  and  $p_1$  cannot be computed.
- \*3. (a) 2.45454  
(b) 2.44444  
(c) Part (a) is better.
4. (a)  $-1.25208$   
(b)  $-0.841355$
5. (a) For  $p_0 = 2$ , we have  $p_5 = 2.69065$ .  
(b) For  $p_0 = -3$ , we have  $p_3 = -2.87939$ .
- \*3. (c) For  $p_0 = 0$ , we have  $p_4 = 0.73909$ .  
(d) For  $p_0 = 0$ , we have  $p_3 = 0.96434$ .
6. (a) For  $p_0 = 1$ , we have  $p_8 = 1.829384$ .  
(b) For  $p_0 = 1.5$ , we have  $p_4 = 1.397748$ .  
(c) For  $p_0 = 2$ , we have  $p_4 = 2.370687$ ; and for  $p_0 = 4$ , we have  $p_4 = 3.722113$ .  
(d) For  $p_0 = 1$ , we have  $p_4 = 1.412391$ ; and for  $p_0 = 4$ , we have  $p_5 = 3.057104$ .  
(e) For  $p_0 = 1$ , we have  $p_4 = 0.910008$ ; and for  $p_0 = 3$ , we have  $p_9 = 3.733079$ .  
(f) For  $p_0 = 0$ , we have  $p_4 = 0.588533$ ; for  $p_0 = 3$ , we have  $p_3 = 3.096364$ ; and for  $p_0 = 6$ , we have  $p_3 = 6.285049$ .
7. Using the endpoints of the intervals as  $p_0$  and  $p_1$ , we have:  
(a)  $p_{11} = 2.69065$   
(b)  $p_7 = -2.87939$   
\*3. (c)  $p_6 = 0.73909$

- (d)  $p_5 = 0.96433$
8. Using the endpoints of the intervals as  $p_0$  and  $p_1$ , we have:
- (a)  $p_7 = 1.829384$   
 (b)  $p_9 = 1.397749$   
 (c)  $p_6 = 2.370687; p_7 = 3.722113$   
 (d)  $p_8 = 1.412391; p_7 = 3.057104$   
 (e)  $p_6 = 0.910008; p_{10} = 3.733079$   
 (f)  $p_6 = 0.588533; p_5 = 3.096364; p_5 = 6.285049$
9. Using the endpoints of the intervals as  $p_0$  and  $p_1$ , we have:
- (a)  $p_{16} = 2.69060$   
 (b)  $p_6 = -2.87938$   
 \*(c)  $p_7 = 0.73908$   
 (d)  $p_6 = 0.96433$
10. Using the endpoints of the intervals as  $p_0$  and  $p_1$ , we have:
- (a)  $p_8 = 1.829383$   
 (b)  $p_9 = 1.397749$   
 (c)  $p_6 = 2.370687; p_8 = 3.722112$   
 (d)  $p_{10} = 1.412392; p_{12} = 3.057099$   
 (e)  $p_7 = 0.910008; p_{29} = 3.733065$   
 (f)  $p_9 = 0.588533; p_5 = 3.096364; p_5 = 6.285049$
11. (a) Newton's method with  $p_0 = 1.5$  gives  $p_3 = 1.51213455$ .  
 The Secant method with  $p_0 = 1$  and  $p_1 = 2$  gives  $p_{10} = 1.51213455$ .  
 The Method of False Position with  $p_0 = 1$  and  $p_1 = 2$  gives  $p_{17} = 1.51212954$ .
- (b) Newton's method with  $p_0 = 0.5$  gives  $p_5 = 0.976773017$ .  
 The Secant method with  $p_0 = 0$  and  $p_1 = 1$  gives  $p_5 = 10.976773017$ .  
 The Method of False Position with  $p_0 = 0$  and  $p_1 = 1$  gives  $p_5 = 0.976772976$ .
12. (a) We have

	Initial Approximation	Result	Initial Approximation	Result
Newton's	$p_0 = 1.5$	$p_4 = 1.41239117$	$p_0 = 3.0$	$p_4 = 3.05710355$
Secant	$p_0 = 1, p_1 = 2$	$p_8 = 1.41239117$	$p_0 = 2, p_1 = 4$	$p_{10} = 3.05710355$
False Position	$p_0 = 1, p_1 = 2$	$p_{13} = 1.41239119$	$p_0 = 2, p_1 = 4$	$p_{19} = 3.05710353$

