SOLUTIONS FOR PART I

1. NUMBERS, SETS, AND FUNCTIONS

1.1. *"We have at least four times as many chairs as tables."* The number of chairs (c) is at least ($\geq$) four times the number of tables (t). Hence $c \geq 4t$.

1.2. *Fill in the blanks.* The equation $x^2+bx+c=0$ has exactly one solution when $b^2 = 4c$, and it has no solutions when $b^2 < 4c$. These statements follow from the quadratic formula.

1.3. *Given that $x+y=100$, the maximum value of xy is 2500.* By the AGM Inequality, $xy \leq (\frac{x+y}{2})^2 = (\frac{1000}{2})^2 = 2500$. This is achieved by $x=y=50$.

1.4. *The square has the largest area among all rectangles with a given perimeter.* With side-lengths x, y and perimeter p, we have $x+y=p/2$. By the AGM Inequality, $xy \leq (\frac{x+y}{2})^2 = (p/4)^2$. The bound is achieved with equality when $x=y$, which is the case of a square.

1.5. *Translation of "The temperature was 10° C and increased by 20° C".* "The temperature was 50° F and increased by 36° F". (One converts a *change* of 20 degrees C to a change of 36 degrees F, not to a temperature of 68 degrees.)

1.6. *Temperature scales.* If f denotes the current temperature in Fahrenheit degrees and c denotes the current temperature in Celsius degrees, then we always have $f = (9/5)c + 32$.

a) Equality in the values occurs at -40 *degrees Fahrenheit,* since -40 is the solution to $f = (9/5)f + 32$.

b) Equal magnitude with opposite signs occurs at $80/7$ *degrees Fahrenheit,* since $80/7$ is the solution to $f = (9/5)(-f) + 32$.

c) The Fahrenheit value is twice the Celsius value at 320 *degrees Fahrenheit,* since 320 is the solution to $f = (9/5)(f/2) + 32$.

1.7. *Correction of "If x and y are nonzero real numbers and $x > y$, then $(-1/x) > (-1/y)$."* If y is negative and x is positive, then $-1/x$ is negative and $-1/y$ is positive, so $(-1/x) < (-1/y)$.

Adding the condition $y > 0$ makes the statement true. If now x is negative, then $(-1/x) > 0 > (-1/y)$. If now x is also positive, then $1/x < 1/y$, and multiplying by 1 yields the desired inequality.

In fact, the statement is true whenever $y > 0$ or $x < 0$, which is a more general situation than $y > 0$.

1.8. *Simpson's Paradox.* The tables below confirm the paradox. The explanation is that the bulk of the men are in the afternoon class, where receiving an A is easier, while half of the women come from each class. This makes it easier on average for men to receive A grades.

A grades	Men	Women
Morning	2/10	2/9
Afternoon	9/14	6/9
Total	11/24	8/18

A grades	Men	Women
Morning	.20	.22
Afternoon	.64	.67
Total	.46	.44

1.9. *Percentage changes.* In either case, (20% decline and then 23% rise) or (20% rise and then 18% decline), the original amount is multiplied by $.984 = .80 \cdot 1.23 = 1.20 \cdot .82 = .984$, producing a loss.

1.10. *If 25% more PhD degrees are produced than the economy can absorb, then there is a 1 in 5 chance of underemployment.* The economy can absorb x PhD's, but $(5/4)x$ are produced. The fraction unused is $\frac{(5/4)x - x}{(5/4)x} = \frac{1}{5}$.

1.11. *Promotional discount.* When a 15% discount is applied to an amount x, the actual cost is $.85x$. When 5% tax is computed on an amount y, the tax is $.05y$, and the paid total is $1.05y$. If the price of the item is z, then applying the discount before the tax yields a total cost of $1.05(.85z)$. Applying the tax first yields a total cost of $.85(1.05z)$. By the commutativity of multiplication, these are equal.

1.12. *Installment plan.* If the first of thirteen payments toward \$1000 is half the others, then the total will be 12.5 times the usual payment. We set $12.5x = 1000$ to obtain \$80 as the regular payment and \$40 as the first payment.

1.13. *If $A = \{2k-1 \colon k \in \mathbb{Z}\}$ and $B = \{2k+1 \colon k \in \mathbb{Z}\}$, then $A = B$.* If $n = 2k-1 \in A$ for $k \in \mathbb{Z}$, then $n = 2(k-1)+1$. Since $k-1 \in \mathbb{Z}$, we have $n \in B$. Similarly, $n = 2k+1$ when $k \in \mathbb{Z}$ yields $n = 2(k+1)-1$, and thus $n \in B$ implies $n \in A$.

1.14. *$[a,b] \cup [c,d]$ using set difference.* If $a < b < c < d$, then $[a,b] \cup [c,d]$ consists of all numbers in the closed interval $[a,d]$ except those between b and c. Thus $[a,b] \cup [c,d] = [a,d] - (b,c)$.

1.15. *For sets, $A - B = B - A$ if and only if $A = B$.* If $A = B$, then both differences are empty. Conversely, each element of $A - B$ is not in B and hence not in $B - A$. Similarly, no element of $B - A$ belongs to $A - B$. Hence equality requires that both differences are empty, and thus that $A = B$.

1.16. *Iteration of the Penny Problem operation.*

$$5 \to 41 \to 32 \to 221 \to 311 \to 32, \text{ reaching a cycle of length 3.}$$
$$6 \to 51 \to 42 \to 321 \to 312, \text{ reaching a fixed point.}$$

1.17. *Domain and image of the absolute value function.* The domain is the set of real numbers. The image is the set of nonnegative real numbers: $\{x \in \mathbb{R}: x \geq 0\}$.

1.18. *Real numbers exceeding their reciprocals by 1.* If x is such a real number, then $x = 1 + 1/x$. Since x cannot be 0, we can multiply by x and obtain the quadratic equation $x^2 - x - 1 = 0$ (without changing the solutions). The solutions of this equation are $(1 \pm \sqrt{5})/2$.

1.19. *Perimeter and area.* The perimeter of a rectangle is twice the sum of the lengths of its sides. Perimeter 48 and area 108 leads to $x + y = 24$ and $xy = 108$; the solution is an 18 by 6 region. More generally, $xy = a$ and $x + y = p/2$. This yields $x(p/2 - a) = a$, and thus $x^2 - (p/2)x + a = 0$. The solutions are $\frac{1}{2}[(p/2) \pm \sqrt{p^2/4 - 4a}]$. Existence of a solution requires $p^2/4 - 4a \geq 0$; that is, $p^2 \geq 16a$. The extreme case $p^2 = 16a$ occurs for a square with sides of length $p/4$.

1.20. *If r and s are distinct real solutions of the equation $ax^2 + bx + c = 0$, then $r + s = -b/a$ and $rs = c/a$.* Specifying the leading coefficient and two distinct zeros of a quadratic polynomial determines the polynomial; similarly, two polynomials are equal if and only if corresponding coefficients are equal (the proof of these statements appears in Chapter 3).

The quadratic polynomial whose value at x is $a(x - r)(x - s)$ has zeros r and s and leading coefficient a. Thus $ax^2 - a(r + s)x + ars = 0$ when $x \in \{r, s\}$. Equating corresponding coefficients yields $r + s = -b/a$ and $rs = c/a$.

Alternatively, the quadratic formula implies that $\{r, s\} = \{(-b + \sqrt{b^2 - 4ac})/(2a), (-b - \sqrt{b^2 - 4ac})/(2a)\}$. Computing the sum and product of these two numbers yields $r + s = -b/a$ and $rs = c/a$.

1.21. *Flawed "proof" that $-b/2a$ is a solution to $ax^2 + bx + c = 0$.*

> Let x and y be solutions to the equation. Subtracting $ay^2 + by + c = 0$ from $ax^2 + bx + c = 0$ yields $a(x^2 - y^2) + b(x - y) = 0$, which we rewrite as $a(x + y)(x - y) + b(x - y) = 0$. Hence $a(x + y) + b = 0$, and thus $x + y = -b/a$. Since x and y can be any solutions, we can apply this computation letting y have the same value as x. With $y = x$, we obtain $2x = -b/a$, or $x = -b/(2a)$.

The problem arises when we cancel $x - y$ from $a(x + y)(x - y) + b(x - y) = 0$. The validity of this step requires $x - y \neq 0$. Thus we cannot use the resulting $a(x + y) + b = 0$ in the case where $x = y$.

1.22. *Mixing wine and water.* Let (a, b) denote amounts of wine and water. Initially, glass 1 is $(x, 0)$ and glass 2 is $(0, x)$. After the first step, they are $(x - 1, 0)$ and $(1, x)$. The amount moved in the second step is $(\frac{1}{x+1}, \frac{x}{x+1})$. Thus the final outcome is $(\frac{x^2}{x+1}, \frac{x}{x+1})$ in glass 1 and $(\frac{x}{x+1}, \frac{x^2}{x+1})$ in glass 2.

Alternatively, one can observe that all wine leaving the first glass winds up in the second, and all water leaving the second winds up in the first. The total wine and water is x each, and the total in each glass is x at each step. Thus if y is the amount of water in glass 1 at the end, then the amount of water in glass 2 at the end is $x - y$, and the amount of wine in glass 2 at the end is y.

1.23. *Broken clock.* A digital 12-hour clock broken so that the readings for minutes and for hours are always the same can be correct every 61 minutes, except that between 12:12 and 1:01 there are only 49 minutes.

The analogous problem for analog clocks is different. Suppose that the minute and hour hand must always point in the same direction. In a normal clock, the minute hand revolves twelve times while the hour hand revolves once, and the speeds are steady. Thus, there is agreement every 1 and 1/12 hours. They agree 11 times in every 12 hours.

1.24. *The missing dollar.* There is no missing dollar. The correct accounting is $3 \cdot 9 - 2 = 25$, not $3 \cdot 9 + 2 \neq 30$.

1.25. *The Census Problem (daughters ages).* We assume that the ages are positive integers. Let them be a, b, c with $a \leq b \leq c$. We are told that $abc = 36$, but that knowing $a + b + c$ is not enough to determine a, b, c. Of the possibilities (1 1 36), (1 2 18), (1 3 12), (1 4 9), (1 6 6), (2 2 9), (2 3 6), (3 3 4), the only case where the sum is not unique is $1+6+6 = 2+2+9 = 13$. The extra information that there is a "well-defined" eldest daughter eliminates the possibility $1 + 6 + 6$, where there are eldest twins. Thus the ages are 9, 2, and 2.

1.26. *The mail carrier's sons' ages.*

Let m be the age of mail carrier A, and let a, b, c be the ages of the sons. The first clue yields $m = abc$. Since that is not enough, m must have more than one expression as a product of three numbers.

The second condition, being insufficient, implies that m has two expressions as a product of three numbers that have the same sum. The third condition states that the middle son is uniquely identified, and hence the three ages are different. Furthermore, since this resolves the prior ambiguities, m must have two expressions as a product of three numbers with the same sum so that one such triple consists of distinct numbers and all others do not. Call these two expressions the "twin triple" (repeated element) and the "solo triple" (no repeated element).

First, we prove that no two triples with the same sum and product can have a common number. If they do, then the remaining two from each triple have the same sum and product, as in $ab = rs$ and $a + b = r + s$. Let

$a = r + k$, so $b = s - k$. Now $ab = rs + k(s - r) - k^2$. Since $ab = rs$, this yields $k^2 = k(s-r)$. If $k \neq 0$, then $k = s - r$, and we obtain $a = s$ and $b = r$. If $k = 0$, then $a = r$ and $b = s$. In either case, $\{a, b\} = \{r, s\}$.

Suppose that m is a power of a prime p. The largest power in the two triples, p^k, is only in one triple. Hence the other triple sums to at most $3p^{k-1}$. The first triple sums to at least $p^k + 2$. Having equal sum requires $p^k + 2 \le 3p^{k-1}$. Hence $p + 2/p^{k-1} \le 3$. This requires $p = 2$. Hence m is expressed as a sum of powers of 2 in two ways. One way is distinct powers of 2, so m has three terms in its binary expansion. The other expression has a repeated power of 2, so m has at most two terms in its binary expansion. The contradiction implies that m is not a power of a prime.

Next suppose that the twin triple is $(m, 1, 1)$. Since the triples have the same sum, some element in the solo triple exceeds $m/3$. Also every element of the solo triple divides m. Hence the only possibilities for the solo triple are $(m/2, m/3, m/4)$ and $(m/2, m/3, m/5)$. These lead to $m + 2 = m(13/12)$ with $m = m^3/24$ and $m + 2 = m(31/30)$ with $m = m^3/30$, respectively. Both cases lead to contradictions, so we forbid $(m, 1, 1)$ as the twin triple.

Since the twin triple cannot repeat 1, m must have a repeated prime factor. If $m = p^2q$, where p and q are primes, then the twin triple must be (p, p, q). The possible solo triples are $(p^2, q, 1)$ and $(pq, p, 1)$, but each shares an element with (p, p, q).

We have shown that m has at least four prime factors, counting multiplicity, and they are not all the same or all different. Suppose that $m = p^3q$, where p and q are primes. The twin triple must be (pq, p, p). The possible solo triples are $(p^2q, p, 1)$, $(pq, p^2, 1)$, $(q, p^3, 1)$, and (q, p, p^2). Avoiding shared elements leaves only $(q, p^3, 1)$.

The condition of equal sum is $pq + 2p = p^3 + q + 1$. Rewrite this as $(p^3 - 2p + 1)/(p - 1) = q$. Whenever prime p on the left yields prime q, we have a solution. Possibilities for (p, q) are $(2, 5)$, $(3, 11)$, $(5, 29)$ (when $p = 7$, the resulting q is not prime). The resulting ages for the mail carrier are $2^3 5 = 40$, $3^3 11 = 297$, and $5^3 29 = 3625$.

The next possibility is $m = p^2q^2$. By symmetry, the possible twin triples are (p^2, q, q) and $(pq, pq, 1)$. The possible solo triples are $(p^2q, q, 1)$, $(q^2p, p, 1)$, $(q^2, p^2, 1)$, (qp, q, p). Avoiding shared numbers leaves only the case (p^2, q, q), $(q^2p, p, 1)$. Now $q > q^2p/3$ yields $qp < 3$, so we may assume that $p^2 > q^2p/3$, which requires $p > q^2/3$. With this we study $p^2 + 2q = q^2p + p + 1$. Now $q = 2$ requires $p^2 - 5p + 3 = 0$, which has no rational solution, and $q = 3$ requires $p^2 - 10p + 5 = 0$, and $q = 5$ requires $p^2 - 26p + 9 = 0$, . . . Already $q \ge 7$ and $p > 16$, so the mail carrier is at least $49 \cdot 17^2$ years old with no solution yet in sight.

If $m = p^2qr$ with p, q, r prime, then the only allowed twin triple is (p, p, qr). The solo triples avoiding p and qr are (p^2, q, r), $(p^2q, r, 1)$, $(p^2r, q, 1)$, $(pq, pr, 1)$, (pq, p, r) and (pr, p, q). Instead of considering cases for the form of the triple, let use consider cases for (p, q, r) that keep the mail carrier to a reasonable age. The only cases that keep the mail carrier under 100 have the following values for $(p, q, r, m, p + p + qr)$: $(2, 3, 5, 60, 19)$, $(2, 3, 7, 84, 25)$, and $(3, 2, 5, 90, 16)$. In the first two cases, none of the possible solo triples have sum 19. However, the last case leads to $3 + 3 + 10 = 9 + 2 + 5$, so the mail carrier could be 90.

With at least five factors in the factorization of m, only $2^4 3$ and $2^4 5$ keep the mail carrier under 100. The allowed twin triples are (p^2, p^2, q) and (p, p, p^2q). Neither when $m = 48$ and nor when $m = 80$ does the sum of an allowed twin triple match the sum of a solo triple.

Thus the possible ages under 100 for the mail carrier are 40 and 90.

1.27. *The set of real solutions to* $|x/(x + 1)| \le 1$ *is* $T = \{x \in \mathbb{R}\colon x \ge -1/2\}$. We transform the inquality without changing the set of solutions to obtain $x \ge -1/2$. (We consider only $x \neq -1$). We have $|x/(x + 1)| \le 1$ equivalent to $x^2/(x + 1)^2 \le 1$ equivalent to $x^2 \le x^2 + 2x + 1$ equivalent to $0 \le 2x + 1$ equivalent to $-1/2 \le x$. The first step uses that the absolute value of a number is nonnegative.

1.28. *Optimizing quadratics without calculus.* For $c > 0$, the value $x(c-x)$ is positive only when $0 < x < c$, so we may restrict our attention to that interval. By the Arithmetic-Geometric mean inequality, $xy \le (x + y)^2/4$ whenever $x, y > 0$. Using $y = c - x$, this tells us that $x(c - x) \le c^2/4$. This bound on $x(c - x)$ is attained when $x = c/2$, so $c^2/4$ is the maximum, occurring at $x = c/2$.

As a function of y, $y(c - ay)$ is maximized at the same value of y where $ay(c - ay)$ is maximized, since the ratio between these is the constant a. Letting $z = ay$, we known that $z(c - z)$ is maximized when $z = c/2$. At this value of z, we have $y = c/(2a)$.

1.29. *If* x, y, z *are nonnegative real numbers such that* $y + z \ge 2$*, then* $(x + y + z)^2 \ge 4x + 4yz$*, with equality if and only if* $x = 0$ *and* $y = z$.

Proof 1. Expanding the square and collecting like terms rewrites the inequality as $x^2 + (2(y + z) - 4)x + (y - z)^2 \ge 0$. Since $y + z \ge 2$, all three terms are nonnegative, and the inquality holds. Equality happens only when all three terms equal 0, which occurs if and only if $y = z$ and $x = 0$.

Proof 2. We expand the square and use the AGM and the inequalities $x^2 \ge 0$ and $y + z \ge 4$ to obtain $(x + y + z)^2 = x^2 + 2x(y + z) + (y + z)^2 \ge x^2 + 2x(y + z) + 4yz \ge 2x(y + z) + 4yz \ge 4x + 4yz$. Equality requires equality at each step, which requires $y = z$ in the first inequality and $x = O$ in the second, after which the third is always an equality.

1.30. Let x, y, u, v be real numbers.

a) $(xu+yv)^2 \le (x^2+y^2)(u^2+v^2)$. The AGM Inequality yields $2xvyu \le (xv)^2+(yu)^2$; alternatively, this follows immediately from the nonnegativity of squares: $(xu-yv)^2 \ge 0$. Adding $x^2u^2+y^2v^2$ to both side of the inequality yields $x^2u^2+2xuyv+y^2v^2 \le x^2u^2+x^2v^2+y^2u^2+y^2v^2$, which is equivalent to the desired inequality.

b) Equality holds in part (a) if and only if $xu=yv$. When equality holds, both sides reduce to $4x^2u^2$. When $xu \ne yv$, we have $(xu-yv)^2>0$, and the steps of part (a) yields strict inequality in the final expression.

1.31. *Extensions of the AGM Inequality.*

a) $4xyzw \le x^4+y^4+z^4+w^4$. The equality holds immediately when an odd number of $\{x,y,z,w\}$ are negative and reduces to the case of all positive when an even number are positive. This allows us to assume that all four variables are positive.

Recall that $2tu \le t^2+u^2$ always (Proposition 1.4). Thus $2xy \le x^2+y^2$ and $2wz \le w^2+z^2$. We multiply these inequalities together (justified by the variables being positive). We then apply $2a^2b^2 \le a^4+b^4$ to each of the products of squares. Thus

$$\begin{aligned}4xyzw &\le x^2w^2+y^2w^2+x^2z^2+y^2z^2\\ &\le \frac{x^4+y^4}{2}+\frac{y^4+w^4}{2}+\frac{x^4+z^4}{2}+\frac{y^4+z^4}{2}=x^4+y^4+z^4+w^4\end{aligned}$$

b) $3abc \le a^3+b^3+c^3$. Consider part (a) with w,x,y,z positive. Setting $w=(xyz)^{1/3}$ yields $4(xyz)^{4/3} \le x^4+y^4+z^4+(xyz)^{4/3}$, and thus $3(xyz)^{4/3} \le x^4+y^4+z^4$. Setting $x=a^{3/4}$, $y=b^{3/4}$, $z=c^{3/4}$ yields the result.

The inequality of part (a) has four variables and fourth powers, while that of part (b) has three variables and third powers. The first substitution eliminates the extra variable, while the second scales fourth powers into third powers.

The inequality fails when a,b,c are negative and not all equal, and often also when two of $\{a,b,c\}$ are negative.

1.32. $\{x \in \mathbb{R}\colon x^2-2x-3<0\} = \{x \in \mathbb{R}\colon -1<x<3\}$. Let S be the first set and T the second. If $x \in T$, then $x+1>0$ and $x-3<0$. Hence $(x+1)(x-3)<0$, which is the same as $x^2-2x-3<0$. Thus $T \subseteq S$.

If $x \in S$, so that $x^2-2x-3<0$, then $(x+1)(x-3)<0$. The product of two numbers is negative only when exactly one factor is negative. Hence $x<3$ and $x>-1$. Thus $-1<x<3$ is needed, and hence $S \subseteq T$.

A rephrasing. Since $x^2-2x-3=(x-3)(x+1)$ and the product of two numbers is negative if and only if exactly one of them is negative, S is the set of real numbers x such that exactly one of $x-3$ and $x+1$ is negative. Since $x-3<x+1$, the negative one must be $x-3$, and the condition is

equivalent to $x-3<0$ and $x+1>0$. This becomes $x<3$ and $x>-1$, which is the condition defining the set T.

1.33. *If $S=\{(x,y) \in \mathbb{N}^2\colon (2-x)(2+y)>2(y-x)\}$ and $T=\{(1,1),(1,2),(1,3),(2,1),(3,1)\}$, then $S=T$.* By the properties of inequalities, the pairs (x,y) satisfying $(2-x)(2+y)>2(y-x)$ are the pairs satisfying $4>xy$. Since $2\cdot 2 \ge 4$, the pairs of natural numbers satisfying this are those where the smaller number is 1 and the larger is at most 3. These pairs form the set T.

1.34. *Description of $S=\{(x,y)\in\mathbb{R}^2 : (1-x)(1-y) \ge 1-x-y\}$.* Expanding the product and canceling like terms shows that the pairs (x,y) satisfying this inequality are those satisfying $xy \ge 0$. These are the pairs for which at least one of $\{x,y\}$ is 0 or x and y have the same sign.

1.35. *$x/y+y/x \ge 2$ if and only if x and y have the same sign.* If x or y is 0, then the expression is undefined. If they have opposite signs, then the left side is negative. If they have the same sign, then multiplying by xy yields $x/y+y/x \ge 2$, equivalent to $x^2+y^2 \ge 2xy$, equivalent to $x^2-2xy+y^2 \ge 0$, equivalent to $(x-y)^2 \ge 0$. The last inequality holds whenever x and y have the same sign, so this necessary condition is also sufficient.

1.36. *If $S=[3]\times[3]$ and $T=\{(x,y)\in\mathbb{Z}\times\mathbb{Z}\colon\ 0 \le 3x+y-4 \le 8\}$, then $S \subseteq T$.* Since $3x+y-4$ increases when x or y increases, it suffices to check the minimum and maximum values for x and y. Since $3\cdot 1+1-4=0$ and $3\cdot 3+3-4=8$, we obtain $T \subseteq S$. The set T also contains other pairs, such as $(1,4)$, so equality does not hold.

1.37. *Solution to the general quadratic inequality $ax^2+bx+c \le 0$.* If $a=b=0$, then the solution set is $\mathbb{R}$ if $c \le 0$ and $\varnothing$ if $c>0$. If $a=0$ and $b>0$, then the solution set is $\{x \in \mathbb{R}\colon x \le -c/b$. If $a=0$ and $b<0$, then the solution set is $\{x \in \mathbb{R}\colon x \ge -c/b$.

