A Modern Introduction to Differential Equations

Second Edition

Instructor's Solutions Manual

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CHAPTER 1

Introduction to Differential Equations

1.1 BASIC TERMINOLOGY

If students have seen an introduction to differential equations as part of the calculus sequence, this section can be covered quickly. However, I've found that review is always helpful. In particular, students who have recently taken two or more semesters of calculus often have trouble distinguishing dependent variables from independent variables in differentiation problems and have trouble with dummy variables in integration. I want students to focus on the *form* of a differential equation, not on the particular variables used or on the derivative notation employed. Throughout the book, I have deliberately mixed the Leibniz (d/dx), Newton (dot), and Lagrange (prime) notations for derivatives, although the dot notation becomes dominant in later chapters as the dynamical systems interpretation becomes more pronounced.

Parameters in equations often cause difficulty, but the student should become more comfortable with this concept as the course progresses. Students will have trouble with the general form(s) of an nth-order differential equation, especially if they are not familiar with functions of several variables. Concrete examples are necessary. Many students need time to understand the idea of a *linear* differential equation. Later discussions of linearity (in particular, the Superposition Principle) should help.

The idea of a *system* of differential equations is introduced early because of its importance in chapters 4–7, but sometimes I postpone any classroom mention of systems until Chapter 4. If the students are reading the book (*Ha!*, you say), they'll see this on their own.

The text is dedicated to the proposition that technology is a valuable tool that can aid a student's understanding and that may be essential in solving certain problems. The November, 1994 issue (Vol. 25, No. 5) of the *College Mathematics Journal* is devoted to the teaching of differential equations. However, I don't want to spend a great deal of valuable class time teaching the intricacies of the syntax of any CAS or other software I may be using. For example, in using *Maple* I've found that a few basic commands should be mastered and used over and over again, making minimal changes to accommodate different problems. I've handed out summaries of these commands and have encouraged students to use the "Help" facility. Getting comfortable with the various options (numerical, graphical, and analytic) may take some time, and I have learned to avoid embarrassment in class by preparing ahead of time, saving examples on flash memory. I hand out hard copies of certain *Maple* worksheets for the students to use as templates. Students sometimes make their own electronic copies of worksheets that may be on a departmental server. There are many books dealing with ODEs and various computer algebra systems.

Also, there is a wealth of ODE information on the Internet. See, for example, my article (May/June, 2002; August/September, p. 24) in the MAA's newsletter *Focus*. Some of the links may no longer be viable, but there are still some good search tips in the article. It's important for students to realize that computers don't have all the answers. I've found that showing problems that the CAS can't solve or can only solve incompletely is a sound pedagogical technique.

A

1. (a) The independent variable is x and the dependent variable is y; (b) first-order; (c) linear
2. (a) The independent variable is x and the dependent variable is y; (b) first-order; (c) linear
3. (a) The independent variable is not indicated, but the dependent variable is x; (b) second-order; (c) nonlinear because of the term $\exp(-x)$—the equation cannot be written in the form (1.1.1), where y is replaced by x and x is replaced by the independent variable.
4. (a) The independent variable is x and the dependent variable is y; (b) first-order; (c) nonlinear because of the term $(y')^2 = y'(x) \cdot y'(x)$—the equation cannot be written in the form (1.1.1).
5. (a) The independent variable is x and the dependent variable is y; (b) first-order; (c) nonlinear because you get the terms $x^2(y')^2$ and $xy'y$ when you remove the parentheses.
6. (a) The independent variable is t and the dependent variable is r; (b) second-order; (c) linear
7. (a) The independent variable is x and the dependent variable is y; (b) fourth-order; (c) linear
8. (a) The independent variable is t and the dependent variable is y; (b) second-order; (c) nonlinear because of the term $-y'(y^2 - 1)$
9. (a) The independent variable is t and the dependent variable is x; (b) third-order; (c) linear
10. (a) The independent variable is t and the dependent variable is x; (b) seventh-order; (c) linear
11. (a) The independent variable is x and the dependent variable is y; (b) first-order; (c) nonlinear because of the term $e^{y'}$
12. (a) The independent variable is t and the dependent variable is R; (b) third-order; (c) linear

13. a. Nonlinear; the first equation is nonlinear because of the term $4xy = 4x(t)y(t)$.
b. Linear
c. Nonlinear; the first and second equations are nonlinear because each contains a product of dependent variables.
d. Linear

B

1. The terms $(a^2 - a)x\frac{dx}{dt}$ and $te^{(a-1)x}$ make the equation nonlinear. If $a^2 - a = 0$—that is, if $a = 0$ or $a = 1$—then the first troublesome term disappears. However, only the value $a = 1$ makes the second nonlinear term vanish as well. Thus $a = 1$ is the answer.
2. a. $\frac{dx}{dt} = \ln(2^x) = x\ln 2 = (\ln 2)x$, a linear equation

b. $$x' = \begin{cases} \frac{x^2-1}{x-1} & \text{for } x \neq 1 \\ 2 & \text{for } x = 1 \end{cases} = \begin{cases} x+1 & \text{for } x = 1 \\ 2 & \text{for } x = 1 \end{cases} = x + 1 \text{ for all } x, \text{ which is linear}$$

c. $$x' = \begin{cases} \frac{x^4-1}{x^2-1} & \text{for } x \neq 1 \\ 2 & \text{for } x = 1 \end{cases} = \begin{cases} x^2+1 & \text{for } x \neq 1 \\ 2 & \text{for } x = 1 \end{cases} = x^2 + 1 \text{ for all } x, \text{ which is nonlinear}$$

