2.1 Early & Modern Numeration Systems

KEY TERMS	
Additive System	An additive system is a numeration system where the number represented is the sum of the values of each individual numeral.
Babylonian System	The Babylonian system is one of the oldest place-value systems and uses a base of 60 with 2 cuneiform numerals to represent each number.
Base 10 System	A base 10 decimal system has place values increasing by powers of ten and is called positional because the value of the symbol is understood by its position in the number.
Chinese System	The Chinese system of numeration uses separate symbols for the numerals 0–9 as well as separate symbols for various multiples of ten. It is a multiplicative system.
Cuneiform	Cuneiform is one of the earliest forms of writing where a stylus made of reed was used to form symbols in either wood or a wet clay tablet.
Decimal System	A number system that has place values increasing by powers of ten and is positional.
Egyptian System	The Egyptian system of numeration is an additive base 10 system using hieroglyphs to represent the digits 0–9 as well as the powers of ten.
Expanded form	Expanded form is a way to write a number to show the value of each digit. It is shown as a sum of each digit multiplied by its matching place value (ones, tens, hundreds, etc.)
Hieroglyphs	Hieroglyphs are Images used to represent numerals in the Egyptian system of numeration.

Hindu-Arabic System	The Hindu-Arabic numeration system is composed of ten symbols (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) named for the Indian scholars who invented it at least as early as 800 BC and for the Arabs who transmitted it to the western world. It is a Base 10 or Decimal system.
Multiplicative System	A multiplicative system is a numeration system that consists of two sets of numerals, one set representing the digits and the other set representing positions.
Numerals	Numerals are the symbols used to represent a number.
Numeration System	A numeration system consists of a set of symbols (numerals) to represent numbers along with a set of rules for combining numerals.
Place Value	The place value assigns a value to a digit depending on its place or position in a numeral.
Roman System	The Roman system is an additive base 10 numeration system with single numerals representing the powers of ten as well as the halves of each power of ten.
Tally System	The tally system is a numeration system where the value of a certain number (i.e. 4) is the sum of the values of each individual numeral (or tally marks).

<u>Objective 1 Understand and Use the Hindu-Arabic System</u>

Concept video questions and answers

1. **What are the key features of the Hindu-Arabic number system?**
    - Ten symbols called digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
    - Place value
    - Numbers can be written in expanded form

2. **Fill in the following place value chart:**

Millions			Thousands			Ones		

EXAMPLE 1 Write a Hindu-Arabic Number in Expanded Form

Write 13,448 in expanded form.

Each digit represents a power of 10:

1	3	4	4	8
10^4	10^3	10^2	10^1	10^0
10,000	1,000	100	10	1

Multiplying each digit by the corresponding value:

13,448 = 1(10,000) + 3(1,000) + 4(100) + 4(10) + 8(1)

Additional problem

1. Write 27,777 in expanded form. **2(10,000) + 7(1,000) + 7(100) + 7(10) + 7(1)**

EXAMPLE 2 Change from Expanded Form into a Hindu-Arabic Numeral

Write $3(10^4) + 0(10^3) + 7(10^2) + 9(10^1) + 0(10^0)$ as a Hindu-Arabic numeral.

$$3(10^4) + 0(10^3) + 7(10^2) + 9(10^1) + 0(10^0)$$

$$= 3(10{,}000) + 0(1{,}000) + 7(100) + 9(10) + 0(1)$$
$$= 30{,}000 + 0 + 700 + 90 + 0$$

$$= 30{,}790$$

In-class activity

First choose a list of numbers. Use index cards to create a template like the one below. Make sure the answer for the “who has” is the next card’s “I have”. Make sure the last card’s “who has” corresponds to the first card’s “I have.”

I have **335** Who has **2(100) + 3(10) + 1**	I have **231** Who has **5(1000) + 7(100) + 0(10) + 5**
I have **5,705** Who has **9(1000) + 8(100) + 8(10) + 0**	I have **9,880** Who has **3(100) + 3(10) + 5**

Hand out a card to each student. Some students may need to have 2 depending upon how many in a set. It is important to use all the cards in a set.

Randomly pick a student to start. They will say “I have 335, who has 2 times 100 plus 3 times 10 plus 1? The student with the matching value to the “who has” will say “I have 231, who has 5 times 1000 plus 7 times 700 plus 0 times 10 plus 5?” This will continue until every student has gone.

Objective 2 Understand and Use the Egyptian System

Hindu-Arabic Numeral	Egyptian Numeral	Description
1	𓏺	Staff
10	𓎆	Heel bone
100	𓍢	Scroll
1,000	𓆼	Lotus flower
10,000	𓂭	Pointed finger
100,000	𓆐 or 𓆏	Tadpole or frog
1,000,000	𓁨	Astonished person

EXAMPLE 3 Write an Egyptian Numeral as a Hindu-Arabic Numeral

Find the Hindu-Arabic numerical value of the following Egyptian numeral:

Use the table above to determine each of the values for the symbols:

1000	1000	100	100	100	10	10	1	1

1,000 + 1,000 + 100 + 100 + 100 + 10 + 10 + 1 + 1 = 2,322

Additional problem

1. Find the Hindu-Arabic numeral for the following Egyptian numeral

728

EXAMPLE 4 Write a Hindu-Arabic Numeral as an Egyptian Numeral

Write 10,454 as an Egyptian numeral.

10,454 in expanded form is:

$10{,}000 + 400 + 50 + 4$
$1(10{,}000) + 0(1{,}000) + 4(100) + 5(10) + 4(1)$

Now determine the symbols needed for each power of 10:

Hindu-Arabic Numeral	Egyptian Numeral	Description
1	𓏺	Staff
10	𓎆	Heel bone
100	𓍢	Scroll
1,000	𓆼	Lotus Flower
10,000	𓂭	Pointed Finger
100,000	𓆐 or 𓆏	Tadpole or Frog
1,000,000	𓁨	Astonished Person

Hindu-Arabic	Egyptian
1(10,000)	𓂭
4(100)	𓍢 𓍢 𓍢 𓍢
5(10)	𓎆 𓎆 𓎆 𓎆 𓎆
4(1)	𓏺 𓏺 𓏺 𓏺

10,454 is equivalent to

𓂭 𓍢 𓍢 𓍢 𓍢 𓎆 𓎆 𓎆 𓎆 𓎆 𓏺 𓏺 𓏺 𓏺

Additional problem

1. Write 12,144 as an Egyptian numeral

Objective 3 Understand and use the Roman system

Roman Numeral	I	V	X	L	C	D	M
Hindu-Arabic Numeral	1	5	10	50	100	500	1,000

Roman Numeral	$\overline{\text{IV}}$	$\overline{\text{V}}$	$\overline{\text{IX}}$	$\overline{\text{X}}$
Hindu-Arabic Numeral	4,000	5,000	9,000	10,000

EXAMPLE 5 Write a Roman Numeral as a Hindu-Arabic Numeral

a) Write the Hindu-Arabic number that LXXVI represents.

L	X	X	V	I

We want to use the following table to determine the corresponding Hindu-Arabic numerals:

Roman Numeral	I	V	X	L	C	D	M
Hindu-Arabic Numeral	1	5	10	50	100	500	1000

L	X	X	V	I
50	10	10	5	1

Because each symbol goes in decreasing order, we just add each Hindu-Arabic numeral to get the corresponding value.

$$50 + 10 + 10 + 5 + 1 = 76$$
$$\text{LXXVI} = 76$$

We want to use the following table to determine the corresponding Hindu-Arabic numerals:

Roman Numeral	I	V	X	L	C	D	M
Hindu-Arabic Numeral	1	5	10	50	100	500	1,000

L	X	X	V	I
50	10	10	5	1

Because each symbol goes in decreasing order, we just add each Hindu-Arabic numeral to get the corresponding value.

$$50 + 10 + 10 + 5 + 1 = 76$$
LXXVI = 76

b) Write the Hindu-Arabic number that DXLIX represents.

We want to use the following table to determine the corresponding Hindu-Arabic numerals:

Roman Numeral	I	V	X	L	C	D	M
Hindu-Arabic Numeral	1	5	10	50	100	500	1,000

D	X	L	I	X
500	10	40	1	10

The Hindu-Arabic equivalents do not all go in descending order, so we need to use subtraction to determine the corresponding values before we can add.

D	X	L	I	X
500	10	50	1	10
500	50−10 = 40		10−1 = 9	

Adding each of the values we have:

$500 + 40 + 9 = 549$

DXLIX = 549

Additional problems

1. Write the Hindu-Arabic numeral MCMXC represents **1,990**
2. Write the Hindu-Arabic numeral MDXXIV represents.**1,524**

EXAMPLE 6 Write a Hindu-Arabic Numeral as a Roman Numeral

a. Write 254 as a Roman numeral.

Break apart into expanded form:

$$= 200 + 50 + 4$$

$$= 100 + 100 + 50 + 5 - 1$$

Using the following table:

Roman Numeral	I	V	X	L	C	D	M
Hindu-Arabic Numeral	1	5	10	50	100	500	1,000

We have the corresponding symbols:

100	100	50	4 = (5 − 1)
C	C	L	IV

254 = CCLIV

b. Write 13,448 as a Roman numeral.

Break apart into expanded form:

$$= 10{,}000 + 3{,}000 + 400 + 40 + 8$$
$$= 10(1{,}000) + 3(1{,}000) + (500 - 100) + (50 - 10) + (5 + 3)$$

Using the following table:

Roman Numeral	I	V	X	L	C	D	M
Hindu-Arabic Numeral	1	5	10	50	100	500	1,000

We have the corresponding symbols:

10(1,000)	3,000	500−100	50−10	5 + 3
$\overline{\text{X}}$	MMM	CD	XL	VIII

$13{,}448 = \overline{\text{X}}\text{MMMCDXLVIII}$

Additional problems

1. Write 89 as a Roman numeral **LXXXIX**
2. Write 6,893 as a Roman numeral. **$\overline{\text{VI}}$DCCCXCIII**

Teacher note

Depending on the source, when working with values over 3,000, you may find different ways to write it. For example, some sources have 4,000 as MMMM whereas others will write it as $\overline{\text{IV}}$.

<u>**Objective 4 Understand and use the Babylonian system**</u>

Concept video question and answer

1. **How would a Babylonian distinguish between the Babylonian numeral for 60 and the Babylonian numeral for 1?**

   Counting was done in a context, so a person would know if they were counting just 1 item or 60.

Babylonian Numerals

Converting from Hindu-Arabic to Babylonian

1. Determine the highest power of the Babylonian base 60 that will divide into the given Hindu-Arabic numeral at least once and then divide.
2. Keep the whole number part and divide the remainder by the next lower power of base 60.
3. Repeat steps 1 and 2 until the remainder is 0.
4. The number in base 60 will be each of the quotients from the highest power of base 60 descending to the quotient when dividing by 1.
5. Write the symbols for each quotient in separate columns.

EXAMPLE 7 Write a Babylonian Numeral as a Hindu-Arabic Numeral

a) Find the Hindu-Arabic numeral represented by the following Babylonian numeral.

 $= 4(10) = 40$

 $= 3(1) = 3$

 $= 43$

b) Find the Hindu-Arabic numeral represented by the following Babylonian numeral.

There are three groupings separated by spaces, so we have three powers of 60 represented.

The Babylonian numerals for 1 and 10.