In the remaining cases, $a \ne 0$. Visually, the graph of the quadratic polynomial is a parabola, and we want to determine for which x the graph is at or below the horizontal axis. The quadratic formula yields the points where the polynomial is zero; these must have the form. $(-b \pm \sqrt{b^2-4ac})/(2a)$.

If $b^2-4ac<0$, then the left side is never 0. If $a>0$, then the solution set is empty. If $a<0$, then the solution set is $\mathbb{R}$.

If $b^2-4ac=0$, then the left side is 0 only at $-b/(2a)$. If $a>0$, then this value is the only solution. If $a<0$, then the solution set is $\mathbb{R}$.

If $b^2-4ac=0$, then the left side is 0 at two points. If $a>0$, then the solution set is the interval $[\frac{-b-\sqrt{b^2-4ac}}{2a}, \frac{-b+\sqrt{b^2-4ac}}{2a}]$. If $a>0$, then the solution set is

$$\{x \in \mathbb{R}\colon x \le \tfrac{-b-\sqrt{b^2-4ac}}{2a}\} \cup \{x \in \mathbb{R}\colon x \ge \tfrac{-b+\sqrt{b^2-4ac}}{2a}\}.$$

1.38. *If* $S = \{x \in \mathbb{R}: x(x-1)(x-2)(x-3) < 0\}$, $T = (0, 1)$, *and* $U = (2, 3)$, *then* $S = T \cup U$. The sign of $x(x-1)(x-2)(x-3)$ depends on how many negative factors it has; the product is positive or negative when the number of negative factors is even or odd, respectively. Thus it is positive when x is large or small or between 1 and 2. It is 0 at $\{0, 1, 2, 3\}$ and negative on $T \cup U$. Thus $S = T \cup U$.

1.39. *Solution of the inequality* $(x - a_1)(x - a_2) \cdots (x - a_n) < 0$. The left side of the inequality is negative if and only if an odd number of the factors are even, because the product of two negative numbers is positive and the product of a negative and a positive is negative. With $a_1 < \cdots < a_n$, an odd number of factors will be negative if and only if x is less than an odd number of the breakpoints. Hence the largest solutions to the inequality are in the numbers in the interval (a_{n-1}, a_n).

The next interval (a_{n-2}, a_{n-1}) doesn't work, but (a_{n-3}, a_{n-2}) does ($x = a_i$ never works because it yields 0). Within an interval (a_i, a_{i+1}), the parity of the number of breakpoints above x doesn't change. The successive intervals alternate between consisting of solutions and consisting of non-solutions. Recording this discussion yields the following expression for the set of solutions of the inequality:

$$\begin{array}{ll} (a_1, a_2) \cup (a_3, a_4) \cup \cdots \cup (a_{n-1}, a_n) & \text{for } n \text{ even} \\ (-\infty, a_1) \cup (a_2, a_3) \cup \cdots \cup (a_{n-1}, a_n) & \text{for } n \text{ odd} \end{array}$$

1.40. *If* A *and* B *are sets, then* $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$. By definition, $A - B$ consists of the elements in A but not in B, and $B - A$ consists of the elements in B but not in A, so the left side is the set of elements in exactly one of A, B. On the right side, we start with all elements in at least one of A, B and delete the elements belonging to both of A, B, so again we are left with the set of elements belonging to exactly one of A and B. This is the symmetric difference of A and B.

In the example, A is the set of U.S. state names beginning with a vowel and B is the set of U.S. state names with at most six letters. We have $A =$ {Alabama, Alaska, Arizona, Arkansas, Idaho, Illinois, Indiana, Iowa, Ohio, Oklahoma, Oregon, Utah}, $B =$ {Alaska, Hawaii, Idaho, Iowa, Kansas, Maine, Nevada, Ohio, Oregon, Texas, Utah}, $A - B =$ {Alabama, Arizona, Arkansas, Illinois, Indiana, Oklahoma}, $B - A =$ {Hawaii, Kansas, Maine, Nevada, Texas}, $A \cup B =$ {Alabama, Alaska, Arizona, Arkansas, Hawaii, Idaho, Illinois, Indiana, Iowa, Kansas, Maine, Nevada, Ohio, Oklahoma, Oregon, Texas, Utah}, and $A \cap B =$ {Alaska, Idaho, Iowa, Ohio, Oregon, Utah}. The symmetric difference is {Alabama, Arizona, Arkansas, Hawaii, Illinois, Indiana, Kansas, Maine, Nevada, Oklahoma Texas}.

1.41. *Relationships among sets* A, B, C.

a) $A \subseteq A \cup B$, *and* $A \cap B \subseteq A$. The union consists of everything in A plus everything in B, so every member of A is included. The intersection consists of those elements of A that are also in B, so the elements of $A \cap B$ do belong to A.

b) $A - B \subseteq A$. $A - B$ consists of the elements of A that are not in B, so the elements of $A - B$ are all in A.

c) $A \cap B = B \cap A$, *and* $A \cup B = B \cup A$. The definitions of intersection and union are independent of the order of the arguments; the intersection consists of the elements in both sets, and the union consists of the elements in at least one of the two sets.

d) $A \subseteq B$ *and* $B \subseteq C$ *imply* $A \subseteq C$. If every element of A is an element of B, and every element of B is an element of C, then an element $x \in A$ must be in B and therefore also in C.

e) $A \cap (B \cap C) = (A \cap B) \cap C$. The elements that are in A and in both B and C are the elements in all three of the sets. The same characterization holds for those that are in C and in both A and B.

f) $A \cup (B \cup C) = (A \cup B) \cup C$. The elements that are in A or in at least one of B and C are the elements in at least one of the three sets. The same characterization holds for those that are in C or in at least one of A and B.

1.42. *Counting the days in each month does not define a function from the set of months to* $\mathbb{N}$. The value for February depends on whether the year is a leap year. Thus we have not assigned exactly one element of the target to the element "February" in the domain.

1.43. *The graph of* $S = \{(x, y) \in \mathbb{R}^2: 2x + 5y \le 10\}$. The set S consists of the points in the Cartesian plane such that $y \le 2 - (2/5)x$. This is the set of points on or below the line defined by $\{(x, y) \in \mathbb{R}^2: y = 2 - (2/5)x\}$. When the constraint is $2x + 5y < 10$, the points must be strictly below the line.

1.44. *Analysis of* $S \cap T$ *when* $S = \{(x, y) \in \mathbb{R}^2: x^2 + y^2 \le 100\}$ *and* $T = \{(x, y) \in \mathbb{R}^2: x + y \le 14\}$.

a) The graph of $S \cap T$ consists of the points that are on or inside the circle with radius 10 centered at the origin (this is S) and also lie on or below the line through $(0, 14)$ and $(14, 0)$.

b) There are 317 points in $S \cap T$ whose coordinates are both integers. It suffice to count the integer points within the circle and subtract the number above the line. First we count points in S with $|x| + |y| \le 10$. With $|x| + |y| = k$, there are $4k$ such points, except 1 when $k = 0$. Thus this part of S has $1 + 4(1 + 2 + \cdots + 10) = 221$ points. When the sum is 11, 12, 13, or 14, the number of positive integer points in S is 8, 7, 6, or 3, respectively, so there are $4(8 + 7 + 6 + 3) = 96$ such integer points in S. No integer points in S have coordinates summing to at least 15. Thus T contains all the integer points of S, and the count is $221 + 96 = 317$.

1.45. *Well-defined functions from $\mathbb{R}$ to $\mathbb{R}$.*

a) $f(x) = |x-1|$ if $x < 4$ and $f(x) = |x| - 1$ if $x > 2$—TRUE. When $2 < x < 4$, both x and $x-1$ are positive, and thus $|x-1| = x-1 = |x|-1$ in the interval of overlap.

b) $f(x) = |x-1|$ if $x < 2$ and $f(x) = |x| - 1$ if $x > -1$—FALSE. When $0 < x < 1$, we have $|x-1| = -(x-1) = -x+1$, but $|x|-1 = x-1$. In this interval the definitions conflict

c) $f(x) = ((x+3)^2-9)/x$ if $x \neq 0$ and $f(x) = 6$ if $x = 0$—TRUE. When $x \neq 0$, there is no division by 0, so the formula for $f(x)$ yields a real number. There is no overlap between the sets with $x \neq 0$ and $x = 0$, so each real number has been assigned a unique real number, and f is well-defined.

d) $f(x) = ((x+3)^2-9)/x$ if $x > 0$ and $f(x) = x+6$ if $x < 7$—TRUE. When $x > 0$, we have $((x+3)^2-9)/x = x+6$.

e) $f(x) = \sqrt{x^2}$ if $x \in \mathbb{Z}$ and $f(x) = x$ if $x < 1$—FALSE. The notation $\sqrt{x^2}$ denotes the positive square root; thus $\sqrt{x^2} = -x$ when x is a negative integer. Thus the function is not well-defined. Furthermore, the function has not been defined at all at real numbers at least 1 that are not integers.

1.46. *Images of functions.* Let S denote the image of f. In each case, we specify T and show that $S = T$.

a) $f: \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = x^2/(1+x^2)$. Let $T = \{y \in \mathbb{R}: 0 \le y < 1\}$.

In the formula defining the function, the numerator is always nonnegative and the denominator is always positive, so the image is nonnegative. Also the numerator is always less than the denominator, so the image is always less than 1. Thus $S \subseteq T$.

For each $y \in T$, we seek $x \in \mathbb{R}$ such that $y = f(x)$. Solving for x shows that when x is $\pm\sqrt{y(1-y)}$, the image is y. Note that the square root is defined when $y \in T$, because $0 \le y < 1$ yields $y(1-y) \ge 0$. Thus $T \subseteq S$.

b) $f: \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = x/(1+|x|)$. Let $T = (-1, 1)$.

In the defining formula, the absolute value of the numerator is always less than the absolute value of the denominator, so $S \subseteq T$.

For $y \in T$, we know that the sign of x must be the same as the sign of y if $y = f(x)$. For $0 \le y < 1$, we solve $y = x/(1+x)$ to obtain $x = y/(1-y)$. For $-1 < y \le 0$, we solve $y = x/(1-x)$ to obtain $x = y/(1+y)$. The resulting x has the right sign, so we have proved $T \subseteq S$.

1.47. *The image of the function $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(a,b) = (a+1)(a+2b)/2$ is the set of all natural numbers that are not powers of 2.* We check first that this defines a function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. We need that $(a+1)(a+2b)/2$ is a natural number when $a, b \in \mathbb{N}$. Since we only add, multiply and divide positive numbers, the result is positive. It is an integer because $a+2b$ has opposite parity from $a+1$. With one odd and one even, the product is divisible by 2.

Now we determine the image. Since exactly one of $a+1$ and $a+2b$ is odd, and it exceeds 1, we know that $f(a,b)$ is the product of two positive integers, one of which is odd and exceeds 1. Thus the image does not contain any power of 2.

We must also show that all other natural numbers are in the image. Let s be an odd factor of n greater than 1.

When $s > \sqrt{2n}$, we desire $a+2b = s$ and $(a+1)/2 = n/s$; the product is n. We set $a = 2(n/s) - 1$ and $b = \frac{s-a}{2} = \frac{1}{2}(s+1-[2n/s])$. Since $s \le n$, a is positive. Since s and a are odd, b is an integer. Since $s > \sqrt{2n}$, b is positive. Hence $n = f(a,b)$ and n is in the image.

When $s \le \sqrt{2n}$, we desire $a+1 = s$ and $(a+2b)/2 = n/s$; the product is n. We set $a = s-1$ and $b = (n/s) - (a/2)$. Since $s > 1$, $a \in \mathbb{N}$. Since a is even, b is an integer. Since $\frac{n}{s} - \frac{a}{2} \ge \frac{n}{\sqrt{2n}} - \frac{\sqrt{2n}-1}{2} > 0$, b is positive. Hence again $n = f(a,b)$ and n is in the image.

1.48. *Descriptions of the function $f: [0,1] \to [0,1]$ defined by $f(x) = 1-x$.* The graph of f is the line segment in $\mathbb{R}^2$ joining $(1,0)$ and $(0,1)$. The function can also be described as giving the amount of water left after x gallons are removed from a full 1-gallon jug. Note that with this description, the domain of the function is the interval $[0,1]$.

1.49. *Properties of functions $f, g: \mathbb{R} \to \mathbb{R}$.*

a) If f and g are bounded, then $f+g$ is bounded—TRUE. By the definition of bounded function, there exist positive constants $M_1, M_2 \in \mathbb{R}$ such that, for $x \in \mathbb{R}$, $|f(x)| \le M_1$ and $|g(x)| \le M_2$. The constant $M = M_1 + M_2$ works to show that $f+g$ is bounded, because applying the triangle inequality yields, for $x \in \mathbb{R}$,

$$|(f+g)(x)| = |f(x)+g(x)| \le |f(x)| + |g(x)| \le M_1 + M_2 = M.$$

b) If f and g are bounded, then fg is bounded—TRUE. Using the same approach as in (a), let $M = M_1, M_2$. Now

$$|(fg)(x)| = |f(x)g(x)| = |f(x)|\,|g(x)| \le M_1 M_2 = M.$$

c) If $f+g$ is bounded, then f and g are bounded—FALSE. The functions f, g defined by $f(x) = x$ and $g(x) = -x$ provide a counterexample. Here f and g have unbounded image, but $f(x) + g(x) = 0$ for all x.

d) If fg is bounded, then f and g are bounded—FALSE. Define f by $f(x) = x$. Define g by $g(x) = 1/x$ for $x \neq 0$, and $g(0) = 0$. In this example, $fg(x) = 1$ for $x \in \mathbb{R} - \{0\}$, and $fg(0) = 0$. Thus fg is bounded, but f and g are unbounded.

e) If both $f+g$ and fg are bounded, then f and g are bounded—TRUE. We are given $M, N \in \mathbb{R}$ such that for all x, $|f(x)+g(x)| \le M$ and

$|f(x)g(x)| \le N$. We show that f and g are bounded by showing that f^2 and g^2 are bounded. We have

$$\begin{aligned} \left|f(x)^2+g(x)^2\right| &= \left|(f(x)+g(x))^2-2f(x)g(x)\right| \\ &\le \left|(f(x)+g(x))^2\right| + 2\,|f(x)g(x)| \le M^2+2N. \end{aligned}$$

Since $f(x)^2$ and $g(x)^2$ are both nonnegative, we have $f(x)^2$ and $g(x)^2$ both bounded by $f(x)^2+g(x)^2$. Thus $|f(x)| \le \sqrt{M^2+2N}$ and $|g(x)| \le \sqrt{M^2+2N}$.

1.50. *Images of subsets of the domain of $f\colon A \to B$.* (**Note**: The original printing incorrectly stated the problem using unions. Part (b) is valid only for intersections.) For a subset S of the domain of f, let $f(S) = \{f(x)\colon x \in S\}$. Let C and D be subsets of the domain.

a) $f(C\cap D) \subseteq f(C)\cap f(D)$. If some $b \in B$ belongs to $f(C\cap D)$, then $f(x) = b$ for some element x in $C \cap D$. Since $x \in C$, $b \in f(C)$. Since $x \in D$, $b \in f(D)$. Thus $b \in f(C\cap D)$ implies $b \in f(C)\cap f(D)$.

b) *Equality need not hold.* Consider $f\colon A \to B$ with $A = \{-1, 1\}$, $B = \{1\}$, and $f(-1) = f(1) = 1$. Let $C = \{-1\}$ and $D = \{1\}$. Now $C \cap D$ and hence also $f(C\cap D)$ is empty, but $1 \in f(C)\cap f(D)$.

1.51. *"Preimage" of subsets of the target of $f\colon A \to B$.* For $S \subseteq B$, let $I_f(S) = \{x \in A\colon f(x) \in S\}$. Let X and Y be subsets of B.

a) $I_f(X\cup Y) = I_f(X)\cup I_f(Y)$. An element of A has its image in $X\cup Y$ if and only if its image is in X or its image is in Y.

b) $I_f(X\cap Y) = I_f(X)\cap I_f(Y)$. An element of A has its image in $X\cap Y$ if and only if its image is in X and its image is in Y.

1.52. *For nonnegative M, N, the maximum x among pairs (x, y) such that $|x+y| \le M$ and $|xy| \le N$ is $x = (M+\sqrt{M^2+4N})/2$.* As in Application 1.38, graphing of level sets shows that the maximum occurs when $x + y = M$ and $xy = -N$. Solving these by $x(M-x)+N = 0$ and taking the larger zero yields $x = (M+\sqrt{M^2+4N})/2$.

1.53. *Maximization of x such that $|x+y| \le 8$ and $|xy| \le 20$, using inequalities.* We avoid case analysis by squaring the first inequality to get $x^2+2xy+y^2 \le 64$. The second inequality implies $-4xy \le 20$. The sum of these is $(x-y)^2 \le 144$, and hence $|x-y| \le 12$.

By the triangle inequality, $2\,|x| \le |x+y|+|x-y| \le 8+12 = 20$. Hence $|x| \le 10$. Since $(x, y) = (10, -2)$ satisfies both inequalities, the answer is 10.

Comment: By symmetry, we have the constraints $-10 \le x \le 10$ and $-10 \le y \le 10$, but not all pairs $(x, y) \in [-10, 10]\times[-10, 10]$ satisfy the inequalities.

1.54. *The set $S = \{(x, y) \in \mathbb{R}^2\colon y \le x$ and $x+3y \ge 8$ and $x \le 8\}$.*

a) The graph of S is a triangle with corners $(8,0)$, $(8,8)$, and $(2,2)$. Replacing the inequalities with equalities yields three lines that form the boundary of this triangle. The inequalities restrict the solution points to the side of each line that includes the interior of the triangle.

b) The minimum value of $x+y$ such that $(x, y) \in S$ is 4. The level sets of $f(x, y) = x + y$ are lines at an angle of 45 degrees to the horizontal axis. The first one to hit S hits S at the point $(2, 2)$.

1.55. *If $\mathbf{F}$ is a field consisting of exactly three elements $0, 1, x$, then $x+x = 1$ and $x\cdot x = 1$.* We are given that x is different from both 0 and 1.

If $y \ne z$, then $y+x \ne z+x$, since otherwise adding the additive inverse $-x$ to both sides yields $y = z$. Thus $0+x$, $1+x$, and $x+x$ are distinct. We have $0+x = x$, and $1+x$ cannot equal 1 since $x \ne 0$. Thus $1+x = 0$, which leaves $x+x = 1$.

Since nonzero elements have multiplicative inverses, it follows that products of nonzero elements are nonzero; hence $x\cdot x \ne 0$. If $x\cdot x = x$, then multiplication by x^{-1} yields $x = 1$, which is forbidden. Thus $x\cdot x = 1$.

$+$	0	1	x
0	0	1	x
1	1	x	0
x	x	0	1

$\cdot$	0	1	x
0	0	0	0
1	0	1	x
x	0	x	1

1.56. *There is a field of size four but none of size six.*

Let $0, 1, x, y$ be the elements of a field $\mathbf{F}$ with four elements. Multiplying distinct elements by a nonzero element produces distinct elements. Since always $0\cdot z = 0$ and $1\cdot z = 1$, this determines the multiplication table for $\mathbf{F}$: $xy = y$ is forbidden by $x \ne 1$, and hence we must have $xy = 1 = yx$, $x\cdot x = y$, $y\cdot y = x$.

Similarly, adding an element to distinct elements produces distinct elements, so $1+x \notin \{1, x\}$. If $1+x = 0$, then $0 = x\cdot 0 = x(1+x) = x+x\cdot x$. This yields $x\cdot x = 1$, but we have shown that $x\cdot x = y$. Thus $1+x = y$. Interchanging x and y in this argument yields $1+y = x$. Also, if $0 = x+y$, then $0 = 0x = (x+y)x = x\cdot x+y\cdot x = y+1$, which we have just forbidden.

We have shown that the only possibility for the arithmetic operations in $\mathbf{F}$ is that given below. With this specification of addition and multiplication in $\mathbf{F}$, it is straightforward (but perhaps tedious) to verify that all the field axioms hold.

+	0	1	x	y
0	0	1	x	y
1	1	0	y	x
x	x	y	0	1
x	y	x	1	0

$\cdot$	0	1	x	y
0	0	0	0	0
1	0	1	x	y
x	0	x	y	1
y	0	y	1	x

Now suppose that $\mathbf{F}$ is a field with six elements, with 0 and 1 being the additive and multiplicative identity elements. We derive a contradiction.

Consider successive powers of an element $x \notin \{0, 1\}$. The list $x, xx, xxx, \ldots$ must eventually repeat, since $\mathbf{F}$ has only six elements. If the first repetition is $x^i = x^j$ with $i < j$, then cancellation yields $x^{j-i} = 1$. Let k be the least positive integer such that $x^k = 1$. For any $y \neq 0$, the elements $y, yx, \ldots, yx^{k-1}$ are distinct, by the choice of k. Multiplying by a power of x leaves this set unchanged, so every nonzero element is in exactly one such set. This partitions $\mathbf{F} - \{0\}$ into sets of size k. Hence k divides 5, and therefore $k = 5$.

Thus for each $x \in \mathbf{F}$, we can write the elements of $\mathbf{F}$ as $\{0, 1, x, x^2, x^3, x^4\}$. Let y be the additive inverse of 1. From $y + 1 = 0$, the distributive law yields $0 = (1 + y)(1 + y) = 1 + y + y + y^2$, and hence $0 = y + y^2$. Thus y^2 is the additive inverse of y, and hence $y^2 = 1$. Our earlier conclusion about powers now implies that $y = 1$.

We now have $1 + 1 = 0$, and multiplying by z yields $z + z = 0$ for all $z \in \mathbf{F}$. Now consider $(1 + x)(1 + x + x^2 + x^3 + x^4)$ for some $x \notin \{0, 1\}$. By the distributive law and our observation about additive inverses, the product is $1 + x^5$. Since $x^5 = 1$, the product is 0. This requires that $1 + x$ or $1 + x + x^2 + x^3 + x^4$ is 0. Since additive inverses are unique, $1 + x \neq 0$, and therefore $1 + x + x^2 + x^3 + x^4 = 0$.

Finally, let $z = 1 + x$. Since $z \notin \{0, 1, x\}$, we have $z = x^r$ for some $r \in \{2, 3, 4\}$. Substituting $1 + x = z$ and applying $x^r + x^r = 0$ in $1 + x + x^2 + x^3 + x^4 = 0$ yields $x^s + x^t = 0$, where $\{s, t\} = \{2, 3, 4\} - \{r\}$. This contradicts the property that each element is its own additive inverse.

2. LANGUAGE AND PROOFS

2.1. *A flawed argument for* $2 = 1$.

Let x, y be real numbers, and suppose that $x = y$. This yields $x^2 = xy$, which implies $x^2 - y^2 = xy - y^2$ by subtracting y^2 from both sides. Factoring yields $(x + y)(x - y) = y(x - y)$, and thus $x + y = y$. In the special case $x = y = 1$, we obtain $2 = 1$.

The step where $x - y$ is cancelled from both sides is not valid when $x = y$.

2.2. *Analysis of "If a and b are integers, then there are integers m, n such that $a = m + n$ and $b = m - n$."* The statement is false, since summing the

two equations implies that a necessary condition for the existence of such integers m, n is that $a + b$ be even. Thus $(a, b) = (0, 1)$ is a counterexample.

Adding to the hypothesis the requirement that a and b have the same parity makes the statement true. In this case $m = (a + b)/2$ and $n = (a - b)/2$ are integers that solve the equations.