1.2 SOLUTIONS OF DIFFERENTIAL EQUATIONS

If students have worked with differential equations before, much of this material can be covered quickly, as a review. However, before the formal solution methods are discussed in Chapter 2, I want the students to develop some facility in guessing and verifying solutions of differential equations. *Implicit* solutions are important, and students usually need a quick review of implicit differentiation. Also, you may want to introduce/review the use of technology in plotting implicit functions.

A

1. $y = \sin x, y' = \cos x, y'' = -\sin x;$ thus $y'' + y = -\sin x + \sin x = 0$.
2. $x = -\pi e^{3t} + \frac{2}{3}e^{2t}, x' = -3\pi e^{3t} + \frac{4}{3}e^{2t}, x'' = -9\pi e^{3t} + \frac{8}{3}e^{2t}$; thus $x'' - 5x' + 6x$

$$= \left(-9\pi e^{3t} + \tfrac{8}{3}e^{2t}\right) - 5\left(-3\pi e^{3t} + \tfrac{4}{3}e^{2t}\right) + 6\left(-\pi e^{3t} + \tfrac{2}{3}e^{2t}\right)$$

$$= -9\pi e^{3t} + \tfrac{8}{3}e^{2t} + 15\pi e^{3t} - \tfrac{20}{3}e^{2t} - 6\pi e^{3t} + \tfrac{12}{3}e^{2t} = 0.$$

3. $y = x^2, dy/dx = 2x$; thus $(1/4)(dy/dx)^2 - x(dy/dx) + y = (1/4)(2x)^2 - x(2x) + x^2 = (1/4)(4x^2) - 2x^2 + x^2 = x^2 - 2x^2 + x^2 = 0$.

4. $R = t(c - \cos t), dR/dt = t(\sin t) + (c - \cos t)$; thus $t(dR/dt) - R = t(t \sin t + c - \cos t) - t(c - \cos t) = t^2 \sin t + ct - t \cos t - tc + t \cos t = t^2 \sin t$.

5. $y = at^3 + bt^2 + ct + d, dy/dt + 3at^2 + 2bt + c, d^2y/dt^2 = 6at + 2b, d^3y/dt^3 = 6a, d^4y/dt^4 = 0$.

6. $r = ce^{bt} - (a/b)t - a/b^2, dr/dt = bce^{bt} - a/b$; thus $at + br = at + b\left(ce^{bt} - (a/b)t - a/b^2\right)$ $= at + bce^{bt} - at - a/b = bce^{bt} - a/b = dr/dt$.

7. $y = \ln x^2, y' = \left(1/x^2\right)(2x) = 2/x$; thus $xy' - 2 = x(2/x) - 2 = 2 - 2 = 0$.

8. $y = \left(e^{ax} + e^{-ax}\right)/2a, y' = \left(ae^{ax} - ae^{-ax}\right)/2a = \left(e^{ax} - e^{-ax}\right)/2$,

$y'' = \left(a^2e^{ax} + a^2e^{-ax}\right)/2a = \left(ae^{ax} + ae^{-ax}\right)/2$; thus $a\sqrt{1 + (y')^2}$

$$= a\sqrt{1 + \left(e^{ax} - e^{-ax}\right)^2/4} = a\sqrt{\frac{4 + \left(e^{2ax} - 2 + e^{-2ax}\right)}{4}} = a\sqrt{\frac{2 + \left(e^{2ax} + e^{-2ax}\right)}{4}}$$

$$= a\sqrt{\frac{\left(e^{ax} + e^{-ax}\right)^2}{4}} = a\sqrt{\left(\frac{\left(e^{ax} + e^{-ax}\right)}{2}\right)^2} = a\frac{\left(e^{ax} + e^{-ax}\right)}{2} = \left(ae^{ax} + ae^{-ax}\right)/2 = y''.$$

9. $y = \int_1^x \frac{\sin t}{t} dt$; By the FTC, we have $y' = \frac{\sin x}{x}$, so that $xy' - \sin x = x(\sin x/x) - \sin x = \sin x - \sin x = 0$. (See Appendix A.4, statement (B), for the FTC.)

10. $y = \int_3^x e^{-t^2} dt$, so that $y' = e^{-x^2}$ and $y'' = -2xe^{-x^2}$. Thus $y'' + 2xy' = \left(-2xe^{-x^2}\right) + 2x\left(e^{-x^2}\right) = -2xe^{-x^2} + 2xe^{-x^2} = 0$. (See Appendix A.4, statement (B), for the FTC.)