(b) We have

	Initial Approximation	Result	Initial Approximation	Result
Newton's	$p_0 = 0.25$	$p_4 = 0.206035120$	$p_0 = 0.75$	$p_4 = 0.681974809$
Secant	$p_0 = 0, p_1 = 0.5$	$p_9 = 0.206035120$	$p_0 = 0.5, p_1 = 1$	$p_8 = 0.681974809$
False Position	$p_0 = 0, p_1 = 0.5$	$p_{12} = 0.206035125$	$p_0 = 0.5, p_1 = 1$	$p_{15} = 0.681974791$

\*13. For  $p_0 = 1$ , we have  $p_5 = 0.589755$ . The point has the coordinates  $(0.589755, 0.347811)$ .

14. For  $p_0 = 2$ , we have  $p_2 = 1.866760$ . The point is  $(1.866760, 0.535687)$ .

15. The equation of the tangent line is

$$y - f(p_{n-1}) = f'(p_{n-1})(x - p_{n-1}).$$

To complete this problem, set  $y = 0$  and solve for  $x = p_n$ .

\*16. Newton's method gives  $p_{15} = 1.895488$ , for  $p_0 = \frac{\pi}{2}$ ; and  $p_{19} = 1.895489$ , for  $p_0 = 5\pi$ . The sequence does not converge in 200 iterations for  $p_0 = 10\pi$ . The results do not indicate the fast convergence usually associated with Newton's method.

17. (a) For  $p_0 = -1$  and  $p_1 = 0$ , we have  $p_{17} = -0.04065850$ , and for  $p_0 = 0$  and  $p_1 = 1$ , we have  $p_9 = 0.9623984$ .

(b) For  $p_0 = -1$  and  $p_1 = 0$ , we have  $p_5 = -0.04065929$ , and for  $p_0 = 0$  and  $p_1 = 1$ , we have  $p_{12} = -0.04065929$ .

(c) For  $p_0 = -0.5$ , we have  $p_5 = -0.04065929$ , and for  $p_0 = 0.5$ , we have  $p_{21} = 0.9623989$ .

18. (a) The Bisection method yields  $p_{10} = 0.4476563$ .

(b) The method of False Position yields  $p_{10} = 0.442067$ .

(c) The Secant method yields  $p_{10} = -195.8950$ .

\*19. This formula involves the subtraction of nearly equal numbers in both the numerator and denominator if  $p_{n-1}$  and  $p_{n-2}$  are nearly equal.