2.3. *Analysis of "If a is a real number, then $ax = 0$ implies $x = 0$".* With $P(a, x)$ being "$ax = 0$" and $Q(x)$ being "$x = 0$", the sentence is $(\forall a \in \mathbb{R})(P(a, x) \Rightarrow Q(x))$. When $a = 0$, the implication fails. When $a \neq 0$, it is true. Thus $(\exists a \in \mathbb{R})(P(a, x) \Rightarrow Q(x))$ is true.

2.4. *Negation of sentences,* where $A, B \subseteq \mathbb{R}$, $f\colon \mathbb{R} \to \mathbb{R}$, and $P = \{x \in \mathbb{R}\colon x > 0\}$.

a) For all $x \in A$, there is a $b \in B$ such that $b > x$. *Negation:* Some $x \in A$ is as large as every element of B.

b) There is an $x \in A$ such that, for all $b \in B$, $b > x$. *Negation:* For all $x \in A$, some $b \in B$ satisfies $b \leq x$.

c) For all $x, y \in \mathbb{R}$, $f(x) = f(y) \Rightarrow x = y$. *Negation:* Some real number is the image of two different elements of $\mathbb{R}$.

d) For all $b \in \mathbb{R}$, there is an $x \in \mathbb{R}$ such that $f(x) = b$. *Negation:* Some real number does not occur in the image of f.

e) For all $x, y \in \mathbb{R}$ and all $\epsilon \in P$, there is a $\delta \in P$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. *Negation:* There is some choice of x, y, ε such that, for every positive number δ, both $|x - y| < \delta$ and $|f(x) - f(y)| < \varepsilon$ are true. Comment: For every function f, the original statement (e) is true, since whenever $x = y$ the conclusion of the inner conditional is true, and whenever $x - y$ one can choose δ between 0 and $|x - y|$ to make the hypothesis of the conditional false. The negated statement is nonsense.

f) For all $\epsilon \in P$, there is a $\delta \in P$ such that, for all $x, y \in \mathbb{R}$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. *Negation:* There is some positive number ε such that, for every positive number δ, some pair of real numbers that differ by at most δ satisfy $|f(x) - f(y)| \geq \varepsilon$.

2.5. *Statements about real numbers.*

a) For all real numbers y, b, m with $m \neq 0$, there is a unique real number x such that $y = mx + b$. Since $m \neq 0$, the number $(y - b)/m$ exists, and the properties of real numbers imply that it satisfies the equation for x. Hence there is at least one solution.

To prove that there is always at most one solution, suppose that $y = mx + b$ and $y = mx' + b$. We conclude that $mx + b = mx' + b$, which implies $mx = mx'$, which implies $x = x'$ (since $m \neq 0$). Hence the solution is unique.

b) For all real numbers y, m, there exist $b, x \in \mathbb{R}$ such that $y = mx + b$. Given the values of y and m, we can set $x = 0$ and $b = y$ to obtain a solution.

2.6. *Usage of language.*

a) Under the mathematical convention about order of quantifiers, the sentence "Please note that every alternative may not be available at this time" states that there may be no food available. Probably they mean "Please note that some alternative may be unavailable at this time".

b) (student-supplied example of an English sentence that has different meaning depending on inflection, pronunciation, or context.)

2.7. *Alibis and conditional statements.* An *alibi* is a (true) statement that a suspect was in a different location from the crime at the time that the crime was committed. Assuming the truth of "If A committed the crime, then A was present when the crime was committed," an alibi allows us to conclude that A did not commit the crime, since otherwise the hypothesis of our conditional is true and its conclusion is false.

2.8. *Student-supplied example of statements A, B, C such that A and B together imply C, but such that neither A nor B alone implies C.* For example, "If our team scores at least 100 points and our opponents score fewer than 100 points, then we win the game," or "If it rains and my car is parked on the street, then my car will get wet.""

2.9. *The negation of the statement "No slow learners attend this school" is:*

c) Some slow learners attend this school.

This option given on the 1955 exam is not completely correct, because it suggests that more than one attendee is needed. The best response would be "Some slow learner attends this school" or "At least one slow learner attends this school".

2.10. *Logical statements.* We list the given statement, a rephrasing as a conditional or a quantification, and the negation.

a) Every odd number is prime. (It is not relevant that this is false.) If x is an odd number, then x is prime. Some odd number is not prime.

b) The sum of the angles of a triangle is 180 degrees. For every triangle T, the sum of the angles in T is 180 degrees. Some triangle has angle-sum not equal to 180 degree.

c) Passing the test requires answering all the problems. If the test was passed, then all the problems were solved. It is possible to pass the test without solving all the problems.

d) Being first guarantees getting a good seat. If I am first, then I will get a good seat. I may be first and not get a good seat.

e) Lockers must be turned in by the last day of class. If classes have ended, then lockers must have been turned in. Someone may keep a locker past the end of classes.

f) Haste makes waste. If haste, then waste. Haste might not always

lead to waste.

g) I get mad whenever you do that. If you do that, then I get mad. You might do that without me getting mad.

h) I won't say that unless I mean it. If I say that, then I mean it. I may say that without meaning it.

2.11. *The \$100 statement.* From a \$1 bill, a \$10 bill, and a \$100 bill, a true statement gets a bill and a false statement gets nothing. To guarantee receiving the \$100 bill, one may say, "You will give me neither the \$1 bill nor the \$10 bill."

2.12. *Telephone bill.* The problem defines f on $\mathbb{N}\cup\{0\}$ by $f(x)=mx+b$ and states that $f(8)=548$ and $f(12)=572$. It is not necessary to determine b. We have $24=f(12)-f(8)=12m+b-(8m+b)=4m$. Thus $m=6$. We now have $f(20)=f(12)+(f(20)-f(12))=572+8\cdot 6=620$.

Alternatively, after computing $m=6$, $f(8)=548$ yields $b=500$, and now $f(20)=6\cdot 20+500=620$.

2.13. *A word problem.* Let m, w, h denote the ages of the man, the woman, and the house. The three sentences establish three equations among these values: $w+1=(h+1)/3$, $m+9=(h+9)/2$, $m=w+10$. Solving by substitution yields $w=27$, $m=37$, $h=83$.

2.14. *Circles.* The circle specified by a, b, c with $c>-a^2+b^2/4$ is $\{(x,y)\in\mathbb{R}^2\colon x^2+y^2+ax+by=c\}$.

a) Circles with various intersections. Keeping a, b fixed and changing c yields circles that do not intersect (they are different level curves of the function $f(x,y)$ defined by $f(x,y)=x^2+y^2+ax+by$.)

The circles determined by $(a,b,c)=(2,0,0)$ *and* $(a,b,c)=(-2,0,0)$ *share only the point* $(0,0)$. If (x,y) lies on both circles, then $x^2+y^2+2x=x^2+y^2-2x$, which yields $x=0$. Setting $x=0$ in the equation for the first circle $y^2=0$, so the only such point is $(0,0)$.

The circles determined by $(a,b,c)=(1,0,0)$ *and* $(a,b,c)=(0,1,0)$ *share exactly the points* $(0,0)$ *and* $(-1/2,-1/2)$. If (x,y) lies on both circles, then $x^2+y^2+x=x^2+y^2+y$, which yields $x=y$. Setting $x=y$ on the first circle yields $2x^2+x=0$. The solutions of this are $x=0$ and $x=-1/2$, which yields $(0,0)$ and $(-1/2,-1/2)$ as the points of intersection.

b) The parameter c is restricted as given in order to permit solutions. We write $c=x^2+y^2+ax+by=(x-a/2)^2-a^2/4+(y-b/2)^2-b^2/4$. Since the contributions of squares are nonnegative, we deduce that $c\ge-(a^2+b^2)/4$ if there is any solution. When equality holds, there is only a single solution point, which we usually don't view as a circle.

2.15. *Alternative derivation of the quadratic formula.* Suppose $a, b, c\in\mathbb{R}$

with $a \neq 0$, and assume that $ax^2 + bx + c$ can be factored as $a(x - r)(x - s)$ for real numbers r, s, so that r and s are solutions to $ax^2 + bx + c = 0$.

a) Sum and product of the roots. From $ax^2 + bx + c = a(x - r)(x - s) = ax^2 - a(r + s)x + ars$, we equate coefficients of powers of x to obtain $b = -a(r + s)$ and $c = ars$, or $r + s = -b/a$ and $rs = c/a$.

b) Expression for $(r - s)^2$. Since $(r - s)^2 = (r + s)^2 - 4rs$, we can substitute the expressions from a for the sum and product of r and s to obtain $(r - s)^2 = (-b/a)^2 - 4c/a = (b^2 - 4ac)/a^2$.

c) Solution for r, s. Taking the square root of both sides in (b), we obtain $r - s = \sqrt{b^2 - 4ac}/a$. Together with $r + s = -b/a$ from (a), we have a system of linear equations in r and s. The sum of the two equations yields $2r = (-b + \sqrt{b^2 - 4ac})/a$, and the difference yields $2s = (-b - \sqrt{b^2 - 4ac})/a$. Dividing by 2 yields the solutions.

d) Effect of the negative square root. Taking the square root of (b) could also yield $r - s = -\sqrt{b^2 - 4ac}/a$. Letting $S = r$ and $R = s$ then yields $R - S = -\sqrt{b^2 - 4ac}/a$ and $R + S = -b/a$ (from (a)), which are the same equations as before. Hence the negative square root interchanges r and s and does not change the set of solutions.

2.16. *a) Every function* $f : \mathbb{R} \to \mathbb{R}$ *has a unique expression as* $g + h$ *such that* $g(-x) = g(x)$ *and* $h(-x) = -h(x)$ *for all* $x \in \mathbb{R}$. The value of f is known at each real number; the values of g and h must be determined. The equation $f(x) = g(x) + h(x)$ has two unknowns; we need another equation involving $g(x)$ and $h(x)$. The desired expression for f in terms of g and h yields $f(-x) = g(-x) + h(-x)$. The properties required for g and h transform this to $f(-x) = g(x) - h(x)$. This yields the system

$$\begin{aligned} f(x) &= g(x) + h(x) \\ f(-x) &= g(x) - h(-x) \end{aligned}$$

We now have two equations for the two unknowns $g(x)$ and $h(x)$. Adding them yields $2g(x) = f(x) + f(-x)$; subtracting them yields $2h(x) = f(x) - f(-x)$. Hence we have determined g and h in terms of f by computing $g(x) = (f(x) + f(-x))/2$ and $h(x) = (f(x) - f(-x))/2$.

b) Even and odd parts of polynomials. By the definition of polynomial, $f(x) = \sum_{j=0}^{k} c_j x^j$. Thus $f(-x) = \sum_{j=0}^{k} c_j (-1)^j x^j$. Summing these two formulas (and dividing by 2) cancels the terms with odd powers of x. Subtracting them (and dividing by 2) cancels the terms with even powers of x. Therefore, the formulas in (a) yield $g(x) = \sum_{i \ge 1} c_{2i} x^{2i}$ and $h(x) = \sum_{i \ge 1} c_{2i-1} x^{2i-1}$. Thus, g is the polynomial obtained by taking the even terms of f, and h is the polynomial obtained by taking the odd terms.

Because of the special case for polynomials, the function g in this problem is in general called the *even part* of f and the function h is called the *odd part* of f.

2.17. *If* $g(x) = \frac{x}{2} + \frac{x}{f(x)-1}$ *and* $g(x) = g(-x)$ *for all* x *such that* $f(x) \neq 1$, *then* $f(x)f(-x) = 1$ *for all such* x. The given conditions yield $\frac{x}{2} + \frac{x}{f(x)-1} = \frac{-x}{2} + \frac{-x}{f(-x)-1}$. Collecting like terms yields $x = -x\left[\frac{1}{f(x)-1)} + \frac{1}{f(-1)-1}\right]$. After further simplification, $[f(x)-1][f(-x)-1] = -[f(x)-1+f(-x)-1]$. After multiplying out and canceling like terms, what remains is $f(x)f(-x) = 1$.

2.18. *If* A *is the sum of the coefficients of the even powers and* B *is the sum of the coefficients of the odd powers in a polynomial* p, *then* $A^2 - B^2 = p(1)p(-1)$. Let $p(x) = \sum_{i=0}^{k} c_i x^i$ be the formula for the polynomial. Note that $A^2 - B^2 = (A + B)(A - B)$. Thus we need the sum of all the coefficients $(A + B)$ and the alternating sum of the coefficients ($A - B = c_0 - c_1 + c_2 - c_3 + \cdots$). These are $A + B = p(1)$ and $A - B = p(-1)$; setting x to be 1 or -1 yields the desired quantities.

2.19. *"You can fool all of the people some of the time, and you can fool some of the people all of the time, but you can't fool all of the people all of the time."* Let P be the set of people, T the set of times, and $F(p, t)$ the sentence "you can fool person p at time t". The sentence is

$$(\forall p \in P)(\exists t \in T)(F(p,t)) \wedge (\exists p \in P)(\forall t \in T)(F(p,t)) \wedge \neg(\forall p \in P)(\forall t \in T)(F(p,t))$$

The negation is

$$\neg(\forall p \in P)(\exists t \in T)(F(p,t)) \vee \neg(\exists p \in P)(\forall t \in T)(F(p,t)) \vee (\forall p \in P)(\forall t \in T)(F(p,t))$$

The first two parts of the negation become $(\exists p \in P)(\forall t \in T)(\neg F(p,t))$ and $(\forall p \in P)(\exists t \in T)(\neg F(p,t))$. Thus we might interpret the negation in English as "There is someone you can never fool, or every person sometimes cannot be fooled, or everyone can always be fooled."

Which statement is true? One might argue that no one can always be fooled (rather, everyone at some time cannot be fooled), and that therefore the negation is more believable than the original statement. This is perhaps a matter of opinion.

2.20. *The notion of a "winning strategy".* The first player has a winning strategy if there is some move for the first player such that, no matter what the second player does in response, the first player will have a winning strategy in what remains of the game. Let $M(x_1, \ldots, x_k)$ be the set of moves available for the person making the $k + 1$th move after the first k moves have been $x_1, \ldots, x_k$. If the game has already ended, we let $M(x_1, \ldots, x_k)$ be "pass". The statement that the first player has a winning strategy is then

$$\begin{aligned} &(\exists x_1 \in M_0)(\forall x_2 \in M(x_1))(\exists x_3 \in M(x_1, x_2)) \cdots \\ &\qquad (\forall x_8 \in M(x_1, \ldots, x_7))(\exists x_9 \in M(x_1, \ldots, x_8))(\text{Player 1 wins}) \end{aligned}$$

2.21. *Negation of a quantified sentence.* The sentence "For every $n \in \mathbb{N}$ there exists a real $x > 0$ such that $x < 1/n$" can be formalized as $(\forall n \in \mathbb{N})(\exists x > 0)P(x, n)$, where $P(x, n)$ is the sentence $x < 1/n$. Existential quantifiers are usually followed by "such that". We can negate the statement as follows: $\neg(\forall n \in \mathbb{N})(\exists x > 0)P(x, n) \Leftrightarrow (\exists n \in \mathbb{N})[\neg((\exists x > 0)P(x, n))] \Leftrightarrow (\exists n \in \mathbb{N})(\forall x > 0)(\neg P(x, n))$. In words, this is "There exists a natural number n such that every positive number x is at least $1/n$." There is no such natural number, because the real number $1/(2n)$ is less than $1/n$. Hence this negation is false, and the true statement is "For every $n \in \mathbb{N}$ there exists $x > 0$ such that $x < 1/n$." This can be seen directly; for each n, the number $1/2n$ can be chosen as the desired x.

2.22. *Negation of the definition of increasing function.* The definition of f being increasing is on domain S is $(\forall x, x' \in S)(x < x' \Rightarrow f(x) < f(x'))$. The negation is $(\exists x, x' \in S)[(x < x') \wedge (f(x) \geq f(x')]$. In words, this is "for some pair x, x' with $x < x'$, the function values satisfy $f(x) \geq f(x')$.

2.23. *The meaning of "$g \notin S$", where $S = \{g\colon \mathbb{R} \to \mathbb{R}\colon (\exists c, a \in \mathbb{R})(x > a \Rightarrow |g(x)| \leq c\,|f(x)|)\}$.* Note that S depends of f. The meaning of "$g \notin S$" is $(\forall c, a \in \mathbb{R})(\exists x > a)(|g(x)| > c\,|f(x)|)$. In other words, for each $c \in \mathbb{R}$, requiring x to be large does not make $|g(x)| \leq c\,|f(x)|$ true.

2.24. *Two statements about a set S of natural numbers.*

a) There is a number M such that, for every $x \in S$, $|x| \leq M$.

b) For every $x \in S$, there is a number M such that $|x| \leq M$.

Statement (a) says that there exists M such that M is a bound for S, so this statement says that S is finite. Statement (b) says that every element of S is bounded by a number, such as itself, but the number can be different for different choices of x. Statement (b) is always true and places no restriction on S. Hence if (a) is true, then (b) is true; i.e. (a) implies (b).

2.25. *For $a \in \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$, the statements (a) and (b) below have different meanings.*

a) $(\forall \varepsilon > 0)(\exists \delta > 0)[(|x - a| < \delta) \Rightarrow (|f(x) - f(a)| < \varepsilon)]$

b) $(\exists \delta > 0)(\forall \varepsilon > 0)[(|x - a| < \delta) \Rightarrow (|f(x) - f(a)| < \varepsilon)]$

Statement (b) is a stronger requirement satisfied only by those functions satisfying (a) that also are constant in a neighborhood of a. For example, the function defined by $f(x) = x$ satisfies (a) (for each $\varepsilon > 0$, simply choose δ equal to ε), but it does not satisfy (b). On the other hand, the function defined by $f(x) = 0$ satisfies both. (Comment: Statement (a) is the definition of continuity at a—see Chapter 15).

2.26. *Order of quantifiers.* Omitting the specifications of universes, the statements symbolically become (a): $\forall(\varepsilon)\forall(a)\exists(\delta)\forall(x)(|f(x) - f(a)| < \varepsilon)$ and (b): $\forall(\varepsilon)\exists(\delta)\forall(a)\forall(x)(|f(x) - f(a)| < \varepsilon)$. Statement (b) is stronger (more restrictive on f), because here a single choice of δ must work for all values of a, while in (a) different δ can be chosen for different values of a.

Comment: This is the distinction between continuity at a and uniform continuity, which is discussed in Chapter 15.

2.27. *Interpretation of statements about $c \in \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$.*

a) For all $x \in \mathbb{R}$ and all $\delta > 0$, there exists $\epsilon > 0$ such that $|x| < \delta$ implies $|f(x) - c| < \epsilon$. This states that on every interval, f is bounded.

b) For all $x \in \mathbb{R}$, there exists $\delta > 0$ such that, for all $\epsilon > 0$, we have $|x| < \delta$ implies $|f(x) - c| < \epsilon$. This is the statement that $f(x) = c$ on some open interval containing 0.

2.28. *The equation $x^4y + ay + x = 0$.*

a) It is false that "For all $a, x \in \mathbb{R}$, there is a unique y such that $x^4y + ay + x = 0$." A counterexample to this statement is the pair $(a, x) = (0, 0)$. For this example, all $y \in \mathbb{R}$ satisfy the equation.

b) The statement "For $x \in \mathbb{R}$, there is a unique y such that $x^4y + ay + x = 0$" is true if and only if a is positive. If the sentence holds for a, then the equation must have a unique solution $y$when $x = 0$. Thus $ay = 0$ must have a unique solution; this requires $a \neq 0$. Also, if $a < 0$, then $x = (-a)^{1/4}$ is a choice of x for which the equation has no solution.

If $a > 0$, then for every $x \in \mathbb{R}$ we can solve the equation for y to obtain $y = -x/(x^4 + a)$. This computes a unique value for y that makes the equation true. Thus the most general condition on a that makes the sentence true is "$a > 0$".

2.29. *Extremal problems.*

a) *Characterization of "minimum".* To prove that $\beta = \min\{f(x)\colon x \in S\}$, it must be shown that $(\forall x \in S)(f(x) \geq \beta)$ and $(\exists x \in S)(f(x) = \beta)$.

b) *The minimum of $f(x, y) = \max\{x, y, \frac{1}{x} + \frac{1}{y}\}$, over the set of ordered pairs (x, y) of positive real numbers, is $\sqrt{2}$.* We prove that $f(x, y) \geq \sqrt{2}$ always, and that this value is achieved. If $\max\{x, y\} \geq \sqrt{2}$, then $f(x, y) \geq \max\{x, y\} \geq \sqrt{2}$. If $x < \sqrt{2}$ and $y < \sqrt{2}$, then $f(x, y) \geq \frac{1}{x} + \frac{1}{y} \geq \frac{2}{\sqrt{2}} = \sqrt{2}$. Finally, when $x = y = \sqrt{2}$, we have $f(x, y) = \sqrt{2}$.

The paragraph above is a complete proof, but it requires knowing the answer. How can the answer be found if not known? If x or y is larger than $\frac{1}{x} + \frac{1}{y}$, then we can reduce the maximum by reducing the larger element of $\{x, y\}$. Hence a natural candidate for the minimum of the maximum occurs when the three quantities are required to be equal, which yields $x = y = \sqrt{2}$.

2.30. Each card has an integer on one side and a letter on the other. Cards are mixed up arbitrarily and then laid out.

a) "Whenever the letter side is a vowel, the number side is odd." This is a conditional statement: "If one side is a vowel, then the other is odd." The statement is false only if there is a card with one side a vowel and the other side even. The statement is true if this never happens. To check this, we must look at the other side of every card showing vowel or even.

b) "The letter side is a vowel if and only if the number side is odd." This is a biconditional statement, requiring both the statement of (a) and its converse. The converse is "Number side odd implies letter side vowel." To check the converse, we must look at the other side of every card showing odd or consonant. To check the conditional in (a), we must look at the other side of every card showing vowel or even. Hence we must look at the other side of every card to test (b).

2.31. *Quantification over empty sets.* The set of my 5-legged dogs is empty. Given any condition, everything in this set satisfies it, but there does not exist an element of this set that satisfies it. In other words, every statement quantified universally over the empty set is true, and every statement quantified existentially over the empty set is false.

a) "All of my 5-legged dogs can fly"—TRUE.
b) "I have no 5-legged dog that cannot fly"—TRUE.
c) "Some of my 5-legged dogs cannot fly"—FALSE.
d) "I have a 5-legged dog that cannot fly"—FALSE.

2.32. *Fraternity pledges.* Each person always tells the truth or always lies:
A) All three of us are liars.
B) Only two of us are liars.
C) The other two are liars.

If the statement of A is true, then it must be false. Hence it is false and A is a liar. If the statement of C is true, then A,B and only A,B are liars. This makes the statement of B true, which is a contradiction. Hence the statement of C is false and C is a liar. Now the statement of B is true, and B is a truth-teller.

2.33. *Three children in line.* The hats are from a set of two red and three black hats. The third child sees the first two hats, the second child sees the first, and the first child sees none. If the first two were both red, the third would know she wore black. Since she is silent, at least one of the first two is black. The second also knows this reasoning. Thus if she saw red on the first, she would know she wore black. Since she is silent, the first child's hat must be black. Thus she names black.

2.34. *Solution of equations in consecutive natural numbers.*

a) *(3,4,5) is the only solution to $n^2 + (n+1)^2 = (n+2)^2$ in natural numbers.* The equality holds if and only if $n^2 - 2n - 3 = 0$. The quadratic factors as $(n+1)(n-3)$, which is 0 only when n is 3 or -1. Hence the only triple of consecutive natural numbers solving the equation is $3, 4, 5$.

b) *There are no consecutive natural numbers such that the cube of one is the sum of the cubes of the other two.* Now our equation is $(n-1)^3 + n^3 = (n+1)^3$, which holds if and only if $n^3 - 6n^2 = 2$. This requires $n^2(n-6) = 2$. Since this expresses 2 as a product of integers, they must all divide 2. We must have $n^2 = 1$, but then $n - 6$ cannot equal 2. Hence the equation is not satisfied by any positive integer n.