11. **a.** For example, $y' = 1/c$, so that $cy' = 1$ is a possible differential equation satisfied by y.

b. Note that $y' = be^{ax} \cos bx + ae^{ax} \sin bx = be^{ax} \cos bx + a\,y$, so that $y' - ay = be^{ax} \cos bx$.

c. We have $y' = (A + Bt)\, e^t + Be^t = y + Be^t$, so that $y' - y = Be^t$.
Other possibilities are the equations $y'' - y = 2Be^t$ and $y'' - y' = Be^t$.

d. We have $\dot{y} = -3\,e^{-3t} + t\,y(t)$, or $\dot{y} - t\,y = -3\,e^{-3t}$, for example. Other possibilities include $\ddot{y} - t\,\dot{y} - y = 9\,e^{-3t}$.

12. Differentiating implicitly, we find that $xy' + y - y'/y = 0$, so that $xyy' + y^2 - y' = (xy - 1)y' + y^2 = 0$, a first-order nonlinear equation, is a possible answer.

13. We get $y' + y'/\left(1 + y^2\right) = 1 + 1/\left(1 + x^2\right)$, so that $y'\left(\frac{y^2+2}{y^2+1}\right) = \frac{x^2+2}{x^2+1}$, or $y' = \left(\frac{y^2+1}{y^2+2}\right)\left(\frac{x^2+2}{x^2+1}\right)$, a first-order nonlinear equation.

14. We get $3y^2y' - 3 + 3y' = 0$, so $y' = 3/\left(3y^2 + 3\right)$ is a possible answer.

15. We have $x^2y' + 2xy + 4y' = 0$, or $y' = -2xy/\left(x^2 + 4\right)$, a linear equation.

16. Differentiating implicitly, we get $2x + 2yy' - 6 + 10y' = 0, 2yy' + 10y' = 6 - 2x$, $2(y + 5)y' = 2(3 - x)$, so $y' = (3 - x)/(y + 5)$ *for those values of* x *for which* $y \neq -5$.

B

1. $y = x^2/2 + (x/2)\sqrt{x^2+1} + \ln\sqrt{x+\sqrt{x^2+1}}, y' = x + (x/2)(1/2)\left(2x\big/\sqrt{x^2+1}\right)$
$+ (1/2)\sqrt{x^2+1} + (1/2)\left(1 + x\big/\sqrt{x^2+1}\right)\big/\left(x+\sqrt{x^2+1}\right)$
$= x + \sqrt{x^2+1}$; thus $xy' + \ln(y') = \left(x^2 + x\sqrt{x^2+1}\right) + \ln\left(x+\sqrt{x^2+1}\right)$
$= \left(x^2 + x\sqrt{x^2+1}\right) + \ln\left(x+\sqrt{x^2+1}\right) = \left(x^2 + x\sqrt{x^2+1}\right) + 2\ln\sqrt{x+\sqrt{x^2+1}}$
$= 2\left(x^2/2 + (x/2)\sqrt{x^2+1} + \ln\sqrt{x+\sqrt{x^2+1}}\right) = 2y.$
2. If you differentiate a polynomial of degree n, you get a polynomial of degree $n-1$, so that the derivative can't be a constant multiple of the original function. The derivative of any basic trigonometric functions is another trigonometric function that is not a constant multiple of the function you started with. Finally, if you differentiate the logarithm function to any base, you'll get a multiple of the *reciprocal* of the original function.
3. a. The given equation is equivalent to $(y')^2 = -1$. Since there is no real-valued function y' whose square is negative, there can be no real-valued function y satisfying the equation.
   b. The only way that two absolute values can have a sum equal to zero is if each absolute value is itself zero. This says that y is identically equal to zero, so that the zero function is the only solution. The graph of this solution is the x-axis (if the independent variable is x).
4. If $x(t) \neq t$, the expression $-|x-t|$ is always negative, so that $\sqrt{-|x-t|}$ is not a real number. If $x(t) = t$, then the equation becomes $dx/dt = 0$, which contradicts the fact that dx/dt must equal 1 in this case.
5. If $y = \pm\sqrt{c^2-x^2} = \pm\left(c^2-x^2\right)^{1/2}$, then $dy/dx = \pm\frac{1}{2}\left(c^2-x^2\right)^{-1/2}\cdot(-2x) = \mp x\left(c^2-x^2\right)^{-1/2}$ and $y\,dy/dx + x = \pm\left(c^2-x^2\right)^{1/2}\cdot\mp x(c^2-x^2)^{1/2} + x = -x + x = 0$. If $x > c$ or $x < -c$, then $c^2 - x^2 < 0$ and then the functions $y = \pm\sqrt{c^2-x^2}$ do not exist as real-valued functions. If $x = \pm c$, then each function is the zero function, which is not a solution of the differential equation.
6. a. If $y = \ln(|C_1x|) + C_2$, then $y' = C_1/C_1x = 1/x$ for all values of C_1 and C_2 (with $C_1 \neq 0$).
   b. Note that $y = \ln(|C_1x|) + C_2 = \ln(|C_1|) + \ln(|x|) + C_2 = \ln(|x|) + \left(\ln(|C_1|) + C_2\right) = \ln(|x|) + C$, where $C = \ln(|C_1|) + C_2$.
7. If $y(x) = c_1\sin x + c_2\cos x$, then $dy/dx + y = (c_1\cos x - c_2\sin x) + (c_1\sin x + c_2\cos x) = (c_1 - c_2)\sin x + (c_2 + c_1)\cos x$. If this last expression must equal $\sin x$, then we must have $c_1 - c_2 = 1$ and $c_2 + c_1 = 0$. Adding these last equations, we find that $c_1 = 1/2$ and therefore $c_2 = -1/2$. Therefore, the solution is $y(x) = (1/2)(\sin x - \cos x)$.
8. Suppose the polynomial is $y(x) = ax^2 + bx + c$, so that $y' = 2ax + b$. Then $2y' - y = 2(2ax+b) - (ax^2+bx+c) = -ax^2 + (4a-b)x + (2b-c) = 3x^2 - 13x + 7$, which implies