20. Newton's method for the various values of  $p_0$  gives the following results.

(a)  $p_8 = -1.379365$

(b)  $p_7 = -1.379365$

(c)  $p_7 = 1.379365$

(d)  $p_7 = -1.379365$

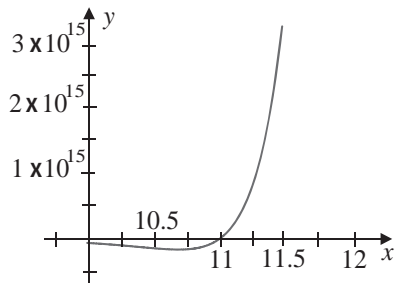
(e)  $p_7 = 1.379365$

(f)  $p_8 = 1.379365$

21. Newton's method for the various values of  $p_0$  gives the following results.

- (a)  $p_0 = -10, p_{11} = -4.30624527$   
 (b)  $p_0 = -5, p_5 = -4.30624527$   
 (c)  $p_0 = -3, p_5 = 0.824498585$   
 (d)  $p_0 = -1, p_4 = -0.824498585$   
 (e)  $p_0 = 0, p_1$  cannot be computed because  $f'(0) = 0$   
 (f)  $p_0 = 1, p_4 = 0.824498585$   
 (g)  $p_0 = 3, p_5 = -0.824498585$   
 (h)  $p_0 = 5, p_5 = 4.30624527$   
 (i)  $p_0 = 10, p_{11} = 4.30624527$
- \*22. The required accuracy is met in 7 iterations of Newton's method.
- \*23. For  $f(x) = \ln(x^2 + 1) - e^{0.4x} \cos \pi x$ , we have the following roots.
- (a) For  $p_0 = -0.5$ , we have  $p_3 = -0.4341431$ .  
 (b) For  $p_0 = 0.5$ , we have  $p_3 = 0.4506567$ .  
 For  $p_0 = 1.5$ , we have  $p_3 = 1.7447381$ .  
 For  $p_0 = 2.5$ , we have  $p_5 = 2.2383198$ .  
 For  $p_0 = 3.5$ , we have  $p_4 = 3.7090412$ .  
 (c) The initial approximation  $n - 0.5$  is quite reasonable.  
 (d) For  $p_0 = 24.5$ , we have  $p_2 = 24.4998870$ .
24. We have  $\lambda \approx 0.100998$  and  $N(2) \approx 2,187,950$ .
25. The two numbers are approximately 6.512849 and 13.487151.
- \*26. The minimal annual interest rate is 6.67%.
27. The borrower can afford to pay at most 8.10%.
- \*28. (a)  $\frac{1}{3}e, t = 3$  hours  
 (b) 11 hours and 5 minutes  
 (c) 21 hours and 14 minutes
- \*29. (a) First define the function by
- $$f := x \rightarrow 3^{3x+1} - 7 \cdot 5^{2x}$$
- $$f := x \rightarrow 3^{(3x+1)} - 7 \cdot 5^{2x}$$
- $solve(f(x) = 0, x)$
- $$-\frac{\ln(3/7)}{\ln(27/25)}$$
- $fsolve(f(x) = 0, x)$
- $$fsolve(3^{(3x+1)} - 7 \cdot 5^{(2x)} = 0, x)$$
- The procedure *solve* gives the exact solution, and *fsolve* fails because the negative  $x$ -axis is an asymptote for the graph of  $f(x)$ .

- (b) Using the Maple command  $\text{plot}\{f(x), x = 10.5..11.5\}$  produces the following graph.



- (c) Define  $f'(x)$  using  
 $fp := x \rightarrow (D)(f)(x)$

$$fp := x \rightarrow 3 \cdot 3^{(3x+1)} \ln(3) - 14 \cdot 5^{(2x)} \ln(5)$$

$Digits := 18; p0 := 11$

$Digits := 18$

$p0 := 11$

for  $i$  from 1 to 5 do

$p1 := \text{evalf}(p0 - f(p0)/fp(p0))$

$err := \text{abs}(p1 - p0)$

$p0 := p1$

od

The results are given in the following table.

$i$	$p_i$	$ p_i - p_{i-1} $
1	11.0097380401552503	.0097380401552503
2	11.0094389359662827	.0002991041889676
3	11.0094386442684488	.2916978339 $10^{-6}$
4	11.0094386442681716	.2772 $10^{-2}$
5	11.0094386442681716	0

- (d) We have  $3^{3x+1} = 7 \cdot 5^{2x}$ . Taking the natural logarithm of both sides gives

$$(3x + 1) \ln 3 = \ln 7 + 2x \ln 5.$$

Thus

$$3x \ln 3 - 2x \ln 5 = \ln 7 - \ln 3, \quad x(3 \ln 3 - 2 \ln 5) = \ln \frac{7}{3},$$

and

$$x = \frac{\ln 7/3}{\ln 27 - \ln 25} = \frac{\ln 7/3}{\ln 27/25} = -\frac{\ln 3/7}{\ln 27/25}.$$

This agrees with part (a).

30. (a)  $\text{solve}(2^{x^2} - 3 \cdot 7^{(x+1)}, x)$  fails and  $\text{fsolve}(2^{x^2} - 3 \cdot 7^{(x+1)}, x)$  returns  $-1.118747530$ .  
 (b)  $\text{plot}(2^{x^2} - 3 \cdot 7^{(x+1)}, x = -2..4)$  shows there is also a root near  $x = 4$ .  
 (c) With  $p_0 = 1$ ,  $p_4 = -1.1187475303988963$  is accurate to  $10^{-16}$ ; with  $p_0 = 4$ ,  $p_6 = 3.9261024524565005$  is accurate to  $10^{-16}$ .  
 (d) The roots are

$$\frac{\ln(7) \pm \sqrt{[\ln(7)]^2 + 4 \ln(2) \ln(4)}}{2 \ln(2)}.$$

31. We have  $P_L = 265816$ ,  $c = -0.75658125$ , and  $k = 0.045017502$ . The 1980 population is  $P(30) = 222,248,320$ , and the 2010 population is  $P(60) = 252,967,030$ .  
 32.  $P_L = 290228$ ,  $c = 0.6512299$ , and  $k = 0.03020028$ ;  
 The 1980 population is  $P(30) = 223,069,210$ , and the 2010 population is  $P(60) = 260,943,806$ .  
 33. Using  $p_0 = 0.5$  and  $p_1 = 0.9$ , the Secant method gives  $p_5 = 0.842$ .  
 34. (a) We have, approximately,

$$A = 17.74, \quad B = 87.21, \quad C = 9.66, \quad \text{and} \quad E = 47.47$$

With these values we have

$$A \sin \alpha \cos \alpha + B \sin^2 \alpha - C \cos \alpha - E \sin \alpha = 0.02.$$

- (b) Newton's method gives  $\alpha \approx 33.2^\circ$ .