2.35. *If x and y are distinct real numbers, then $(x+1)^2 = (y+1)^2$ if and only if $x + y = -2$.* If $x + y = -2$, then $x + 1 = -(y+1)$, and then we square both sides. Starting with $(x+1)^2 = (y+1)^2$, we must consider the two possibilities $x + 1 = \pm(y+1)$. If x and y must be distinct, then only the possibility $x + y = -2$, but otherwise the solution set consists of these points together with those where $x = y$.

2.36. *If x is a real number such that $|x-1| < 1$, then $\left|x^2 - 4x + 3\right| < 3$.* Given $|x-1| < 1$, the triangle inequality yields $|x-3| = |x-1-2| \le |x-1| + |-2| < 1 + 2 = 3$. (Geometrically, if x is within 1 of 1, then x is within 3 of 3.) This yields $\left|x^2 - 4x + 3\right| = |x-1|\,|x-3| < 1 \cdot 3 = 3$.

2.37. *Conditional statements for real numbers.* For a given real number x, let A be the statement "$\frac{1}{2} < x < \frac{5}{2}$", let B be the statement "$x \in \mathbb{Z}$", let C be the statement $x^2 = 1$, and let D be the statement "$x = 2$".

a) $A \Rightarrow C$—FALSE. Every number in $(1/2, 5/2)$ other than 1 is a counterexample.

b) $B \Rightarrow C$—FALSE. Every integer not in $\{1, -1\}$ is a counterexample.

c) $(A \wedge B) \Rightarrow C$—FALSE. The hypothesis is satisfied by 1 and by 2, but the conclusion is not satisfied by 2.

d) $(A \wedge B) \Rightarrow (C \vee D)$—TRUE. The hypothesis is satisfied by 1 and by 2. Since 1 satisfies C and 2 satisfies D, each satisfies the conclusion $C \vee D$.

e) $C \Rightarrow (A \wedge B)$—FALSE. The set of numbers satisfying the hypothesis is $\{1, -1\}$. Among these, both satisfy B, but -1 does not satisfy A. Thus -1 is a counterexample.

f) $D \Rightarrow [A \wedge B \wedge (\neg C)]$—TRUE. The hypothesis is satisfied only by the number 2. This number is in $(1/2, 5/2)$, is an integer, and does not yield 1 when squared, so it also satisfies the conclusion, and the conditional statement is true.

g) $(A \vee C) \Rightarrow B$—FALSE. The hypothesis is satisfied by -1 and by all numbers in the interval $(1/2, 5/2)$. Of these, only -1, 1, and 2 are integers; all other numbers in the interval are counterexamples.

2.38. *Parity of products.*

a) xy is odd if and only if x and y are odd—TRUE. If x and y are odd, then $x = 2k+1$ and $y = 2l+1$ for some integers k and l. The product is $(2k+1)(2l+1) = 4kl+2k+2l+1 = 2(kl+k+l)+1$; being one more than twice an integer, this is odd. We also prove the contrapositive of the converse. If x and y are not both odd, then at least one is even; by symmetry, we may assume that $x = 2k$, where k is an integer. Now $xy = 2(ky)$, which is even.

b) xy is even if and only if x and y are even—FALSE. If $x = 2k$ and $y = 2l+1$, then $xy = 2(2kl+k)$, and xy is even but y is odd.

2.39. *Conditions on the position of a moving particle.* Starting from the origin, the particle moves one unit horizontally or vertices each day. Thus it is always at an integer point, and the sum of its coordinates changes by one each day. Thus the sum of the magnitudes of the coordinates of its position on day k is at most k, and the parity of the sum is the parity of k. These conditions are necessary if the particle can be at (a, b) on day k.

The conditions are also sufficient. Suppose that $|a| + |b| \le k$ and that $a+b$ has the same parity as k. To get the particle to position (a, b) on day k, move it to $(a, 0)$ on the first $|a|$ days and then to (a, b) at day $|a| + |b|$. Since $|a|+|b|$ has the same parity as k, $k-|a|-|b|$ is even, and we can now alternate between (a, b) and $(a, b+1)$, ending at (a, b) on day k.

2.40. *Checkerboard problems.* To prove the nonexistence of a tiling, we assume that one exists and obtain a contradiction.

a) *If two opposite corner squares are removed from an eight by eight checkerboard, then the remaining squares cannot be covered exactly by dominoes.* Each domino covers one black square and one white square, so a board covered by dominoes has the same number of black squares as white squares. Removing opposite corners leaves 32 squares of one color and 30 of the other color.

b) *If two squares of each color are removed from the checkerboard, then the remaining squares cannot be covered exactly by copies of the "T-shape" and its rotations.* Since 60 squares remain, 15 T-shapes must be used. Each T-shape covers an odd number of squares of each color. Since the sum of 15 odd numbers is always odd, any board formed from 15 T-shapes has an odd number of squares of each color. Our remaining board has 30 squares of each color, so it can't be covered by 15 T-shapes.

Alternatively, since the region has the same number of squares of each color, one can conclude that there must be the number of tiles covering 3 black and 1 white must be the same as the number covering 3 white and 1 black. Thus an even number of tiles must be used, which contradicts the total of 60 squares, since 60 is not 4 times an even number.

2.41. *When n hats are returned to n people, it is possible for exactly k people to have the wrong hat if and only if $0 \le k \le n$ and $k \ne 1$.* When a person has the wrong hat, the owner of that hat also has the wrong hat. Thus we must exclude $k = 1$.

When $k = 0$, we can give all people their own hats. For all other values with $0 \le k \le n$, give hat i to person $i+1$, for $1 \le i \le k-1$, and give hat k to person 1. Given the other hats to their owners. Thus all specified values are achievable.

2.42. *If a closet contains n pairs of shoes, then $n+1$ shoes must be extracted to guarantee that at least one pair of matching shoes is obtained.* It is possible to avoid having a pair when choosing n shoes by getting one from each pair. If more than n shoes are chosen, then the average number of shoes chosed from a pair is more than 1, so some pair must be chosen more than once. An alternative way of arguing that more than n shoes force a pair is to prove the contrapositive: if no pair is obtained, then each pair (group of two shoes) is selected at most once, so the total number of shoes selected is at most n.

To guarantee that two matching pairs are obtained, $n+2$ shoes must be extracted. Choosing two of one pair and one each from the others yields a set of size $n+1$ without two pairs. Conversely, if two pairs are not obtained, then the maximum possible is one from each of $n-1$ incomplete pairs, and at most two from one complete pair.

2.43. *Logical equivalence of $P \Leftrightarrow Q$ and $Q \Leftrightarrow P$.* Writing **iff** ("if and only if") to mean logical equivalence, we have

$$P \Leftrightarrow Q \text{ iff } (P \Rightarrow Q) \wedge (Q \Rightarrow P) \text{ iff } (Q \Rightarrow P) \wedge (P \Rightarrow Q) \text{ iff } Q \Leftrightarrow P.$$

2.44. *Conditional statements that are true for all statements P, Q.*

a) $(Q \wedge \neg Q) \Rightarrow P$. For every statement Q, the hypothesis of this conditional statement is false. Thus the conditional statement is true regardless of whether P is true, since the conditional is false only when the hypothesis is true and the conclusion is false.

b) $P \wedge Q \Rightarrow P$. When P and Q are not both true, the hypothesis is false, and the conditional is true. When P and Q are both true, the conclusion P is true. Hence the conditional statement is always true.

c) $P \Rightarrow P \vee Q$. When the hypothesis is true, P is true, which means that P or Q is true regardless of whether Q is true. Since the conclusion is true whenever the hypothesis is true, the conditional statement is true.

2.45. *$P \Rightarrow Q$ and $Q \Rightarrow R$ imply $P \Rightarrow R$.* One interpretation of the hypothesis is $(\neg P \vee Q) \wedge (\neg Q \vee R)$. Given this, if Q is true, then R is true. If Q is false, then $\neg P$ is true. Regardless of whether Q is true or false, we thus have $\neg P \vee R$, which is the same as $P \Rightarrow R$.

$P \Leftrightarrow Q$ and $Q \Leftrightarrow R$ imply $P \Leftrightarrow R$. This follows by two applications of the first part.

2.46. $S \Leftrightarrow [\neg S \to (R \wedge \neg R)]$. If S is true, then any conditional that has $\neg S$ as its hypothesis is true, by the definition of when the conditional is true. Conversely, suppose the conditional on the right above is true. Since its conclusion is always false, the truth of the conditional requires that the hypothesis is always false, which means $\neg S$ is false, and hence S is true.

When S is the statement $P \to Q$, the conditional on the right describes the method of contradiction, because in this case $\neg S$ is $P \wedge \neg Q$, and the statement then says that $P \to Q$ is equivalent to $P \wedge \neg Q$ yielding a contradiction.

2.47. *Quantifiers and conditional statements.* Let $P(x)$ be "x is odd", and let $Q(x)$ be "$x^2 - 1$ is divisible by 8".

a) $(\forall x \in \mathbb{Z})(P(x) \Rightarrow Q(x))$—*TRUE*. Consider an integer x. Under the hypothesis "x is odd", we have $x = 2k + 1$ for some integer k, and hence $x^2 - 1 = 4k^2 + 4k + 1 - 1 = 4k(k+1)$. When k is an integer, one of k and $k+1$ is even, and hence this product is divisible by 8.

b) $(\forall x \in \mathbb{Z})(Q(x) \Rightarrow P(x))$—*TRUE*. For (b), we prove for each x the contrapositive $\neg P(x) \Rightarrow \neg Q(x)$. If x is not odd, then x is even, so x^2 is even, so $x^2 - 1$ is odd and hence not divisible by 8.

2.48. *Quantifiers and conditional statements.* Let $P(x)$ be the assertion "x is odd", and let $Q(x)$ be the assertion "x is twice an integer".

a) $(\forall x \in \mathbb{Z})(P(x) \Rightarrow Q(x))$—*FALSE*. We need only exhibit a single integer x where the statement is false, which happens when the hypothesis is true and the conclusion is false. Each odd integer is such a counterexample.

b) $(\forall x \in \mathbb{Z})(P(x)) \Rightarrow (\forall x \in \mathbb{Z})(Q(x))$—*TRUE*. This is a single conditional. The hypothesis is the statement "All integers are odd". The conclusion is the statement "All integers are even. The hypothesis is false. Hence the conditional is true, regardless of whether the conclusion is true.

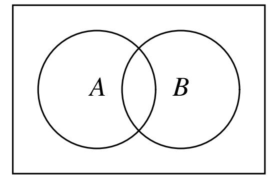
2.49. *Comparison of* $S = \{x \in \mathbb{R}\colon x^2 > x + 6\}$ *and* $T = \{x \in \mathbb{R}\colon x > 3\}$. We rewrite S as $\{x \in \mathbb{R}\colon (x-3)(x+2) > 0\}$. The quadratic inequality holds when $|x|$ is "large". The set T consists of the positive numbers where it holds, but not the negative numbers.

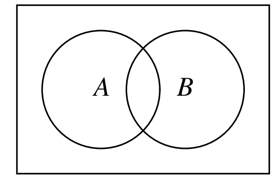
a) $T \subseteq S$. If $x > 3$, then $x - 3$ and $x + 2$ are both positive, and thus $x \in S$.

b) $S \not\subseteq T$. When $x < -2$, both $x - 3$ and $x + 2$ are negative, and thus $x \in S$. However, these elements of S are not in T.

2.50. *Identities about sets.*

a) $(A \cup B)^c = A^c \cap B^c$. The expression $(A \cup B)^c$ denotes the set of everything that is not in A or B. This consists of everything that is outside A and outside B, which is precisely the set described by the expression $A^c \cap B^c$.





b) $A \cap [(A \cap B)^c] = A - B$. $A \cap [(A \cap B)^c] = A \cap (A^c \cup B^c) = (A \cap A^c) \cup (A \cap B^c) = \varnothing \cup (A \cap B^c) = A \cap B^c = A - B$.

c) $A \cap [(A \cap B^c)^c] = A \cap B$. $A \cap (A \cap B^{cc}) = A \cap (A^c \cup B) = (A \cap A^c) \cup (A \cap B) = A \cap B$.

d) $(A \cup B) \cap A^c = B - A$. $(A \cup B) \cap A^c = (A \cap A^c) \cup (B \cap A^c) = \varnothing \cup (B - A) = B - A$.

2.51. *Distributive laws for intersection and union.*

a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. An element in the set on the left must belong to A, and it must also belong to B or to C. In the first case, it belongs to A and to B; in the second, it belongs to A and to C. Hence it is in the set on the right. Similar discussion shows that every element of the set on the right belongs to the set on the left.

2.52. *If* A, B, C *are sets, then* $A \cap (B - C) = (A \cap B) - (A \cap C)$. Elements of $B - C$ belong to B and not to C, so every element in both A and $B - C$ belongs to $A \cap B$ and not to $A \cap C$. Hence $A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$.

Conversely, every element of $(A \cap B) - (A \cap C)$ is in A. Also it is in B. Since we discard all elements of C that are also in A, we keep no elements of $A \cap B$ in C. Hence our element is in B but not in C, and we have $(A \cap B) - (A \cap C) \subseteq A \cap (B - C)$.

2.53. *If* A, B, C *are sets, then* $(A \cup B) - C \subseteq [A - (B \cup C)] \cup [B - (A \cap C)]$, *but equality need not hold.* From a Venn diagram with circles for A, B, C forming eight regions, one can see that $(A \cup B) - C$ consists of the regions $A - (B \cup C)$, $(A \cap B) - C$, and $B - (A \cup C)$. The set $[A - (B \cup C)] \cup [B - (A \cap C)]$ consists of these together with $(B \cap C) - A$.

Thus inclusion holds, and the sets differ whenever there is an element that is in B and C but not in A. The smallest example of this is $A = \varnothing$, $B = \{0\}$, $C = \{0\}$.

2.54. *When the seven bounded regions formed by three circles in the plane each have a black/white token, the operations of (a) flipping the tokens inside one circle or (b) making the tokens inside one circle all white CANNOT turn the all-white configuration into the configuration that is all white except for the region common to all three circles.* The desired configuration has an odd number of blacks in every circle, which can begin to happen only via operation (a). Since (a) flips two or four tokens in each circle, it does not change the parity of the number of black tokens in any circle. Hence a configuration with an odd number of blacks in every circle arises only from another such configuration. Since the initial configuration is not of this type, the desired configuration cannot be reached.

3. INDUCTION

3.1. *A sequence of statements where the 100th statement is the first one false.* If $P(n)$ is "$n < 100$", then $P(1), \dots, P(99)$ are true but $P(100)$ is false.

3.2. *Falsity of a sequence of statements.* We are given $P(1), P(2), \dots$ such that $P(1)$ is false, and such that whenever $P(k)$ is false, also $P(k+1)$ is false. Define $Q(n)$ by $Q(n) = \neg P(n)$. The hypotheses imply that $Q(1)$ is true and that whenever $Q(k)$ is true, also $Q(k+1)$ is true. By the principle of induction, all $Q(n)$ are true, and hence all $P(n)$ are false.

3.3. *Induction in both directions.* We are given statements with an integer parameter such that $P(0)$ is true, and such that whenever $P(n)$ is true, also both $P(n+1)$ and $P(n-1)$ are true. Since $P(n) \Rightarrow P(n+1)$, ordinary induction implies that $P(n)$ is true when $n \geq 0$.

Let $Q(n) = P(-n)$. Since $P(n) \Rightarrow P(n-1)$, ordinary induction implies that $Q(n)$ is true when $n \geq 0$, and hence $P(n)$ is true when $n \leq 0$.

3.4. *If $P(0)$ is true, and the truth of $P(n)$ implies the truth of $P(n+1)$ or $P(n-1)$, then possibly only two of the indexed statements are true.* Since $P(0)$ is true, $P(1)$ or $P(-1)$ must be true. However, the truth of $P(0)$ and $P(1)$ does not imply that any other statements among those indexed are true, and neither does the truth of $P(0)$ and $P(-1)$.

3.5. *For $n \in \mathbb{N}$, $\sum_{k=1}^{n}(2k+1) = n^2 + 2n$—TRUE.* For $n = 1$, $2 \cdot 1 + 1 = 3 = 1^2 + 2 \cdot 1$. If $\sum_{k=1}^{n}(2k+1) = n^2 + 2n$, then

$$\sum_{k=1}^{n+1}(2k+1) = (n^2 + 2n) + (2(n+1)+1) = (n+1)^2 + 2(n+1).$$

3.6. *If $P(2n)$ is true for all $n \in \mathbb{N}$, and $P(n) \Rightarrow P(n+1)$ for all $n \in \mathbb{N}$, then $P(n)$ is true for all $n \in \mathbb{N}$—FALSE.* The statement $P(1)$ need not be true. For example, suppose that $P(n)$ is "$n > 1$". Here $P(n)$ is true when n is an even natural number, and $n > 1$ implies $n+1 > 1+1 > 1$, so this sequence of statements is a counterexample.

3.7. *For $n \in \mathbb{N}$, $2n - 8 < n^2 - 8n + 17$—FALSE.* The inequality holds when $n \in \{1, 2, 3, 4\}$, but it fails for $n = 5$. In fact, the inequality fails only when $n = 5$, since it is equivalent to $0 < n^2 - 10n + 25 = (n-5)^2$. One can prove that $2n - 8 < n^2 - 8n + 17$ implies $2(n+1) - 8 < (n+1)^2 - 8(n+1) + 17$ when $n \geq 5$.

3.8. *For $n \in \mathbb{N}$, $2n - 18 < n^2 - 8n + 8$—TRUE.* The inequality is equivalent to $0 < n^2 - 10n + 26 = (n-5)^2 + 1$, which is positive for all n.

Alternatively, one can use induction. Let $P(n)$ be "$2n - 18 < n^2 - 8n + 8$". If $P(n)$ is true, then $2(n+1) - 18 <= 2n - 18 + 2 < n^2 - 8n + 10 = (n+1)^2 - 8(n+1) + 8 - (2n+1) + 10$. The last expression is less than or equal to $(n+1)^2 - 8(n+1) + 8$ when $-(2n+1) + 10 \leq 0$, which is true when $n \geq 9/2$. We can check explicitly that $P(1), P(2), P(3), P(4), P(5)$ are true and then use the computation above to complete a proof by induction.

3.9. *For $n \in \mathbb{N}$, $\frac{2n-18}{n^2-8n+8} < 1$—FALSE.* The inequality differs from that in the preceding problem when $n^2 - 8n + 8 \leq 0$. It is false for $n \in \{2, 3, 4, 5, 6\}$.

3.10. *For an odd number of odd integers, the sum and the product are odd.* We prove this for $2n+1$ odd integers, where $n \geq 0$. For the basis step, one odd integer is an odd integer. The induction step uses the direct computations that the sum of two odd integers is even, while the product of two odd integers is odd. Thus when we add on the last two odd integers to an odd sum, the sum remains odd, and when we multiply on the last two odd integers to an odd product, the product remains odd.

3.11. *Every set of n elements has 2^n subsets.* We use induction on n to prove this for $n \geq 0$. Basis step: The empty set $\varnothing$ is the only set of 0 elements, and $\varnothing$ is the only subset of $\varnothing$, so the formula 2^0 is correct when $n = 0$.

Induction step: Suppose that the claim is true when $n = k$. Let S be a set of $k+1$ elements, and let x be an element of S. The subsets of S consist of those containing x and those not containing x. The subsets not containing x are subsets of $S - \{x\}$; by the induction hypothesis, there are 2^k of these. The subsets containing x consist of x together with a subset of $S - \{x\}$; again the induction hypothesis implies that there are 2^k. Thus altogether there are $2^k + 2^k = 2^{k+1}$ subsets of S. Since S was chosen as any set with $k+1$ elements, the claims also holds when $n = k+1$.

3.12. *If $x \in \mathbb{R}$ and $n \in \mathbb{N}$, then $\sum_{i=1}^{n} x = nx$.* Let $P(n)$ be "$\sum_{i=1}^{n} x = nx$". We use induction on n. Basis step ($P(1)$ is true): $x = 1 \cdot x$.

Induction step ($P(k) \Rightarrow P(k+1)$). The induction hypothesis is $\sum_{i=1}^{k} = kx$. Using this and the distributive law yields $\sum_{i=1}^{k+1} x = kx + x = (k+1)x$.

3.13. *The sum and the difference of two polynomials are polynomials.* Let f and g be polynomials, so that $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{i=0}^{m} b_i x^i$ for some coefficients $a_0, \dots, a_n$ and $b_0, \dots, b_m$. We may assume that $n \geq m$ and let $b_{m+1} = \cdots = b_n = 0$. Writing $g(x) = \sum_{i=0}^{n} b_i x^i$ does not change g.

Since we can reorder terms in a sum (proved by induction), we have

$$(f+g)(x) = f(x)+g(x) = \sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i = \sum_{i=0}^{n} (a_i x^i + b_i x^i) = \sum_{i=0}^{n} (a_i + b_i) x^i.$$

Dropping high indices if $a_n + b_n = 0$, we have now expressed $f+g$ as a polynomial. A similar argument holds for $f-g$, which is a polynomial with coefficients of the form $a_i - b_i$.

3.14. *Summation formulas.* We reduce each summation to a known summation. In each case, induction can also be used directly.

a) $\sum_{i=1}^{n}(4i-1) = 4\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 = 2n(n+1) - n = n(2n+1)$.
b) $\sum_{i=0}^{n}(4i+1) = 4\sum_{i=0}^{n} i + \sum_{i=0}^{n} 1 = 2n(n+1)+(n+1) = (n+1)(2n+1)$.
c) $-1+2-3+4-\cdots-(2n-1)+2n = \sum_{i=1}^{n}[-(2i-1)+2i] = \sum_{i=1}^{n} 1 = n$.
d) $1-3+5-7+\cdots+(4n-3)-(4n-1) = \sum_{i=1}^{n}[(4i-3)-(4i-1)] = \sum_{i=1}^{n}(-2) = -2n$.

3.15. $\sum_{i=1}^{n}(-1)^i i^2 = (-1)^n \frac{n(n+1)}{2}$. When $n=1$, $(-1)^1 1^2 = (-1)^1 \frac{1\cdot 2}{2}$, so the formula holds. For the induction step, suppose that the formula holds when $n=k$. By the induction hypothesis, $\sum_{i=1}^{k+1}(-1)^i i^2 = (-1)^{k+1}(k+1)^2 + (-1)^k \frac{k(k+1)}{2} = (-1)^{k+1}(k+1)[(k+1)-\frac{k}{2}] = (-1)^{k+1}(k+1)\frac{k+2}{2}$. Thus the formula also holds when $n=k+1$, which completes the induction step.

3.16. *For* $n \in \mathbb{N}$, $\sum_{i=1}^{n} i^3 = (\frac{n(n+1)}{2})^2$. We use induction on n. For $n=1$, we have $\sum_{i=1}^{1} i^3 = 1 = (\frac{1\cdot 2}{2})^2$, which completes the basis step.

For the induction step, suppose that the claim holds when n is k. Using the induction hypothesis after isolating the last term, we have

$$\sum_{i=1}^{k+1} i^3 = (k+1)^3 + \sum_{i=1}^{k} i^3 = (k+1)^3 + (\tfrac{k(k+1)}{2})^2$$

$$= \tfrac{(k+1)^2}{4}[4(k+1)+k^2] = \tfrac{(k+1)^2}{4}(k+2)^2.$$

Hence the claim holds also when n is $k+1$.