$60^2 \times 3600$	$60^1 \times 60$	$60^0 \times 1$	Value
 2(1)(3600) 7200	 2(10)(60) (20)(60) 1200	 2(10) + 2(1) 22(1) 22	7200 + 1200 + 22 = 8,422

Additional problems

1. Find the Hindu-Arabic numeral represented by the following Babylonian numeral.

1,218

2. Find the Hindu-Arabic numeral represented by the following Babylonian numeral.

39,064

EXAMPLE 8 Write a Hindu-Arabic Numeral as a Babylonian Numeral

a) Write 12,156 as a Babylonian numeral.

1. Determine the highest power of the Babylonian base 60 that will divide into given Hindu-Arabic numeral at least once and then divide.

   The power of 60 less than 12,156 is $60^2 = 3{,}600$

$60^3 = 216{,}000$	$60^2 = 3{,}600$	$60^1 = 60$	$60^0 = 1$
Too big			

2. Divide. Keep the whole number part and divide the remainder by the next lower power of base 60.
3. Repeat step 2 until the remainder is zero.

	$60^2 = 3{,}600$	$60^1 = 60$	$60^0 = 1$
	$12{,}156 \div 3{,}600$	$1{,}356 \div 60$	$36 \div 1$
Quotient	3	22	36
Remainder	1,356	36	0

4. The number in base 60 will be each of the quotients from the highest power of base 60 descending to the quotient when dividing by 1.

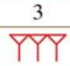
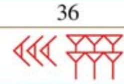
	$60^2 = 3{,}600$	$60^1 = 60$	$60^0 = 1$
	$12{,}156 \div 3{,}600$	$1{,}356 \div 60$	$36 \div 1$
Quotient	3	22	36
Remainder	1,356	36	0

5. Arrange the quotients in order along with the last remainder.

3	22	36

6. Use the Babylonian symbols that correspond to each of the quotients.

 = 1  = 10

3	22	36
		

b) Write 227,352 as a Babylonian numeral.

1. Determine the highest power of the Babylonian base 60 that will divide into the given Hindu-Arabic numeral at least once and then divide.

   The power of 60 less than 227,352 is $60^3 = 216,0000$

$60^4 = 12,960,000$	$60^3 = 216,000$	$60^2 = 3,600$	$60^1 = 60$	$60^0 = 1$
Too big				

2. Divide. Keep the whole number part and divide the remainder by the next lower power of base 60.

3. Repeat step 2 until the remainder is zero.

	$60^3 = 216,000$	$60^2 = 3,600$	$60^1 = 60$	$60^0 = 1$
	$227,352 \div 216,000$	$11,352 \div 3,600$	$552 \div 60$	$12 \div 1$
Quotient	1	3	9	12
Remainder	11,352	552	12	0

4. The number in base 60 will be each of the quotients from the highest power of base 60 descending to the quotient when dividing by 1.

	$60^3 = 216,000$	$60^2 = 3,600$	$60^1 = 60$	$60^0 = 1$
	$227,352 \div 216,000$	$11,352 \div 3,600$	$552 \div 60$	$12 \div 1$
Quotient	1	3	9	12
Remainder	11,352	552	12	0

5. Arrange the quotients in order along with the last remainder

1	3	9	12

6. Use the Babylonian symbols that correspond to each of the quotients.

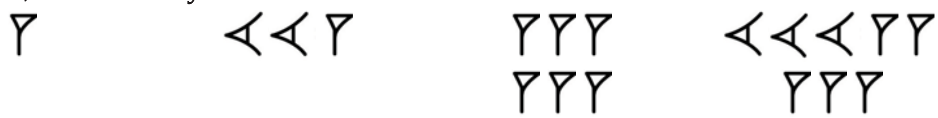
   𒁹 = 1   𒌋 = 10

1	3	9	12
𒁹	𒐗	𒐝	𒌋𒐖

Additional problems

1. Write 13,221 as Babylonian numeral.

2. Write 291,995 as Babylonian numeral.



Objective 5 Understand and use the Traditional Chinese System

Digits

Chinese	Hindu-Arabic
零/〇	= 0
一	= 1
二	= 2
三	= 3
四	= 4
五	= 5
六	= 6
七	= 7
八	= 8
九	= 9

Position

十	= 10
百	= 100
千	= 1000
萬	= 10,000

Concept video question and answer

1. **How would you write the Hindu-Arabic numeral 893 in the Chinese system?**

八
百
九
十
三

EXAMPLE 9 Write a Hindu-Arabic Numeral as a Chinese Numeral

Write 4,254 as a Chinese numeral

$4{,}254 = 4{,}000 + 200 + 50 + 4$

$4{,}254 = 4(1000) + 2(100) + 5(10) + 4$

Write the digit symbol and position symbol for each with Chinese numerals.

4,000	4	四
	1,000	千
200	2	二
	100	百
50	5	五
	10	十
4	4	四

Additional problems

1. Write 737 as Chinese numeral.
   七
   百
   三
   十
   七
2. Write 10,302 as Chinese numeral
   .一
   萬
   三
   百
   二

EXAMPLE 10 Write a Chinese Numeral as a Hindu-Arabic Numeral

Write the Hindu-Arabic numeral represented by the following Chinese numerals:

三
千
一
百
六
十
四

Digits

Chinese	Hindu-Arabic
零/〇	= 0
一	= 1
二	= 2
三	= 3
四	= 4
五	= 5
六	= 6
七	= 7
八	= 8
九	= 9

Position

十	= 10
百	= 100
千	= 1,000
萬	= 10,000

三 = 3	
千 = 1,000	$3 \times 1{,}000 = 3{,}000$
一 = 1	
百 = 100	$1 \times 100 = 100$
六 = 6	
十 = 10	$6 \times 10 = 60$
四 = 4	4

$3{,}000 + 100 + 60 + 4 = 3{,}164$

Additional problems

1. Write the Hindu-Arabic numeral 一千三 represents. **1,003**
2. Write the Hindu-Arabic numeral 六千几百几十三 represents. **6,883**

In class activity

Source: https://maya.nmai.si.edu/maya-sun/maya-math-game
Introduce the Mayan number system.

In class activity

Number System Bingo
Have each student create a bingo card using the numbers 1 – 100 in whichever number system you would like (Roman, Egyptian, Chinese, or Babylonian) Call out a number and if they have the matching symbol, they can mark it off. Winner can be to blackout or just one line.

2.2 Base Number Systems

KEY TERMS	
Binary	The binary system is a base 2 number system used in computer operations where 1 represents "on" and 0 represents "off".
Hexadecimal	A hexadecimal system is a base 16 number system used in computer operations.
Octal	An octal system is a base 8 number system used in computer operations.

Objective 1 Convert Other Base Numbers to Base 10

Concept Video - Characteristics of Different Base Systems

1. **How do you count to 5 in base 5?**
   1, 2, 3, 4, 10
2. **Fill in the following chart showing the letters which represent the numbers 10 – 35.**

A	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**	**K**	**L**	**M**
10	11	12	13	14	15	16	17	18	19	20	21	22
N	**O**	**P**	**Q**	**R**	**S**	**T**	**U**	**V**	**W**	**X**	**Y**	**Z**
23	24	25	26	27	28	29	30	31	32	33	34	35

In class activity

- Go to the following website: http://www.shodor.org/interactivate/activities/NumberBaseClocks/
- Click on "Learner" tab and then print out "Number Base Clocks Exploration Questions" for each student.
- Use a camera projection to go through the activity with your students, or if they have access to computers do with a partner and then have a whole class discussion.

EXAMPLE 1 Converting From Base 5 to Base 10

Convert 3224_5 to base 10.

Step 1: Write each digit of the given number along with the corresponding power of the given base. The powers should increase going from right to left, starting at a power of zero.

5^3	5^2	5^1	5^0
3	2	2	4

Step 2: Multiply each digit by the corresponding power.

5^3	5^2	5^1	5^0
3	2	2	4
$3(5^3)$	$2(5^2)$	$2(5^1)$	$4(5^0)$
3(125)	2(25)	2(5)	4(1)
375	50	10	4

Step 3: Add the products together to get the result in base 10.

$$375 + 50 + 10 + 4 = 439$$

$$3224_5 = 439$$

Additional problems

1. Convert 340_5 to base 10. **95**
2. Convert 11112_3 to base 10.**122**

EXAMPLE 2 Converting From Base 16 to Base 10

Convert $5ED8_{16}$ to base 10.

Step 1: Write each digit of the given number along with the corresponding power of the given base. The powers should increase going from right to left, starting at a power of zero.

16^3	16^2	16^1	16^0
5	E	D	8

Because we are in base 16, we need 16 numerals to represent each digit. For numbers greater than 9, we use letters:

A	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**	**K**	**L**	**M**
10	11	12	13	14	15	16	17	18	19	20	21	22
N	**O**	**P**	**Q**	**R**	**S**	**T**	**U**	**V**	**W**	**X**	**Y**	**Z**
23	24	25	26	27	28	29	30	31	32	33	34	35

16^3	16^2	16^1	16^0
5	E = 14	D = 13	8

Step 2: Multiply each digit by the corresponding power.

16^3	16^2	16^1	16^0
5	E = 14	D = 13	8
$5(16^3)$	$14(16^2)$	$13(16^1)$	$8(16^0)$
5(4,096)	14(256)	13(16)	8(1)
20,480	3,584	208	8

Step 3: Add the products together to get the result in base 10.

$$20{,}480 + 3{,}584 + 208 + 8 = 24{,}280$$
$$5ED8_{16} = 24{,}280$$

Additional problems

1. Convert $8A21_{12}$ to base 10. **15,289**
2. Convert $9B47_{14}$ to base 10. **26,915**

Objective 2 Convert Base 10 Numbers to Numbers in Other Bases

Concept video question and answer

1. **How do you write 105 base 10 in base 5?**

   410_5

Converting from Base 10 to Base *b*

1. Determine the highest power of the base b that will divide into the given number at least once and then divide.
2. Keep the whole number part and divide the remainder by the next lower power of the base b.
3. Repeat steps 1 and 2 until the remainder is being divided by 1.
4. The number in base b will be each of the quotients from the highest power of base b descending to the quotient when dividing by 1.

EXAMPLE 3 Converting from Base 10 to Base 2

Write 97 in base 2.

1. Determine the highest power of base 2 that will divide into the given numeral at least once and then divide.The power of 2 that is less than 97 is 26 = 64 because 27 = 128 is too big.

$2^7 = 128$	$2^6 = 64$	$2^5 = 32$	$2^4 = 16$	$2^3 = 8$	$2^2 = 4$	$2^1 = 2$	$2^0 = 1$
Too big							

2. Divide. Keep the whole number part and divide the remainder by the next lower power of base 2.

3. Repeat step 2 until the remainder is zero.

	$2^6 = 64$	$2^5 = 32$	$2^4 = 16$	$2^3 = 8$	$2^2 = 4$	$2^1 = 2$	$2^0 = 1$
	$97 \div 64$	$33 \div 32$	$1 \div 16$	$1 \div 8$	$1 \div 4$	$1 \div 2$	$1 \div 1$
Quotient	1	1	0	0	0	0	1
Remainder	33	1	1	1	1	1	0

97 converted to base 2 can be written by writing each of the quotients from left to right starting at the highest power of the base.

$$97 = 1100001_2$$

Additional problems

1. Write 743 in base nine. **1015_9**
2. Write 1345 in base six. **10121_6**

EXAMPLE 4 Converting from Base 10 to Base 16

Write 19,442 in base 16.

1. Determine the highest power of base 16 that will divide into the given base 10 number at least once and then divide. The power of 16 that is less than 19,442 is 163 = 4,096 because 164 = 65,536 is too big.