that $a=-3$, $4a-b=-13$, and $2b-c=7$. Thus $a=-3$, $b=1$, and $c=-5$, so that the solution is $y(x)=-3x^2+x-5$.

9. We have $y=Cx\pm\sqrt{C^2+1}$, so that $y'=C$. Then $(xy'-y)^2-(y')^2-1=(Cx-Cx\mp\sqrt{C^2+1})^2-C^2-1=\left(C^2+1\right)-C^2-1=0$. But if we assume that a function y is defined implicitly and we differentiate the relation $x^2+y^2=1$ implicitly with respect to x, we get $2x+2yy'=0$, or $y'=-x/y$.

10. We have $y(t)=\cos t+\int_0^t(t-u)y(u)du=\cos t+t\int_0^t y(u)du-\int_0^t u\,y(u)du$, so (using the Product Rule and the FTC) $y'(t)=-\sin t+t\,y(t)+\int_0^t y(u)du-t\,y(t)=-\sin t+\int_0^t y(u)du$. Differentiating again, we get $y''(t)=-\cos t+y(t)$, or $y''-y=-\cos t$. In this problem it is important to distinguish between t and the "dummy variable" u.

C

1. **a.** We have $y=e^x=y'=y''$. Therefore $xy''-(x+n)y'+ny=xy-(x+n)y+ny=0$.

b. We have $y=\sum_{k=0}^{n}\frac{x^k}{k!}$, $y'=\sum_{k=0}^{n}\frac{k\cdot x^{k-1}}{k!}=\sum_{k=1}^{n}\frac{x^{k-1}}{(k-1)!}=y-\frac{x^n}{n!}$, and $y''=y'-\frac{n\,x^{n-1}}{n!}$

$=y'-\frac{x^{n-1}}{(n-1)!}$. Therefore, $xy''-(x+n)y'+n\,y=x\left[y'-\frac{x^{n-1}}{(n-1)!}\right]-(x+n)y'+ny$

$=xy'-\frac{x^n}{(n-1)!}-xy'-ny'+n\,y=-\frac{x^n}{(n-1)!}-ny'+ny=-\frac{x^n}{(n-1)!}-n\left[y-\frac{x^n}{n!}\right]+ny$

$=-\frac{x^n}{(n-1)!}-ny+\frac{x^n}{(n-1)!}+ny=0.$

1.3 INITIAL-VALUE PROBLEMS AND BOUNDARY-VALUE PROBLEMS

I emphasize the fact that an initial-value problem or a boundary-value problem may have *no* solution, *one* solution, or *many* solutions, anticipating the formal discussion of existence and uniqueness in Section 2.8.

A

1. $R(t)=t(c-\cos t)$, so that $R(\pi)=\pi(c-\cos\pi)=\pi(c+1)=0$ implies that $c=-1$. Thus the solution of the IVP is $R(t)=\pi(-1-\cos t)=-\pi(1+\cos t)$.

2. Since $y=at^3+bt^2+ct+d$, $y(0)=1$ implies that $d=1$, $y'(0)=0$ implies that $c=0$, $y''(0)=1$ implies that $2b=1$, or $b=1/2$, and $y'''(0)=6$ tells us that $6a=6$, or $a=1$. Thus the solution of the IVP is $y=t^3+(1/2)t^2+1$.

3. $r(t)=ce^{bt}-(a/b)t-a/b^2$, so that $r(0)=ce^0-(a/b)(0)-a/b^2=c-a/b^2=0$ implies that $c=a/b^2$. Thus the solution of the IVP can be written as $r(t)=(a/b^2)\,e^{bt}-(a/b)t-a/b^2=(a/b)\left(e^{bt}/b-t-1/b\right)$.

4. We have $y=(e^{ax}+e^{-ax})/2a$ and $y'=\left(ae^{ax}-ae^{-ax}\right)/2a=\left(e^{ax}-e^{-ax}\right)/2$, so that $y(0)=\left(e^0+e^0\right)/2a=1/a=2$ implies that $a=1/2$ Noting that $y'(0)=\left(ae^0-ae^0\right)/2=0$ for *all* values of a, we conclude that $y(x) = e^{x/2} + e^{-x/2}$ is the solution of the initial value problem.

