Chapter 2

Euclidean Space

2.1 Practice Problems

1. $\mathbf{u}-\mathbf{w}=\begin{bmatrix}-4\\3\\4\end{bmatrix}-\begin{bmatrix}5\\0\\-2\end{bmatrix}=\begin{bmatrix}-4-5\\3-0\\4-(-2)\end{bmatrix}=\begin{bmatrix}-9\\3\\6\end{bmatrix}$

   $\mathbf{v}+3\mathbf{w}=\begin{bmatrix}-1\\6\\2\end{bmatrix}+3\begin{bmatrix}5\\0\\-2\end{bmatrix}=\begin{bmatrix}-1+3(5)\\6+3(0)\\2+3(-2)\end{bmatrix}=\begin{bmatrix}14\\6\\-4\end{bmatrix}$

   $-2\mathbf{w}+\mathbf{u}+3\mathbf{v}=-2\begin{bmatrix}5\\0\\-2\end{bmatrix}+\begin{bmatrix}-4\\3\\4\end{bmatrix}+3\begin{bmatrix}-1\\6\\2\end{bmatrix}=\begin{bmatrix}-2(5)+(-4)+3(-1)\\-2(0)+3+3(6)\\-2(-2)+4+3(2)\end{bmatrix}=\begin{bmatrix}-17\\21\\14\end{bmatrix}$

2. (a) $\begin{array}{rcrcr}-x_1&+&4x_2&=&3\\7x_1&+&6x_2&=&10\\2x_1&-&6x_2&=&5\end{array}$

   (b) $\begin{array}{rcrcrcr}3x_1&&&-&x_3&=&4\\4x_1&-&2x_2&+&2x_3&=&7\\&-&5x_2&+&9x_3&=&11\\2x_1&+&6x_2&+&5x_3&=&-6\end{array}$

3. (a) $x_1\begin{bmatrix}1\\-5\\4\end{bmatrix}+x_2\begin{bmatrix}1\\7\\0\end{bmatrix}+x_3\begin{bmatrix}-2\\6\\-8\end{bmatrix}=\begin{bmatrix}3\\12\\0\end{bmatrix}$

   (b) $x_1\begin{bmatrix}4\\0\\3\end{bmatrix}+x_2\begin{bmatrix}-3\\2\\12\end{bmatrix}+x_3\begin{bmatrix}-1\\5\\6\end{bmatrix}+x_4\begin{bmatrix}5\\-2\\0\end{bmatrix}=\begin{bmatrix}0\\6\\10\end{bmatrix}$

4. (a) $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}5\\7\\0\end{bmatrix}+s_1\begin{bmatrix}3\\-2\\1\end{bmatrix}$

   (b) $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}1\\0\\17\\0\end{bmatrix}+s_1\begin{bmatrix}2\\0\\1\\1\end{bmatrix}+s_2\begin{bmatrix}13\\1\\0\\0\end{bmatrix}$

5. (a) $x_1\mathbf{a}_1+x_2\mathbf{a}_2=\mathbf{b} \quad\Leftrightarrow\quad x_1\begin{bmatrix}1\\-5\end{bmatrix}+x_2\begin{bmatrix}3\\6\end{bmatrix}=\begin{bmatrix}5\\9\end{bmatrix} \quad\Leftrightarrow\quad \begin{bmatrix}x_1+3x_2\\-5x_1+6x_2\end{bmatrix}=\begin{bmatrix}5\\9\end{bmatrix} \quad\Leftrightarrow$

the augmented matrix $\begin{bmatrix} 1 & 3 & 5 \\ -5 & 6 & 9 \end{bmatrix}$ has a solution:

$$\begin{bmatrix} 1 & 3 & 5 \\ -5 & 6 & 9 \end{bmatrix} \overset{5R_1+R_2\to R_2}{\sim} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 21 & 34 \end{bmatrix}$$

From row 2, $21x_2 = 34 \Rightarrow x_2 = \frac{34}{21}$. From row 1, $x_1 + 3(\frac{34}{21}) = 5 \Rightarrow x_1 = \frac{1}{7}$. Thus, $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$, with $\mathbf{b} = \frac{1}{7}\mathbf{a}_1 + \frac{34}{21}\mathbf{a}_2$.

(b) $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_2\mathbf{a}_2 = \mathbf{b} \quad \Leftrightarrow \quad x_1 \begin{bmatrix} 1 \\ -3 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ -4 \end{bmatrix} \quad \Leftrightarrow$

$\begin{bmatrix} x_1 - 2x_2 \\ -3x_1 + 3x_2 \\ 8x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ -4 \end{bmatrix} \Leftrightarrow$ the augmented matrix $\begin{bmatrix} 1 & -2 & 7 \\ -3 & 3 & 5 \\ 8 & -3 & -4 \end{bmatrix}$ yields a solution.

$$\begin{bmatrix} 1 & -2 & 7 \\ -3 & 3 & 5 \\ 8 & -3 & -4 \end{bmatrix} \overset{\substack{3R_1+R_2\to R_2 \\ -8R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 1 & -2 & 7 \\ 0 & -3 & 26 \\ 0 & 13 & -60 \end{bmatrix}$$

$$\overset{(\frac{13}{3})R_2+R_3\to R_3}{\sim} \begin{bmatrix} 1 & -2 & 7 \\ 0 & -3 & 26 \\ 0 & 0 & \frac{158}{3} \end{bmatrix}$$

From the third equation, we have $0 = \frac{158}{3}$, and thus the system does not have a solution. Thus, $\mathbf{b}$ is *not* a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

6. (a) False. Addition of vectors is associative and commutative.
   (b) True. The scalars may be any real number.
   (c) True. The solutions to a linear system with variables $x_1, \ldots, x_n$ can be expressed as a vector $\mathbf{x}$, which is the sum of a fixed vector with n components and a linear combination of k vectors with n components, where k is the number of free variables.
   (d) False. The Parallelogram Rule gives a geometric interpretation of vector addition.

2.1 Vectors

1. $\mathbf{u} - \mathbf{v} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} -4 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3-(-4) \\ -2-1 \\ 0-5 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ -5 \end{bmatrix}$;

   $6\mathbf{w} = 6 \begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix} = \begin{bmatrix} (6)\,2 \\ (6)\,(-7) \\ (6)\,(-1) \end{bmatrix} = \begin{bmatrix} 12 \\ -42 \\ -6 \end{bmatrix}$

2. $\mathbf{w} - \mathbf{u} = \begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2-3 \\ -7-(-2) \\ -1-0 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ -1 \end{bmatrix}$;

   $-5\mathbf{v} = (-5) \begin{bmatrix} -4 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} (-5)\,(-4) \\ (-5)\,1 \\ (-5)\,5 \end{bmatrix} = \begin{bmatrix} 20 \\ -5 \\ -25 \end{bmatrix}$

3. $\mathbf{w} + 3\mathbf{v} = \begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} -4 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 2+3\,(-4) \\ -7+3\,(1) \\ -1+3\,(5) \end{bmatrix} = \begin{bmatrix} -10 \\ -4 \\ 14 \end{bmatrix}$;

   $2\mathbf{w} - 7\mathbf{v} = 2 \begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix} - 7 \begin{bmatrix} -4 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 2\,(2) - 7\,(-4) \\ 2\,(-7) - 7\,(1) \\ 2\,(-1) - 7\,(5) \end{bmatrix} = \begin{bmatrix} 32 \\ -21 \\ -37 \end{bmatrix}$

4. $4\mathbf{w}-\mathbf{u}=4\begin{bmatrix}2\\-7\\-1\end{bmatrix}-\begin{bmatrix}3\\-2\\0\end{bmatrix}=\begin{bmatrix}4(2)-3\\4(-7)-(-2)\\4(-1)-0\end{bmatrix}=\begin{bmatrix}5\\-26\\-4\end{bmatrix};$