3.17. $\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$. Using known formulas,

$$\sum_{i=1}^{n} i(i+1) = \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i = \tfrac{n(n+1)(2n+1)}{6} + \tfrac{n(n+1)}{2}$$

$$= \tfrac{n(n+1)(2n+1+3)}{6} = \tfrac{n(n+1)(n+2)}{3}.$$

Using induction, the basis step is $1\cdot 2 = 1\cdot 2\cdot 3/3$. For the induction step, we assume that the formula holds when $n=k$ and compute

$$\sum_{i=1}^{k+1} i(i+1) = (k+1)(k+2) + \sum_{i=1}^{k} i(i+1) = (k+1)(k+2) + \tfrac{k(k+1)(k+2)}{3}$$

$$= (k+1)(k+2)(1+\tfrac{k}{3}) = \tfrac{(k+1)(k+2)(k+3)}{3}.$$

3.18. *If* $0 \le a_i \le b_i$ *for all* $i \in \mathbb{N}$, *then* $\prod_{i=1}^{n} a_i \le \prod_{i=1}^{n} b_i$. We use induction on n. Basis step ($n=1$): given by hypothesis.

Induction step ($n>1$): The induction hypothesis states that $\prod_{i=1}^{n-1} a_i \le \prod_{i=1}^{n-1} b_i$. We use this and Proposition 1.45(F2) (twice, with commutativity of multiplication) to obtain

$$\prod_{i=1}^{n} a_i = \left(\prod_{i=1}^{n-1} a_i\right) a_n \le \left(\prod_{i=1}^{n-1} b_i\right) a_n \le \left(\prod_{i=1}^{n-1} b_i\right) b_n = \prod_{i=1}^{n} b_i.$$

3.19. *If* $k \in \mathbb{N}$ *and* $x < y < 0$, *then* $x^{2k+1} < y^{2k+1}$. Using induction on k, we prove that $x^{2k+1} < y^{2k+1} < 0$ for each nonnegative integer k. Basis step ($k=0$): given by hypothesis.

Induction step ($k>0$). We use commutativity and associativity of multiplication. By Proposition 1.46a and $x < y < 0$, we have $-x > -y > 0$. If $a > b > 0$ and $c > d > 0$, then two applications of Proposition 1.45(F2) yield $ac > bc > bd > 0$. With this and Proposition 1.43e, $x^2 > y^2 > 0$. By the induction hypothesis, $x^{2k-1} < y^{2k-1} < 0$. By Proposition 1.46a, $-x^{2k-1} > -y^{2k-1} > 0$. Combining this with $x^2 > y^2 > 0$ yields $-x^{2k+1} > -y^{2k+1} > 0$, by our earlier computation. Now Proposition 1.46a yields $x^{2k+1} < y^{2k+1} < 0$.

Alternatively, we can verify by induction that the product of an odd number of negative numbers is negative, and that inequalities $a_i > b_i > 0$ yield $\prod_{i=1}^{j} a_i > \prod_{i=1}^{j} b_i > 0$. Since $-x > -y > 0$, this yields $(-x)^{2k+1} > (-y)^{2k+1} > 0$. We transform this to $(-1)^{2k+1}x^{2k+1} > (-1)^{2k+1}y^{2k+1} > 0$. Since $(-1)^{2k+1} < 0$, we obtain $x^{2k+1} < y^{2k+1} < 0$.

3.20. *The proof of Lemma 3.13 in summation notation.*

$$(x-y)\sum_{j=1}^{n} x^{n-j}y^{j-1} = \sum_{j=1}^{n} x^{n-j+1}y^{j-1} - \sum_{j=1}^{n} x^{n-j}y^{j} = \sum_{j=0}^{n-1} x^{n-j}y^{j} - \sum_{j=1}^{n} x^{n-j}y^{j} = x^n - y^n$$

3.21. *The square of a sum.* When expanding the product $\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} x_i\right)$, each term in the first factor is multiplied by each term in the second factor. Thus $\left(\sum_{i=1}^{n} x_i\right)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j$. After collecting like terms, this can also be written as $\left(\sum_{i=1}^{n} x_i\right)^2 = \sum_{i=1}^{n} x_i^2 + 2\sum_{1\le i<j\le n} x_i x_j$.

3.22. *For* $a_1,\dots,a_n \in \mathbb{R}$, $\left|\sum_{i=1}^{n} a_i\right| \le \sum_{i=1}^{n} |a_i|$. We use induction on n. When $n=1$, the two sides are equal. When $n=2$, the statement is the ordinary triangle inequality (Proposition 1.3).

For the induction step, suppose that the inequality holds when $n=k$; this is the induction hypothesis. We prove that if $k \ge 2$, then the inequality also holds when $n = k+1$, using the ordinary triangle inequality and the induction hypothesis applied to the first k numbers. We compute

$$\left|\sum_{i=1}^{k+1} a_i\right| = \left|a_{k+1} + \sum_{i=1}^{k} a_i\right| \le |a_{k+1}| + \left|\sum_{i=1}^{k} a_i\right| \le |a_{k+1}| + \sum_{i=1}^{k} |a_i| = \sum_{i=1}^{k=1} |a_i|$$

3.23. *Flaw in induction proof that $a^n = 1$ for every nonnegative integer n,* where$a$ is a nonzero real number. "Basis step: $a^0 = 1$. Induction step: $a^{n+1} = a^n \cdot a^n / a^{n-1} = 1 \cdot 1/1 = 1$."

In the induction step, the induction hypothesis is applied for the two previous values of the induction parameter. Thus the argument of the induction step is not valid when $n = 0$ (proving $a^1 = 1$), because we do not have the statement for a^{-1}. Thus we need the statement for a^1 in the basis, and then the proof for a^2 can use the statements for a^0 and a^1. However, the statement for a^1 is false.

3.24. *If T is a set of integers such that 1) $x \in T$ and 2) $y \in T$ implies $y+1 \in T$, then it need not hold that $T = \{y \in \mathbb{Z}\colon\ y \ge x\}$.* The hypothesis of this statement does imply that T contains every integer greater than or equal to x, by induction on $y - x$. It does not imply that T equals this set, because T may contain numbers less than x. For example, if $T = \mathbb{N}$, the hypothesis is true with $x = 4$, but the conclusion is not. Changing the equality symbol to $\supseteq$ produces a true statement.

3.25. *The sum and product of natural numbers are natural numbers.* First consider the sum. For $n \in \mathbb{N}$, let $S_n = \{m \in \mathbb{N}\colon n + m \in \mathbb{N}\}$. It suffices to prove that $S_n = \mathbb{N}$ for all n. We use induction on n, omitting some details.

Basis step ($n = 1$). By the definition of $\mathbb{N}$, every real number that is one more than a natural number is also a natural number. Since $1 \in \mathbb{N}$, also $1 + m \in \mathbb{N}$ when $m = 1$. This is the basis step for a proof by induction on m that every natural number m is in S_1. Thus $S_1 = \mathbb{N}$.

Induction step. Suppose that $S_k = \mathbb{N}$. Given that $m \in S_k$, we have $k + m \in \mathbb{N}$, and hence also $k + m + 1 = (k+1) + m \in \mathbb{N}$, which yields $m + 1 \in S_{k+1}$. Hence $\mathbb{N} = S_k \subseteq S_{k+1}$, so also $S_{k+1} = \mathbb{N}$.

3.26. *If $\langle a \rangle$ is a sequence such that $a_1 = 1$ and $a_{n+1} = a_n + 3n(n+1)$ for $n \in \mathbb{N}$, then $a_n = n^3 - n + 1$ for $n \in \mathbb{N}$.* We use induction on n. Basis step: $a_1 = 1 = 1^3 - 1 + 1$. Induction step: Given that $a_k = k^3 - k + 1$, we have

$$a_{k+1} = a_k + 3k(k+1) = k^3 - k + 1 + 3k^2 + 3k = (k+1)^3 - k = (k+1)^3 - (k+1) + 1.$$

3.27. $\sum_{i=1}^{n} \frac{1}{(3i-2)(3i+1)} = \frac{n}{3n+1}$. Induction can be used. Alternatively, recognizing that $\frac{1}{(3i-2)(3i+1)} = \frac{1}{3}[\frac{1}{3i-2} - \frac{1}{3i+1}]$ leads to a telescoping sum.

$$\sum_{i=1}^{n} \frac{1}{(3i-2)(3i+1)} = \frac{1}{3}\sum_{i=1}^{n} \frac{1}{3i-2} - \frac{1}{3i+1} = \frac{1}{3}\left[\frac{1}{1} - \frac{1}{3n+1}\right] = \frac{n}{3n+1}$$

3.28. $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$. Induction can be used. Alternatively, recognizing that $\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$ leads to a telescoping sum.

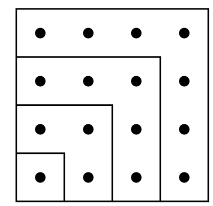
$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) = \frac{1}{1} - \frac{1}{n+1} = \frac{n}{n+1}$$

3.29. $\sum_{i=1}^{n}(2i-1) = n^2$.

Proof 1 (using a previous result). $\sum_{i=1}^{n}(2i-1) = 2\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 = 2n(n+1)/2 - n = n^2$.

Proof 2 (induction on n). $\sum_{i=1}^{1}(2i-1) = 1^2$. If $\sum_{i=1}^{k}(2i-1) = k^2$, then $\sum_{i=1}^{k+1}(2i-1) = 2k + 1 + \sum_{i=1}^{k}(2i-1) = 2k + 1 + k^2 = (k+1)^2$.

Proof 3 ("counting two ways"). Arrange n^2 dots in an n by n square. We can count these in layers from a corner, starting with 1 in the corner, then 3 around it, then the next 5, and so on. Each successive rim has two more dots than the one before it, so the rim sizes are the first n odd numbers, which counts all n^2 dots.



3.30. $\sum_{i=1}^{n}(2i-1)^2 = \frac{n(2n-1)(2n+1)}{3}$.

Proof 1 (induction). Basis Step: the formula holds for $n = 1$ since $2 \cdot 1 - 1 = 1 = 1 \cdot 1 \cdot 3/3$. Induction Step: we prove that the formula holds when $n = k+1$ under the hypothesis that it holds when $n = k$. Splitting off the last term of the summation when $n = k+1$ and applying the induction hypothesis to what remains yields

$$\sum_{i=1}^{n+1}(2i-1)^2 = (2(n+1)-1) + \sum_{i=1}^{n}(2i-1)^2 = (2n+1)^2 + \tfrac{1}{3}n(2n-1)(2n+1)$$

$$= \frac{2n+1}{3}[3(2n+1) + n(2n-1)] = \frac{2n+1}{3}[2n^2 + 5n + 3] = \frac{(2n+1)(n+1)(2n+3)}{3}.$$

Proof 2 (known formulas). We have proved that $\sum_{i=1}^{m} i = m(n+1)/2$ and $\sum_{i=1}^{m} i^2 = m(m+1)(2m+1)/6$. Thus

$$\sum_{i=1}^{n}(2i-1)^2 = \sum_{i=1}^{n}(4i^2 - 4i + 1) = 4\frac{n(n+1)(2n+1)}{6} - 4\frac{n(n+1)}{2} + n$$

$$= \frac{2n(n+1)(2n+1)}{3} - n(2n+1) = n(2n+1)\left[\frac{2n+2}{3} - 1\right] = \frac{n(2n-1)(2n+1)}{3}.$$

3.31. *For $n \in \mathbb{N}$ and $n \geq 2$, $\prod_{i=2}^{n}(1-\frac{1}{i^2}) = \frac{n+1}{2n}$.* The first few values are $\frac{3}{4}, \frac{4}{6}, \frac{5}{8}$; the pattern suggests the formula $\frac{n+1}{2n}$, which we prove by induction on n. The key observation is that $1-\frac{1}{i^2} = \frac{(i-1)(i+1)}{i \cdot i}$. For $n=2$, $1-1/4 = 3/4 = \frac{3}{2 \cdot 2}$, as desired. For the induction step, suppose that the claim holds when n is k. Using the induction hypothesis for $n=k$, we have

$$\prod_{i=2}^{k+1}\left(1-\tfrac{1}{i^2}\right) = \tfrac{k+1}{2k}\left(1-\tfrac{1}{(k+1)^2}\right) = \tfrac{k+1}{2k}\,\tfrac{k(k+2)}{(k+1)\cdot(k+1)} = \tfrac{k+2}{2(k+1)}.$$

Hence the claim also holds when n is $k+1$.

3.32. *If $a_n = \prod_{i=2}^{n}(1-(-1)^i/i)$, then $a_n = 1/2$ if n is even, and $a_n = (n+1)/(2n)$ if n is odd.* After guessing the formula by computing small instances explicitly, we prove the formula by induction. For $n=2$, direct computation yields $a_n = 1/2$. For the induction step, we use $a_n = a_{n-1}(1-(-1)^n/n)$ for $n \geq 3$, with the value of a_{n-1} given by the induction hypothesis. When n is odd, this yields $a_n = (1/2)(1+1/n) = (n+1)/(2n)$. When n is even, it yields $a_n = (n/(2n-2))(1-1/n) = 1/2$.

3.33. *The number of closed intervals with integer endpoints contained in the interval $[1,n]$ (including one-point intervals) is $(n+1)n/2$.* There are $n-i$ intervals of length i, for $0 \leq i \leq n-1$. Thus the total count is the sum of the integers from 1 (when $i=n-1$) to n (when $i=0$).

3.34. *The defective box.* We have 20 boxes, each with 20 balls, each ball weighing one pound except that the balls in one box are one ounce too heavy or one ounce too light. To identify the defective box, we make one weighing consisting of i balls from the ith box for each $1 \leq i \leq 20$. The result differs by j ounces from 190 pounds if and only if the jth box contains the defective balls, and they are too heavy if and only if the total weight errs to the positive side of 190 pounds.

3.35. *Inductive proof that $\sum_{i=0}^{n-1} q^i = (q^n-1)/(q-1)$ when $q \neq 1$.* When $n=1$, the formula reduces to 1, which indeed equals $\sum_{i=0}^{0} q^i$. To prove the formula for a positive integer $n=k+1$ assuming it holds for when $n=k$, we have

$$\sum_{i=0}^{k} q^i = x^k + \sum_{i=0}^{k-1} q^i = q^k + \tfrac{q^k-1}{q-1} = \tfrac{q^{k+1}-q^k+q^k-1}{q-1} = \tfrac{q^{k+1}-1}{q-1}.$$

3.36. *A polynomial f such that $\sum_{i=2}^{n} x^i = f(x)/(x-1)$.* Factoring out x^2 from the terms in the sum yields a standard geometric sum. Thus $\sum_{i=2}^{n} x^i = x^2 \sum_{i=0}^{n-2} x^i = x^2(x^{n-1}-1)/(x-1)$. Thus the desired polynomial f is given by $f(x) = x^{n+1} - x^2$.

3.37. *A sum.* We have $\sum_{i=1}^{n} n^i = (n^{n+1}-1)/(n-1) - 1 = (n^{n+1}-n)/(n-1)$, by the Geometric Sum.

3.38. *The second player wins the "1000" game.* Starting with 0, two players play a game by alternately adding 1, 2, or 3 to the previous total. The first player to bring the total exactly to 1000 wins. The second player can win if the desired total is $4k$, for any $k \in \mathbb{N}$. This is true for $k=1$, because the second player responds to 1, 2, or 3 by adding 3, 2, or 1 and making the total 4. For $k>1$, the second player first plays the game for $k-1$, which (s)he can win, by the induction hypothesis. This means the second player completes a move on which the total becomes $4(k-1)$. Now the first player must add 1, 2, or 3, and the second player adds 3, 2, or 1 to reach $4k$.

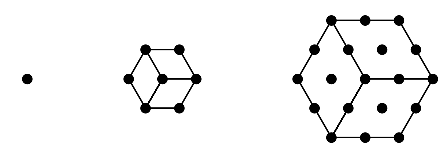
Comments. The second player can guarantee a total of 4 in each round. Thus the claim can be proved using multiplication, but actually multiplication is defined from addition using induction.

The argument generalizes further. Let S be the allowed set of numbers to add. If B wins the S-game to the total r and to the total s, then B also wins to the total $r+s$. If $S = \{1,\ldots,k\}$, then the set of values to which B wins is exactly the multiples of $k+1$. The proof that B wins to these totals is as above. For a total t of the form $t = p(k+1)+q$ with $1 \leq q \leq k$, A can start with q and then follow B's strategy for the game to $p(k+1)$, so A wins in the remaining cases.

3.39. *Hexagonal numbers.* Let a_n be the number of dots in the hexagonal array S_n with n rings. We use the summation formulas for the first m integers and the first m squares to compute a_n and $\sum_{k=0}^{n} a_k$. As illustrated, $a_1 = 1$. Beyond that, ring i adds $6(i-1)$ dots, so $a_n = 1 + \sum_{i=2}^{n} 6(i-1) = 1 + 6\sum_{i=1}^{n-1} i = 1 + 3n(n-1)$ for $n \geq 1$. Furthermore,

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(1-3k+3k^2) = n - 3\tfrac{n(n+1)}{2} + 3\tfrac{n(n+1)(2n+1)}{6}.$$

This simplifies algebraically to n^3. This answer n^3 can be explained directly by viewing S_n as the nth shell of a cubical array of dots.



3.40. *The number of cubes of all positive integer sizes in a cubical array of size n is $\frac{1}{4}n^2(n+1)^2$.* The number of cubes with edges of length $n+1-i$ is i^3. Hence the desired value is $\sum_{i=1}^{n} i^3$. We prove by induction on n that the value of the sum is the given formula. Basis step: $1^3 = \frac{1}{4}1^2 2^2$.

Induction step: If the formuls is correct when $n=k$, then

$$\sum_{i=1}^{k+1} i^3 = (\sum_{i=1}^{k} i^3) + (k+1)^3 = \frac{1}{4}k^2(k+1)^2 + (k+1)^3$$
$$= \frac{1}{4}(k+1)^2[k^2+4(k+1)] = \frac{1}{4}(k+1)^2(k+2)^2$$

3.41. *A function $f: \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ for $x, y \in \mathbb{R}$.*

a) $f(0) = 0$. When $x = y = 0$, we obtain $f(0+0) = f(0) + f(0)$. Thus $f(0) = 2f(0)$, which requires $f(0) = 0$.

b) $f(n) = nf(1)$ *for* $n \in \mathbb{N}$. We use induction on n. Since $f(1) = 1 \cdot f(1)$, the claim holds when $n = 1$. For the induction step, suppose that $f(n-1) = (n-1)f(1)$. Since $f(n-1+1) = f(n-1)+f(1) = (n-1)f(1)+f(1) = nf(1)$, the claim holds also at the next value.

3.42. *The sum of n numbers is independent of the order of addition.* A strict interpretation of this statement considers only summation by adding summands to the total, one by one. We use induction on n. For $n = 1$, there is nothing to do. For $n = 2$, this is the statement of the commutative property. For $n > 2$, consider two possible orderings for accumulating the sum. If the last number is the same in both orderings, then the induction hypothesis says that the sum accumulated before adding the last number is the same. If the last numbers differ, let the last number be x in the first order and y in the second order. Let t be the total of the other $n-2$ numbers; by the induction hypothesis, this is independent of the order. Thus we may assume that the first sum is obtained as $(t+y)+x$ and that the second is obtained as $(t+x)+y$. Now the associative and commutative properties yield $(t+y)+x = t+(y+x) = t+(x+y) = (t+x)+y$.

A more general interpretation allows arbitrary additions, always adding numbers that were obtained by summing smaller lists. We use strong induction to prove that all resulting sums are the same. The basis step is the same as above. For the induction step, suppose that $n \geq 3$.

When we sum a list S of fewer than n numbers, the induction hypothesis yields a common sum $\sigma(S)$ for any order of summation. Under any addition scheme, some last addition is performed. The two numbers combined are $\sigma(S)$ and $\sigma(T)$ for some partition of the n numbers into lists S and T, each with fewer than n numbers. We must show that $\sigma(S)+\sigma(T)$ is the same as $\sigma(S')+\sigma(T')$, where S', T' is another such partition.

Suppose that the S, T partition differs from the S', T' by $S = S' \cup \{x_i\}$ and $T' = T \cup \{x_i\}$. The induction hypothesis allows us to include x_i last when we sum fewer than n elements, or we can also write it first and sum the rest to it. Using this at the ends and associativity in the middle,

$$\sigma(S)+\sigma(T) = (\sigma(S')+x_i)+\sigma(T) = \sigma(S')+(x_i+\sigma(T)) = \sigma(S')+\sigma(T').$$

Thus the claim holds when S and S' differ by a single element. Repeating this argument allows us to switch numbers one by one to turn the partition S, T into the partition S', T' without changing the overall sum.

3.43. *If $f: \mathbb{R} \to \mathbb{R}$ satisfies $f(xy) = xf(y) + yf(x)$ for all $x, y \in \mathbb{R}$,then $f(1) = 0$, and $f(u^n) = nu^{n-1}f(u)$ for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$.* With $y = 1$, the hypothesis yields $f(x) = xf(1) + f(x)$. Thus $xf(1) = 0$ for all $x \in \mathbb{R}$, which requires $f(1) = 0$.

For the second statement, the proof is by induction on n. For $n = 1$, we have $f(x^1) = f(x) = 1x^0 f(x)$. For $n > 1$, we use the induction hypothesis for $n-1$ to compute $f(x^n) = f(xx^{n-1}) = xf(x^{n-1}) + x^{n-1}f(x) = x(n-1)x^{n-2}f(x) + x^{n-1}f(x) = nx^{n-1}f(x)$.

3.44. *The set of natural numbers that can be expressed as the sum of some nonnegative number of 3's and some nonnegative number of 10's.* For $n \leq 20$, we can consider the numbers achievable using at most two 10's to achieve first $\{3, 6, 9, 12, 15, 18\}$, then $\{10, 13, 16, 19\}$, and then $\{20\}$. This omits $S = \{1, 2, 4, 5, 7, 8, 11, 14, 17\}$ and achieves $\{18, 19, 20\}$. By induction on n with basis step $n \in \{18, 19, 20\}$, every number larger than 17 is achievable. For $n > 20$, the induction step achieves n by adding one 3 to the set achieving $n-3$. Thus every natural number can be expressed in this way except the numbers in S.

3.45. *A natural number n has the property that every sum of n consecutive natural numbers is divisible by n if and only if n is odd.* The sum of n consecutive natural numbers starting with s is $\sum_{i=1}^{n}(s+i-1) = ns + n(n-1)/2$. This is divisible by n if and only if $s + (n-1)/2$ is an integer, which is true if and only if n is odd.

3.46. *If $f(n) = n^2 - 8n + 18$, then the natural numbers n for which $f(n) > f(n-1)$ are* $\{n \in \mathbb{N}: n \geq 5\}$. First compute $g(n) = f(n) - f(n-1) = 2n - 9$. Since $2n > 9$ for $n \geq 5$ and $2n < 9$ for $n \leq 4$, the claim follows. This doesn't need induction, but it can be proved using induction.

3.47. $5^n + 5 < 5^{n+1}$ *for all* $n \in \mathbb{N}$. Proof by induction: For the basis step, $5^1 + 5 = 10 < 25 = 5^2$, so the claim holds when $n = 1$. For $n > 1$, we factor out 5 and then use the induction hypothesis to obtain

$$5^n + 5 = 5(5^{n-1} + 5 - 4) < 5(5^n - 4) = 5^{n+1} - 20 < 5^{n+1}.$$

Alternatively, $1 + 1/5^{n-1} < 5$ when $n > 1$, since $5^{n-1} > 1$. Multiplying both sides by 5^n then yields the desired inquality without induction.