5. We have $y' = -\frac{1}{4} + \frac{66}{296}e^{6x} + 2Ax + B\cos x - C\sin x$, $y'' = \frac{396}{296}e^{6x} + 2A - B\sin x - C\cos x$, $y''' = \frac{6(396)}{296}e^{6x} - B\cos x + C\sin x$. Substituting these derivatives in the original differential equation and simplifying, we get $(-B + 6C)\cos x + (C + 6B)\sin x - 12A = 3 - \cos x$. Equating coefficients of like functions—a technique that will come in handy in Chapter 4—we get the system $\{-B + 6C = -1, C + 6B = 0, -12A = 3\}$, which has the solution $A = -1/4$, $B = 1/37$, $C = -6/37$. [Of course, the boundary conditions yield the same result.]

B

1. As was illustrated in Example 1.3.1, the velocity function is the derivative of the position function, so that we have $\frac{dx}{dt} = \frac{1}{t^2+1}$. Integrating both sides, we get $x(t) - x(0) = x(t) = \int_0^t \frac{1}{u^2+1}du = \arctan(t)$. (Also see equation (1.3.1).) For $t \geq 0$, $\arctan(t) \leq \pi/2$, so that $x(t) \leq \pi/2$. (Look at the graph of the arctangent.)

2. If $y_1(x) \equiv 0$, then $\frac{dy_1}{dx} = 0 = 3(0)^{2/3}$. Also, $y_1(x_0) = 0$. If $y_2(x) = (x - x_0)^3$, then $\frac{dy_2}{dx} = 3(x - x_0)^2 = 3[(x - x_0)^3]^{2/3} = 3\,y_2^{2/3}$. Furthermore, $y_2(x_0) = 3(x_0 - x_0)^3 = 0$.

3. We calculate that $y' = e^{x^2}\left(\int_1^x e^{-t^2}\,dt\right)' + \left(e^{x^2}\right)'\int_1^x e^{-t^2}\,dt = e^{x^2}\left(e^{-x^2}\right) + 2x\,e^{x^2}\int_1^x e^{-t^2}\,dt = 1 + 2x\,e^{x^2}\int_1^x e^{-t^2}\,dt = 1 + 2x\left(e^{x^2}\int_1^x e^{-t^2}\,dt\right) = 1 + 2xy$. Also, $y(1) = e^{1^2}\int_1^1 e^{-t^2}\,dt = e\,(0) = 0$.

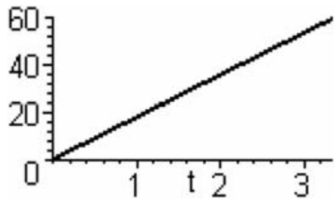
4. a. Integrating successively, we see that $y''' = -24\cos\left(\frac{\pi}{2}x\right) \Rightarrow y'' = -\frac{48}{\pi}\sin\left(\frac{\pi}{2}x\right) + C_1 \Rightarrow y' = \frac{96}{\pi^2}\cos\left(\frac{\pi}{2}x\right) + C_1x + C_2 \Rightarrow y = \frac{192}{\pi^3}\sin\left(\frac{\pi}{2}x\right) + C_1\frac{x^2}{2} + C_2x + C_3$. There are **3** parameters involved.

   b. Now $y(0) = -4$ implies that $\boxed{-4} = y(0) = \frac{192}{\pi^3}\sin(0) + C_1\frac{0^2}{2} + C_2(0) + C_3\ \boxed{= C_3}$; and $y(1) = 0$ implies that $0 = y(1) = \frac{192}{\pi^3}\sin\left(\frac{\pi}{2}\right) + C_1\frac{1^2}{2} + C_2(1) + C_3 = \frac{192}{\pi^3} + \frac{C_1}{2} + C_2 + C_3$, or $\boxed{\frac{192}{\pi^2} + \frac{C_1}{2} + C_2 = 4}$. Also, $y'(1) = 6$ implies that $\boxed{6} = y'(1) = \frac{96}{\pi^2}\cos\left(\frac{\pi}{2}\right) + C_1 + C_2\ \boxed{= C_1 + C_2}$. Solving the last two simultaneous equations, we find that $C_1 = \left(4\pi^3 + 384\right)/\pi^3$ and $C_2 = \left(2\pi^3 - 384\right)/\pi^3$. Thus $y = \frac{192}{\pi^3}\sin\left(\frac{\pi}{2}x\right) + \left(\frac{2\pi^3+192}{\pi^3}\right)x^2 + \left(\frac{2\pi^3-384}{\pi^3}\right)x - 4$.

5. No. An equation of order n requires an n-parameter family of solutions. Essentially, to solve a differential equation of order 4 requires 4 integrations, each of which introduces a constant of integration (parameter).