$16^4 = 65{,}536$	$16^3 = 4{,}096$	$16^2 = 256$	$16^1 = 16$	$16^0 = 1$
Too big				

2. Divide. Keep the whole number part and divide the remainder by the next lower power of base 16.
3. Repeat step 2 until the remainder is zero.

	$16^3 = 4{,}096$	$16^2 = 256$	$16^1 = 16$	$16^0 = 1$
	$19{,}442 \div 4{,}096$	$3{,}058 \div 256$	$242 \div 16$	$2 \div 1$
Quotient	4	11	15	2
Remainder	3,058	242	2	0

Use the following chart to evaluate the quotients that are greater than 9.

A	B	C	D	E	F	G	H	I	J	K	L	M
10	11	12	13	14	15	16	17	18	19	20	21	22
N	**O**	**P**	**Q**	**R**	**S**	**T**	**U**	**V**	**W**	**X**	**Y**	**Z**
23	24	25	26	27	28	29	30	31	32	33	34	35

	$16^3 = 4{,}096$	$16^2 = 256$	$16^1 = 16$	$16^0 = 1$
	$19{,}442 \div 4{,}095$	$3{,}058 \div 256$	$242 \div 16$	$2 \div 1$
Quotient	4	11 = B	15 = F	2
Remainder	3,058	242	2	0

19,442 converted to base 16 can be written by writing each of the quotients from left to right starting at the highest power of the base.

$$19{,}442 = 4\mathrm{BF}2_{16}$$

Additional problems

1. Write 1345 in base fifteen. **5EA$_{15}$**
2. Write 1345 in base thirteen. **7C6$_{13}$**

Objective 3 Convert between Binary and Octal and Binary and Hexadecimal Number Systems

Table - Octal to Binary

Octal	Binary
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

Hexadecimal to Binary Table

Hexadecimal	Binary
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001
A	1010
B	1011
C	1100
D	1101
E	1110
F	1111

In class activity

The game of Nim

Source: **http://ocw.mit.edu/courses/urban-studies-and-planning/11-124-introduction-to-education-looking-forward-and-looking-back-on-education-fall-2011/math-science-education/week-7/MIT11_124F11_nim_handout.pdf**

Source:**http://education.jlab.org/nim/s_gamepage.html**

In class discussion

Source: http://www.businessinsider.com/the-martian-hexidecimal-language-2015-9

The link above is to an article talking about the use of hexadecimal in the 2015 movie *The Martian*. If you have the movie you can show the clip and talk about how this was used to relay messages from Mars to Earth by Mark Watney.

EXAMPLE 5 Convert Between Binary and Octal

Convert 1101110011_2 to octal.

Separate the base 2 number from right to left into groups of three digits, adding zeros if necessary.

1	101	110	011
001	101	110	011

Use the table below to change each group of three binary numbers to octal numbers.

Octal	Binary
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

Binary Number	1	101	110	011
Binary Number (zeros added)	001	101	110	011
Corresponding Octal Number	1	5	6	3

$$1101110011_2 = 1{,}563_8$$

Additional problems

1. Convert 110001101_2 to octal. **615_8**
2. Convert 10000100001_2 to octal.**2041_8**

EXAMPLE 6 Convert Between Binary and Hexadecimal

Convert 1101110011_2 to hexadecimal.

When going from base 2 to hexadecimal, we want to separate the base 2 number into groups of four digits, going from right to left adding zeros as necessary.

Base 2 Number	0011	0111	0011
Corresponding hexadecimal number	3	7	3

Hexadecimal to Binary Table

Hexadecimal	Binary
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001
A	1010
B	1011
C	1100
D	1101
E	1110
F	1111

$$1101110011_2 = 373_{16}$$

In class activity

Source: **http://cse4k12.org/binary/bitmaps.html**

Source: **http://csunplugged.org/binary-numbers/**

If you have the space for a physical activity, have two (or more) teams (of 4 or 5 students each) line up.

- Each student represents a bit, with the student on one end being the bit in the 1's place, the next student representing the 2's place, the next the 4's place, etc.
- The students start in a standing position, which represents neither 1 nor 0. To represent a 1, the student's arms must be stretched straight overhead; to represent a 0, the student must squat down.
- You then call a number (one that can be represented using that many bits). The two teams then race to get their team to represent that number. The first team to have it correct gets a point. They normalize (all stand with no arms up) and a new number is called.
- Ask them about patterns they find during the game. (The 1's place student should notice that if the number is odd, their arms are up, but if the number is even, they are squatting. The student representing the highest bit should notice that their arms are up if the number is larger than or equal to their place value.)

In class activity

http://cse4k12.org/

Several activities for converting between binary and octal and binary and hexadecimal.

2.3 Computation in Other Bases

KEY TERMS	
Dividend	A dividend is the quantity to be divided.
Divisor	A divisor is the quantity by which another quantity, the dividend, is to be divided by.
Quotient	A quotient is the answer to a division problem.
Regrouping	Regrouping is a process of shifting a place value of the base from one place value column to the next. Regrouping is also called carrying or borrowing
Remainder	A remainder is the amount left over after division.

Objective 1 Add in Bases Other than 10

Adding in Bases Other than 10

When ADDING with other bases, follow these steps:

1. If the numbers are not already arranged vertically, place them vertically, with each place value in the same column.
2. Add the ones digits first, just as in base 10.
3. Evaluate the sum.
   a. If the sum is less than the base, write it under that column.
   b. If the sum is greater than or equal to the base, we must carry.

Find the remainder after dividing by the base and write it under the column.
To find the amount to carry, find out how many times the base will go into the sum evenly.

4. Continue adding the next digits as described in step 2 until all numbers are added. Remember to add in any carried numbers.

EXAMPLE 1 Add in Different Bases: Base 8

Add $376_8 + 334_8$.

$$\begin{array}{rrrr} & 3 & 7 & 6_8 \\ + & 3 & 3 & 4_8 \\ \hline \end{array}$$

Step 1:

$6 + 4 = 10$

$10 > 8$

$10 \div 8 = 1\,R2$

$$\begin{array}{rrrr} & & 1 & \\ & 3 & 7 & 6_8 \\ + & 3 & 3 & 4_8 \\ \hline & & & 2_8 \end{array}$$

Step 2:

$1 + 7 + 3 = 11$

$11 > 8$

$11 \div 8 = 1\,R3$

$$\begin{array}{rrrr} & 1 & 1 & \\ & 3 & 7 & 6_8 \\ + & 3 & 3 & 4_8 \\ \hline & & 3 & 2_8 \end{array}$$

Step 3:

$1 + 3 + 3 = 7$

$7 < 8$

$$\begin{array}{rrrr} & 1 & 1 & \\ & 3 & 7 & 6_8 \\ + & 3 & 3 & 4_8 \\ \hline & 7 & 3 & 2_8 \end{array}$$

Additional problems

1. $146317_7 + 6532_7$ **24463_7**
2. $101101_2 + 110010_2$ **1011111_2**

EXAMPLE 2 Add in Different Bases: Base 12

Add $445_{12} + 3A6_{12}$

$$\begin{array}{r} 4\ 4\ 5_{12} \\ +\ 3\ A\ 6_{12} \\ \hline \end{array}$$

Step 1:
5 + 6 = 11
11 < 12, so we can just write the value.
11 corresponds to B.

A	B	C	D	E	F	G	H	I	J	K	L	M
10	11	12	13	14	15	16	17	18	19	20	21	22
N	**O**	**P**	**Q**	**R**	**S**	**T**	**U**	**V**	**W**	**X**	**Y**	**Z**
23	24	25	26	27	28	29	30	31	32	33	34	35

$$\begin{array}{r} 4\ 4\ 5_{12} \\ +\ 3\ A\ 6_{12} \\ \hline B_{12} \end{array}$$

Step 2:
4 + A

A	B	C	D	E	F	G	H	I	J	K	L	M
10	11	12	13	14	15	16	17	18	19	20	21	22
N	**O**	**P**	**Q**	**R**	**S**	**T**	**U**	**V**	**W**	**X**	**Y**	**Z**
23	24	25	26	27	28	29	30	31	32	33	34	35

A corresponds to 10.
4 + 10 = 14
14 > 12
14 ÷ 12 = 1 *R*2

$$\begin{array}{r} 1\phantom{\ 4\ 5_{12}} \\ 4\ 4\ 5_{12} \\ +\ 3\ A\ 6_{12} \\ \hline 2\ B_{12} \end{array}$$

Step 3:
1 + 4 + 3 = 8
8 < 12

$$\begin{array}{r} 1\phantom{\ 4\ 5_{12}} \\ 4\ 4\ 5_{12} \\ +\ 3\ A\ 6_{12} \\ \hline 8\ 2\ B_{12} \end{array}$$

Additional problems

1. $AB1_{12} + 315_{12}$ **1206_{12}**
2. $8C51_{16} + 947B_{16}$ **$120CC_{16}$**

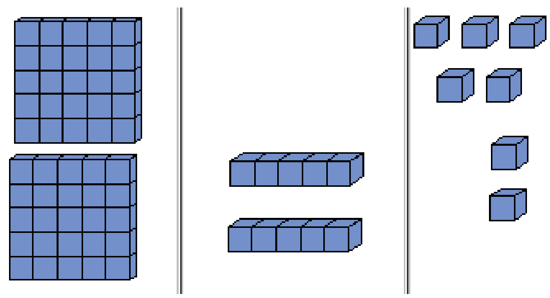
Objective 2 Subtract in Bases Other Than 10

Subtract 906 – 457 without a calculator showing your steps:

$$\begin{array}{rccc} & 8 & 9 & 16 \\ & \cancel{9} & \cancel{0} & \cancel{6} \\ - & 4 & 5 & 7 \\ \hline & \mathbf{4} & \mathbf{4} & \mathbf{9} \end{array}$$

Concept video questions and answers

1. **Draw the model for 232_5 labeling the columns.**



2. **When regrouping (or borrowing) what are you doing?**
   Taking a group that is the size of the base to add to another place value column.

Subtracting in Bases Other Than 10

When SUBTRACTING with other bases, follow these steps:

1. Align the numbers vertically, with each place value in the same column.
2. Subtract the ones digits first, just as in base 10. If you need to borrow from the first nonzero digit, make sure to borrow the amount of the base each time.
3. Borrow in the amount of the base until each digit in the top row is greater than the digits in the bottom row.
4. Subtract each column.
5. Check using addition in the given base.