$-2\mathbf{v}+5\mathbf{w}=(-2)\begin{bmatrix}-4\\1\\5\end{bmatrix}+5\begin{bmatrix}2\\-7\\-1\end{bmatrix}=\begin{bmatrix}(-2)(-4)+5(2)\\(-2)(1)+5(-7)\\(-2)(5)+5(-1)\end{bmatrix}=\begin{bmatrix}18\\-37\\-15\end{bmatrix}$

5. $-\mathbf{u}+\mathbf{v}+\mathbf{w}=-\begin{bmatrix}3\\-2\\0\end{bmatrix}+\begin{bmatrix}-4\\1\\5\end{bmatrix}+\begin{bmatrix}2\\-7\\-1\end{bmatrix}=$

$\begin{bmatrix}-3-4+2\\-(-2)+1-7\\-0+5-1\end{bmatrix}=\begin{bmatrix}-5\\-4\\4\end{bmatrix};$

$2\mathbf{u}-\mathbf{v}+3\mathbf{w}=2\begin{bmatrix}3\\-2\\0\end{bmatrix}-\begin{bmatrix}-4\\1\\5\end{bmatrix}+3\begin{bmatrix}2\\-7\\-1\end{bmatrix}=$

$\begin{bmatrix}2(3)-(-4)+3(2)\\2(-2)-1+3(-7)\\2(0)-5+3(-1)\end{bmatrix}=\begin{bmatrix}16\\-26\\-8\end{bmatrix}$

6. $3\mathbf{u}-2\mathbf{v}+5\mathbf{w}=3\begin{bmatrix}3\\-2\\0\end{bmatrix}-2\begin{bmatrix}-4\\1\\5\end{bmatrix}+5\begin{bmatrix}2\\-7\\-1\end{bmatrix}=$

$\begin{bmatrix}3(3)-2(-4)+5(2)\\3(-2)-2(1)+5(-7)\\3(0)-2(5)+5(-1)\end{bmatrix}=\begin{bmatrix}27\\-43\\-15\end{bmatrix};$

$-4\mathbf{u}+3\mathbf{v}-2\mathbf{w}=-4\begin{bmatrix}3\\-2\\0\end{bmatrix}+3\begin{bmatrix}-4\\1\\5\end{bmatrix}-2\begin{bmatrix}2\\-7\\-1\end{bmatrix}=$

$\begin{bmatrix}(-4)(3)+3(-4)-2(2)\\(-4)(-2)+3(1)-2(-7)\\(-4)(0)+3(5)-2(-1)\end{bmatrix}=\begin{bmatrix}-28\\25\\17\end{bmatrix}$

7. $\begin{array}{rcrcr}3x_1&-&x_2&=&8\\2x_1&+&5x_2&=&13\end{array}$

8. $\begin{array}{rcrcr}-x_1&+&9x_2&=&-7\\6x_1&-&5x_2&=&-11\\-4x_1&&&=&3\end{array}$

9. $\begin{array}{rcrcrcr}-6x_1&+&5x_2&&&=&4\\5x_1&-&3x_2&+&2x_3&=&16\end{array}$

10. $\begin{array}{rcrcrcrcr}2x_1&&&+&5x_3&+&4x_4&=&0\\7x_1&+&2x_2&+&x_3&+&5x_4&=&4\\8x_1&+&4x_2&+&6x_3&+&7x_4&=&3\\3x_1&+&2x_2&+&x_3&&&=&5\end{array}$

11. $x_1\begin{bmatrix}2\\-1\end{bmatrix}+x_2\begin{bmatrix}8\\-3\end{bmatrix}+x_3\begin{bmatrix}-4\\5\end{bmatrix}=\begin{bmatrix}-10\\4\end{bmatrix}$

12. $x_1\begin{bmatrix}-2\\1\\7\end{bmatrix}+x_2\begin{bmatrix}5\\-2\\-17\end{bmatrix}+x_3\begin{bmatrix}-10\\3\\34\end{bmatrix}=\begin{bmatrix}4\\-1\\-16\end{bmatrix}$

13. $x_1\begin{bmatrix}1\\-2\\-3\end{bmatrix}+x_2\begin{bmatrix}-1\\2\\-3\end{bmatrix}+x_3\begin{bmatrix}-3\\6\\10\end{bmatrix}+x_4\begin{bmatrix}-1\\2\\0\end{bmatrix}=\begin{bmatrix}-1\\-1\\5\end{bmatrix}$

14. $x_1\begin{bmatrix}-5\\3\\1\end{bmatrix}+x_2\begin{bmatrix}9\\-5\\-2\end{bmatrix}=\begin{bmatrix}13\\-9\\-2\end{bmatrix}$

15. $\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}-4\\0\end{bmatrix}+s_1\begin{bmatrix}3\\1\end{bmatrix}$

16. $\begin{bmatrix}x_1\\x_2\end{bmatrix}=s_1\begin{bmatrix}-2\\1\end{bmatrix}$

17. $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}7\\-3\\0\end{bmatrix}+s_1\begin{bmatrix}-2\\0\\1\end{bmatrix}$

18. $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}1\\-2\\0\\0\end{bmatrix}+s_1\begin{bmatrix}3\\0\\0\\1\end{bmatrix}+s_2\begin{bmatrix}-4\\5\\1\\0\end{bmatrix}$

19. $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}4\\0\\-9\\0\end{bmatrix}+s_1\begin{bmatrix}6\\0\\3\\1\end{bmatrix}+s_2\begin{bmatrix}-5\\1\\0\\0\end{bmatrix}$

20. $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}1\\0\\0\\-12\\0\end{bmatrix}+s_1\begin{bmatrix}-7\\0\\0\\1\\1\end{bmatrix}+s_2\begin{bmatrix}14\\0\\1\\0\\0\end{bmatrix}+s_3\begin{bmatrix}-1\\1\\0\\0\\0\end{bmatrix}$

21. $1\mathbf{u}+0\mathbf{v}=\mathbf{u}=\begin{bmatrix}3\\-2\end{bmatrix}$, $0\mathbf{u}+1\mathbf{v}=\mathbf{v}=\begin{bmatrix}-1\\-4\end{bmatrix}$, $1\mathbf{u}+1\mathbf{v}=\begin{bmatrix}3\\-2\end{bmatrix}+\begin{bmatrix}-1\\-4\end{bmatrix}=\begin{bmatrix}2\\-6\end{bmatrix}$

22. $1\mathbf{u}+0\mathbf{v}=\mathbf{u}=\begin{bmatrix}7\\1\\-13\end{bmatrix}$, $0\mathbf{u}+1\mathbf{v}=\mathbf{v}=\begin{bmatrix}5\\-3\\2\end{bmatrix}$,

$1\mathbf{u}+1\mathbf{v}=\begin{bmatrix}7\\1\\-13\end{bmatrix}+\begin{bmatrix}5\\-3\\2\end{bmatrix}=\begin{bmatrix}12\\-2\\-11\end{bmatrix}$.

23. $1\mathbf{u}+0\mathbf{v}+0\mathbf{w}=\mathbf{u}=\begin{bmatrix}-4\\0\\-3\end{bmatrix}$, $0\mathbf{u}+1\mathbf{v}+0\mathbf{w}=\mathbf{v}=\begin{bmatrix}-2\\-1\\5\end{bmatrix}$, $0\mathbf{u}+0\mathbf{v}+1\mathbf{w}=\mathbf{w}=\begin{bmatrix}9\\6\\11\end{bmatrix}$.

24. $1\mathbf{u}+0\mathbf{v}+0\mathbf{w}=\mathbf{u}=\begin{bmatrix}1\\8\\2\\2\end{bmatrix}$, $0\mathbf{u}+1\mathbf{v}+0\mathbf{w}=\mathbf{v}=\begin{bmatrix}4\\-2\\5\\-5\end{bmatrix}$, $0\mathbf{u}+0\mathbf{v}+1\mathbf{w}=\mathbf{w}=\begin{bmatrix}9\\9\\0\\1\end{bmatrix}$.