3.48. *Given $x > 0$, the inequality $x^n + x < x^{n+1}$ holds for all n if and only if $x > 2$.* For $n = 1$, the condition is $x + x < x^2$; when x is positive this is equivalent to $x > 2$. Thus the condition $x > 2$ is necessary. We give two proofs that $x > 2$ is sufficient.

Proof 1 (induction on n). Basis step ($n = 1$): checked above. Induction step: suppose that $x^{n+1} > x^n + x$. Since $x > 2$, we have $x^2 > x$. Thus

$$x^{n+2} = x(x^{n+1}) > x(x^n + x) = x^{n+1} + x^2 > x^{n+1} + x$$

Proof 2 (direct proof for all $n \in \mathbb{N}$). Since $x > 2$, we have $1/x^{n-1} \le 1$, and thus $1 + 1/x^{n-1} \le 2 < x$. Since $x > 0$, we can multiply both sides by x^n to obtain $x^n + x < x^{n+1}$.

3.49. *Inequalities by induction.*

a) $3^n \ge 2^{n+1}$. By explicit computation, this fails for $n = 1$, but 9>8. With truth for $n = 2$ as the basis step, we prove by induction on n that the inequality holds for all $n \ge 2$. For the induction step, suppose it holds when $n = k$. Then when $n = k + 1$ we $3^{k+1} = 3 \cdot 3^k \ge 3 \cdot 2^{k+1} > 2 \cdot 2^{k+1} = 2^{k+1}$, which yields the desired inequality for $n = k + 1$.

b) $2^n \ge (n+1)^2$. By explicit computation, this fails for $n \in \{1, 2, 3, 4, 5\}$, but $2^6 = 64 > 49 = 7^2$. With truth for $n = 6$ as the basis step, we prove by induction on n that the inequality holds for all $n \ge 6$. For the induction step, suppose it holds when $n = k$. Then when $n = k + 1$ we use the induction hypothesis to compute $2^{k+1} = 2 \cdot 2^k \ge 2 \cdot (k+2)^2 = k^2 + 4k + 2 + k^2$. Because we are consider values of k with $k \ge 6$, we have $k^2 > 2$, so we can replace k^2 by 2 in the last expression to obtain $2^{k+1} \ge (k+2)^2$.

Alternatively, $n \ge 2$ implies $1 + 1/(n+1) < \sqrt{2}$. Thus $2 > (\frac{n+2}{n+1})^2$ when $n \ge 2$, so the induction step can also start with the induction hypothesis $2^k \ge 2 \cdot (k+1)^2$ and multiply by $2 > (\frac{n+2}{n+1})^2$ to obtain $2^{k+1} \ge (k+2)^2$.

c) $3^n > n^4$. By explicit computation, this fails for $n \in \{1, 2, 3, 4, 5, 6, 7\}$ ($3^7 = 2187 < 2401 = 7^4$), but $3^8 = 6561 > 4096 = 8^4$. With truth for $n = 8$ as the basis step, we prove by induction on n that the inequality holds for all $n \ge 8$. The hypothesis of the induction step is that it holds when $n = k$. When $n = k+1$, we use this hypothesis to compute $3^{k+1} = 3 \cdot 3^k > 3 \cdot k^4$. To prove that $3 \cdot k^4 \ge (k+1)^4$, observe that $2k > 15$ for the values $k \ge 8$ that interest us. Also $k^3 > k^2 > k > 1$. Hence

$$3 \cdot k^4 = k^4 + 2k^4 > k^4 + 15k^3 > k^4 + 4k^3 + 6k^2 + 4k + 1 = (k+1)^4.$$

d) $n^3 + (n+1)^3 > (n+2)^3$. By explicit computation, this fails for $n \in \{1, 2, 3, 4, 5\}$ ($5^3 + 6^3 = 341 < 343 = 7^3$), but $6^3 + 7^3 = 559 > 512 = 8^3$. With truth for $n = 6$ as the basis step, we prove by induction on n that the inequality holds for all $n \ge 6$. For the induction step, we assume that the inequality holds for $n = k$. We have $(k+1)^3 = k^3 + (3k^2 + 3k + 1)$ and

$$(k+2)^3 = (k+1)^3 + [3(k+1)^2 + 3(k+1) + 1].$$

When we sum these equations and apply the induction hypothesis to replace $k^3 + (k+1)^3$ by $(k+2)^3$, we obtain

$$(k+1)^3 + (k+2)^3 > (k+2)^3 + (3k^2 + 3k + 1) + [3(k+1)^2 + 3(k+1) + 1].$$

Expanding the right side yields $(k+1)^3 + (k+2)^3 > k^3 + 12k^2 + 24k + 16$. We need to prove that the right side is at least $(k+3)^3$, which is $k^3 + 9k^2 + 27k + 27$. This holds if and only if $3k^2 > 3k + 11$, which is equivalent to $3k(k-1) > 11$ and does hold for the range $k \ge 6$ where we are interested.

3.50. *If $f(x - y) = f(x)/f(y)$ for $x, y \in \mathbb{Z}$ and $f(1) = c$, then $f(n) = c^n$ for* $n \in \mathbb{N}$. The formula holds by hypotheses when $n = 1$. If $f(k) = c^k$, then we let $x = k + 1$ and $y = 1$ to obtain $f(k + 1 - 1) = f(k+1)/f(1)$. This yields $f(k+1) = f(1)f(k) = c^{k+1}$.

3.51. *A cubic polynomial such that the set of natural numbers where its value is at least 3 is* $\{1\} \cup \{n \in \mathbb{N}\colon n \ge 5\}$. We add 3 to a cubic polynomial with zeros at 1, 1, 5: let $f(x) = (x-1)^2(x-5) + 3$.

We obtained f from an understanding of the graphs of polynomials, and we can use induction to prove that it has the desired properties. Check the values up to $x = 5$, and prove that $f(x+1) > f(x)$ when $x \ge 5$.

3.52. *Partial fraction expansion of $\frac{1}{x^2+x-6}$ is $\frac{-1/5}{x+3} + \frac{1/5}{x-2}$.* Since $x^2 + x - 6 = (x+3)(x-2)$, we seek the equality $\frac{1}{x^2+x-6} = \frac{A}{x+3} + \frac{B}{x-2}$. Multiplying by $(x+3)(x-2)$ yields $1 = Ax - 2A + Bx + 3B$. Since equal polynomials have equal coefficients, we require $A + B = 0$ (from the linear term) and $1 = 3B - 2A$ (from the constant term). The solution is $A = -1/5$, $B = 1/5$.

3.53. *If f is a polynomial of degree n and the values $f(0), f(1), \ldots., f(n)$ are known, then f can be determined by an inductive procedure.* When $n = 0$, f is a constant function, and we are given $c = f(0)$, so f is defined by $f(x) = c$. This provides the basis step for induction.

Suppose that $n \ge 1$. Given a polynomial f such that $f(n) = c$, let g be the polynomial defined by $g(x) = f(x) - c$. Since $g(n) = 0$, Theorem 3.24 implies that $g(x) = (x-n)h(x)$, where h is a polynomial of degree $n - 1$. If we can determine h, then we can determine f by $f(x) = (x-n)h(x) + c$.

If we can compute the values $h(0), \ldots, h(n-1)$, then the induction hypothesis allows us to determine h. Since $h(x) = g(x)/(x-n)$ when $x \ne n$, we have $h(i) = [f(i) - c]/(i - n)$ for $i \in \{0, \ldots, n-1\}$. We are given these values of f, so we obtain the values $h(0), \ldots, h(n-1)$.

Comment: The computation in this proof says nothing directly about f until we work down to a constant polynomial, but then we work back up, computing one polynomial of each degree until we get f from h.

Alternatively, one can use n linear equations. The problem is to obtain the coefficients $c_0, \ldots, c_n$ such that $f(x) = \sum_{i=0}^n c_i x^i$ for all x. Evaluating this expression for $x \in \{0, \ldots, n\}$ yields $n+1$ linear equations for the $n+1$ coefficients. The equation for $x = k$ is $f(k) = \sum_{i=0}^n k^i c_i$. What is needed to ensure that this works is a proof that a linear system with the special coefficients $a_{k,i} = k^i$ has a unique solution for each choice of the constants $f(0), \ldots, f(n)$. Even with a proof of this, the method above is faster.

3.54. *If F is defined by $f(x) = \sum_{i=0}^{n} c_i x^i$ and has zeros $\alpha_1, \ldots, \alpha_n$ (all nonzero), then $\sum_{i=1}^{n}(1/\alpha_i) = -c_1/c_0$.* If α is a zero of a polynomial f, then $f(x) = (x-\alpha)h(x)$ for some polynomial h of degree less then f. The other zeros of f are zeros of h. Induction on the degree of f thus yields $f(x) = c\prod(x-\alpha_i)$, where c is a constant. Multiplying out the product (using induction) shows that c must be c_n, the leading coefficient of f.

Let $\beta = \prod_{i=1}^{n} \alpha_i$. The constant term in the expansion of the product is $c_0 = c_n(-1)^n \prod_{i=1}^{n} \alpha_i = c_n(-1)^n \beta$. The linear term is $c_1 = c_n(-1)^{n-1}\frac{\beta}{\alpha_1} + \ldots + c_n(-1)^{n-1}\frac{\beta}{\alpha_n}$. Thus the ratio $-c_1/c_0$ simplifies to the desired sum.

Comment. Starting with the the desired sum and placing the terms over the common denominator β leads to the introduction of c_0 and c_1. A more general result is proved in Exercise 17.40.

3.55. *If $a_1 = 1$, $a_2 = 8$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n \ge 3$, then $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for $n \in \mathbb{N}$.* Basis step ($n \le 2$). For $n = 1$, $a_1 = 1 = 3 \cdot 2^0 + 2(-1)^1$. For $n = 2$, $a_2 = 8 = 3 \cdot 2^1 + 2(-1)^2$. (We need to check two values in the basis step because the induction step always uses the statement for the two previous values.)

Induction step ($n \ge 2$). Suppose that the statement is true for $n-2$ and for $n-1$. We compute

$$\begin{aligned} a_n = a_{n-1} + 2a_{n-2} &= 3 \cdot 2^{n-2} + 2(-1)^{n-1} + 6 \cdot 2^{n-3} + 4(-1)^{n-2} \\ &= (3 \cdot 2 + 6)2^{n-3} + (2(-1) + 4)(-1)^{n-2} = 3 \cdot 2^{n-1} + 2(-1)^n \end{aligned}$$

3.56. *Properties of $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \ge 2$.*

a) If a_1 and a_2 are odd, then a_n is odd for all $n \in \mathbb{N}$. Proof by induction on n. By hypothesis, a_1 and a_2 are odd, which forms the basis of the induction. For the induction step, consider $n \ge 3$, and suppose that a_{n-1} and a_{n-2} are odd. By the recurrence, $a_n = 2a_{n-1} + 3a_{n-2}$, which has the same parity as a_{n-2}. Since a_{n-2} is odd, we conclude that a_n is odd.

b) If $a_1 = a_2 = 1$, then $a_n = \frac{1}{2}(3^{n-1} - (-1)^n)$. Proof by induction on n. Since $\frac{1}{2}(1+1) = 1$ and $\frac{1}{2}(3-1) = 1$, the formula holds for $n = 1$ and $n = 2$, which forms the basis. For the induction step, suppose the formula holds for $n \le k$, in particular for a_k and a_{k-1}, where $k \ge 2$. We can now apply the recurrence to compute

$$a_{k+1} = 2 \cdot \tfrac{1}{2}(3^{k-1} - (-1)^k) + 3 \cdot \tfrac{1}{2}(3^{k-2} - (-1)^{k-1}) = \tfrac{1}{2}(3^k - (-1)^{k+1}),$$

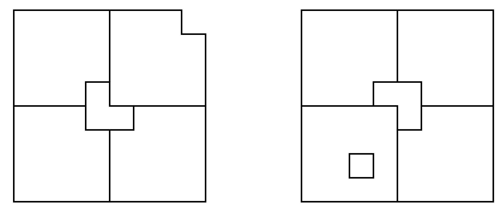
so the formula is also valid when $n = k+1$.

3.57. *If $a_1 = a_2 = 1$ and $a_n = \frac{1}{2}(a_{n-1} + 2/a_{n-2})$ for $n \ge 3$, then $1 \le a_n \le 2$ for $n \in \mathbb{N}$.* Basis step ($n \le 2$). Since $a_1 = a_2 = 1$, these values lie in the interval $[1, 2]$. (We need to check two values in the basis step because the induction step always uses the statement for the two previous values.)

Induction step ($n \ge 2$). Suppose that the statement is true for $n-2$ and for $n-1$. Since $a_n = \frac{1}{2}(a_{n-1} + 2/a_{n-2}$, we have $a_n \le \frac{1}{2}(2 + 2/1) = 2$ and $a_n \ge \frac{1}{2}(1 + 2/2) = 1$.

3.58. *L -tilings.*

a) A 2^n by 2^n chessboard with one corner square removed can be tiled by L. Proof by induction on n. For $n = 1$, the region R_n is a single copy of L. For the induction step, suppose that R_{n-1} can be tiled by L. If we split R_n down the middle horizontally and vertically, we obtain one copy of R_{n-1} and three copies of a full 2^{n-1} by 2^{n-1} board. Using one copy of L, we can cover one square from each of these boards to leave three more copies of R_{n-1}. Now we can apply the induction hypothesis to each of the four copies of R_{n-1} to complete the decomposition of R_n.



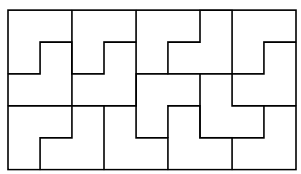
b) A 2^n by 2^n chessboard with any one square removed can be tiled by L. Proof by induction on n. For $n = 1$, the region is a single copy of L. For the induction step, suppose that the previous statement $P(n-1)$ holds, and let R be a 2^n by 2^n region missing one square. If we split R down the middle horizontally and vertically, we obtain one region that contains the missing square plus three copies of a full 2^{n-1} by 2^{n-1} board. By the induction hypothesis, the quarter containing the missing square can be tiled. Using one copy of L, we can cover one square from each of the other quarters to leave three copies of the region in part (a). By part (a), these regions can also be tiled. *Alternatively*, the three squares together form a large L, which by Solution 3.27 can be tiled by the small L.

3.59. *The m-by-n rectangle $R(m, n)$ is L-tileable if and only if mn is divisible by 3, except when $\min\{m, n\} = 3$ and mn is odd.* Since the L-tile has area three, a necessary condition for tileability is that the area mn is divisible by 3, and hence m or n is divisible by 3. By symmetry, we may restrict our attention to the case where m is divisible by 3.

Note that $R(3, 2)$ is L-tileable. Also $R(3k, 2l)$ is L-tileable, since it can be partitioned into kl copies of $R(3, 2)$. It remains to consider $R(3k, 2l + 1)$.

If $k = 1$, then an end of the rectangle can be filled only by two copies of L forming $R(3, 2)$ at the end, leaving $R(3, 2l - 1)$. Since $R(3, 1)$ is not L-tileable, this implies by induction on l that $R(3, 2l + 1)$ is not L-tileable.

For the remaining cases of $R(3k, 2l + 1)$, it suffices by induction on $k+l$ to show that $R(6, 5)$ and $R(9, 5)$ are L-tileable, since $R(3k, 2l+1)$ can be partitioned into $R(3k, 2l - 1)$ and $R(3k, 2)$ when $l >= 3$, and $R(3k, 5)$ can be partitioned into $R(3k - 6, 5)$ and $R(6, 5)$ if $k >= 4$. Since we have shown that $R(6, 2)$ and $R(6, 3)$ are L-tileable, we conclude that $R(6, 5)$ is L-tileable. However, $R(9, 5)$ cannot be partitioned into L-tileable rectangles; we need an *ad hoc* decomposition such as indicated on the left below, where five copies of $R(2, 3)$ and five other copies of L are used.



3.60. *Binary search—It is possible to search for a number x in a sorted list of length n using k probes if and only if $n < 2^k$.* We prove each statement by induction on k.

a) $n < 2^k$ suffices. When $k = 1$, we can answer the question if there is at most one location. For $k > 1$, we examine the middle location; let y be its contents. If $x = y$, then we are done. If $x < y$, then we look for x among the locations before the middle. If $x > y$, then we look for x among the locations after the middle. In each case, $k - 1$ probes remain. Since $n < 2^k$, there are fewer than 2^{k-1} locations before the middle and fewer than 2^{k-1} locations after it. The induction hypothesis guarantees that we can search for x in the appropriate part of the list with the remaining $k - 1$ probes.

b) When $n \geq 2^k$, no strategy suffices. When $k = 1$, one probe will not suffice when there is more than one location. For $k > 1$, we must check some first location; let y be its contents. It may happen that $y \neq x$. If the location we check is before the middle, it may happen that $y < x$. If it is after the middle, it may happen that $y > x$. Wherever we look, it may happen that the remaining list where x may be located has length at least $(n - 1)/2$. Since n is an integer at least 2^k, the remaining list may have length at least 2^{k-1}. The induction hypothesis states that no strategy will guarantee completing the search with the remaining $k-1$ probes. Since we obtain this conclusion for each possible initial probe, there is no strategy that guarantees completing the search.

3.61. *Removing all the heads.* The rule is to remove heads and flip neighbors. The string $HTHTHHHTHHH$ has an odd number of heads, so the game is winnable. Since we always remove one coin at each step, the number of steps needed is the number of coins, 10. One winning strategy, as shown in the text, is to always remove the leftmost head. The sequence is then $HTHTHHHTHHH$, $.HHTHHHTHHH$, $..TTHHHTHHH$, $..TH.TTHHH$, $..H..TTHHH$, $.....TTHHH$, $.....TH.TH$, $.....H..TH$, $........TH$, $........H.$, ...

3.62. *The December 31 Game—Starting with Jan. 1, players alternately increase the month or the day (not both). By always leaving the distance to Dec. 31 the same in both coordinates, the first player guarantees winning.* A *winning position* is a pair (x, y) such that a player who moves the remainder to (x, y) can guarantee winning by proper play thereafter.

Proof 1. Observe that $(12, 31)$ is a winning position. This is the basis step ($n = 0$) for a proof by strong induction that every position of the form $(12-n, 31-n)$ is a winning position, where n is a nonnegative integer. For the induction step, suppose that a player has said $(12 - n, 31 - n)$. The other player must say a date of the form $(12 - n + j, 31 - n)$ or the form $(12-n, 31-n+j)$; advancing the month or the day but not both. Now the original player can say $(12-n+j, 31-n+j)$. By the induction hypothesis, this is a winning position, since it can be written as $(12-(n-j), 31-(n-j))$, and $n - j < n$.

Knowing the winning positions, we find that the first player can win by saying Jan. 20. This is the only position in the winning set that can be reached on the first move; all other first moves yield losing positions.

Proof 2. The game ends with the point $(12, 31)$ on the line $y = x + 19$. We prove that every point on this line is a winning position. From a point on this line, the other player must move off the line, rightward or upward. The original player can then make the opposite move to return to the line. Thus a player who reaches a position on the line can maintain being the only one to reach the line $y = x + 19$. *Comment*. This geometric phrasing actually uses strong induction on the distance from the point $(12, 31)$.

3.63. *Playing along the line $y = 5x$.* Play begins at the origin. When the token is at (x, y), the player chooses a natural number n and moves either to $(x + n, y)$ or to $(x, y + 5n)$. In order to stay along the line $y = 5x$, the second player chooses the same natural number that the first player used on the previous move but moves in the other coordinate.

3.64. *Derivation of the Well-Ordering Property for natural numbers from the principle of induction.* The Well-Ordering Property states that every nonempty set of natural numbers has a least element. Its contrapositive states the a set of natural numbers with no least element must be empty. A set S of natural numbers is empty if and only if $S \cap [n] = \varnothing$ for all $n \in \mathbb{N}$. Thus it suffices to prove that if S has no least element, then $S \cap [n] = \varnothing$ for all $n \in \mathbb{N}$. We prove the conclusion by induction on n.

Since $S \subseteq \mathbb{N}$ and S has no least element, $1 \notin S$, so $S \cap [1] = \varnothing$. For the induction step, suppose that $S \cap [n] = \varnothing$. Since S has no least element, we therefore have $n + 1 \notin S$, since $n + 1$ is the least natural number among those not in S. Now we have $S \cap [n + 1] = \varnothing$.

3.65. *Employers and thieves.* Each employer has one apprentice. When an apprentice is a thief, everyone knows except the thief's employer. The mayor declares: "At least one apprentice is a thief. Each thief is known to be a thief by everyone except his/her employer, and all employers reason perfectly. If during the ith day from now you are able to conclude that your apprentice is a thief, you must come to the village square at the next noon to denounce your apprentice." The villages gather at noon every day thereafter to see what will happen. If in fact $k \geq 1$ of the apprentices are thieves, then their employers denounce them on the kth day.

The proof is by induction on k. Basis step ($k = 1$). When there is exactly one thief, the thief's employer knows of no thieves. Since the employer knows there is at least one thief, his apprentice must be a thief.

Induction step ($k = n + 1$). The induction hypothesis states that when there are actually n thieves, they will be denounced on the nth day. When there are $n+1$ thieves, every employer knows of $n+1$ thieves or of n thieves. An employer who knows of n thieves knows that there must actually be n thieves or $n+1$ thieves, depending on whether his/her apprentice is a thief. If there were actually n thieves, then by the induction hypothesis they would be denounced on the nth day. Since this doesn't happen (there is no one who knows of fewer than n thieves), there can't be only n thieves. Hence there must be $n + 1$ thieves. The employers who know of only n thieves conclude this after waiting past noon on the nth day, so they denounce their employees on the $n + 1$th day.

4. BIJECTIONS AND CARDINALITY

4.1. *Summation of* $(120102)_3$ *and* $(110222)_3$ *in base 3, with check in base 10.* When the sum of the entries in a column is at least 3, the number of 3s "carries" to the next column, as in decimal addition.

base 3	conversion to base 10	base 10
120102	$1 \cdot 243 + 2 \cdot 81 + 0 \cdot 27 + 1 \cdot 9 + 0 \cdot 3 + 2 \cdot 1$	416
110222	$1 \cdot 243 + 1 \cdot 81 + 0 \cdot 27 + 2 \cdot 9 + 2 \cdot 3 + 2 \cdot 1$	350
1001101	$1 \cdot 729 + 0 \cdot 243 + 0 \cdot 81 + 1 \cdot 27 + 1 \cdot 9 + 0 \cdot 3 + 1 \cdot 1$	766

4.2. $333_{(12)}$ *is larger than* $3333_{(5)}$. Let $x = 333_{(12)} = 3 \cdot 111_{(12)}$ and $y = 3333_{(5)} = 3 \cdot 1111_{(5)}$. It suffices to compare $144+12+1$ and $125+25+5+1$. The first is larger (by 1!), so $x > y$.

4.3. *Squares in base 10.* The square of the number obtained by appending 5 to the base 10 representation of n is $(10n+5)^2 = 100n^2 + 100n + 25$. The last two digits are 25. The number obtained by appending 25 to the base 10 representation of $n(n+1)$ is $100n(n+1)+25$. These are the same number.

4.4. *Another temperature scale.* If the conversion of Fahrenheit temperature x to T-temperature is $ax + b$, then changes of fixed amount in x correspond to changes of fixed amount on the T scale. Thus the Fahrenheit temperature corresponding to T-temperature 50 is the average of the Fahrenheit temperatures corresponding to T-temperatures 20 and 80. If water freezes at T-temperature 20 and boils at T-temperature 80, then the Fahrenheit temperature corresponding to 50 is the average of the Fahrenheit temperatures 32 (freezing) and 212 (boiling). The answer is 90.