EXAMPLE 3 Subtract in Different Bases: Base 5

Subtract $400_5 - 244_5$

25's 5's 1's 3 5 ~~4~~ $^{5+}0$ 0 − 2 4 4 25's 5's 1's 4 3 ~~5~~ 5 ~~4~~ $^{5+}0$ $^{5+}0$ − 2 4 4	In base 5 our place value columns are 1's, 5's and 25's. Each time we borrow we are borrowing a group of 5. **1) Right most column - do you need to borrow?** **A.** Because 4 is more than 0 we need to borrow. **B.** We cannot take anything from the 5's column because it is 0. **C.** We have to borrow from the 4 in the 25's column. **D.** Take one from the 4. This turns the 4 into a 3. We are adding 5 − 5's to the 0 in the 5's column, giving us 5 in this column. **E.** Take one from the 5 in the 5's column. This turns the 5 into a 4. We are adding 5 − 1's to the ones column, giving us 5.
25's 5's 1's 4 3 ~~5~~ 5 ~~4~~ $^{5+}0$ $^{5+}0$ − 2 4 4 1 0 1 $400_5 - 244_5 = 101_5$	2) Subtract each column. A. $5 - 4 = 1$ B. $4 - 4 = 0$ C. $3 - 2 = 1$

Additional problems

1. $1001_2 - 110_2$
2. $476_8 - 267_8$

EXAMPLE 4 Subtract in Different Bases: Base 16

Subtract $157A_{16} - 21C_{16}$

A	B	C	D	E	F	G	H	I	J	K	L	M
10	11	12	13	14	15	16	17	18	19	20	21	22
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
23	24	25	26	27	28	29	30	31	32	33	34	35

$$\begin{array}{rcccc} & 1 & 5 & 7 & A_{16} \\ - & & 2 & 1 & C_{16} \\ \hline \end{array}$$

$$\begin{array}{rcccc} & & & 6 & 16+10 \\ & 1 & 5 & \not{7} & \not{A}(\not{10}) \\ - & & 2 & 1 & C(12) \\ \hline & & & & \mathbf{E_{16}} \end{array}$$

$$\begin{array}{rcccc} & & & 6 & 16+10 \\ & 1 & 5 & \not{7} & \not{A}(\not{10}) \\ - & & 2 & 1 & C(12) \\ \hline & & & \mathbf{5} & \mathbf{E_{16}} \end{array}$$

$$\begin{array}{rcccc} & & & 6 & 16+10 \\ & 1 & 5 & \not{7} & \not{A}(\not{10}) \\ - & & 2 & 1 & C(12) \\ \hline & \mathbf{1} & \mathbf{3} & \mathbf{5} & \mathbf{E_{16}} \end{array}$$

Additional problems

1. $4C6_{16}$ - 198_{16} **$32E_{16}$**
2. $97A_{12}$ – $3B8_{12}$ **582_{12}**

Objective 3 Multiply in Bases Other than 10

Multiplying in Bases Other Than 10

When MULTIPLYING with other bases, follow these steps:

1. If numbers are not already arranged vertically, place them vertically, aligning each one's place.
2. Carry out the multiplication for each column using the base multiplication table and carrying when necessary.
3. Add each product to get the final answer.

EXAMPLE 5 Multiply in Different Bases: Base 6

Multiply: $314_6 \times 23_6$

Step 1: Complete a multiplication table for the base in which you are multiplying.

X	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	10	12	14
3	0	3	10	13	20	23
4	0	4	12	20	24	32
5	0	5	14	23	32	41

Step 2: Carry out the multiplication using the base multiplication table.

$$\begin{array}{rrrr} & & 2 & \\ & 3 & 1 & 4_6 \\ \times & & 2 & 3_6 \\ \hline & & & 0_6 \end{array}$$

Multiply 3×4 which from our table is 20. Write the 0 in the ones column and carry the 2.

$$\begin{array}{rrrr} & & 2 & \\ & 3 & 1 & 4_6 \\ \times & & 2 & 3_6 \\ \hline & & 5 & 0_6 \end{array}$$

Multiply $3 \times 1 = 3$ and add the 2 to get 5.

$$\begin{array}{rrrr} & & 2 & \\ & 3 & 1 & 4_6 \\ \times & & 2 & 3_6 \\ \hline 1 & 3 & 5 & 0_6 \\ & & & 0_6 \end{array}$$

Multiply 3×3 which is 13 from our table. We write 13 and then bring down a placeholder zero.

$$\begin{array}{rrrr} & & 1 & \\ & & \cancel{2} & \\ & 3 & 1 & 4_6 \\ \times & & 2 & 3_6 \\ \hline 1 & 3 & 5 & 0_6 \\ & & 2 & 0_6 \end{array}$$

Now multiply 2×4 which is 12 from our table. Write the 2 below the 5 and carry the 1.

$$\begin{array}{rrrr} & & 1 & \\ & & \cancel{2} & \\ & 3 & 1 & 4_6 \\ \times & & 2 & 3_6 \\ \hline 1 & 3 & 5 & 0_6 \\ & 3 & 2 & 0_6 \end{array}$$

Now multiply 2×1 which is 2 and add the 1 to get 3. Write this below the 3.

$$\begin{array}{rrrrr} & & & 1 & \\ & & & \cancel{2} & \\ & & 3 & 1 & 4_6 \\ & \times & & 2 & 3_6 \\ \hline & 1 & 3 & 5 & 0_6 \\ 1 & 0 & 3 & 2 & 0_6 \end{array}$$

Multiply 2×3 which is 10 from the multiplication table. Write this below the 1.

$$\begin{array}{rrrrr} & & & 1 & \\ & & & \cancel{2} & \\ & & 3 & 1 & 4_6 \\ & \times & & 2 & 3_6 \\ \hline & & 1 & & \\ & 1 & 3 & 5 & 0_6 \\ 1 & 0 & 3 & 2 & 0_6 \\ \hline & & & 1 & 0_6 \end{array}$$

Now we want to add each column, remembering we are adding in base 6.

$$0 + 0 = 0$$
$$5 + 2 = 11_6$$

Write the 1 and carry a 1.

$$\begin{array}{rrrrr} & & & 1 & \\ & & & \cancel{2} & \\ & & 3 & 1 & 4_6 \\ & \times & & 2 & 3_6 \\ \hline & 1 & 1 & & \\ & 1 & 3 & 5 & 0_6 \\ 1 & 0 & 3 & 2 & 0_6 \\ \hline & & 1 & 1 & 0_6 \end{array}$$

$$1 + 3 + 3 = 11_6$$

Write the 1 and carry a 1.

$$
\begin{array}{rrrrr}
 & & & 1 & \\
 & & & 2 & \\
 & & 3 & 1 & 4_6 \\
 & \times & & 2 & 3_6 \\
\hline
 & 1 & 1 & & \\
 & 1 & 3 & 5 & 0_6 \\
1 & 0 & 3 & 2 & 0_6 \\
\hline
1 & 2 & 1 & 1 & 0_6
\end{array}
$$

$1 + 1 = 2_6$

Bring down the last 1.

ALTERNATE METHOD
Convert both numbers to base 10, multiply them normally, then convert that number back to the desired base.

$$314_6 \times 23_6$$

- Convert each number to base 10.

$$314_6 = 3(6^2) + 1(6) + 4 = 118$$

$$23_6 = 2(6) + 3 = 15$$

- Multiply the base 10 numbers.

$$118 \times 15 = 1{,}770$$

- Convert the answer back to base 6.
  $1{,}770 = 12110_6$

Additional problems

1. 21_3 x 2_3 **112_3**
2. $6A3_{16}$ x 24_{16} **$EEEC_{16}$**

Objective 4 Divide in Bases Other Than 10

Dividing in Bases Other Than 10

When DIVIDING with other bases, follow these steps:

1. Determine how many times the divisor can divide into the dividend without going over and write this quotient in the correct place value column.
2. Multiply the quotient by the divisor and write the answer under the dividend.
3. Subtract and bring down the next digit on the right.
4. Repeat until there are no more digits to bring down. There will be a remainder if, after the last subtraction, the difference is not equal to zero.
5. Check by multiplying the quotient by the divisor and adding the remainder in the given base.

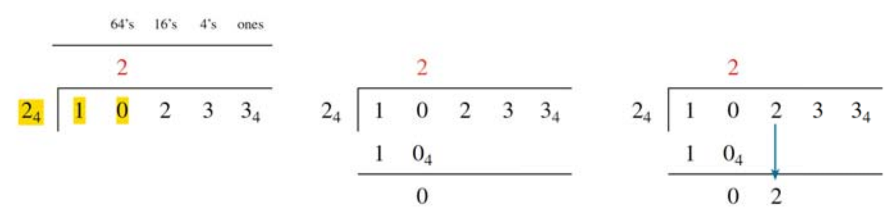
EXAMPLE 6 Divide in Different Bases: Base 4

Divide 10233_4 by 2_4

Find all the multiplication facts for 2_4. Because this problem is in base 4 which only includes the numbers 0, 1, 2, and 3, we will only need the multiplication facts up to 2×3.

$$2 \times 0 = 0_4$$
$$2 \times 1 = 2_4$$
$$2 \times 2 = 10_4$$
$$2 \times 3 = 12_4$$

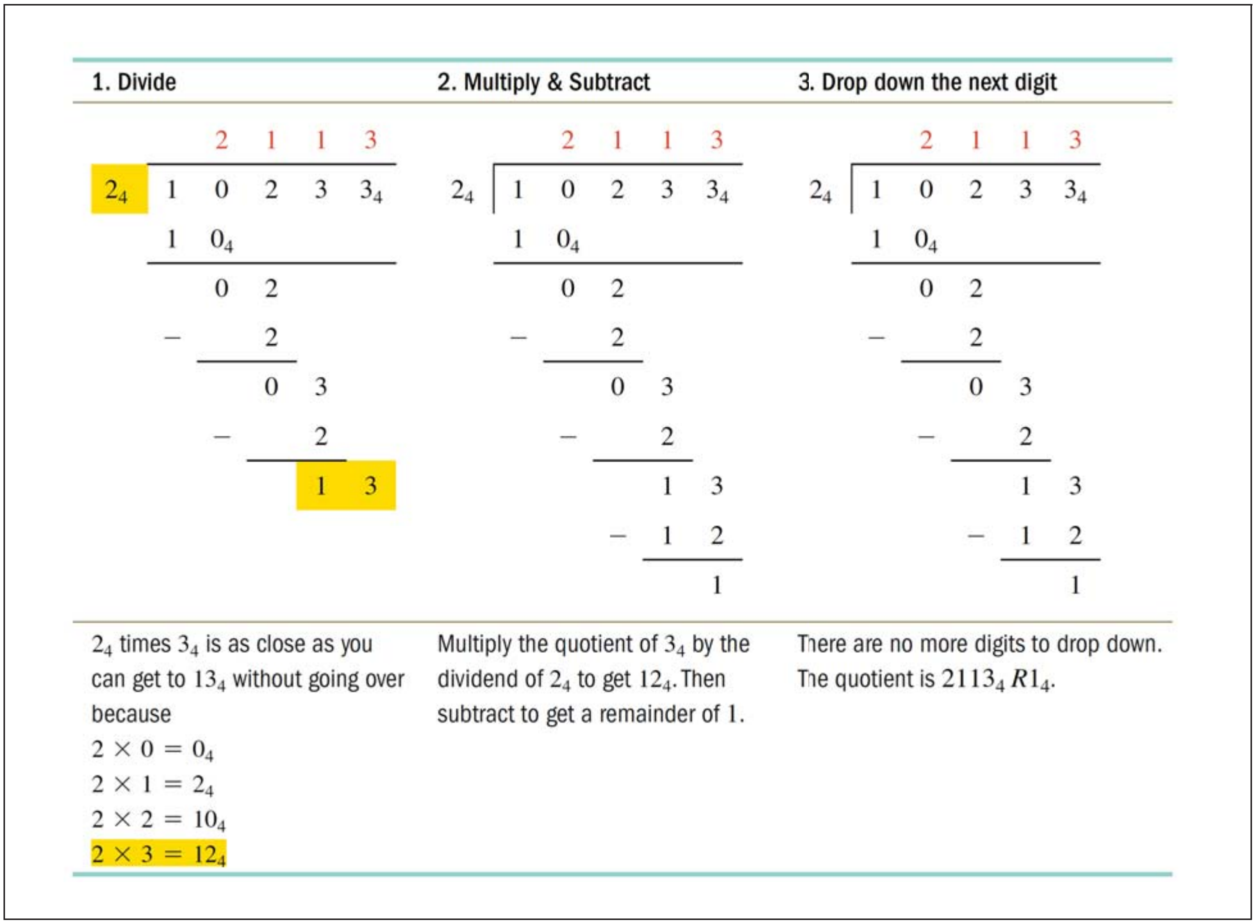
1. Divide	2. Multiply & Subtract	3. Drop down the next digit



2_4 times 2_4 is as close as you can get to 10_4 without going over because $2 \times 0 = 0_4$ $2 \times 1 = 2_4$ $2 \times 2 = 10_4$ $2 \times 3 = 12_4$	Multiply the quotient of 2_4 by the dividend 2_4 to get 10_4. Subtract to find the remainder of zero.	Drop down the 2.