25. $-3\begin{bmatrix}a\\3\end{bmatrix}+4\begin{bmatrix}-1\\b\end{bmatrix}=\begin{bmatrix}-10\\19\end{bmatrix}\Rightarrow\begin{bmatrix}-3a-4\\-9+4b\end{bmatrix}=\begin{bmatrix}-10\\19\end{bmatrix}\Rightarrow -3a-4=-10$ and $-9+4b=19$. Solving these equations, we obtain $a=2$ and $b=7$.

26. $4\begin{bmatrix}4\\a\end{bmatrix}+3\begin{bmatrix}-3\\5\end{bmatrix}-2\begin{bmatrix}b\\8\end{bmatrix}=\begin{bmatrix}-1\\7\end{bmatrix}\Rightarrow\begin{bmatrix}16-9-2b\\4a+15-16\end{bmatrix}=\begin{bmatrix}-1\\7\end{bmatrix}\Rightarrow$
$7-2b=-1$ and $4a-1=7$. Solving these equations, we obtain $a=2$ and $b=4$.

27. $-\begin{bmatrix} -1 \\ a \\ 2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ -2 \\ b \end{bmatrix} = \begin{bmatrix} c \\ -7 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1+6 \\ -a-4 \\ -2+2b \end{bmatrix} = \begin{bmatrix} c \\ -7 \\ 8 \end{bmatrix} \Rightarrow$
$7 = c$, $-a-4 = -7$, and $-2+2b = 8$. Solving these equations, we obtain $a = 3$, $b = 5$, and $c = 7$.

28. $-\begin{bmatrix} a \\ -3 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ b \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ c \end{bmatrix} \Rightarrow \begin{bmatrix} -a-1 \\ 3-b \\ -5 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ c \end{bmatrix} \Rightarrow$
$-a-1 = 4$, $3-b = 2$, and $-5 = c$. Solving these equations, we obtain $a = -5$, $b = 1$, and $c = -5$.

29. $-\begin{bmatrix} 1 \\ 2 \\ a \\ 1 \end{bmatrix} + 2\begin{bmatrix} b \\ 1 \\ -2 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ c \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \\ 3 \\ d \end{bmatrix} \Rightarrow \begin{bmatrix} 2b-3 \\ -c \\ -a-9 \\ 5 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \\ 3 \\ d \end{bmatrix} \Rightarrow$
$2b-3 = -3$, $-c = -4$, $-a-9 = 3$, and $5 = d$. Solving these equations, we obtain $a = -12$, $b = 0$, $c = 4$, and $d = 5$.

30. $-\begin{bmatrix} a \\ 4 \\ -2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 1 \\ b \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ c \\ -3 \\ -6 \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \\ 3 \\ d \end{bmatrix} \Rightarrow \begin{bmatrix} -a+10-2 \\ -4+2-c \\ 2+2b+3 \\ 1+6+6 \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \\ 3 \\ d \end{bmatrix} \Rightarrow$
$-a+8 = 11$, $-2-c = -4$, $5+2b = 3$, and $13 = d$. Solving these equations, we obtain $a = -3$, $b = -1$, $c = 2$, and $d = 13$.

31. $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \Leftrightarrow x_1\begin{bmatrix} -2 \\ 5 \end{bmatrix} + x_2\begin{bmatrix} 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -2x_1+7x_2 \\ 5x_1-3x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix} \Leftrightarrow$ the augmented matrix $\begin{bmatrix} -2 & 7 & 8 \\ 5 & -3 & 9 \end{bmatrix}$ has a solution:

$$\begin{bmatrix} -2 & 7 & 8 \\ 5 & -3 & 9 \end{bmatrix} \overset{(5/2)R_1+R_2\to R_2}{\sim} \begin{bmatrix} -2 & 7 & 8 \\ 0 & \frac{29}{2} & 29 \end{bmatrix}$$

From row 2, $\frac{29}{2}x_2 = 29 \Rightarrow x_2 = 2$. From row 1, $-2x_1 + 7(2) = 8 \Rightarrow x_1 = 3$. Hence $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$, with $\mathbf{b} = 3\mathbf{a}_1 + 2\mathbf{a}_2$.

32. $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \Leftrightarrow x_1\begin{bmatrix} 4 \\ -6 \end{bmatrix} + x_2\begin{bmatrix} -6 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 4x_1-6x_2 \\ -6x_1+9x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix} \Leftrightarrow$ the augmented matrix $\begin{bmatrix} 4 & -6 & 1 \\ -6 & 9 & -5 \end{bmatrix}$ has a solution:

$$\begin{bmatrix} 4 & -6 & 1 \\ -6 & 9 & -5 \end{bmatrix} \overset{(3/2)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 4 & -6 & 1 \\ 0 & 0 & -\frac{7}{2} \end{bmatrix}$$

Because no solution exists, $\mathbf{b}$ is not a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

33. $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \Leftrightarrow x_1\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2x_1 \\ -3x_1+3x_2 \\ x_1-3x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -2 \end{bmatrix}$. The first equation $2x_1 = 1 \Rightarrow x_1 = \frac{1}{2}$. Then the second equation $-3\left(\frac{1}{2}\right) + 3x_2 = -5 \Rightarrow x_2 = -\frac{7}{6}$. We check the third equation, $\frac{1}{2} - 3\left(-\frac{7}{6}\right) = 4 \neq -2$. Hence $\mathbf{b}$ is *not* linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

34. $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \Leftrightarrow x_1\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -9 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2x_1 \\ -3x_1+3x_2 \\ x_1-3x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -9 \end{bmatrix}$. The first equation $2x_1 = 6 \Rightarrow x_1 = 3$. Then the second equation $-3(3) + 3x_2 = 3 \Rightarrow x_2 = 4$. We check the third equation, $3 - 3(4) = -9$. Hence $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$, with $\mathbf{b} = 3\mathbf{a}_1 + 4\mathbf{a}_2$.

35. $x_1\mathbf{a}_1+x_2\mathbf{a}_2+x_2\mathbf{a}_2=\mathbf{b} \quad\Leftrightarrow\quad x_1\begin{bmatrix}1\\2\\1\end{bmatrix}+x_2\begin{bmatrix}-3\\5\\-3\end{bmatrix}+x_3\begin{bmatrix}2\\2\\4\end{bmatrix}=\begin{bmatrix}1\\-2\\3\end{bmatrix} \quad\Leftrightarrow \begin{bmatrix}x_1-3x_2+2x_3\\2x_1+5x_2+2x_3\\x_1-3x_2+4x_3\end{bmatrix}$ $=\begin{bmatrix}1\\-2\\3\end{bmatrix}$ $\Leftrightarrow$ the augmented matrix $\begin{bmatrix}1&-3&2&1\\2&5&2&-2\\1&-3&4&3\end{bmatrix}$ yields a solution.

$$\begin{bmatrix}1&-3&2&1\\2&5&2&-2\\1&-3&4&3\end{bmatrix}\underset{\sim}{\substack{-2R_1+R_2\to R_2\\-R_1+R_3\to R_3}}\begin{bmatrix}1&-3&2&1\\0&11&-2&-4\\0&0&2&2\end{bmatrix}$$

From row 3, we have $2x_3=2 \;\Rightarrow\; x_3=1$. From row 2, $11x_2-2(1)=-4 \;\Rightarrow\; x_2=-\frac{2}{11}$. From row 1, $x_1-3(-\frac{2}{11})+2\,(1)=1 \;\Rightarrow\; x_1=-\frac{17}{11}$. Hence $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, with $\mathbf{b}=-\frac{\mathbf{17}}{\mathbf{11}}\mathbf{a}_1-\frac{2}{11}\mathbf{a}_2+\mathbf{a}_3$.