4.5. *A finite set A has a nonidentity bijection to itself if and only if it has at least two elements.* With one element, the only function is the identity. When A has at least two elements, we let x, y be distinct elements in A. Let $f(x) = y$, $f(y) = x$, and $f(a) = a$ for every $a \in A$ other than x, y. By construction, the image is all of A, and no two elements of A are mapped to the same element of A, so f is a bijection other than the identity.

4.6. *The function giving each day of the week the number of letters in its English name is not injective.* Two days are mapped to the same integer: $f(\text{Sunday}) = f(\text{Monday}) = 6$, but Sunday$\neq$Monday.

4.7. *Injectivity and surjectivity of functions from $\mathbb{R}^2$ to $\mathbb{R}$.*

a) $A(x, y) = x + y$. The addition function is surjective. For each $b \in \mathbb{R}$, we have $A(b, 0) = b$. It is not injective, since also $A(b-1, 1) = b$.

b) $M(x, y) = xy$. The multiplication function is surjective. For each $b \in \mathbb{R}$, we have $M(b, 1) = b$. It is not injective, since also $M(b/2, 2) = b$.

c) $D(x, y) = x^2 + y^2$. This function is not surjective, since no negative number belongs to the image. It is not injective, since $D(0, a) = D(a, 0)$ even though $(0, a) \neq (a, 0)$ when $a \neq 0$.

4.8. *Examples of composition.* If $f(x) = x - 1$ and $g(x) = x^2 - 1$, then $f \circ g$ and $g \circ f$ are defined by $(f \circ g)(x) = x^2 - 2$ and $(g \circ f)(x) = x^2 - 2x$.

4.9. *If f and g are monotone functions from $\mathbb{R}$ to $\mathbb{R}$, then $g \circ f$ is also monotone—TRUE.* The composition is decreasing if one of $\{f, g\}$ is indecreasing and the other is decreasing. The composition is increasing if f

and g are both increasing or both decreasing. Given $x < y$, application of the functions reverses the order for each of $\{f, g\}$ that is decreasing and preserves the order for each of $\{f, g\}$ that is increasing. Since f and g are monotone, this is independent of the choice of x and y, so the claimed statements hold.

4.10. *Linear functions and their composition.* Let $f(x) = ax+b$ and $g(x) = cx+d$ for constants a, b, c, d with a and c not zero.

Both f and g are bijections. For each real number y, the number $(y-b)/a$ is defined and is the only choice of x such that $f(x) = y$. Thus f is both surjective and injective. The same analysis applies to g.

The function $g \circ f - f \circ g$ is neither injective nor surjective. Note that $(g \circ f)(x) = c(ax+b)+d$ and $(f \circ g)(x) = a(cx+d)+b$. The difference h is defined by $h(x) = cax+cb+d-acx-ad-b = cb-ad+d-b$. Thus h is a constant function. It maps all of $\mathbb{R}$ to a single element of $\mathbb{R}$, so it is neither injective nor surjective.

4.11. *Multiplication by 2 defines a bijection from $\mathbb{R}$ to $\mathbb{R}$ but not from $\mathbb{Z}$ to $\mathbb{Z}$.* Let f denote the doubling function. For $y \in \mathbb{R}$, the number $x = y/2$ is the unique real number such that $f(x) = y$. When $y \in \mathbb{Z}$ and y is odd, $y/2 \notin \mathbb{Z}$. Hence odd numbers are not in the image of $f: \mathbb{Z} \to \mathbb{Z}$.

4.12. *Properties of functions.*

a) Every decreasing function from $\mathbb{R}$ to $\mathbb{R}$ is surjective—FALSE.
Let $f(x) = \begin{cases} -x & \text{if } x \le 0 \\ 0 & \text{if } x > 0 \end{cases}$.

b) Every nondecreasing function from $\mathbb{R}$ to $\mathbb{R}$ is injective—FALSE. The constant function f defined by $f(x) = 0$ is nondecreasing but not injective.

c) Every injective function from $\mathbb{R}$ to $\mathbb{R}$ is monotone—FALSE. The function f defined by $f(0) = 0$ and $f(x) = 1/x$ for $x \neq 0$ is injective but not monotone. The function is decreasing on every interval not containing 0, but $f(x)$ is positive when x is positive and negative when x is negative.

d) Every surjective function from $\mathbb{R}$ to $\mathbb{R}$ is unbounded—TRUE. When f is surjective to $\mathbb{R}$, every real number appears in the image, which means that there is no bound on the absolute value of numbers in the image.

e) Every unbounded function from $\mathbb{R}$ to $\mathbb{R}$ is surjective—FALSE. Define f by $f(x) = 0$ for $x \le 0$ and $f(x) = x$ for $x > 0$. This function is unbounded but has no negative numbers in its image.

4.13. *The difference between abc and cba, added to its own reverse, yields 1089 (given that $a \neq c$).* We may assume that $a > c$. The digits of $abc - cba$ are $(a-c-1), 9, (10+c-a)$, so $abc - cba = 100(a-c-1)+90+(10+c-a)$. The reverse of this is $100(10+c-a)+90+(a-c-1)$. Summing the two expressions yields $100(10-1)+180+(10-1) = 1089$.

4.14. *Finding the q-ary expansion of $n+1$ from the q-ary expansion of n.* The idea is to add 1 in base q. Let $a_m, \dots, a_0$ be the q-ary expansion of n. If $a_0 = q-1$, let $b_0 = a0+1$, and let $b_i = a_i$ for $i > 0$. Otherwise, let j be the greatest index such that $a_i = q-1$ for $0 \le i \le j$. Let $b_i = 0$ for $0 \le i \le j$, let $b_{j+1} = a_{j+1}+1$, and let $b_i = a_i$ for $i > j+1$.

By construction, $0 \le b_i \le q-1$ for all i, so b is the q-ary expansion of some number. The contribution from indices greater than $j+1$ is the same. By the geometric sum, the value of the expansion b is one more than the value of the expansion a.

4.15. *By induction on k, the known weights $\{1, 3, \dots, 3^{k-1}\}$ suffice to measure the weights 1 through $(3^k-1)/2$ on a balance scale.* Basis Step: For $k = 1$, the single known weight 1 balances 1. Induction Step: Suppose that the statement holds when the parameter is k. When we add 3^k as the $k+1$th known weight, we can still weigh the numbers $1, \dots, (3^k-1)/2$ as done previously, without using the new weight.

The new weight by itself can balance 3^k. We can balance $3^k-1, \dots, 3^k-(3^k-1)/2$ by putting the new weight on the light side of earlier configurations. Since $3^k-(3^k-1)/2 = (3^k+1)/2$, this fills the gap between the earlier configurations and 3^k. We can balance weights $3^k+1, \dots, 3^k+(3^k-1)/2$ by putting the new weight 3^k on the heavy side of earlier configurations. Since $3^k+(3^k-1)/2 = (3^{k+1}-1)/2$ and we left no gaps, we have balanced all the desired weights. Thus the claim holds also for $k+1$.

Comment: The largest weight balanced by k weights occurs when all the known weights are on the same side. This value is $\sum_{i=0}^{k-1} 3^i$, which by the geometric sum equals $(3^k-1)/2$.

4.16. *Using weights $w_1 \le \cdots \le w_n$ on a two-pan balance, where $S_j = \sum_{i=1}^{j} w_i$, every integer weight from 1 to S_n can be weighed if and only if $w_1 = 1$ and $w_{j+1} \le 2S_j + 1$ for $1 \le j < n$.* For sufficiency, we use induction on n. When $n = 1$, the condition forces $w_1 = 1$, and the weight 1 can be balanced. For the induction step, consider $n > 1$, and suppose that the condition is sufficient for $n-1$ weights. For $1 \le i \le S - w_n$, the induction hypothesis implies that we can weigh i using $\{w_1, \dots, w_{n-1}\}$. With w_n also available, we can also weigh $w_n - i$ and $w_n + i$, so we can weigh every weight from $w_n - S_{n-1}$ to $w_n + S_{n-1} = S_n$ using $\{w_1, \dots, w_n\}$. Since $w_n - S_{n-1} \le S_{n-1} + 1$ by hypothesis, we can weigh every weight up to S_n.

For necessity, suppose we can balance all weights from 1 to S_n. The second largest possibility is $S_n - w_1$, required to be $S_n - 1$, so $w_1 = 1$. If $w_{j+1} > 2S_j + 1$ for some j, then let $W = S_n - 2S_j - 1$; we claim that W cannot be weighed. The largest weight achievable without putting all of $\{w_{j+1}, \dots, w_n\}$ in one pan is $S_n - w_{j+1} < W$, but the smallest weight achievable using all of $\{w_{j+1}, \dots, w_n\}$ in one pan is $S_n - 2S_j$, which exceeds W.

4.17. *Winning positions in Nim.* We prove by strong induction on the total number of coins that a position is winning (for the second player who leaves it) if and only if for all j, the number of pile-sizes whose binary representation has a 1 in the $j$thplace is even. By *jth place* we mean contributions of 2^j. Let s_j be the number of pile-sizes whose binary representation has a 1 in the jth place. Let (*) denote the condition that each s_j is even.

The condition (*) holds when the (starting) number of coins is 0. Since Player 1 cannot move, we view this as Player 2 having taken the last coin. This is the only position with 0 coins. For every j, we have $s_j = 0$. Thus the position is winning and satisfies (*).

When the (starting) number of coins is larger, suppose first that some of the s_j's are odd. We show that some amount can be taken from some pile to leave them all even. By the induction hypothesis, Player 1 thus leaves a winning position, and therefore Player 2 loses. To find a winning move, let J be the largest j such that s_j is odd, and let $S = \{j: s_j \text{ is odd}\}$. Since s_J is odd, some pile-size has a 1 in position J. We want to take coins from this pile P change its binary representation b in the positions indexed by S.

For each position $j \in S$ where b has a 1 in position j, we take 2^j coins from P. For each position $j \in S$ where b has a 0 in position j, we add 2^j coins to P. Because $\sum_{i=1}^{j-1} 2^i$ is less than 2^j, the total of these adjustments is a positive number less that the size of P, so we have obtained a legal move that achieves (*). As we have remarked, the induction hypothesis implies that Player 1 wins.

When each s_j is even, every move changes the binary representation of one pile. Thus it changes the parity of some s_j, and therefore Player 1 cannot produce a smaller position that satisfies (*). By applying the method describedabove, Player 2 can now produce a position satisfying (*). By the induction hypothesis, such a position is winning, so the original position is a winning position to leave.

4.18. *Exponentiation to a positive odd power is a strictly increasing function.* We use induction on k to prove this for the power $2k-1$. Basis step ($k = 1$). Here exponentiation is the identity function: $x < y$ implies $x < y$.

Induction step. Suppose that exponentiation to the power $2n-1$ is strictly increasing. Thus $x^{2n-1} < y^{2n-1}$ when $x < y$. If $0 < x < y$, then $0 < x^2 < y^2$, and multiplying the two inequalities yields $x^{2n+1} < y^{2n+1}$. If $x < 0 \le y$, then x^{2n+1} is negative and y^{2n+1} is nonnegative, so $x^{2n+1} < y^{2n+1}$. If $x < y \le 0$, then $0 \le -y < -x$, and we have proved that $(-y)^{2n+1} < (-x)^{2n+1}$. Since an odd power of -1 is -1, this yields $-y^{2n+1} < -x^{2n+1}$, and thus $x^{2n+1} < y^{2n+1}$.

Solutions to $x^n = y^n$. All pairs with $x = y$ are solutions. When n is odd, the exponentiation is strictly increasing, and hence in this case there are

no other solutions. When n is even, the solutions are $x = \pm y$. To show that there are no other solutions, it suffices to show that exponentiation to the nth power is injective from the set of positive real numbers to itself. This follows by an induction like that above.

4.19. *For $k \in \mathbb{N}$, the only solution to $\sum_{j=0}^{2k} x^{2k-j} y^j = 0$ is $(x, y) = (0, 0)$.* For (x, y) satisfying the equation, multiplying both sides by $(x - y)$ yields $x^{2k+1} - y^{2k+1} = 0$. Since exponentiation to an odd power is injective, this requires $x = y$. Among solution pairs with $x = y$, the equation reduces to $(2k+1)x^{2k} = 0$. The only solution of this is $x = 0$, so the only solution of the original equation is $(x, y) = (0, 0)$, which indeed works.

4.20. *Properties of the map $f: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (ax - by, bx + ay)$, where a, b are fixed parameters with $a^2 + b^2 \ne 0$.*

a) f is a bijection. As proved in Example 4.12, the function $f: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (ax+by, cx+dy)$ is a bijection if and only if $ad - bc \ne 0$. In this problem, the values taken by a, b, c, d are $a, -b, b, a$, respectively, and hence $ad - bc$ becomes $a^2 + b^2$, which by hypothesis is non-zero. Hence the function given here is a bijection, by Example 4.12.

To prove directly that f is a bijection, we show directly that f is both surjective and injective, meaning that for every element (r, s) in the target there is exactly one element (x, y) in the domain such that $f(x, y) = (r, s)$. To prove injectivity, suppose $f(x, y) = f(x', y')$. This requires $ax - by = ax' - by'$ and $bx + ay = bx' + ay'$. Subtracting b times the first equation from a times the second yields $(a^2 + b^2)y = (a^2 + b^2)y'$, and hence $y = y'$, since $a^2 + b^2 \ne 0$. Similarly, adding a times the first equation to b times the second yields $(a^2 + b^2)x = (a^2 + b^2)x'$, so $x = x'$. We have prove that if element $(x, y), (x', y')$ of the domain have the same image, then they must be the same element (no collapsing).

To prove surjectivity, we show that every element (r, s) in the target is the image of some element of the domain. A suitable element (x, y) must satisfy $r = ax - by$ and $s = bx + ay$. Because $a^2 + b^2 \ne 0$, we can solve this system of equations to find such a pair (x, y). The formula for (x, y) appears in part (b).

b) Formula for f^{-1}. When f is a bijection, the inverse function f^{-1} gives for each element of the target the unique element of the domain that maps to it. Computing the inverse function may allow us to prove surjectivity and injectivity simultaneously. In this example, the inverse image of the element (r, s) in the target is the set of solutions (x, y) to the system $r = ax - by$ and $s = bx + ay$. Because $a^2 + b^2 \ne 0$, there is a unique solution (existence implies surjectivity of f, uniqueness implies injectivity of f). The unique solution of the system is $x = \frac{ra+bs}{a^2+b^2}$ and $y = \frac{-br+as}{a^2+b^2}$. Hence the inverse function is $f^{-1}(r, s) = (\frac{ra+bs}{a^2+b^2}, \frac{-br+as}{a^2+b^2})$.

c) A geometric interpretation of f when $a^2+b^2=1$. This uses the distance from the origin to a point (x, y) in $\mathbb{R}^2$, defined to be $\sqrt{x^2+y^2}$. The distance from the origin to the image point $(ax-by, bx+ay)$ is $\sqrt{(a^2+b^2)(x^2+y^2)}$, which equals the distance from the origin to (x, y) if $a^2+b^2=1$. Hence the effect of f on the vector (x, y) is to rotate it around the origin. Every vector is rotated through the same angle; in particular, when $a=0$ and $b=1$, the function rotates everything by 90 degrees counterclockwise. Proving that every vector is rotated by the same amount relies on knowing that the angle between two vectors is determined by their dot product divided by the product of their lengths. Considering the old vector (x, y) and the new vector $(ax-by, bx+ay)$, their dot product is $ax^2-bxy+bxy+ay^2=a(x^2+y^2)$, and the product of their lengths is x^2+y^2. The ratio is a, independent of the element (x, y), so every point in the plane is rotated by the same amount.

4.21. *The number of subsets of $[n]$ with odd size equals the number of subsets of $[n]$ with even size, where $n \in \mathbb{N}$, bijectively.*

Let A be the collection of even subsets of $[n]$, and let B be the collection of odd subsets. For each $x \in A$, define $f(x)$ as follows:

$$f(x)=\begin{cases} f(x)-\{n\} & \text{if } n \in x \\ f(x)\cup\{n\} & \text{if } n \notin x \end{cases}$$

By this definition, $|x|$ and $|f(x)|$ differ by one, so $f(x)$ is a set of odd size, and f maps A to B.

We claim that f is a bijection. Consider distinct $x, y \in A$. If both contain or both omit n, then $f(x)$ and $f(y)$ agree on whether they contain n but differ outside $\{n\}$. If exactly one of $\{x, y\}$ contains n, then exactly one of $\{f(x), f(y)\}$ contains n. Thus $x \neq y$ implies $f(x) \neq f(y)$, and f is injective. If $z \in B$, then flipping whether n is present in z yields a subset x such that $f(x)=z$, so f also is surjective. Thus f is a bijection.

When $n=0$, there is one even subset and no odd subset. The bijection fails because $[0]=\varnothing$ and there is no element n to change.

Alternatively, one can define a function $g\colon B \to A$ by the same rule used to define f (switching the domain and target), and observe that $g \circ f$ is the identity function on A and $f \circ g$ is the identity function on B. This implies that g is the inverse of f and thus that f is a bijection and $|A|=|B|$. Without knowing $|A|=|B|$, it does not suffice to show that only one of the compositions is the identity.

4.22. *The formula $f(x)=\frac{2x-1}{2x(1-x)}$ defines a bijection from $(0,1)$ to $\mathbb{R}$.*

f is injective. Suppose that $f(x)=f(y)$. From $\frac{2x-1}{2x(1-x)}=\frac{2y-1}{2y(1-y)}$, we obtain $(2x-1)2y(1-y)=(2y-1)2x(1-x)$, which simplifies to $2y^2-2y-$

$4xy^2=2x^2-2x-4x^2y$ and then $2(y^2-x^2)-2(y-x)=4xy(y-x)$. If $y \neq x$, then we can divide by $2(y-x)$ to obtain $y+x-1=2xy$. Rewriting this as $-xy=(x-1)(y-1)$ makes it clear that there is no solution when $x, y \in (0,1)$, since the left side is negative and the right side is positive.

f is surjective. Suppose that $f(x)=b$; we solve for x to obtain $x \in (0,1)$ such that $f(x)=b$. Observe that $b=0$ is achieved by $x=1/2$, so we may assume that $b \neq 0$. Clearing fractions leads to $xb-x^2b=x-1/2$, or $bx^2+(1-b)x-1/2=0$. The quadratic formula yields

$$x=\frac{b-1\pm\sqrt{b^2+1}}{2b}.$$

The magnitude of the square root is larger than $|b|$. Therefore, choosing the negative sign in the numerator yields a negative x, which is not in the domain of f. We therefore choose the positive sign.

If $b>0$, then the square root is less than $b+1$, and we obtain $x<\frac{b-1+b+1}{2b}=1$. Also the square root is bigger than 1, so $x>0$. If $b<0$, then let $b'=-b$. The formula for x becomes $x=\frac{b'+1-\sqrt{b'^2+1}}{2b'}$, where $b'>0$. The square root is strictly between 1 and $b'+1$, so x is strictly between $1/2$ and 0. In each case, we have found x in the domain $(0,1)$ such that $f(x)=b$.

4.23. *Functions from $\mathbb{R}$ to $\mathbb{R}$.*

a) $f(x)=x^3-x+1$. This function is surjective, like all cubic polynomials, but it is not injective, since $f(1)=f(-1)=1$. The formula defines a bijection from S to S, where $S=[1,\infty)$.

b) $f(x)=\cos(\pi x/2)$. This function is not surjective, since the value of cosine is always between -1 and 1. Also it is not injective; the value at every odd integer is 0. Nevertheless, when the domain and target are restricted to the interval $[0,1]$, f is a bijection.

4.24. *If f and g are surjective functions from $\mathbb{Z}$ to $\mathbb{Z}$, then the pointwise product of f and g need not be surjective.* If f and g are defined by $f(x)=x$ and $g(x)=x$, then f and g are surjective, but $fg(x)=x^2$, and fg does not map onto any negative integer. (Many other examples can be given.)

4.25. *Formulas defining surjections from $\mathbb{N}\times\mathbb{N}$ to $\mathbb{N}$.*

a) $f(a,b)=a+b$—NO. When $a, b \in \mathbb{N}$, $a+b \geq 2$, so the image does not contain 1.

b) $f(a,b)=ab$—YES. For $n \in \mathbb{N}$, $f(n,1)=n$, so n is in the image.

c) $f(a,b)=ab(b+1)/2$—YES. For $n \in \mathbb{N}$, $f(n,1)=n$, so n is in the image.

d) $f(a,b)=(a+1)b(b+1)/2$—NO. When $a, b \in \mathbb{N}$, $(a+1)b(b+1)/2 \geq 2$, so the image does not contain 1.

e) $f(a,b) = ab(a+b)/2$*—NO.* We have $f(1,1) = 1$. When $\min\{a,b\} = 1$ and $\max\{a,b\} \geq 2$, we have $ab(a+b)/2 \geq 3$. When $a, b \geq 2$, we have $ab(a+b)/2 \geq 8$. Thus the image does not contain 2.

4.26. *If there are positive constants c, α such that, for all $x, y \in \mathbb{R}$, $|f(x) - f(y)| \geq c|x-y|^\alpha$, then f is injective.* If f is not injective, then there are distinct numbers x, y such that $f(x) = f(y)$. Since $c|x-y|^\alpha > 0$, this contradicts the hypothesized condition.

4.27. *Surjectivity and injectivity of polynomials.* Consider an arbitrary quadratic polynomial, $ax^2 + bx + c$, with $a \neq 0$. As in the derivation of the quadratic formula, we write $ax^2 + bx + c = a(x + b/(2a))^2 + c - b^2/(4a)$. Since $(x + b/(2a))^2 \geq 0$, the value of the polynomial cannot be smaller than $c - b^2/(4a)$ if $a > 0$, and it cannot be larger than $c - b^2/(4a)$ if $a < 0$. Hence the function is not surjective. (Comment: Since equality holds when $x = -b/(2a)$, this is where the extreme value of the quadratic occurs, and the extreme value equals $c - b^2/(4a)$; this is consistent with problem 1 of homework 1).

The polynomial $x^3 - x + 1$ is not injective, since it has the value 1 at more than once place (at $x = 0$ or $x = \pm 1$). Until Chapter 4, we can only sketch a proof that this function is surjective. Note that $x(x^2-1)+1$ is increasing when $x > 1$, because $y > x > 1$ implies $y^2 - 1 > x^2 - 1$, and then $y(y^2-1)+1 > x(x^2-1)+1$. Similarly, it is increasing when $x < -1$. If we believe in continuity and in the values getting arbitrarily far from 0, then the function is surjective.

4.28. *The cubic polynomial defined by $ax^3 + bx^2 + cx + d$ is injective if and only if $b^2 - 3ac < 0$.*

The formula for the value of the general cubic polynomial at x is $f(x) = ax^3 + bx^2 + cx + d$; these coefficients are known. Since multiplying the function by -1 doesn't affect injectivity and quadratics are not injective, we may assume that $a > 0$.

We use a change of variables to reduce the problem to polynomials h of the form $h(y) = y^3 + ry + d'$. We determine constants s, t so that substituting $x = s(y+t)$ expresses $ax^3 + bx^2 + cx + d$ as $y^3 + ry + d'$, where r, d' are constants. That is,

$$as^3(y+t)^3 + bs^2(y+t)^2 + cs(y+t) + d = y^3 + ry + d'.$$

Since polynomials are equal when their coefficients are equal, we set $as^3 = 1$ for the coefficient of y^3 and $3as^3t + bs^2 = 0$ for the coefficient of y^2. This yields $s = (1/a)^{1/3}$ and $t = -b/(3as)$. The resulting coefficient r for y^1 is $3as^3t^2 + 2tbs^2 + cs$, which can be computed using the formulas for s and t.