1. Divide	2. Multiply & Subtract	3. Drop down the next digit
2 1 2_4 ⟌ 1 0 2 3 3_4 1 0_4 0 2	2 1 2_4 ⟌ 1 0 2 3 3_4 1 0_4 0 2 − 2 0	2 1 2_4 ⟌ 1 0 2 3 3_4 1 0_4 0 2 2 0 3
2_4 times 1_4 is as close as you can bet to 2_4 without going over because $2 \times 0 = 0_4$ $2 \times 1 = 2_4$ $2 \times 2 = 10_4$ $2 \times 3 = 12_4$	Multiply the quotient of 1_4 by the dividend 2_4 to get 2_4. Subtract to find the remainder of zero.	Drop down the 3.

1. Divide	2. Multiply & Subtract	3. Drop down the next digit
2 1 1 2_4 ⟌ 1 0 2 3 3_4 1 0_4 0 2 − 2 0 3	2 1 1 2_4 ⟌ 1 0 2 3 3_4 1 0_4 0 2 − 2 0 3 − 2 1	2 1 1 2_4 ⟌ 1 0 2 3 3_4 1 0_4 0 2 − 2 0 3 − 2 1 3
2_4 times 1_4 is as close as you can get to 3_4 without going over because $2 \times 0 = 0_4$ $2 \times 1 = 2_4$ $2 \times 2 = 10_4$ $2 \times 3 = 12_4$	Multiply the quotient of 1_4 by the dividend 2_4 to get 2_4. Subtract to find the remainder of 1.	Drop down the 3.



Teaching note

Source: http://www.dozenal.org/articles/DSA-Mult.pdf
This is a pdf of several different base multiplication tables to use as a reference.

EXAMPLE 7 Divide in Different Bases: Base 6

Divide 2430_6 by 4_6

To divide 2430 base 6 by 4 base 6, we want to first write the problem as a long division where 4 base 6 is the divisor and goes on the outside, and 2430 base 6 is the dividend and goes under the division bar.

$$4_6 \overline{)\,2\ 4\ 3\ 0_6}$$

Find all the multiplication facts for 4_6. We only need to go up to 4×5 because we are working in base 6 which only uses the values 0, 1, 2, 3, 4, and 5.

$4 \times 0 = 0_6$
$4 \times 1 = 4_6$
$4 \times 2 = 12_6$
$4 \times 3 = 20_6$
$4 \times 4 = 24_6$
$4 \times 5 = 32_6$

Looking at the table, we see $4 \times 4 = 24_6$,

$4 \times 0 = 0_6$
$4 \times 1 = 4_6$
$4 \times 2 = 12_6$
$4 \times 3 = 20_6$
$4 \times 4 = 24_6$
$4 \times 5 = 32_6$

4 base 6 divides into 24 base 6 four times. Multiplying 4 times 4 in base 6 is 24. We write this below and then subtract 24 minus 24 to get 0. Now, we want to bring down the 3.

$$\begin{array}{r|cccc} & & 4 & & \\ \hline 4_6 & 2 & 4 & 3 & 0 \\ - & 2 & 4 & & \\ \hline & & 0 & 3 & \end{array}$$

4 base 6 cannot divide into 3, so we write a zero and bring down the zero.

$$\begin{array}{r|cccc} & & 4 & 0 & \\ \hline 4_6 & 2 & 4 & 3 & 0 \\ - & 2 & 4 & & \\ \hline & & 0 & 3 & 0 \end{array}$$

4 base 6 divides into 30 four times. It cannot go five times because 32 is bigger than 30.

$4 \times 0 = 0_6$
$4 \times 1 = 4_6$
$4 \times 2 = 12_6$
$4 \times 3 = 20_6$
$4 \times 4 = 24_6$
$4 \times 5 = 32_6$

Multiplying 4 times 4 gives us 24 in base 6. We write this below the 30 and subtract.

$$
\begin{array}{rrrrr}
 & & 4 & 0 & 4 \\ \cline{2-5}
4_6 & \multicolumn{1}{|r}{2} & 4 & 3 & 0 \\
- & 2 & 4 & & \\ \cline{2-5}
 & & 0 & 3 & 0 \\
 & - & & 2 & 4 \\ \cline{3-5}
\end{array}
$$

We can’t subtract 4 from zero, so we will need to borrow. Remember when you are doing the subtraction that you are doing this in base 6, so when you borrow, you are borrowing groups of 6. When we borrow 1 from the 3, it becomes 2, but then we need to add a group of 6 to the ones place.

$$
\begin{array}{rrrrr}
 & & 4 & 0 & 4 \\ \cline{2-5}
4_6 & \multicolumn{1}{|r}{2} & 4 & 3 & 0 \\
- & 2 & 4 & & \\ \cline{2-5}
 & & 0 & \overset{2}{\not{3}} & 6+0 \\
 & & - & 2 & 4 \\ \cline{4-5}
 & & & & 2
\end{array}
$$

Adding 6 and zero then subtracting 4 gives us a remainder of 2.

Dividing 2430 base 6 by 4 base 6 give us $404_6 R2_6$.

Additional problems

1. $1110_4 \div 3_4$ **130_4**
2. $D80C_{16} \div B_{16}$ **$13A4_{16}$**

2.4 Early Computational Methods

KEY TERMS	
Egyptian Algorithm for Multiplication	The Egyptian algorithm for multiplication is a procedure for multiplying two numbers which uses only the ability to add and multiply by two.
Lattice	A lattice is a grid used in lattice multiplication, which is constructed based on the number of digits being multiplied.
Lattice Method	The lattice method is an alternative way to multiply numbers using a lattice that is constructed based on the number of digits being multiplied.
Multiplicand	A multiplicand is the number that gets multiplied.
Multiplier	A multiplier is the number that gets multiplied to the multiplicand.
Napier's Rods	Napier's rods (bones) use the multiplication tables embedded in the rods to reduce multiplication to addition operations.
Russian Peasant Method	The Russian peasant method for multiplication involves a process of halving the multiplicand while doubling the multiplier to determine a product of two numbers.
Standard Algorithm	The standard algorithm is the algorithm for multiplication known as long multiplication.

Objective 1 Multiply Using the Egyptian Algorithm

Egyptian Algorithm for Multiplication

$$A \times B$$

1. Create two columns with one number (A) at the top of the first column and the other number (B) at the top of the second column. In this example, $A = 21$, $B = 18$.
2. Below the first number (A), write all of the powers of 2 that are smaller than or equal to the first number starting with 1.
3. Below the second number (B), double the second number until you reach the same row corresponding to the highest power of 2.

21	×	18	= 378
1		18	
2		36	
4		72	
8		144	
16		288	

4. In column A, find the numbers that sum to A, using each number at most once. In this example, the number at the top of column A is 21, $21 = 16 + 4 + 1$
5. Mark the rows in column A and the corresponding numbers in column B.

   In this example, we have 1, 4, 16 in column A corresponding to 18, 72, and 288 in column B.
6. Sum the marked numbers from column B.

$$\begin{array}{r} 18 \\ 72 \\ +288 \\ \hline 378 \end{array}$$

EXAMPLE 1 Multiply using the Egyptian Algorithm

Multiply 13 x 23 using the Egyptian algorithm.

13	23

To multiply 13×23 using the Egyptian algorithm, we will first create two columns with 13 in the first column and 23 in the 2nd column.

13	23
1	
2	
4	
8	

Below 13 we will write all of the powers of 2 that are smaller than or equal to 13. Each time you are just multiplying by 2.

13	23
1	23
2	46
4	92
8	184

Below 23, we will start with 23 and double each number until we get to the last row.

13	23
1	23
2	46
4	92
8	184

Now we want to go back to the first column and figure out what numbers add up to give us 13. In this case we have $8 + 4 + 1 = 13$. These are highlighted.

13	23
1	23
2	46
4	92
8	184

Finally we will add the corresponding values in the 2^{nd} column to get $23 + 92 + 184 = 299$.

Using the Egyptian algorithm we get $13 \times 23 = 299$.

Additional problems

1. 18 x 52 **936**

18	52
1	52
2	104
4	208
8	416
16	832

2. 34 x 105 **3570**

34	105
1	105
2	210
4	420
8	840
16	1680
32	3360

Objective 2 Multiply Using the Russian Peasant Method

Russian Peasant Method of Multiplication

$$A \times B$$

1. Write each number (A and B) at the top of its own column
2. Double the number in the first column and halve the number in the second column. If the number in the second column is odd, divide it by two and drop the remainder.
3. If the number in the first column is even, cross out that entire row.
4. Keep doubling, halving, and crossing out until the number in the second column is 1.
5. Add up the remaining numbers in the second column, including the number at the top of column B. The total is the product of your original numbers.

EXAMPLE 2 Multiply Using the Russian Peasant Method

Multiply 29 x 49 using the Russian Peasant method.

Multiply 29×49 using the Russian peasant method.

Write each number at the top of a column in a table. 49 is odd so we will keep the 1st row.

29	49

We want to double the number in the first column, and halve the number in the second column, dropping any remainder. Our goal is to get to 1 in the second column.

29	49
58	24

24 is even so we will cross out the entire row.

29	49
~~58~~	~~24~~

We will double 58 and halve 24 to get 116 and 12. Again, 12 is even so we will cross off the entire row.

29	49
~~58~~	~~24~~
~~116~~	~~12~~

29	49
~~58~~	~~24~~
~~116~~	~~12~~
~~232~~	~~6~~

Doubling 116 we get 232 and halving 12 we get 6. 6 is also even so we cross out the entire row.

29	49
~~58~~	~~24~~
~~116~~	~~12~~
~~232~~	~~6~~
464	3

Doubling 232 we get 464 and halving 6 we get 3. Three is not even so we will not cross out the row this time.

29	49
~~58~~	~~24~~
~~116~~	~~12~~
~~232~~	~~6~~
464	3
928	1

Doubling 464 we get 928 and halving 3 but dropping the remainder, we get 1.

Now that the second column is at 1 we are done and can add up each number in the 1st column that is not crossed out.

29	49
~~58~~	24
~~116~~	12
~~232~~	6
464	3
928	1

$29 + 464 + 928 = 1{,}421$

$29 \times 49 = 1{,}421$

Additional problems

1. 34 x 105 **3,570**

34	105
~~68~~	~~52~~
~~136~~	~~26~~
272	13
~~544~~	~~6~~
1,088	3
2,176	1

2. 16 x 135 **2,160**

16	135
32	67
64	33
~~128~~	~~16~~
~~256~~	~~8~~
~~512~~	~~4~~
~~1,024~~	~~2~~
2,048	1

Objective 3 Multiply Using the Lattice Method

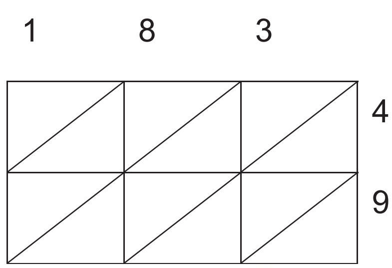
Lattice Method of Multiplication

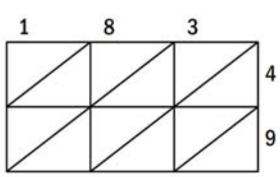
$$A \times B$$

1. Draw a grid with one box for each digit in the product and a diagonal through each box from the upper right corner to the lower left corner (see **lattice**).
2. Write one multiplier across the top and the other down the right side, lining the digits up with the boxes.
3. Record each partial product as a two-digit number with the tens digit in the upper left and the ones digit in the lower right of each box. If the product does not have a tens digit, record a zero in the tens triangle.
4. When all partial products are complete, sum the numbers along the diagonals.
5. Carry double digits to the next place and record the answer.