36. $x_1\mathbf{a}_1+x_2\mathbf{a}_2+x_2\mathbf{a}_2=\mathbf{b} \quad\Leftrightarrow\quad x_1\begin{bmatrix}2\\-3\\1\end{bmatrix}+x_2\begin{bmatrix}0\\3\\-3\end{bmatrix}+x_3\begin{bmatrix}-2\\-1\\3\end{bmatrix}=\begin{bmatrix}2\\-4\\5\end{bmatrix} \quad\Leftrightarrow$ $\begin{bmatrix}2x_1-2x_3\\-3x_1+3x_2-x_3\\x_1-3x_2+3x_3\end{bmatrix}=\begin{bmatrix}2\\-4\\5\end{bmatrix}$ $\Leftrightarrow$ the augmented matrix $\begin{bmatrix}2&0&-2&2\\-3&3&-1&-4\\1&-3&3&5\end{bmatrix}$ yields a solution.

$$\begin{bmatrix}2&0&-2&2\\-3&3&-1&-4\\1&-3&3&5\end{bmatrix}\underset{\sim}{\substack{(3/2)R_1+R_2\to R_2\\(-1/2)R_1+R_3\to R_3}}\begin{bmatrix}2&0&-2&2\\0&3&-4&-1\\0&-3&4&4\end{bmatrix}$$
$$\underset{\sim}{R_2+R_3\to R_3}\begin{bmatrix}2&0&-2&2\\0&3&-4&-1\\0&0&0&3\end{bmatrix}$$

From the third equation, we have $0=3$, and hence the system does not have a solution. Hence $\mathbf{b}$ is *not* a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

37. Using vectors, we calculate
$$(2)\begin{bmatrix}29\\3\\4\end{bmatrix}+(1)\begin{bmatrix}18\\25\\6\end{bmatrix}=\begin{bmatrix}76\\31\\14\end{bmatrix}$$
Hence we have 76 pounds of nitrogen, 31 pounds of phosphoric acid, and 14 pounds of potash.

38. Using vectors, we calculate
$$(4)\begin{bmatrix}29\\3\\4\end{bmatrix}+(7)\begin{bmatrix}18\\25\\6\end{bmatrix}=\begin{bmatrix}242\\187\\58\end{bmatrix}$$
Hence we have 242 pounds of nitrogen, 187 pounds of phosphoric acid, and 58 pounds of potash.

39. Let x_1 be the amount of Vigoro, x_2 the amount of Parker's, and then we need
$$x_1\begin{bmatrix}29\\3\\4\end{bmatrix}+x_2\begin{bmatrix}18\\25\\6\end{bmatrix}=\begin{bmatrix}112\\81\\26\end{bmatrix}$$
Solve using the corresponding augmented matrix:
$$\begin{bmatrix}29&18&112\\3&25&81\\4&6&26\end{bmatrix}\underset{\sim}{\substack{(-3/29)R_1+R_2\to R_2\\(-4/29)R_1+R_3\to R_3}}\begin{bmatrix}29&18&112\\0&\frac{671}{29}&\frac{2013}{29}\\0&\frac{102}{29}&\frac{306}{29}\end{bmatrix}$$
$$\underset{\sim}{(-102/671)R_2+R_3\to R_3}\begin{bmatrix}29&18&112\\0&\frac{671}{29}&\frac{2013}{29}\\0&0&0\end{bmatrix}$$

From row 2, we have $\frac{671}{29}x_2 = \frac{2013}{29} \Rightarrow x_2 = 3$. Form row 1, we have $29x_1 + 18(3) = 112 \Rightarrow x_1 = 2$. Thus we need 2 bags of Vigoro and 3 bags of Parker's.

40. Let x_1 be the amount of Vigoro, x_2 the amount of Parker's, and then we need

$$x_1 \begin{bmatrix} 29 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 18 \\ 25 \\ 6 \end{bmatrix} = \begin{bmatrix} 285 \\ 284 \\ 78 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 29 & 18 & 285 \\ 3 & 25 & 284 \\ 4 & 6 & 78 \end{bmatrix} \underset{\sim}{\substack{(-3/29)R_1+R_2\to R_2 \\ (-4/29)R_1+R_3\to R_3}} \begin{bmatrix} 29 & 18 & 285 \\ 0 & \frac{671}{29} & \frac{7381}{29} \\ 0 & \frac{102}{29} & \frac{1122}{29} \end{bmatrix}$$
$$\underset{\sim}{(-102/671)R_2+R_3\to R_3} \begin{bmatrix} 29 & 18 & 285 \\ 0 & \frac{671}{29} & \frac{7381}{29} \\ 0 & 0 & 0 \end{bmatrix}$$

From row 2, we have $\frac{671}{29}x_2 = \frac{7381}{29} \Rightarrow x_2 = 11$. Form row 1, we have $29x_1 + 18(11) = 285 \Rightarrow x_1 = 3$. Thus we need 3 bags of Vigoro and 11 bags of Parker's.

41. Let x_1 be the amount of Vigoro, x_2 the amount of Parker's, and then we need

$$x_1 \begin{bmatrix} 29 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 18 \\ 25 \\ 6 \end{bmatrix} = \begin{bmatrix} 123 \\ 59 \\ 24 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 29 & 18 & 123 \\ 3 & 25 & 59 \\ 4 & 6 & 24 \end{bmatrix} \underset{\sim}{\substack{(-3/29)R_1+R_2\to R_2 \\ (-4/29)R_1+R_3\to R_3}} \begin{bmatrix} 29 & 18 & 123 \\ 0 & \frac{671}{29} & \frac{1342}{29} \\ 0 & \frac{102}{29} & \frac{204}{29} \end{bmatrix}$$
$$\underset{\sim}{\substack{(29/671)R_2\to R_2 \\ (-102/29)R_2+R_3\to R_3}} \begin{bmatrix} 29 & 18 & 123 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Back substituting gives $x_2 = 2$ and $x_1 = 3$. Hence we need 3 bags of Vigoro and 2 bags of Parker's.

42. Let x_1 be the amount of Vigoro, x_2 the amount of Parker's, and then we need

$$x_1 \begin{bmatrix} 29 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 18 \\ 25 \\ 6 \end{bmatrix} = \begin{bmatrix} 159 \\ 109 \\ 36 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 29 & 18 & 159 \\ 3 & 25 & 109 \\ 4 & 6 & 36 \end{bmatrix} \underset{\sim}{\substack{(-3/29)R_1+R_2\to R_2 \\ (-4/29)R_1+R_3\to R_3}} \begin{bmatrix} 29 & 18 & 159 \\ 0 & \frac{671}{29} & \frac{2684}{29} \\ 0 & \frac{102}{29} & \frac{408}{29} \end{bmatrix}$$
$$\underset{\sim}{\substack{(29/671)R_2\to R_3 \\ (-102/29)R_2+R_3\to R_3}} \begin{bmatrix} 29 & 18 & 159 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Back substituting gives $x_2 = 4$ and $x_1 = 3$. Hence we need 3 bags of Vigoro and 4 bags of Parker's.

43. Let x_1 be the amount of Vigoro, x_2 the amount of Parker's, and then we need

$$x_1 \begin{bmatrix} 29 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 18 \\ 25 \\ 6 \end{bmatrix} = \begin{bmatrix} 148 \\ 131 \\ 40 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 29 & 18 & 148 \\ 3 & 25 & 131 \\ 4 & 6 & 40 \end{bmatrix} \underset{\sim}{\substack{(-3/29)R_1+R_2\to R_2 \\ (-4/29)R_1+R_3\to R_3}} \begin{bmatrix} 29 & 18 & 148 \\ 0 & \frac{671}{29} & \frac{3355}{29} \\ 0 & \frac{102}{29} & \frac{568}{29} \end{bmatrix}$$

$$\underset{\sim}{(-102/671)R_2+R_3\to R_3} \begin{bmatrix} 29 & 18 & 148 \\ 0 & \frac{671}{29} & \frac{3355}{29} \\ 0 & 0 & 2 \end{bmatrix}$$

Since row 3 corresponds to the equation $0 = 2$, the system has no solutions.