6. Barry's steady increase in speed implies that her acceleration was constant: $a(t) = C$. Then $v(t) = \int a(t)dt = \int C dt = Ct + K$ and $v(0) = 0$ implies that $K = 0$. Thus we can write $v(t) = Ct$. But $60 = v(3\frac{1}{3}) = C(10/3)$ tells us that $C = 18$, so that $v(t) = 18t$. Finally, distance equals $\int_0^t v(u)du = \int_0^t 18u\,du = 9t^2$, and after $3\frac{1}{3}$ hours Barry has traveled $9(10/3)^2 = 100$ miles.

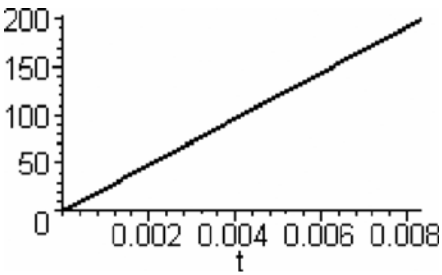
Alternatively, we can just draw the velocity curve (a straight line) and calculate the area under the curve, which is the area of a right triangle:



$$\text{Area} = \frac{1}{2}\left(\frac{10}{3}\text{ hours}\right)(60\text{ miles/hour}) = 100\text{ miles}.$$

7. The given information is that $v(0) = 0, v(30\text{ seconds}) = v(30/3600\text{ hours}) = 200\text{ mph}$, and $a(t) = C$, a constant. Now $a(t) = C \Rightarrow v(t) = \int a(t)dt = Ct + K$. Then $v(0) = 0 \Rightarrow K = 0 \Rightarrow v(t) = Ct$. Therefore $200 = v(30/3600) = v(1/120) = C(1/120) \Rightarrow C = 200(120)$ and $v(t) = 200(120)t$. Finally, distance $= s(1/120) = \int_0^{1/120} v(u)du = \int_0^{1/120} 200(120)udu = 12000(1/120)^2 = 5/6$ mile.

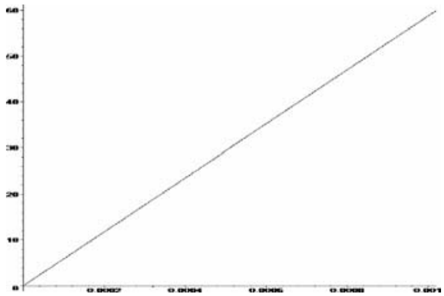
Alternatively, we can just draw the velocity curve (a straight line) and calculate the area under the curve, which is the area of a right triangle:



$$\text{Area} = \frac{1}{2}\left(\frac{1}{120}\text{ hours}\right)(200\text{ miles/hour}) = 5/6\text{ miles}.$$

8. a. This is like Example 1.3.4. We are given that $v(0) = 0, v(3.8\text{ seconds}) = v(19/18000\text{ hour}) = 62\text{ mph}$, and $a(t) = C$, a constant. Then $v(t) = Ct + K$ and $v(0) = 0 \Rightarrow K = 0$, so that $v(t) = Ct$. Now $62 = v(19/18000) = (19/18000)C \Rightarrow C = 62(18000/19) \Rightarrow v(t) = 62(18000/19)t$. Thus the car will reach a speed of **60** mph in $60(19)/(18000)(62) = 19/18600$ of an hour. Finally, distance $= s(19/18600) = \int_0^{19/18600} v(t)dt = \int_0^{19/18600} 62(18000/19)t\,dt = 62(18000/19)(19/18600)^2/2 = 285/9300$ mile $\approx$ **161.8 feet**.

Alternatively, we can just draw the velocity curve (a straight line) and calculate the area under the curve, which is the area of a right triangle:



$$\text{Area} = \frac{1}{2}\left(\frac{19}{18600}\text{ hours}\right)(60\text{ miles/hour}) = 19/620\text{ mile} \approx 161.8\text{ feet.}$$

b. Let t^* be the time (in hours) it takes the car to stop. We are given that $v(0) = 62$ mph, $v(t^*) = 0$, and $a(t) = C$, where C is a negative constant. Then $v(t) = Ct + K$ and $v(0) = 62 \Rightarrow K = 62 \Rightarrow v(t) = Ct + 62$. Also, $v(t^*) = 0 \Rightarrow C = -62/t^* \Rightarrow v(t) = (-62/t^*)t + 62$. Noting that the stopping distance, 114 feet, is 114/5280 mile, we can write $s(t^*) = 114/5280 = \int_0^{t^*} v(t)dt = (-62/t^*)(t^*)^2/2 + 62\,t^* = -31\,t^* + 62\,t^* = 31\,t^*$, so that $t^* = 114/(5280 \cdot 31)$ hour $\approx$ **2.5 seconds**. [As in earlier exercises, we can just interpret the problem in terms of the area of the appropriate right triangle.]