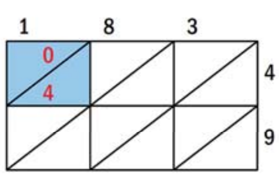
EXAMPLE 3 Multiply using the Lattice Method

Multiply 183 x 49 using the lattice method.



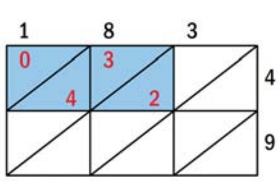


1. Set up lattice with 1 box for each digit in the product and a diagonal in each box. This one will have 3 columns because 183 has 3 digits and 2 rows because 49 has 2 digits.

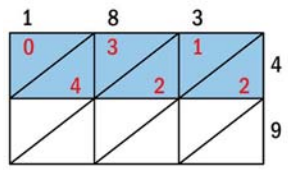


2. Multiply each individual product placing one digit in each part of the box.

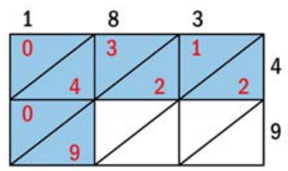
1 times 4 is 4 so we write 0 on top and 4 below.



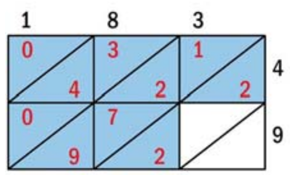
8 times 4 is 32. Write 3 on top and 2 below.



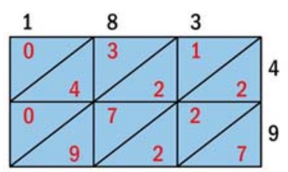
3 times 4 is 12. Write 1 on top and 2 below.



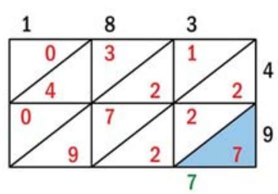
1 times 9 is 9. Write 0 on top and 9 below.



8 times 9 is 72. Write 7 on top and 2 below.

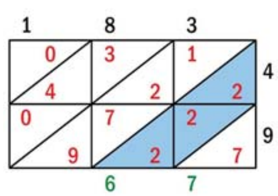


Finally, 3 times 9 is 27. Write 2 on top and 7 below.

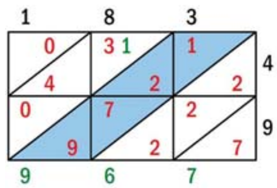


Add along each diagonal.

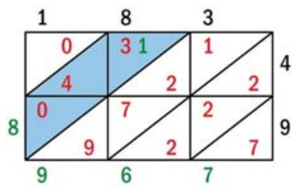
First in the bottom right corner, 7 has nothing to add so we just write a 7.



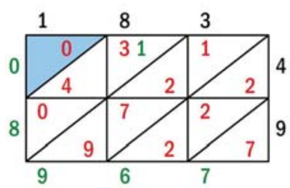
On the next diagonal over we add 2 plus 2 plus 2 to get 6.



The third diagonal we will add 1 plus 2 plus 7 plus 9 to get 19. We will write the nine but carry the one to the next diagonal.



Now we will add along the fourth diagonal. 1 plus 3 plus 4 is 8.

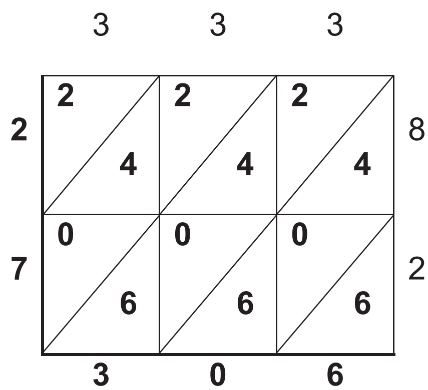


Finally the last diagonal only has a zero so we will just write a zero.

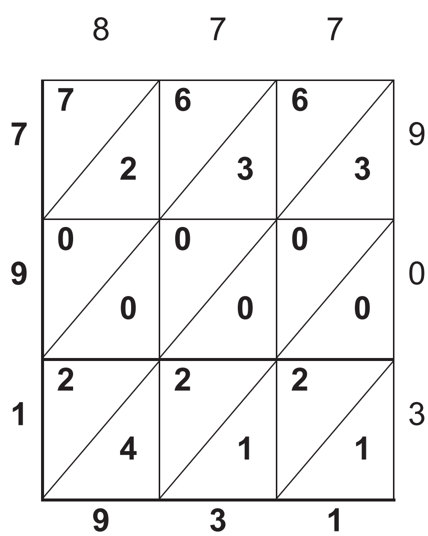
We get the answer to the multiplication problem by reading from left to right. $183 \times 49 = 8{,}967$.

Additional problems

1. 333 x 82 **27,306**



2. 877 x 903 **791,931**



Objective 4 Multiply Using Napier's Rods

Concept video question and answer

1. **Explain how to multiply 497×7 using Napier's rods.**

   Select the rods 4, 9, and 7 and place them next to the index rod.

   In the 7 row, the number on the very right is the ones digit for our answer.

   The next two numbers are together inside a parallelogram, and added together will form the tens digit, which will be 4 +3 = 7.

   The sum of the next two numbers, also in a parallelogram forms the hundreds digit. In this case, 6 + 8 =14 and is not just one single digit. So, we keep the 4 and carry the 1 to the next diagonal.

   The last digit plus the 1 we carried is the thousands digit. So we have 1 + 2 which is 3. Reading from left to right, the result of 497 times 7 is 3,479

EXAMPLE 4 Multiply Using Napier's Rods

Multiply 259 x 42 using Napier's rods.

1	2	5	9
2	0/4	1/0	1/8
3	0/6	1/5	2/7
4	0/8	2/0	3/6
5	1/0	2/5	4/5
6	1/2	3/0	5/4
7	1/4	3/5	6/3
8	1/6	4/0	7/2
9	1/8	4/5	8/1

To multiply using Napier's bones, we want to select the rods 2, 5, and 9 and place them next to the index rod.

$42 = 40 + 2$, so we have

$259(40 + 2) = 259(40) + 259(2)$

We are multiplying by 42 so we want to think about this as 40 plus 2.

Step 1: 259×40

1	2	5	9
2	0/4	1/0	1/8
3	0/6	1/5	2/7
4	0/8	2/0	3/6
5	1/0	2/5	4/5
6	1/2	3/0	5/4
7	1/4	3/5	6/3
8	1/6	4/0	7/2
9	1/8	4/5	8/1

Look at the row that has 4 in it. We will use this to complete the first multiplication of 259 times 40.

We have 1 zero at the end of 40 so we will write the zero

Step 1: $259 \times 40 =$ ______0

1	0 1 / 8	2 / 0	3 / 6
	0	3	6

Now we will add each of the diagonals starting at the right. We have 6, then 3 plus 0 is 3. 2 plus 8 is 10 so we will write a 0 and carry the 1. 1 plus 0 is 1. So 259 times 40 is 10,360

Step 1: $259 \times 40 = 10{,}360$

Step 2: 259×42

$(259 \times 40) + (259 \times 2)$

$10{,}360 + (259 \times 2)$

Now we want to multiply 259 times 2 so we look at the row that has 2.

1	2	5	9
2	0 / 4	1 / 0	1 / 8
3	0 / 6	1 / 5	2 / 7
4	0 / 8	2 / 0	3 / 6
5	1 / 0	2 / 5	4 / 5
6	1 / 2	3 / 0	5 / 4
7	1 / 4	3 / 5	6 / 3
8	1 / 6	4 / 0	7 / 2
9	1 / 8	4 / 5	8 / 1

Step 2: 259×2

	0 / 4	1 / 0	1 / 8
	5	1	8

Adding each of the diagonals going from right to left we first have 8.

Adding 1 plus 0 we get 1.

Adding 1 plus 4 we get 5.

$259 \times 2 = 518$

259 times 2 is 518.

From **Step 1:** $259 \times 40 = 10{,}360$

From **Step 2:** $259 \times 2 = \underline{+\quad 518}$

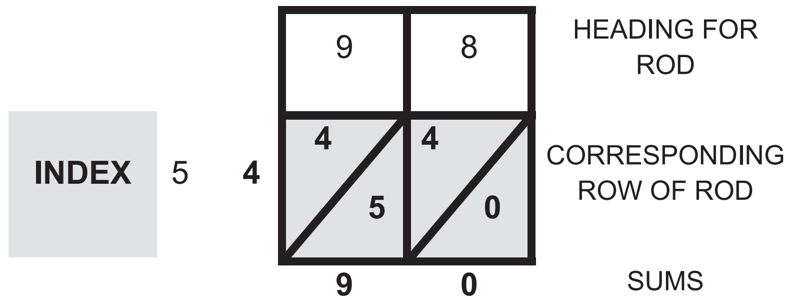
$10{,}878$

Adding the results from each step we add 10,360 plus 518 to get 10,878.

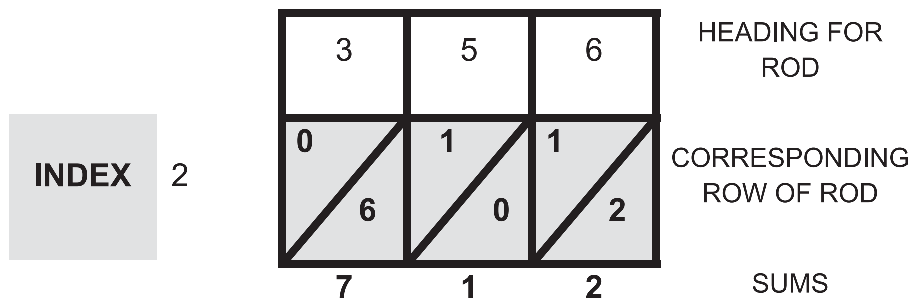
$259 \times 42 = 10{,}878$

Additional problems

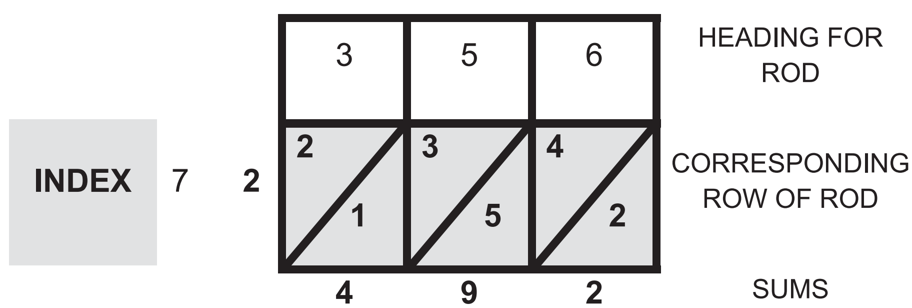
1. 5 x 98 **490**



2. 27 x 356 **9,612**



Because you are multiplying by 20 and not just 2, the answer for this part is 7,120