44. Let x_1 be the amount of Vigoro, x_2 the amount of Parker's, and then we need

$$x_1 \begin{bmatrix} 29 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 18 \\ 25 \\ 6 \end{bmatrix} = \begin{bmatrix} 100 \\ 120 \\ 40 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 29 & 18 & 100 \\ 3 & 25 & 120 \\ 4 & 6 & 40 \end{bmatrix} \underset{\sim}{\substack{(-3/29)R_1+R_2\to R_2 \\ (-4/29)R_1+R_3\to R_3}} \begin{bmatrix} 29 & 18 & 100 \\ 0 & \frac{671}{29} & \frac{3180}{29} \\ 0 & \frac{102}{29} & \frac{760}{29} \end{bmatrix}$$

$$\underset{\sim}{(-102/671)R_2+R_3\to R_3} \begin{bmatrix} 29 & 18 & 100 \\ 0 & \frac{671}{29} & \frac{3180}{29} \\ 0 & 0 & \frac{6400}{671} \end{bmatrix}$$

Since row 3 is $0 = \frac{6400}{671}$, we conclude that we can not obtain the desired amounts.

45. Let x_1 be the amount of Vigoro, x_2 the amount of Parker's, and then we need

$$x_1 \begin{bmatrix} 29 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 18 \\ 25 \\ 6 \end{bmatrix} = \begin{bmatrix} 25 \\ 72 \\ 14 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 29 & 18 & 25 \\ 3 & 25 & 72 \\ 4 & 6 & 14 \end{bmatrix} \underset{\sim}{\substack{(-3/29)R_1+R_2\to R_2 \\ (-4/29)R_1+R_3\to R_3}} \begin{bmatrix} 29 & 18 & 25 \\ 0 & \frac{671}{29} & \frac{2013}{29} \\ 0 & \frac{102}{29} & \frac{306}{29} \end{bmatrix}$$

$$\underset{\sim}{(-102/671)R_2+R_3\to R_3} \begin{bmatrix} 29 & 18 & 25 \\ 0 & \frac{671}{29} & \frac{2013}{29} \\ 0 & 0 & 0 \end{bmatrix}$$

From row 2, we have $\frac{671}{29}x_2 = \frac{2013}{29} \Rightarrow x_2 = 3$. From row 1, we have $29x_1 + 18(3) = 25 \Rightarrow x_1 = -1$. Since we can not use a negative amount, we conclude that there is no solution.

46. Let x_1 be the amount of Vigoro, x_2 the amount of Parker's, and then we need

$$x_1 \begin{bmatrix} 29 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 18 \\ 25 \\ 6 \end{bmatrix} = \begin{bmatrix} 301 \\ 8 \\ 38 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\left[\begin{array}{rrr} 29 & 18 & 301 \\ 3 & 25 & 8 \\ 4 & 6 & 38 \end{array}\right] \begin{array}{c} (-3/29)R_1+R_2\to R_2 \\ (-4/29)R_1+R_3\to R_3 \\ \sim \end{array} \left[\begin{array}{ccc} 29 & 18 & 301 \\ 0 & \frac{671}{29} & -\frac{671}{29} \\ 0 & \frac{102}{29} & -\frac{102}{29} \end{array}\right]$$

$$\begin{array}{c} (-102/671)R_2+R_3\to R_3 \\ \sim \end{array} \left[\begin{array}{ccc} 29 & 18 & 301 \\ 0 & \frac{671}{29} & -\frac{671}{29} \\ 0 & 0 & 0 \end{array}\right]$$

From row 2, we have $\frac{671}{29}x_2 = -\frac{671}{29} \Rightarrow x_2 = -1$. Since we can not use a negative amount, we conclude that there is no solution.

47. Let x_1 be the number of cans of Red Bull, and x_2 the number of cans of Jolt Cola, and then we need

$$x_1 \begin{bmatrix} 27 \\ 80 \end{bmatrix} + x_2 \begin{bmatrix} 94 \\ 280 \end{bmatrix} = \begin{bmatrix} 148 \\ 440 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\left[\begin{array}{rrr} 27 & 94 & 148 \\ 80 & 280 & 440 \end{array}\right] \begin{array}{c} (-80/27)R_1+R_2\to R_2 \\ \sim \end{array} \left[\begin{array}{ccc} 27 & 94 & 148 \\ 0 & \frac{40}{27} & \frac{40}{27} \end{array}\right]$$

From row 2, we have $\frac{40}{27}x_2 = \frac{40}{27} \Rightarrow x_2 = 1$. From row 1, $27x_1 + 94(1) = 148 \Rightarrow x_1 = 2$. Thus we need to drink 2 cans of Red Bull and 1 can of Jolt Cola.

48. Let x_1 be the number of cans of Red Bull, and x_2 the number of cans of Jolt Cola, and then we need

$$x_1 \begin{bmatrix} 27 \\ 80 \end{bmatrix} + x_2 \begin{bmatrix} 94 \\ 280 \end{bmatrix} = \begin{bmatrix} 309 \\ 920 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\left[\begin{array}{rrr} 27 & 94 & 309 \\ 80 & 280 & 920 \end{array}\right] \begin{array}{c} (-80/27)R_1+R_2\to R_2 \\ \sim \end{array} \left[\begin{array}{ccc} 27 & 94 & 309 \\ 0 & \frac{40}{27} & \frac{40}{9} \end{array}\right]$$

From row 2, we have $\frac{40}{27}x_2 = \frac{40}{9} \Rightarrow x_2 = 3$. From row 1, $27x_1 + 94(3) = 309 \Rightarrow x_1 = 1$. Thus we need to drink 1 can of Red Bull and 3 cans of Jolt Cola.

49. Let x_1 be the number of cans of Red Bull, and x_2 the number of cans of Jolt Cola, and then we need

$$x_1 \begin{bmatrix} 27 \\ 80 \end{bmatrix} + x_2 \begin{bmatrix} 94 \\ 280 \end{bmatrix} = \begin{bmatrix} 242 \\ 720 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\left[\begin{array}{rrr} 27 & 94 & 242 \\ 80 & 280 & 720 \end{array}\right] \begin{array}{c} (-80/27)R_1+R_2\to R_2 \\ \sim \end{array} \left[\begin{array}{ccc} 27 & 94 & 242 \\ 0 & \frac{40}{27} & \frac{80}{27} \end{array}\right]$$

From row 2, we have $\frac{40}{27}x_2 = \frac{80}{27} \Rightarrow x_2 = 2$. From row 1, $27x_1 + 94(2) = 242 \Rightarrow x_1 = 2$. Thus we need to drink 2 cans of Red Bull and 2 cans of Jolt Cola.

50. Let x_1 be the number of cans of Red Bull, and x_2 the number of cans of Jolt Cola, and then we need

$$x_1 \begin{bmatrix} 27 \\ 80 \end{bmatrix} + x_2 \begin{bmatrix} 94 \\ 280 \end{bmatrix} = \begin{bmatrix} 457 \\ 1360 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 27 & 94 & 457 \\ 80 & 280 & 1360 \end{bmatrix} \overset{(-80/27)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 27 & 94 & 457 \\ 0 & \frac{40}{27} & \frac{160}{27} \end{bmatrix}$$

From row 2, we have $\frac{40}{27}x_2 = \frac{160}{27} \Rightarrow x_2 = 4$. From row 1, $27x_1 + 94(4) = 457 \Rightarrow x_1 = 3$. Thus we need to drink 3 cans of Red Bull and 4 cans of Jolt Cola.