9. a. For any values of A and B, $x' = 3(A + Bt)e^{3t} + Be^{3t} = \{3(A + Bt) + B\}e^{3t} = (3A + B + 3Bt)e^{3t} = y$. Now $y' = 3(3A + B + 3Bt)e^{3t} + 3Be^{3t} = (9A + 3B + 9Bt + 3B)e^{3t} = (9A + 6B + 9Bt)e^{3t}$ and $-9x + 6y = (-9A - 9Bt)e^{3t} + (18A + 6B + 18Bt)e^{3t} = (9A + 6B + 9Bt)e^{3t}$, so that $y' = -9x + 6y$.

b. The initial condition $x(0) = 1$ yields $1 = x(0) = (A + 0)e^0 = A$, and $y(0) = 0$ gives us $0 = (3A + B)$, so that $B = -3$. The solution of this system IVP is therefore $\{x(t) = (1 - 3t)e^{3t}, y(t) = -9te^{3t}\}$.

10. First of all, $dx/dt = e^{-t/10}\cos t - (1/10)e^{-t/10}\sin t = (1/10)e^{-t/10}(10\cos t - \sin t) = -y$. Next, $dy/dt = (1/10)e^{-t/10}(10\sin t + \cos t) - (1/100)e^{-t/10}(-10\cos t + \sin t) = e^{-t/10}\sin t + 0.1e^{-t/10}\cos t + 0.1e^{-t/10}\cos t - 0.01e^{-t/10}\sin t = 0.99e^{-t/10}\sin t + 0.2e^{-t/10}\cos t$. But $1.01x - 0.2y = 1.01e^{-t/10}\sin t - 0.2(0.1e^{-t/10})(-10\cos t + \sin t) = 0.99e^{-t/10}\sin t + 0.2e^{-t/10}\cos t$. Now we see that $x(0) = e^0\sin 0 = 0$ and $y(0) = (1/10)e^0(-10\cos 0 + \sin 0) = -1$.

11. a. Deriving inspiration from Example 1.2.2, we get $u(t) = u(0)e^{kat} = Ae^{kat}$.

b. Substituting the expression for u found in (a) in the differential equation for w, we get $\frac{dw}{dt} = a(1 - k)Ae^{kat}$. If $k = 0$, this last equation is just $\frac{dw}{dt} = aA$, so that $w = aAt + C$. The condition $w(0) = 0$ tells us that $C = 0$, so that $w(t) = aAt$. On the other hand, if $0 < k \le 1$, the differential equation for w is $\frac{dw}{dt} = a(1 - k)Ae^{kat}$

and we can integrate to find that $w(t) = \frac{a(1-k)A}{ka} e^{kat} + C$. Knowing that $w(0) = 0$, we have $0 = \frac{a(1-k)A}{ka} + C$ and therefore $C = -\frac{a(1-k)A}{ka} = \frac{(k-1)A}{k}$. Then we can write $w(t) = \frac{(1-k)}{k} A e^{kat} + \frac{(k-1)A}{k} = \frac{(k-1)A}{k}(1 - e^{kat})$.

C

1. a. If $W = \frac{C}{E} + \left(W_0 - \frac{C}{E}\right) e^{-kEt}$, then $\frac{dW}{dt} = -kE\left(W_0 - \frac{C}{E}\right) e^{-kEt} = -kE\left(W - \frac{C}{E}\right) = -kEW + kC = k(C - EW)$.

b. As $t \to \infty$, $e^{-kEt} \to 0$, so that $W(t) \to \frac{C}{E}$. Note that C/E is in pounds per day.

c. From (a) we know that $W(t) = \frac{2500}{20} + \left(180 - \frac{2500}{20}\right) e^{-\frac{20}{3500}t} = 125 + 55e^{-t/175}$. A loss of 20 pounds means that $W(t) = 160$, so that we must solve the equation $160 = 125 + 55e^{-t/175}$, $35 = 55e^{-t/175}$, $\frac{7}{11} = e^{-t/175}$. Taking the natural logarithm of both sides of this last equation, we get $\ln(7/11) = -t/175$, so that $t = -175 \ln(7/11) \approx 79$ days. Similarly, for a loss of 30 pounds, we must solve $150 = 125 + 55e^{-t/175}$, so that $t = -175 \ln(5/11) \approx 138$ days. For a loss of 35 pounds, we solve $145 = 125 + 55e^{-t/175}$, finding that $t = -175 \ln(4/11) \approx 177$ days.

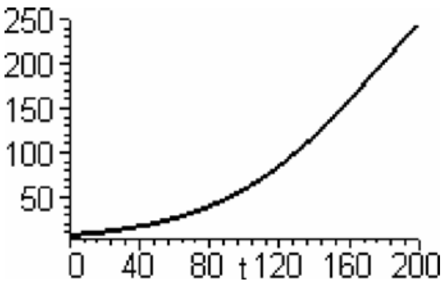
Our conclusion is that dieting can be very frustrating. Although it takes 79 days to lose the first 20 pounds, it takes an extra 59 days to lose 10 more pounds and 39 additional days to lose 5 pounds beyond the first 30. The original differential equation indicates that if C and E are constant, with $C - EW < 0$, then $dW/dt < 0$ and $d^2W/dt^2 = -k^2E(C - EW) > 0$. This says that the weight is a concave up decreasing function of time—that is, the rate of weight loss slows down with time.