Adding together the results of both to get 7,120 + 2,492 = 9,612

<u>Napier's Rods cutouts</u>

1	1	2	3	4	5	6	7	8	9	0
2	0/2	0/4	0/6	0/8	1/0	1/2	1/4	1/6	1/8	0/0
3	0/3	0/6	0/9	1/2	1/5	1/8	2/1	2/4	2/7	0/0
4	0/4	0/8	1/2	1/6	2/0	2/4	2/8	3/2	3/6	0/0
5	0/5	1/0	1/5	2/0	2/5	3/0	3/5	4/0	4/5	0/0
6	0/6	1/2	1/8	2/4	3/0	3/6	4/2	4/8	5/4	0/0
7	0/7	1/4	2/1	2/8	3/5	4/2	4/9	5/6	6/3	0/0
8	0/8	1/6	2/4	3/2	4/0	4/8	5/6	6/4	7/2	0/0
9	0/9	1/8	2/7	3/6	4/5	5/4	6/3	7/2	8/1	0/0

NAME______________________________
CLASS DAY/TIME__________________

1. Express the given Hindu-Arabic numeral in expanded form: 1,588

2. Express the given Hindu-Arabic numeral in expanded form: 17,474

3. Express the given expanded numeral as a Hindu-Arabic numeral.

$$(3 \times 10^1) + (0 \times 1)$$

3. ________________

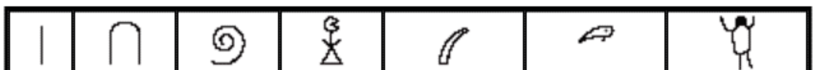
4. Express the given expanded numeral as a Hindu-Arabic numeral.

$$(4 \times 10^2) + (0 \times 10^1) + (7 \times 1)$$

4. ________________

5. Using the table, write the given Egyptian numeral as a Hindu-Arabic numeral.

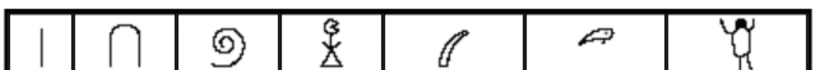
5. ________________



1	10	100	1000	10,000	100,000	1,000,000

6. Using the table, write the given Egyptian numeral as a Hindu-Arabic numeral.

6. ________________



1	10	100	1000	10,000	100,000	1,000,000

7. Using the table, write 332 as an Egyptian numeral.

𓏺	𓎆	𓍢	𓆼	𓂭	𓆐	𓁨
1	10	100	1000	10,000	100,000	1,000,000

8. Using the table, write 32,305 as an Egyptian numeral.

𓏺	𓎆	𓍢	𓆼	𓂭	𓆐	𓁨
1	10	100	1000	10,000	100,000	1,000,000

9. Use the table to write **CXLV** as a Hindu-Arabic numeral. 9. ______________

Roman Numerals	I	V	X	L	C	D	M
Hindu-Arabic Numerals	1	5	10	50	100	500	1000

10. Use the table to write **MMMCLIV** as a Hindu-Arabic numeral. 10. _______________

Roman Numerals	I	V	X	L	C	D	M
Hindu-Arabic Numerals	1	5	10	50	100	500	1000

11. Use the table to write **$\overline{\text{X}}$CDLXVIII** as a Hindu-Arabic numeral. 11. ______________

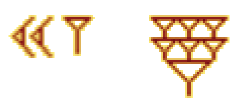
Roman Numerals	I	V	X	L	C	D	M
Hindu-Arabic Numerals	1	5	10	50	100	500	1000

12. Use the table to write $\overline{\mathbf{VIII}}$**CMLII** as a Hindu-Arabic numeral. 12. ____________

Roman Numerals	I	V	X	L	C	D	M
Hindu-Arabic Numerals	1	5	10	50	100	500	1000

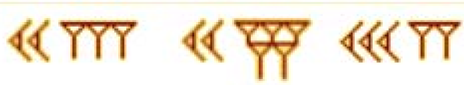
13. Use the table to express the following Babylonian numeral as a Hindu-Arabic
numeral: 13. ____________

Babylonian numerals		
Hindu-Arabic numerals	1	10



14. Use the table to express the following Babylonian numeral as a Hindu-Arabic
numeral: 14. ____________

Babylonian numerals		
Hindu-Arabic numerals	1	10



15. Using the table, write the Hindu-Arabic numeral 1,278 as a Babylonian numeral.

Babylonian numerals		
Hindu-Arabic numerals	1	10

16. Use the table to write 498 as a Chinese Numeral. 16.

一	二	三	四	五	六	七	八	九	十	百	千
1	2	3	4	5	6	7	8	9	10	100	1000

17. Use the table to write 9,408 as a Chinese Numeral. 17.

一	二	三	四	五	六	七	八	九	十	百	千
1	2	3	4	5	6	7	8	9	10	100	1000

18. Use the table to write the following Chinese Numeral as a Hindu-Arabic Numeral. 18. ____________

九
百
三
十
五

一	二	三	四	五	六	七	八	九	十	百	千
1	2	3	4	5	6	7	8	9	10	100	1000

18. Use the table to write the following Chinese Numeral as a Hindu-Arabic Numeral. 18. ____________

一
千
二
百
二
十
三

一	二	三	四	五	六	七	八	九	十	百	千
1	2	3	4	5	6	7	8	9	10	100	1000

19. Convert 1433_5 to a numeral in base ten. 19. ____________

20. Convert $9C51_{16}$ to a numeral in base ten. 20. ____________

21. Convert 477 to base three. 21. ____________

22. Convert 6,565 to base seven. 22. ____________

23. Convert the number 1001011001_2 from binary form to octal form. 23. ____________________

24. Convert the number 1010111001_2 from binary form to octal form. 24. ____________________

25. Convert the number 10000111001_2 from binary form to hexadecimal form. 25. ____________________

26. Convert the number 11100001011001_2 from binary form to hexadecimal form. 26. ____________________

27. Add the given numbers in the indicated base. 27. ____________________

$$5142_8 + 1756_8$$

28. Add the given numbers in the indicated base. 28. ____________________

$101101_2 + 110010_2$

29. Subtract the given numbers in the indicated base. 29. ____________________

$32_7 - 6_7$

30. Subtract the given numbers in the indicated base. 30. ____________________

$1001_3 - 121_3$

31. Multiply the given numbers in the indicated base. 31. ____________________

$457_8 \times 5_8$

32. Multiply the given numbers in the indicated base. 32. ____________________

$3122_4 \times 23_4$

33. Divide the given numbers in the indicated base.

$$534_7 \div 4_7$$

33. ____________________

34. Divide the given numbers in the indicated base.

$$1343_5 \div 4_5$$

34. ____________________

35. Use the Egyptian algorithm to find the product.

$$16 \cdot 187$$

35. ____________________

36. Use the Egyptian algorithm to find the product.

$$27 \cdot 235$$

36. ____________________

37. Multiply 51×12 using the Russian peasant method.

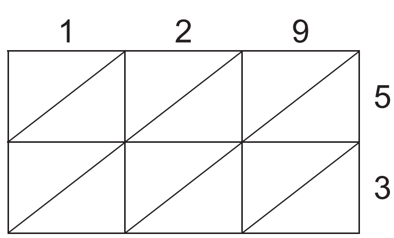
37. ____________________

38. Multiply 233×17 using the Russian peasant method.

38. ____________________

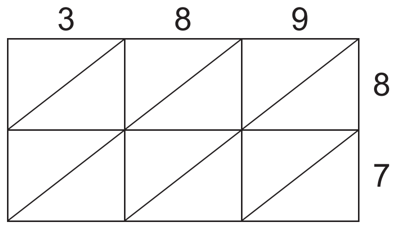
39. Multiply 129×53 using the lattice method.

39. ____________________

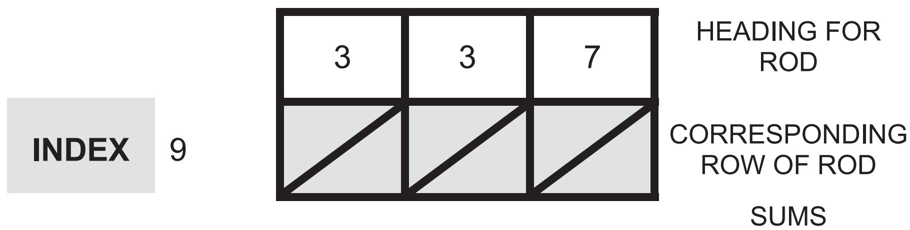


40. Multiply 389×87 using the lattice method.

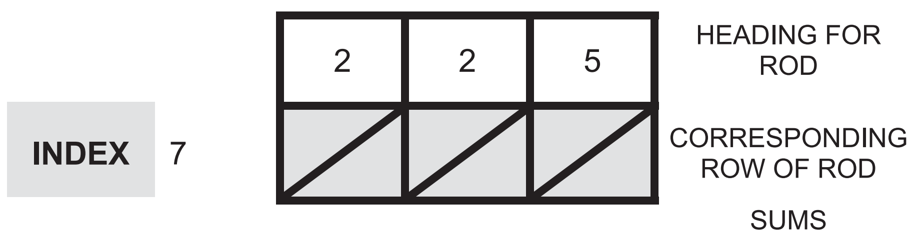
40. ____________________



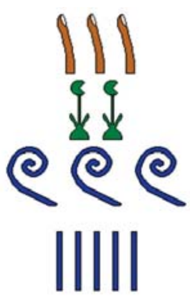
41. Multiply 9×337 using Napier's rods.



42. Multiply 7×225 using Napier's rods.



Module 2 Review Answers

1. $1(1{,}000) + 5(100) + 8(10) + 8$
2. $1(10{,}000) + 7(1{,}000) + 4(100) + 7(10) + 4$
3. 30
4. 407
5. 341
6. 2,300,432
7. 
8. 
9. 145
10. 3,154
11. 10,448
12. 8,952
13. 1,267
14. 84,332
15. 