51. Let x_1 be the number of servings of Lucky Charms and x_2 the number of servings of Raisin Bran, and then we need

$$x_1 \begin{bmatrix} 10 \\ 25 \\ 25 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 25 \\ 10 \end{bmatrix} = \begin{bmatrix} 40 \\ 200 \\ 125 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 10 & 2 & 40 \\ 25 & 25 & 200 \\ 25 & 10 & 125 \end{bmatrix} \overset{\substack{(-5/2)R_1+R_2\to R_2 \\ (-5/2)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 10 & 2 & 40 \\ 0 & 20 & 100 \\ 0 & 5 & 25 \end{bmatrix}$$

$$\overset{(-1/4)R_2+R_3\to R_3}{\sim} \begin{bmatrix} 10 & 2 & 40 \\ 0 & 20 & 100 \\ 0 & 0 & 0 \end{bmatrix}$$

From row 2, we have $20x_2 = 100 \Rightarrow x_2 = 5$. From row 1, $10x_1 + 2(5) = 40 \Rightarrow x_1 = 3$. Thus we need 3 servings of Lucky Charms and 5 servings of Raisin Bran.

52. Let x_1 be the number of servings of Lucky Charms and x_2 the number of servings of Raisin Bran, and then we need

$$x_1 \begin{bmatrix} 10 \\ 25 \\ 25 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 25 \\ 10 \end{bmatrix} = \begin{bmatrix} 34 \\ 125 \\ 95 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 10 & 2 & 34 \\ 25 & 25 & 125 \\ 25 & 10 & 95 \end{bmatrix} \overset{\substack{(-5/2)R_1+R_2\to R_2 \\ (-5/2)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 10 & 2 & 34 \\ 0 & 20 & 40 \\ 0 & 5 & 10 \end{bmatrix}$$

$$\overset{(-1/4)R_2+R_3\to R_3}{\sim} \begin{bmatrix} 10 & 2 & 34 \\ 0 & 20 & 40 \\ 0 & 0 & 0 \end{bmatrix}$$

From row 2, we have $20x_2 = 40 \Rightarrow x_2 = 2$. From row 1, $10x_1 + 2(2) = 34 \Rightarrow x_1 = 3$. Thus we need 3 servings of Lucky Charms and 2 servings of Raisin Bran.

53. Let x_1 be the number of servings of Lucky Charms and x_2 the number of servings of Raisin Bran, and then we need

$$x_1 \begin{bmatrix} 10 \\ 25 \\ 25 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 25 \\ 10 \end{bmatrix} = \begin{bmatrix} 26 \\ 125 \\ 80 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 10 & 2 & 26 \\ 25 & 25 & 125 \\ 25 & 10 & 80 \end{bmatrix} \overset{\substack{(-5/2)R_1+R_2\to R_2 \\ (-5/2)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 10 & 2 & 26 \\ 0 & 20 & 60 \\ 0 & 5 & 15 \end{bmatrix}$$

$$\overset{(-1/4)R_2+R_3\to R_3}{\sim} \begin{bmatrix} 10 & 2 & 26 \\ 0 & 20 & 60 \\ 0 & 0 & 0 \end{bmatrix}$$

From row 2, we have $20x_2 = 60 \Rightarrow x_2 = 3$. From row 1, $10x_1 + 2(3) = 26 \Rightarrow x_1 = 2$. Thus we need 2 servings of Lucky Charms and 3 servings of Raisin Bran.

54. Let x_1 be the number of servings of Lucky Charms and x_2 the number of servings of Raisin Bran, and then we need

$$x_1 \begin{bmatrix} 10 \\ 25 \\ 25 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 25 \\ 10 \end{bmatrix} = \begin{bmatrix} 38 \\ 175 \\ 115 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 10 & 2 & 38 \\ 25 & 25 & 175 \\ 25 & 10 & 115 \end{bmatrix} \underset{(-5/2)R_1+R_3 \to R_3}{\overset{(-5/2)R_1+R_2 \to R_2}{\sim}} \begin{bmatrix} 10 & 2 & 38 \\ 0 & 20 & 80 \\ 0 & 5 & 20 \end{bmatrix}$$

$$\overset{(-1/4)R_2+R_3 \to R_3}{\sim} \begin{bmatrix} 10 & 2 & 38 \\ 0 & 20 & 80 \\ 0 & 0 & 0 \end{bmatrix}$$

From row 2, we have $20x_2 = 80 \Rightarrow x_2 = 4$. From row 1, $10x_1 + 2(4) = 38 \Rightarrow x_1 = 3$. Thus we need 3 servings of Lucky Charms and 4 servings of Raisin Bran.

55. (a) $\mathbf{a} = \begin{bmatrix} 2000 \\ 8000 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3000 \\ 10000 \end{bmatrix}$

(b) $8\mathbf{b} = (8) \begin{bmatrix} 3000 \\ 10000 \end{bmatrix} = \begin{bmatrix} 24000 \\ 80000 \end{bmatrix}$. The company produces 24000 computer monitors and 80000 flat panel televisions at facility B in 8 weeks.

(c) $6\mathbf{a} + 6\mathbf{b} = 6 \begin{bmatrix} 2000 \\ 8000 \end{bmatrix} + 6 \begin{bmatrix} 3000 \\ 10000 \end{bmatrix} = \begin{bmatrix} 30000 \\ 108000 \end{bmatrix}$. The company produces 30000 computer monitors and 108000 flat panel televisions at facilities A and B in 6 weeks.

(d) Let x_1 be the number of weeks of production at facility A, and x_2 the number of weeks of production at facility B, and then we need

$$x_1 \begin{bmatrix} 2000 \\ 8000 \end{bmatrix} + x_2 \begin{bmatrix} 3000 \\ 10000 \end{bmatrix} = \begin{bmatrix} 24000 \\ 92000 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 2000 & 3000 & 24000 \\ 8000 & 10000 & 92000 \end{bmatrix} \overset{(-4)R_1+R_2 \to R_2}{\sim} \begin{bmatrix} 2000 & 3000 & 24000 \\ 0 & -2000 & -4000 \end{bmatrix}$$

From row 2, we have $-2000x_2 = -4000 \Rightarrow x_2 = 2$. From row 1, $2000x_1 + 3000(2) = 24000 \Rightarrow x_1 = 9$. Thus we need 9 weeks of production at facility A and 2 weeks of production at facility B.

56. We assume a 5-day work week.

(a) $\mathbf{a} = \begin{bmatrix} 10 \\ 20 \\ 10 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 20 \\ 30 \\ 40 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 40 \\ 70 \\ 50 \end{bmatrix}$

(b) $20\mathbf{c} = (20) \begin{bmatrix} 40 \\ 70 \\ 50 \end{bmatrix} = \begin{bmatrix} 800 \\ 1400 \\ 1000 \end{bmatrix}$. The company produces 800 metric tons of PE, 1400 metric tons of PVC, and 1000 metric tons of PS at facility C in 4 weeks.

(c) $10\mathbf{a} + 10\mathbf{b} + 10\mathbf{c} = 10 \begin{bmatrix} 10 \\ 20 \\ 10 \end{bmatrix} + 10 \begin{bmatrix} 20 \\ 30 \\ 40 \end{bmatrix} + 10 \begin{bmatrix} 40 \\ 70 \\ 50 \end{bmatrix} = \begin{bmatrix} 700 \\ 1200 \\ 1000 \end{bmatrix}$. The company produces 700 metric tons of PE, 1200 metric tons of PVC, and 1000 metric tons of PS at facilities A,B, and C in 2 weeks.