2. Integrating each side of the given equation successively, we have $EIy^{(3)} = (-W/L)x + A$, $EIy^{(2)} = (-W/2L)x^2 + Ax + B$, and $EIy' = (-W/6L)x^3 + (A/2)x^2 + Bx + C$. If we use the boundary condition $y'(0) = 0$ in the last equation, we find that $C = 0$. Integrating again, we get $EIy = (-W/24L)x^4 + (A/6)x^3 + (B/2)x^2 + D$. Because $y(0) = 0$, we get $D = 0$. Finally, using the conditions $y(L) = 0$ and $y'(L) = 0$ in the equations for y and y', we get the algebraic equations $(-W/6)L^2 + (A/2)L^2 + BL = 0$ and $(-W/24)L^3 + (A/6)L^3 + (B/2)L^2 = 0$, respectively. Solving these simultaneously for A and B, we find $A = W/2$ and $B = -WL/12$. Therefore the solution is $EIy = (-W/24L)x^4 + (W/12)x^3 - (WL/24)x^2 = (-W/24L)x^2(x - L)^2$.

Then $(xy' - y)^2 - (y')^2 - 1 = (-x^2/y - y)^2 - (-x/y)^2 - 1 = \left(\frac{-(x^2+y^2)}{y}\right)^2 - \frac{x^2}{y^2} - 1 = \frac{1}{y^2} - \frac{x^2}{y^2} - 1 = \frac{1-x^2}{y^2} - 1 = \frac{y^2}{y^2} - 1 = 0$. Note that solving the relation $x^2 + y^2 = 1$ for y gives us $y = \pm\sqrt{1 - x^2}$, which does not correspond to a particular value of C in the solution formula given.

3. a. $y = M(1 + Ae^{-kt})^{-1} \Rightarrow y' = -M(1 + Ae^{-kt})^{-2} \cdot (-kAe^{-kt}) = \frac{MkAe^{-kt}}{(1+Ae^{-kt})^2} = \frac{kM}{1+Ae^{-kt}} \cdot \frac{Ae^{-kt}}{1+Ae^{-kt}} = \frac{kM}{1+Ae^{-kt}} \cdot \frac{1+Ae^{-kt}-1}{1+Ae^{-kt}} = \frac{kM}{1+Ae^{-kt}}\left(1 - \frac{1}{1+Ae^{-kt}}\right) = ky\left(1 - \frac{y}{M}\right)$. Clearly $y(0) = M/(1 + A)$.

b. Here's the graph of $y(t) = \frac{387.9802}{1+54.0812e^{-0.02270347t}}$:



c.

	Actual Population	Logistic Pop. Value
1790	3,929,214	7,043,786
1980	226,545,805	225,066,248
1990	248,709,873	246,050,716

d. $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{387.9802}{1+54.0812e^{-0.02270347t}} = 387.9802$ million people.

4. **a.** Using the same reasoning found in Example 1.2.1, we see that $V_1(t) = V_1(0)e^{-ct} = V_0e^{-ct}$.

b. Let $T^*(t) = T^*(0)e^{-\delta t} + \frac{kT_0V_0}{\delta-c}(e^{-ct} - e^{-\delta t})$. Then $\frac{dT^*}{dt} = -\delta T^*(0)e^{-\delta t} + \frac{kT_0V_0}{\delta-c}(-ce^{-ct} + \delta e^{-\delta t}) = -\delta T^*(0)e^{-\delta t} - \frac{c}{\delta-c}(kV_0T_0e^{-ct}) + \frac{\delta}{\delta-c}(kV_0T_0e^{-\delta t}) = -\delta T^*(0)e^{-\delta t} + kV_0T_0e^{-ct}\left(1 - \frac{\delta}{\delta-c}\right) + \frac{\delta}{\delta-c}(kV_0T_0)e^{-\delta t} = -\delta T^*(0)e^{-\delta t} + kV_0T_0e^{-ct} - \frac{\delta kT_0V_0}{\delta-c}(e^{-ct} - e^{-\delta t}) = k(V_0e^{-ct})T_0 - \delta\left[T^*(0)e^{-\delta t} + \frac{kT_0V_0}{\delta-c}(e^{-ct} - e^{-\delta t})\right] =$ [using the result of part (a)] $kV_1T_0 - \delta T^*$, so that T^* is a solution of the differential equation for T^*.

c. $\lim_{t\to\infty} T^*(t) = \lim_{t\to\infty} T^*(0)e^{-\delta t} + \frac{kT_0V_0}{\delta-c}(e^{-ct} - e^{-\delta t}) = 0 + 0 = 0$. The number of infected cells decreases to zero.

5. Suppose $y = y_{GR} + y_P$. Then $x^2y'' + xy' - 4y = x^2(y''_{GR} + y''_P) + x(y'_{GR} + y'_P) - 4(y_{GR} + y_P) = \left(x^2y''_{GR} + xy'_{GR} - 4y_{GR}\right) + \left(x^2y''_P + xy'_P - 4y_P\right) = 0 + x^3 = x^3$. Thus y, having two arbitrary constants because of its term y_{GR}, is the general solution of the original equation (*).

***Project 1–1** anticipates the discussion of the *logistic equation* in Example 2.4.1. *

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