Let $g(y) = s(y+t)$. When $a \neq 0$ and s, t are defined above, g is a bijection from $\mathbb{R}$ to $\mathbb{R}$ and $h = f \circ g$. Thus $f = h \circ g^{-1}$, and f will be injective if and only if h is injective.

The constant d' in the formula for h does not affect injectivity. Replacing it by 0 merely shifts the images. It suffices to consider $y^3 + ry$. If $y^3 + ry = z^3 + rz$ for some distinct y, z, then dividing by $y - z$ yields $y^2 + yz + z^2 = -r$.

If r is negative, then $(y, z) = (0, \sqrt{-r}$ is a solution, and the function is not injective. If r is 0, then there is no solution with $y \neq z$ (since cubing is injective). If r is positive, then again there is no solution, because $y^2 + yz + z^2$ is never negative, which follows from $y^2 + z^2 \geq 2|y||z|$ (AGM Inequality).

Thus h is injective if and only if $r \geq 0$, and this determines whether f is injective. Since we have assumed that $a > 0$, also $s > 0$. Canceling s from the formula for r yields $3a(st)^2 + 2b(st) + c$. It suffices to consider the sign of this. From $3as^3t + bs^2 = 0$, we obtain $st = -b/(3a)$. Thus we are interested in the sign of $b^2/(3a) - 2b^2/(3a) + c$. This is positive if and only if $b^2 - 3ac < 0$.

Comment. The methods of calculus in Part IV would enable us to observe that a differentiable function from $\mathbb{R}$ to $\mathbb{R}$ is injective if and only if its derivative is never 0. The derivative of $ax^3 + bx^2 + cx + d$ is $3ax^2 + 2bx + c$. This is never 0 if and only if $3ax^2 + 2bx + c = 0$ has no solution. By the quadratic formula, the condition for this is $4b^2 - 12ac < 0$, which is the same answer obtained above. This argument is shorter because it relies on the work of defining and studying the derivative.

4.29. *Properties of three functions f, g, h mapping $\mathbb{R}$ to $\mathbb{R}$.*

$$f(x) = x/(1+x^2), \qquad g(x) = x^2/(1+x^2), \qquad h(x) = x^3/(1+x^2).$$

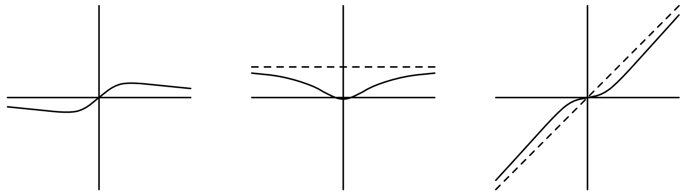
a) The functions f and g are not injective, but h is injective. Since $g(x) = g(-x)$ for all x, g is not injective. For f, this is less obvious. If we do not see immediately something like $f(2) = f(1/2)$, then we try to prove that f is injective. Setting $f(x) = f(y)$ and assuming $x \neq y$ yields $x + xy^2 = y + x^2y$, which simplifies to $x - y = xy(x-y)$ and reduces to $1 = xy$. When $x \neq y$, we have $f(x) = f(y)$ if and only if $xy = 1$.

For h, again we set $h(x) = h(y)$ and assume that $x \neq y$. We obtain $x^3 + x^3y^2 = y^3 + x^2y^3$, which reduces to $x^2 + xy + y^2 = -x^2y^2$ after we rearrange and divide by $x - y$. Rewriting this as $x^2(1+y^2) + yx + y^2 = 0$ yields a quadratic equation for x in terms of y. Since $b^2 - 4ac = y^2 - y^2 4(1+y^2) < 0$, there is no solution for x. Hence there are no distinct x, y with $h(x) = h(y)$.

b) The functions f and g are not surjective. For all x, $g(x) > 0$. (Furthermore, $\frac{x^2}{1+x^2} = \frac{1}{1/x^2+1} < 1$ for $x \neq 0$, so always $0 \leq g(x) < 1$.)

Also $f(x) < 1$ always. If $x < 0$, then $f(x) < 0$. If $0 \leq x < 1$, then $x/(1+x^2) < x < 1$. If $x \geq 1$, then $x/(1+x^2) = 1/(1+1/x) < 1$.

c) *The graphs.* Note that $f(-x) = -f(x)$, $g(-x) = g(x)$, $h(-x) = -h(x)$. All are 0 at 0. For large x, they are asymptotic to $0, 1, x$, respectively.



4.30. *If a, b, c, d are given real numbers, and $f: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $f(x, y) = (ax + by, cx + dy)$, then f is injective if and only if f is surjective.* If $ad - bc \neq 0$, then the system $ax + by = r$ and $cx + dy = s$ has a unique solution pair (x, y) for each choice of (r, s). This implies that f is a bijection. Thus, when $ad - bc \neq 0$, f is both injective and surjective.

In the remaining case, we have $ad - bc = 0$. Given $f(x, y) = (r, s)$, we can multiply the first equation by c and the second by a to obtain $acx + bcy = cr$ and $acx + ady = as$. Because $ad = bc$, the left sides of these two equations are equal. Hence (r, s) belongs to the image if and only if $cr = as$. This does not include all of $\mathbb{R}^2$, so f is not surjective. Also, $ad - bc = 0$ implies that increasing x by b and decreasing y by a does not change $ax+by$ or $cx + dy$. Hence for each (r, s) in the image, there are infinitely many choices of (x, y) such that $f(x, y) = (r, s)$.

By considering the two cases, we have that f is surjective if and only if $ad - bc \neq 0$, and that $ad - bc \neq 0$ if and only if f is injective.

4.31. *If $f: A \to B$ is an increasing function, then f^{-1} is an increasing function.* The contrapositive of the statement $x < y \Rightarrow f(x) < f(y)$ is the statement $f(x) \geq f(y) \Rightarrow x \geq y$. Writing $u = f(x)$ and $v = f(y)$ converts this to $u \geq v \Rightarrow f^{-1}(u) \geq f^{-1}(v)$.

4.32. *When F is a field, negation (f) defines a bijection from F to itself, and reciprocal (g) defines a bijection from $F-\{0\}$ to itself.* The field axioms imply that every element of F has a unique additive inverse, and every nonzero element of F has a unique mulitplicative inverse. Given y in the target, these inverses are the unique elements x' and x such that $f(x') = -x' = y$ and $g(x) = x^{-1} = y$ (the latter applies only for $y \neq 0$).

4.33. *Composition of injections and surjections.* Let $f: A \to B$ and $g: B \to C$, so $(g \circ f)(x) = g(f(x))$ for all $x \in A$.

a) The composition of two injections is an injection. Assume that f and g are injective. Suppose that $(g \circ f)(x) = (g \circ f)(x')$, i.e. $g(f(x)) = g(f(x'))$. Since g is injective, this implies $f(x) = f(x')$. Since f is injective, this in turn implies $x = x'$. Hence $(g \circ f)(x) = (g \circ f)(x')$ implies $x = x'$, and $g \circ f$ is injective.

Alternatively, consider the contrapositive. For $x, x' \in A$ with $x \neq x'$, we have $f(x) \neq f(x')$ because f is injective, and then $g(f(x)) \neq g(f(x'))$ because g is injective. Thus $x \neq x'$ implies $(g \circ f)(x) \neq (g \circ f)(x')$, so $g \circ f$ is injective.

b) The composition of two surjections is a surjection. Assume that f and g are surjective. Let z be an arbitrary element of C. Since g is surjective, there is an element $y \in B$ such that $g(y) = z$. Since f is surjective, there is an element $x \in A$ such that $f(x) = y$. Hence we have found an element of A, namely x, such that $(g \circ f)(x) = z$, and $g \circ f$ satisfies the definition of a surjective function.

c) The composition of two bijections is a bijection. By (a) and (b), $g \circ f$ is both injective and surjective and hence is a bijection, by definition.

d) If $f: A \to B$ and $g: B \to C$ are bijections, then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. By part (c), $g \circ f$ is a bijection from A to C. Thus $g \circ f$ is invertible, and the inverse is defined to be the function that yields the identity function on A when composed with $g \circ f$. Let I_A and I_B denote the identity functions on A and B. Letting $h = f^{-1} \circ g^{-1}$, we use the associativity of composition to obtain $h \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ I_B \circ f = f^{-1} \circ f = I_A$. Thus h is the inverse of $g \circ f$.

One can also argue more explicitly that $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ have the same domain and target and have the same value at each element of the domain, so they are the same function.

4.34. *Composition of functions.* Suppose that $f: A \to B$, $g: B \to C$, and $h = g \circ f$.

a) If h is injective, then f is injective—TRUE. If f is not injective, then there exist two distinct elements $x, y \in A$ such that $f(x) = f(y)$. Since g is a function, this implies that $g(f(x)) = g(f(y))$. Since $h = g \circ f$, we have obtained distinct elements $x, y \in A$ such that $h(x) = h(y)$, and hence h is not injective. We have proved the contrapositive, so the implication is true.

b) If h is injective, then g is injective—FALSE. Let $A = \{1\}$, $B = \{a, b\}$, and $C = \{\alpha\}$. Define $f(1) = a$ and $g(a) = g(b) = \alpha$. Both f and h are injective, but g is not injective.

c) If h is surjective, then f is surjective—FALSE. Let $A = \{1, 2\}$, $B = \{a, b\}$, and $C = \{\alpha\}$. Define $f(1) = f(2) = a$ and $g(a) = g(b) = \alpha$. Then $h(1) = h(2) = \alpha$, and h is surjective, but f is not surjective.

d) If h is surjective, then g is surjective—TRUE. If $z = h(x)$, then $z = g(f(x))$. Thus the image of g contains the image of h, which the hypothesis says is all of C.

4.35. *Composition of functions.* Suppose $f: A \to B$ and $g: B \to A$.

a) If $f(g(y)) = y$ for all $y \in B$, then f need not be a bijection. For each $y \in B$, y is the image under f of some element of A, namely $g(y)$. This guarantees that f is surjective, but f need not be injective. For example, suppose $A = \mathbb{N}$ and $B = \{2n : n \in \mathbb{N}\}$. Suppose $f(n)$ is the least even number as large as n. Suppose $g(m) = m$. Then f and g satisfy the conditions required, but $f(2k-1) = f(2k) = 2k$, and f is not injective.

b) If f is injective and $g(f(x)) = x$ for all $x \in A$, then it need not hold that $f(g(y)) = y$ for all $y \in B$. Let $A = \{2n\colon n \in \mathbb{N}\}$, $B = \mathbb{N}$, and $f(m) = m$. Let $g(n)$ be the least even number as large as n. Then f is injective and $g(f(x)) = x$ for all $x \in A$, but $f(g(y))$ is even when y is odd.

If f is injective but not surjective, then the conclusion no longer holds. Exchange f and g in the earlier example, so that A is the set of even numbers and B the set of all natural numbers. Then f is injective and $g(f(x)) = x$ for all $x \in A$, but $f(g(y))$ is even when y is odd.

4.36. Suppose $f\colon A \to B$ and $g\colon B \to A$.

a) If $f \circ g$ is the identity function on B, then f is surjective. By the hypothesis, $y \in B \Rightarrow f(g(y)) = y$. Hence there is an element of A mapped to y by f; namely, the element $g(y)$. This shows that f satisfies the definition of surjection.

b) If $g \circ f$ is the identity function on A, then f is injective. The hypothesis states that $x \in A$ implies $g(f(x)) = x$. If f is not injective, then distinct elements $x_1, x_2 \in A$ exist such that $f(x_1) = f(x_2)$. If we apply g to both sides of the equality, we obtain $x_1 = g(f(x_1)) = g(f(x_2)) = x_2$, which contradicts our choice of distinct elements. Hence our assumption that f is not injective must be wrong.

4.37. *If $f \circ f$ is injective, then f is injective.* Suppose that $f(x) = f(y)$. Because f is a function, we can apply it to this element to obtain $f(f(x)) = f(f(y))$. By the definition of composition, this yields $(f \circ f)(x) = (f \circ f)(y)$. The hypothesis that $f \circ f$ is injective now implies that $x = y$. We have proved that $f(x) = f(y)$ implies $x = y$, and thus f is injective.

4.38. *Translation and scaling.* Given $f\colon \mathbb{R} \to \mathbb{R}$, the functions $(T_a f)$ and $(M_b f)$ are defined by $(T_a f)(x) = f(x+a)$ and $(M_b f)(x) = f(bx)$. The vertical distance associated with x by $(T_a f)$ is the vertical distance associated with $x + a$ by f, so the graph of $(T_a f)$ is obtained by shifting the graph of f to the left by distance a.

For the graph of $(M_b f)$, the description of the change depends on b. If $b \ge 1$, then the graph shrinks toward the vertical axis by a factor of b. If $0 < b < 1$, then the result is expansion from the vertical axis by a factor of $1/b$. If $b = 0$, then the graph becomes a horizontal line above the horizontal axis by the amount $f(0)$. If $b < 0$, then the horizontal shrinkage or expansion is combined with reflection through the vertical axis.

4.39. *If $f(x) = a(x+b) - b$, then the nth iterate is given by $f^n(x) = a^n(x+b) - b$.*

For $n = 1$, the formula reduces to that for f, which completes the basis step. Assuming that the formula holds when $n = k$, we have $f^{n+1}(x) = f(f^n(x)) = a[f^n(x) + b] - b = a[a^n(x+b) - b + b] - b = a^{n+1}(x + b) - b$, and the formula also holds when $n = k+1$.

In the language of translation and scaling, we have $f = T_{-b} \circ M_a T_b \circ I$, where I is the identity function. Thus this exercise is a special case of the subsequent exercise.

4.40. *a) Iteration of a composition—If $f\colon A \to B$, $g\colon B \to B$, and $h = f^{-1} \circ g \circ f$, then $h^n = f^{-1} \circ g^n \circ f$, for $n \ge 1$.* We use induction on n. For $n = 1$, it holds by the definition of h. For $n > 1$, we use the definition of nth iterate, the induction hypothesis, and associativity of composition to compute

$$\begin{aligned} h^n &= h \circ h^{n-1} = (f^{-1} \circ g \circ f)(f^{-1} \circ g^{n-1} \circ f) \\ &= f^{-1} \circ g \circ (f \circ f^{-1}) \circ g^{n-1} \circ f = f^{-1} \circ g \circ g^{n-1} \circ f = f^{-1} \circ g^n \circ f \end{aligned}$$

4.41. *If $f\colon A \to A$, and n, k are natural numbers with $k < n$, then $f^n = f^k \circ f^{n-k}$.* We use induction on n. When $n = 2$, we have $k = 1$, and the formula $f^2 = f^1 \circ f^1$ is the definition of f^2. For the induction step, suppose that the claim is true when $n = m$; we prove that it also holds for $n = m+1$. For $k = 1$, again the definition of iteration yields $f^{m+1} = f^1 \circ f^m$. Now consider $1 < k < n+1$. Using the definition of iteration, the induction hypothesis, the associativity of composition, and the definition of iteration again, we have

$$f^{m+1} = f \circ f^m = f \circ (f^{k-1} \circ f^{m+1-k}) = (f \circ f^{k-1}) \circ f^{m+1-k} = f^k \circ f^{m+1-k}$$

4.42. *If f is a bijection from $[m]$ to $[n]$, then $m = n$.* We use induction on n. Basis step ($n = 0$). In this case, $[n] = \varnothing$, and a function from A to $\varnothing$ can be defined only if $A = \varnothing$. Hence $m = 0$.

Induction step ($n > 0$). Let f be a bijection from $[m]$ to $[n]$. Let $r = f^{-1}(n)$. Define g by $g(k) = f(k)$ for $k < r$, while $g(k) = f(k+1)$ for $k \ge r$; this function maps $[m-1]$ into $[n-1]$. Since f is a bijection and we have used all images under f except $f(r)$, g is a bijection. By the induction hypothesis, $m - 1 = n - 1$, and hence $m = n$.

4.43. *There is a bijection from a set A to a proper subset B of A only if A is infinite.* If A is finite, then also B is finite. Let $m = |A|$ and $n = |B|$. By the definition of size, there are bijections $f\colon A \to [m]$ and $g\colon B \to [n]$. Let h be a bijection from A to B. Now $g \circ h \circ f^{-1}$ is a bijection from $[m]$ to $[n]$. By Exercise 4.42, $m = n$. This contradicts the hypothesis that B is a proper subset of A. Hence the hypothesis that A is finite must be false.

4.44. *The function h in the proof of Corollary 4.41 is a bijection.* We have $h\colon A \cup B \to [m+n]$ defined by $h(x) = f(x)$ for $x \in A$ and $h(x) = g(x) + m$ for $x \in B$, where $f\colon A \to [m]$ and $g\colon B \to [n]$ are bijections.

The target of h is $[m+n]$. Since f has target $[m]$ and g has target $[n]$, h maps A into $[m]$ and B into $\{m+1, \ldots, n\}$. Since f and g are bijections, we have a well-defined inverse function h^{-1} defined by $h^{-1}(y) = f^{-1}(y)$ for $y \in [m]$ and $h^{-1}(y) = g^{-1}(y-m)$ for $y \in \{m+1, \ldots, n\}$. This defines a function because f and g are injective and surjective.

Alternatively, one can verify separately that h is injective and surjective, using the hypothesis that these properties hold for f and g.

4.45. *If $f\colon A \to A$ and A is finite, then f is injective if and only if f is surjective.* We use the method of contradiction to prove each direction of the claim. First suppose that f is injective, but some $y \in A$ is not in the image of f. Each inverse image has size at most one (since f is injective) and $I_f(y)$ is empty. Hence the total is less than $|A|$. This is a contradiction, because there are $|A|$ elements in the domain.

Now suppose that f is surjective, but $f(x) = f(x') = y$ for some distinct $x, x' \in A$. Each inverse image has size at least one (since f is surjective) and $I_f(y)$ has size at least 2. Hence the total is more than $|A|$. This is a contradiction, because the inverse images partition the domain, which has only $|A|$ elements.

If $A = \mathbb{N}$ and f is defined by $f(x) = 2x$, then f is injective but not surjective. Hence the claim does not hold when A is infinite.

4.46. *Cardinality and functions.* Suppose that A and B are finite, and $f\colon A \to B$.

a) If f is injective, then $|A| \le |B|$. Since f is injective, each element of B is the image of at most 1 element of A. When we sum the contribution 0 or 1 over all elements of B (depending on whether the element is in the image), we obtain $|A|$ (each element of A has an image in B) and the sum is at most $|B|$ (each element of B contributes at most once).

b) If f is surjective, then $|A| \ge |B|$. When f is surjective, each element of B belongs to the image of f. By the definition of **function**, the inverse images of the elements of B are pairwise disjoint subsets of A. Therefore, picking one element from the inverse image of each element of B yields $|B|$ distinct elements of A. This is a subset of A, so $|B| \le |A|$.

c) If A and B are finite and $f\colon A \to B$ and $g\colon B \to A$ are injections, then $|A| = |B|$ and f and g are bijections. Applying (a) to f yields $|A| \le |B|$. Applying (a) to g yields $|B| \le |A|$. Hence $|A| = |B|$. Since f is injective, its image has $|A|$ elements; since $|A| = |B|$, the image is all of B and f is surjective. By the same argument, g is surjective. Being both injective and surjective, f and g are bijections.

4.47. *The even natural numbers, the odd natural numbers, and the set $\mathbb{N}$ itself all have the same cardinality (they are countable).* Every even natural number is obtained by doubling a unique natural number, so doubling is a bijection from $\mathbb{N}$ to the set of even numbers. The operation of adding 1 is a bijection from the set of odd natural numbers to the set of even natural numbers. Using this bijection and the inverse of the first one, we also obtain a bijection from the set of odd natural numbers to $\mathbb{N}$. This assigns to the odd number k the number $(k+1)/2$. It is the inverse of the map that assigns $2n-1$ to the natural number n.

4.48. *Explicit description of a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.* The sequence described in Theorem 4.44 has k points on the diagonal that starts at the point $(k, 1)$. The points (i, j) on this diagonal all satisfy $i + j = k + 1$. The number of points in the sequence before the point $(k, 1)$ that starts the diagonal with $i + j = k + 1$ is $\sum_{r=1}^{k-1} r$, which equals $(k-1)k/2$. The point (i, j) is the jth point in the diagonal starting with $(i, 1)$. Therefore, the position of (i, j) in the sequence is $(i+j-2)(i+j-1)/2 + j$. The function f defined by $f(i, j) = (i+j-2)(i+j-1)/2 + j$ is an explicit bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$, because each point of $\mathbb{N} \times \mathbb{N}$ appears exactly once in the sequence.

4.49. *The union of a countable sequence of countable sets is countable.* Let $\{A_i\colon i \in \mathbb{N}\}$ be the sets, and let B be their union. Since each A_i is countable, for each i there is a sequence $\{a_{i,j}\colon j \in \mathbb{N}\}$ listing the elements of A_i once and only once. View these elements as listed at the points (j, i) in the first quadrant of the Cartesian plane, with the elements of A_i in the ith row.

To show that B is countable, it suffices to construct a sequence listing each element of B once and only once. Each element of B now appears at a point in the first quadrant, but it appears more than once if it belongs to more than one of the sets. The positions with $i + j - 1 = k$ form the kth diagonal of the arrangement; every element appears in some diagonal. We form the sequence by listing the elements of the first diagonal, then the second, and so on in increasing order of k, as in the bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. Within each diagonal, we use increasing order in j. However, whenever we encounter an element that already appears in our list, we skip it to avoid listing elements more than once. Since each diagonal is finite, we eventually reach each specified diagonal and thus each specified element.

Note: When we are given only that each A_i is countable, obtaining the sequences of the form $a_{i,1}, a_{i,2}, \ldots$ relies on the Axiom of Choice. This Axiom states that for any collection of disjoint sets, it is possible to choose an element of each set. We apply this to $B_1, B_2, \ldots$, where B_i is the set of bijections from $\mathbb{N}$ to A_i.

4.50. *Applying the proof of the Shroeder-Bernstein Theorem.* Let $A = (0, 1)$ and $B = \{y \in \mathbb{R}: 0 \le y < 1\}$. Define $f: A \to B$ and $g: B \to A$ by $f(x) = x$ and $g(y) = (y+1)/2$. The Shroeder-Bernstein Theorem provides a bijection $h: A \to B$. The function h constructed in the proof agrees with f on all elements of A except those whose "family" (backing up by alternating g^{-1} and f^{-1}) has an origin in $B - f(A)$.

In this example, $B - f(A) = \{0\}$, so we use g^{-1} instead of f on only one family. The values mapped using g^{-1} are those of the form $g((f^{-1} \circ g)^k)(0)$ for $k \ge 0$. Since f is the identity, this reduces to $g^k(0)$ for $k \in \mathbb{N}$. The resulting sequence begins 1/2, 3/4, 7/8; it follows by induction on k that $g^k(0) = 1 - 1/2^k$, since $g(1 - 1/2^k) = (2 - 1/2^k)/2 = 1 - 2/2^{k+1}$.

Thus $h(x) = 2x - 1$ when $x = 1 - 1/2^k$ for $k \in \mathbb{N}$, and otherwise $h(x) = x$.

4.51. *An explicit bijection from* $[0, 1]$ *to* $(0, 1)$. Define $f : [0, 1] \to (0, 1)$ as follows: $f(0) = 1/2$, $f(1/n) = f(1/(n + 2))$ for all $n \in \mathbb{N}$, $f(x) = x$ otherwise. Every element of $(0, 1)$ is in the image of f, and no element is hit twice, because 1/2 comes only from 0, reciprocals of integers (other than 1/2) come only from reciprocals of other integers, and other elements are fixed points. The construction works because the elements in the sequence $0, 1, 1/2, 1/3, 1/4, \cdots$ are shifted two positions by f, thus omitting $\{0, 1\}$ from the image.