(d) Let x_1 be the number of days of production at facility A, x_2 the number of days of production at facility B, and x_3 the number of days of production at facility C. Then we need

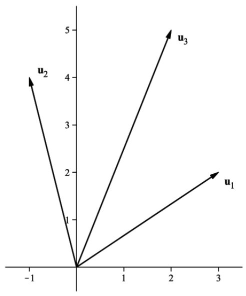
$$x_1 \begin{bmatrix} 10 \\ 20 \\ 10 \end{bmatrix} + x_2 \begin{bmatrix} 20 \\ 30 \\ 40 \end{bmatrix} + x_3 \begin{bmatrix} 40 \\ 70 \\ 50 \end{bmatrix} = \begin{bmatrix} 240 \\ 420 \\ 320 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 10 & 20 & 40 & 240 \\ 20 & 30 & 70 & 420 \\ 10 & 40 & 50 & 320 \end{bmatrix} \underset{\sim}{\overset{-2R_1+R_2\to R_2}{-R_1+R_3\to R_3}} \begin{bmatrix} 10 & 20 & 40 & 240 \\ 0 & -10 & -10 & -60 \\ 0 & 20 & 10 & 80 \end{bmatrix}$$

$$\overset{2R_2+R_3\to R_3}{\sim} \begin{bmatrix} 10 & 20 & 40 & 240 \\ 0 & -10 & -10 & -60 \\ 0 & 0 & -10 & -40 \end{bmatrix}$$

From row 3, we have $-10x_3 = -40 \Rightarrow x_3 = 4$. From row 2, $-10x_2 - 10(4) = -60 \Rightarrow x_2 = 2$. From row 1, $10x_1 + 20(2) + 40(4) = 240 \Rightarrow x_1 = 4$. Thus we need 4 days of production at facility A, 2 days of production at facility B, and 4 days of production at facility C.

57. 

$$\overline{\mathbf{v}} = \frac{5\mathbf{u}_1+3\mathbf{u}_2+2\mathbf{u}_3}{5+3+2} = \frac{1}{10}\left(5\begin{bmatrix} 3 \\ 2 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 4 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \frac{1}{10}\begin{bmatrix} 16 \\ 32 \end{bmatrix} = \begin{bmatrix} \frac{8}{5} \\ \frac{16}{5} \end{bmatrix}$$

58. $$\overline{\mathbf{v}} = \frac{4\mathbf{u}_1+1\mathbf{u}_2+2\mathbf{u}_3+5\mathbf{u}_4}{4+1+2+5} = \frac{1}{12}\left(4\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} + 5\begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}\right) = \frac{1}{12}\begin{bmatrix} 23 \\ 19 \\ 11 \end{bmatrix} = \begin{bmatrix} \frac{23}{12} \\ \frac{19}{12} \\ \frac{11}{12} \end{bmatrix}$$

59. Let x_1, x_2,and x_3 be the mass of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ respectively. Then

$$\overline{\mathbf{v}} = \frac{x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3}{11} = \frac{1}{11}\left(x_1\begin{bmatrix} -1 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ -2 \end{bmatrix} + x_3\begin{bmatrix} 5 \\ 2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -\frac{1}{11}x_1 + \frac{3}{11}x_2 + \frac{5}{11}x_3 \\ \frac{3}{11}x_1 - \frac{2}{11}x_2 + \frac{2}{11}x_3 \end{bmatrix} = \begin{bmatrix} \frac{13}{11} \\ \frac{16}{11} \end{bmatrix}$$

We obtain the 2 equations, $-x_1+3x_2+5x_3 = 13$ and $3x_1-2x_2+2x_3 = 16$. Together with the equation $x_1 + x_2 + x_3 = 11$, we have 3 equations and solve the corresponding augmented matrix:

$$\begin{bmatrix} -1 & 3 & 5 & 13 \\ 3 & -2 & 2 & 16 \\ 1 & 1 & 1 & 11 \end{bmatrix} \underset{\sim}{\overset{3R_1+R_2\to R_2}{R_1+R_3\to R_3}} \begin{bmatrix} -1 & 3 & 5 & 13 \\ 0 & 7 & 17 & 55 \\ 0 & 4 & 6 & 24 \end{bmatrix}$$

$$\overset{(-4/7)R_2+R_3\to R_3}{\sim} \begin{bmatrix} -1 & 3 & 5 & 13 \\ 0 & 7 & 17 & 55 \\ 0 & 0 & -\frac{26}{7} & -\frac{52}{7} \end{bmatrix}$$

From row 3, $-\frac{26}{7}x_3 = -\frac{52}{7} \Rightarrow x_3 = 2$. From row 2, $7x_2 + 17(2) = 55 \Rightarrow x_2 = 3$. From row 1, $-x_1 + 3(3) + 5(2) = 13 \Rightarrow x_1 = 6$.

60. Let x_1, x_2, x_3,and x_4 be the mass of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, and $\mathbf{u}_4$ respectively. Then

$$\overline{\mathbf{v}} = \frac{x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 + x_4\mathbf{u}_4}{11} = \frac{1}{11}\left(x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$
$$= \begin{bmatrix} \frac{1}{11}x_1 + \frac{2}{11}x_2 - \frac{1}{11}x_4 \\ \frac{1}{11}x_1 - \frac{1}{11}x_2 + \frac{3}{11}x_3 \\ \frac{2}{11}x_1 + \frac{2}{11}x_3 + \frac{1}{11}x_4 \end{bmatrix} = \begin{bmatrix} \frac{4}{11} \\ \frac{5}{11} \\ \frac{12}{11} \end{bmatrix}$$

We obtain the 3 equations, $x_1 + 2x_2 - x_4 = 4$, $x_1 - x_2 + 3x_3 = 5$, and $2x_1 + 2x_3 + x_4 = 12$. Together with the equation $x_1 + x_2 + x_3 + x_4 = 11$, we have 4 equations and solve the corresponding augmented matrix:

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 4 \\ 1 & -1 & 3 & 0 & 5 \\ 2 & 0 & 2 & 1 & 12 \\ 1 & 1 & 1 & 1 & 11 \end{bmatrix} \underset{\substack{-R_1+R_2\to R_2 \\ -2R_1+R_3\to R_3 \\ -R_1+R_4\to R_4}}{\sim} \begin{bmatrix} 1 & 2 & 0 & -1 & 4 \\ 0 & -3 & 3 & 1 & 1 \\ 0 & -4 & 2 & 3 & 4 \\ 0 & -1 & 1 & 2 & 7 \end{bmatrix}$$
$$\underset{\substack{(-4/3)R_2+R_3\to R_3 \\ (-1/3)R_2+R_4\to R_4}}{\sim} \begin{bmatrix} 1 & 2 & 0 & -1 & 4 \\ 0 & -3 & 3 & 1 & 1 \\ 0 & 0 & -2 & \frac{5}{3} & \frac{8}{3} \\ 0 & 0 & 0 & \frac{5}{3} & \frac{20}{3} \end{bmatrix}$$

From row 4, $\frac{5}{3}x_4 = \frac{20}{3} \Rightarrow x_4 = 4$. From row 3, $-2x_3 + \frac{5}{3}(4) = \frac{8}{3} \Rightarrow x_3 = 2$. From row 2, $-3x_2 + 3(2) + 4 = 1 \Rightarrow x_2 = 3$. From row 1, $x_1 + 2(3) - 4 = 4 \Rightarrow x_1 = 2$.

61. For example, $\mathbf{u} = (0, 0, -1)$ and $\mathbf{v} = (3, 2, 0)$.

62. For example, $\mathbf{u} = (4, 0, 0, 0)$ and $\mathbf{v} = (0, 2, 0, 1)$.

63. For example, $\mathbf{u} = (1, 0, 0)$, $\mathbf{v} = (1, 0, 0)$, and $\mathbf{w} = (-2, 0, 0)$.

64. For example, $\mathbf{u} = (1, 0, 0, 0)$, $\mathbf{v} = (1, 0, 0, 0)$, and $\mathbf{w} = (-2, 0, 0, 0)$.

65. For example, $\mathbf{u} = (1, 0)$ and $\mathbf{v} = (2, 0)$.

66. For example, $\mathbf{u} = (1, 0)$ and $\mathbf{v} = (-1, 0)$.

67. For example, $\mathbf{u} = (1, 0, 0)$, $\mathbf{v} = (2, 0, 0)$, and $\mathbf{w} = (3, 0, 0)$.

68. For example, $\mathbf{u} = (1, 0, 0, 0)$, $\mathbf{v} = (2, 0, 0, 0)$, $\mathbf{w} = (2, 0, 0, 0)$,and $\mathbf{x} = (4, 0, 0, 0)$.