## Exercise Set 2.4, page 85

- \*(a) For  $p_0 = 0.5$ , we have  $p_{13} = 0.567135$ .  
 (b) For  $p_0 = -1.5$ , we have  $p_{23} = -1.414325$ .  
 (c) For  $p_0 = 0.5$ , we have  $p_{22} = 0.641166$ .  
 (d) For  $p_0 = -0.5$ , we have  $p_{23} = -0.183274$ .
- (a) For  $p_0 = 0.5$ , we have  $p_{15} = 0.739076589$ .  
 (b) For  $p_0 = -2.5$ , we have  $p_9 = -1.33434594$ .  
 (c) For  $p_0 = 3.5$ , we have  $p_5 = 3.14156793$ .  
 (d) For  $p_0 = 4.0$ , we have  $p_{44} = 3.37354190$ .
- Modified Newton's method in Eq. (2.11) gives the following:

\*(a) For  $p_0 = 0.5$ , we have  $p_3 = 0.567143$ .  
 (b) For  $p_0 = -1.5$ , we have  $p_2 = -1.414158$ .  
 (c) For  $p_0 = 0.5$ , we have  $p_3 = 0.641274$ .  
 (d) For  $p_0 = -0.5$ , we have  $p_5 = -0.183319$ .

4. (a) For  $p_0 = 0.5$ , we have  $p_4 = 0.739087439$ .  
 (b) For  $p_0 = -2.5$ , we have  $p_{53} = -1.33434594$ .  
 (c) For  $p_0 = 3.5$ , we have  $p_5 = 3.14156793$ .  
 (d) For  $p_0 = 4.0$ , we have  $p_3 = -3.72957639$ .
5. Newton's method with  $p_0 = -0.5$  gives  $p_{13} = -0.169607$ . Modified Newton's method in Eq. (2.11) with  $p_0 = -0.5$  gives  $p_{11} = -0.169607$ .

6. \*(a) Since

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1,$$

we have linear convergence. To have  $|p_n - p| < 5 \times 10^{-2}$ , we need  $n \geq 20$ .

- (b) Since

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n \rightarrow \infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^2 = 1,$$

we have linear convergence. To have  $|p_n - p| < 5 \times 10^{-2}$ , we need  $n \geq 5$ .

7. (a) For  $k > 0$ ,

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - 0|}{|p_n - 0|} = \lim_{n \rightarrow \infty} \frac{\frac{1}{(n+1)^k}}{\frac{1}{n^k}} = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^k = 1,$$

so the convergence is linear.

- (b) We need to have  $N > 10^{m/k}$ .

- \*8. (a) Since

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - 0|}{|p_n - 0|^2} = \lim_{n \rightarrow \infty} \frac{10^{-2^{n+1}}}{(10^{-2^n})^2} = \lim_{n \rightarrow \infty} \frac{10^{-2^{n+1}}}{10^{-2^{n+1}}} = 1,$$

the sequence is quadratically convergent.

- (b) We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|p_{n+1} - 0|}{|p_n - 0|^2} &= \lim_{n \rightarrow \infty} \frac{10^{-(n+1)^k}}{(10^{-n^k})^2} = \lim_{n \rightarrow \infty} \frac{10^{-(n+1)^k}}{10^{-2n^k}} \\ &= \lim_{n \rightarrow \infty} 10^{2n^k - (n+1)^k} = \lim_{n \rightarrow \infty} 10^{n^k(2 - (\frac{n+1}{n})^k)} = \infty, \end{aligned}$$

so the sequence  $p_n = 10^{-n^k}$  does not converge quadratically.

9. Typical examples are

- (a)  $p_n = 10^{-3^n}$   
 (b)  $p_n = 10^{-\alpha^n}$

- \*10. Suppose  $f(x) = (x - p)^m q(x)$ . Since

$$g(x) = x - \frac{m(x-p)q(x)}{mq(x) + (x-p)q'(x)},$$

we have  $g'(p) = 0$ .