16. 四
百
九
十
八

17. 九
千
四
百
零
八

18. 1,223

19. 243

20. 40,017

21. 122200_3

22. 25066_7

23. 1131_8

24. 1271_8

25. 439_{16}

26. 3859_{16}

27. 7120_8

28. 1011111_2

29. 23_7

30. 110_3

31. 2753_8

32. 211132_4

33. 124_7 R2_7

34. 210_5 R3_5

35. 2,992

16	187
1	187
2	374
4	748
8	1496
16	2992

36. 6,345

27	235
1	235
2	470
4	940
8	1880
16	3760

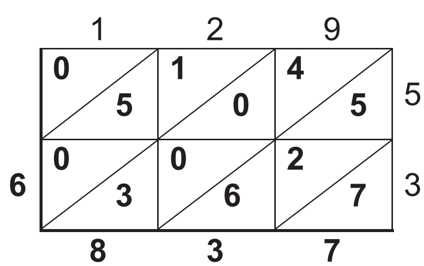
37. 612

~~51~~	~~12~~
~~102~~	~~6~~
204	3
408	1

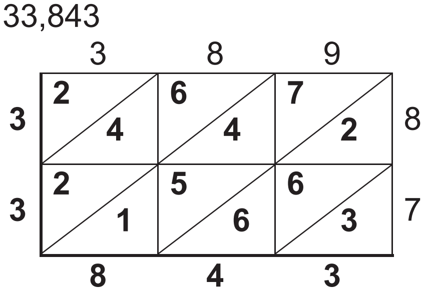
38. 3,961

233	17
~~466~~	~~8~~
~~932~~	~~4~~
~~1864~~	~~2~~
3728	1

39. 6,837



40. 33,843



NAME____________________
CLASS DAY/TIME____________

1. Express the given Hindu-Arabic numeral in expanded form: .
901

2. Write the Hindu-Arabic numeral in expanded form: 10,721

3. Express the given expanded numeral as a Hindu-Arabic numeral. 3. __________

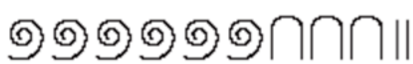
$$(5 \times 10^1) + (9 \times 1)$$

4. Express the given expanded numeral as a Hindu-Arabic numeral. 4. __________

$$(8 \times 10^2) + (6 \times 10^1) + (7 \times 1)$$

5. Using the table, write the given Egyptian numeral as a Hindu-Arabic numeral. 5. __________

1	10	100	1000	10,000	100,000	1,000,000



6. Using the table, write 2,441 as an Egyptian numeral. 6. __________

1	10	100	1000	10,000	100,000	1,000,000

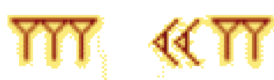
7. Use the table to write **DV** as a Hindu-Arabic numeral. 7. ____________

Roman Numerals	I	V	X	L	C	D	M
Hindu-Arabic Numerals	1	5	10	50	100	500	1000

8. Use the table to write $\overline{\mathbf{V}}$**CDLXVI** as a Hindu-Arabic numeral. 8. ____________

Roman Numerals	I	V	X	L	C	D	M
Hindu-Arabic Numerals	1	5	10	50	100	500	1000

9. Use the table to express the following Babylonian numeral as a Hindu-Arabic numeral: 9. ____________



Babylonian numerals		
Hindu-Arabic numerals	1	10

10. Write 1,3476 as a Babylonian numeral.

Babylonian numerals		
Hindu-Arabic numerals	1	10

11. Use the table to write 567 as a Chinese Numeral. 11.

一	二	三	四	五	六	七	八	九	十	百	千
1	2	3	4	5	6	7	8	9	10	100	1000

12. Use the table to write the following Chinese Numeral as a Hindu-Arabic Numeral. 12. ___________

百
三
十
四
五

一	二	三	四	五	六	七	八	九	十	百	千
1	2	3	4	5	6	7	8	9	10	100	1000

13. Convert $8A21_{12}$ to a numeral in base ten: 13. ____________

14. Convert 3,248 to base eight. 14. ____________

15. Convert the number 10000100001_2 from binary form to octal form. 15. ________________

16. Convert the number 100111001_2 from binary form to hexadecimal form. 16. ________________

17. Add the given numbers in the indicated base. 17. ________________

$$7632_9 + 1656_9$$

18. Subtract the given numbers in the indicated base. 18. ________________

$$4C6_{16} - 198_{16}$$

19. Multiply the given numbers in the indicated base. 19. ________________

$$513_6 \times 23_6$$

20. Divide the given numbers in the indicated base.

$$424_5 \div 3_5$$

20. ______________

21. Use the Egyptian algorithm to find the product.

$$32 \cdot 119$$

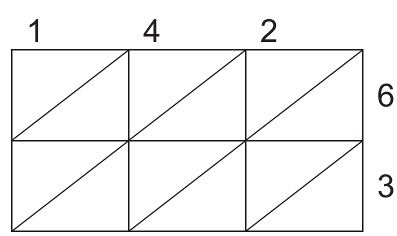
21. ______________

22. Multiply 42×13 using the Russian peasant method.

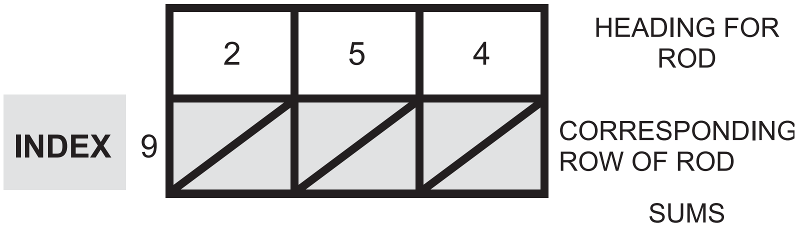
22. ______________

23. Multiply 142×63 using the lattice method.

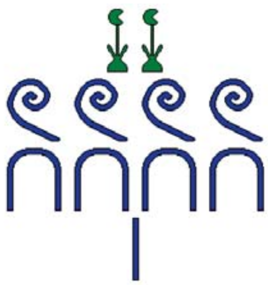
23. ______________



24. Multiply 9×254 using Napier's rods. 24. ______________



Module 2 Test Answers

1. $9(100) + 0(10) + 1$
2. $1(10{,}000) + 0(1{,}000) + 7(100) + 2(10) + 1$
3. 59
4. 867
5. 632
6. 
7. 505
8. 5,446
9. 12,120
10. 
11. 五
百
六
十
七
12. 135
13. 15,289
14. 6260_8
15. 2041_8
16. 139_{16}
17. 9388_9

18. $32E_{16}$

19. 21043_6

20. 123_5

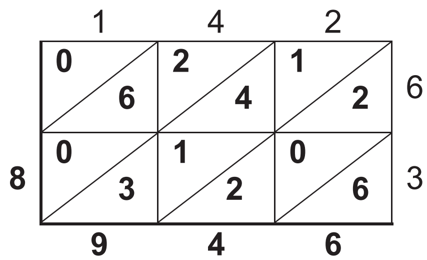
21. 3,808

32	119
1	119
2	238
4	476
8	952
16	3808

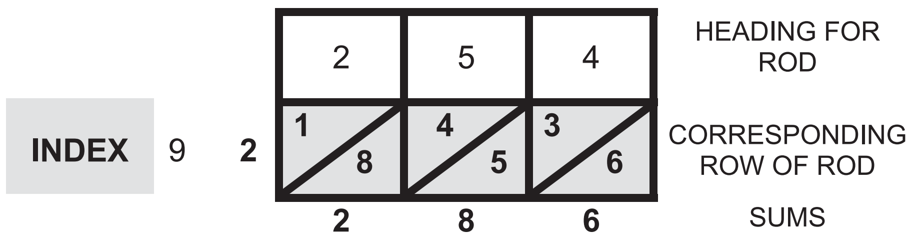
22. 546

42	13
~~84~~	~~6~~
168	3
408	1

23. 8,946



24. 2,286



Module 2 – Project

Numeration Systems

Computers deal in numbers, not letters. To get computers to work, each character needs to be represented as a sequence of numbers. In order for text files to be reliably stored and processed by computers, it is important that the data is interpreted in the same way. The **A**merican **S**tandard **C**ode for **I**nformation **I**nterchange (ASCII) is used to encode characters of the alphabet as binary numbers. Each character is assigned an eight-digit binary number written in two groups of four digits. The capital letters A – 0 start with 0100 and the letters P – Z start with 0101. Lowercase letters a – 0 start with 0110 and lowercase letters p – z start with 0111.

ASCII - Binary Character Table

Letter	ASCII Code	Binary	Letter	ASCII Code	Binary
a	097	01100001	A	065	01000001
b	098	01100010	B	066	01000010
c	099	01100011	C	067	01000011
d	100	01100100	D	068	01000100
e	101	01100101	E	069	01000101
f	102	01100110	F	070	01000110
g	103	01100111	G	071	01000111
h	104	01101000	H	072	01001000
i	105	01101001	I	073	01001001
j	106	01101010	J	074	01001010
k	107	01101011	K	075	01001011
l	108	01101100	L	076	01001100
m	109	01101101	M	077	01001101
n	110	01101110	N	078	01001110

o	111	01101111	O	079	01001111
p	112	01110000	P	080	01010000
q	113	01110001	Q	081	01010001
r	114	01110010	R	082	01010010
s	115	01110011	S	083	01010011
t	116	01110100	T	084	01010100
u	117	01110101	U	085	01010101
v	118	01110110	V	086	01010110
w	119	01110111	W	087	01010111
x	120	01111000	X	088	01011000
y	121	01111001	Y	089	01011001
z	122	01111010	Z	090	01011010

1) Write your first name (Capitalize the first letter of your first name).
2) Write your first name in ASCII binary (base 2) code.
3) Now change your entire first name from base 2 to base 16 (hexadecimal) Recall the following conversions:

0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

4) Change the first letter from base 2 (binary) to base 10.

5) Write the number in the Egyptian, Babylonian, Roman, and traditional Chinese.
6) Pick your two favorite 4-digit base 10 number and convert them to base 5.
7) Add and subtract the two numbers from #6 while they are still in base 5.
8) Make up your own 3 digit by 2 digit multiplication problem and show the answer to that problem using
    a. Egyptian algorithm
    b. Russian peasant method
    c. Lattice method and
    d. Napier's rods.

Module 2 – Project Sample Answers

Numeration Systems

Computers deal in numbers, not letters. To get computers to work, each character needs to be represented as a sequence of numbers. In order for text files to be reliably stored and processed by computers, it is important that the data is interpreted in the same way. The **A**merican **S**tandard **C**ode for **I**nformation **I**nterchange (ASCII) is used to encode characters of the alphabet as binary numbers. Each character is assigned an eight-digit binary number written in two groups of four digits. The capital letters A – 0 start with 0100 and the letters P – Z start with 0101. Lowercase letters a – 0 start with 0110 and lowercase letters p – z start with 0111.

ASCII - Binary Character Table

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e	101	01100101	E	069	01000101
f	102	01100110	F	070	01000110
g	103	01100111	G	071	01000111
h	104	01101000	H	072	01001000
i	105	01101001	I	073	01001001
j	106	01101010	J	074	01001010
k	107	01101011	K	075	01001011
l	108	01101100	L	076	01001100
m	109	01101101	M	077	01001101
n	110	01101110	N	078	01001110

o	111	01101111	O	079	01001111
p	112	01110000	P	080	01010000
q	113	01110001	Q	081	01010001
r	114	01110010	R	082	01010010
s	115	01110011	S	083	01010011
t	116	01110100	T	084	01010100
u	117	01110101	U	085	01010101
v	118	01110110	V	086	01010110
w	119	01110111	W	087	01010111
x	120	01111000	X	088	01011000
y	121	01111001	Y	089	01011001
z	122	01111010	Z	090	01011010

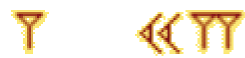
1) Write your first name (Capitalize the first letter of your first name).
   Ron
2) Write your first name in ASCII binary (base 2) code.
   01010010111110
3) Now change your entire first name from base 2 to base 16 (hexadecimal) Recall the following conversions:

0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

   12276

4) Change the first letter from base 2 (binary) to base 10.
   01010010_2 = 82

5) Write the number in the Egyptian, Babylonian, Roman, and traditional Chinese.

Egyptian	Babylonian	Roman	Chinese
		LXXXII	八 十 二

6) Pick your two favorite 4-digit base 10 number and convert them to base 5.
8888 = 241023_5 **2121 = 31441_5**
7) Add and subtract the two numbers from #6 while they are still in base 5.
    a. Addition: 323014_5
    b. Subtraction: 204032_5
8) Make up your own 3 digit by 2 digit multiplication problem and show the answer to that problem using
    a. Egyptian algorithm
    b. Russian peasant method
    c. Lattice method and
    d. Napier's rods.
    See instructor's resource manual for section 2.4 for examples