69. Simply, $x_1 = 3$ and $x_2 = -2$.

70. For example, $x_1 - 2x_2 = 1$ and $x_2 + x_3 = 1$.

71. (a) True, since $-2\begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} (-2)(-3) \\ (-2)(5) \end{bmatrix} = \begin{bmatrix} 6 \\ -10 \end{bmatrix}$.

    (b) False, since $\mathbf{u} - \mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1-(-4) \\ 3-2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \neq \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

72. (a) False. Scalars may be any real number, such as $c = -1$.

    (b) True. Vector components and scalars can be any real numbers.

73. (a) True, by Theorem 2.3(b).

    (b) False. The sum $c_1 + \mathbf{u}_1$ of a scalar and a vector is undefined.

74. (a) False. A vector can have any initial point.

(b) False. They do not point in opposite directions, as there does not exist $c < 0$ such that $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = c\begin{bmatrix} -2 \\ 4 \\ 8 \end{bmatrix}$.

75. (a) True, by Definition 2.1, where it is stated that vectors can be expressed in column or row form.
(b) True. For any vector $\mathbf{v}$, $\mathbf{0} = 0\mathbf{v}$.

76. (a) True, because $-2(-\mathbf{u}) = (-2)((-1)\mathbf{u}) = ((-2)(-1))\mathbf{u} = 2\mathbf{u}$.
(b) False. For example, $x\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ has no solution.

77. (a) False. It works regardless of the quadrant, and can be established algebraically for vectors positioned anywhere.
(b) False. Because vector addition is commutative, one can order the vectors in either way for the Tip-to-Tail Rule.

78. (a) False. For instance, if $\mathbf{u} = (2, 1)$ and $\mathbf{v} = (-1, 3)$, then $\mathbf{u} - \mathbf{v} = (3, -2)$ while $-\mathbf{u} + \mathbf{v} = (-3, 2)$. (The difference $\mathbf{u} - \mathbf{v}$ is found by adding $\mathbf{u}$ to $-\mathbf{v}$.)
(b) True, as long as the vectors have the same number of components.

79. (a) Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$. Then $(a+b)\mathbf{u} = (a+b)\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} (a+b)u_1 \\ (a+b)u_2 \\ \vdots \\ (a+b)u_n \end{bmatrix}$

$$= \begin{bmatrix} au_1 + bu_1 \\ au_2 + bu_2 \\ \vdots \\ au_n + bu_n \end{bmatrix} = \begin{bmatrix} au_1 \\ au_2 \\ \vdots \\ au_n \end{bmatrix} + \begin{bmatrix} bu_1 \\ bu_2 \\ \vdots \\ bu_n \end{bmatrix} = a\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + b\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = a\mathbf{u} + b\mathbf{u}.$$

(b) Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$. Then

$$(\mathbf{u}+\mathbf{v})+\mathbf{w} = \left(\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

$$= \begin{bmatrix} (u_1+v_1)+w_1 \\ (u_2+v_2)+w_2 \\ \vdots \\ (u_n+v_n)+w_n \end{bmatrix} = \begin{bmatrix} u_1+(v_1+w_1) \\ u_2+(v_2+w_2) \\ \vdots \\ u_n+(v_n+w_n) \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1+w_1 \\ v_2+w_2 \\ \vdots \\ v_n+w_n \end{bmatrix}$$

$$= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}\right) = \mathbf{u} + (\mathbf{v}+\mathbf{w}).$$

(c) Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$. Then $a(b\mathbf{u}) = a\left(b\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}\right) = a\left(\begin{bmatrix} bu_1 \\ bu_2 \\ \vdots \\ bu_n \end{bmatrix}\right)$

$$= \begin{bmatrix} a\,(bu_1) \\ a\,(bu_2) \\ \vdots \\ a\,(bu_n) \end{bmatrix} = \begin{bmatrix} (ab)\,u_1 \\ (ab)\,u_2 \\ \vdots \\ (ab)\,u_n \end{bmatrix} = (ab) \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = (ab)\mathbf{u}.$$

(d) Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$. Then $\mathbf{u} + (-\mathbf{u}) = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \left(- \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \right)$

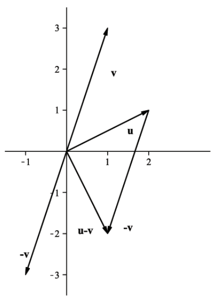
$$= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} -u_1 \\ -u_2 \\ \vdots \\ -u_n \end{bmatrix} = \begin{bmatrix} u_1 - u_1 \\ u_2 - u_2 \\ \vdots \\ u_n - u_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}.$$

(e) Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$. Then $\mathbf{u} + \mathbf{0} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 + 0 \\ u_2 + 0 \\ \vdots \\ u_n + 0 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \mathbf{u}$. Likewise,

$$\mathbf{0} + \mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} 0 + u_1 \\ 0 + u_2 \\ \vdots \\ 0 + u_n \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \mathbf{u}.$$

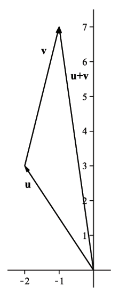
(f) Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$. Then $1\mathbf{u} = (1) \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} (1)\,u_1 \\ (1)\,u_2 \\ \vdots \\ (1)\,u_n \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \mathbf{u}$.

80. Using, for example, $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

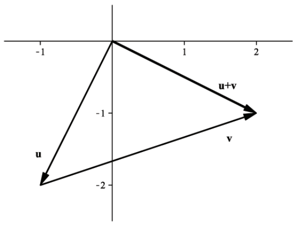


The vector $\mathbf{u} - \mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is the translation of the vector $\mathbf{w}'$ which has initial point the tip of $\mathbf{u}$ and terminal point the tip of $\mathbf{v}$, as in Figure 6.

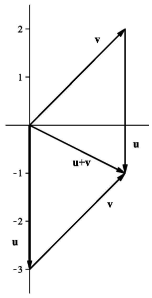
81.



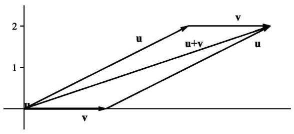
82.



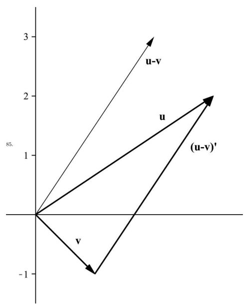
83.



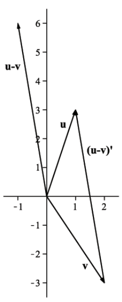
84.



85.



86.



87. We obtain the three equations $2x_1 + 2x_2 + 5x_3 = 0$, $7x_1 + 4x_2 + x_3 = 3$, and $3x_1 + 2x_2 + 6x_3 = 5$. Using a computer algebra system to solve this system, we get $x_1 = 4$, $x_2 = -6.5$, and $x_3 = 1$.

88. We obtain the four equations $x_1+4x_2-4x_3+5x_4 = 1$, $-3x_1+3x_2+2x_3+2x_4 = 7$, $2x_1+2x_2-3x_3-4x_4 = 2$, and $x_2 + x_3 = -6$. Using a computer algebra system to solve this system, we get $x_1 = -7.5399$, $x_2 = -1.1656$, $x_3 = -4.8344$, and $x_4 = -1.2270$. (Solving this system exactly, we obtain $x_1 = -\frac{1229}{163}$, $x_2 = -\frac{190}{163}$, $x_3 = -\frac{788}{163}$, and $x_4 = -\frac{200}{163}$.)

2.2 Practice Problems

1. (a) $0\mathbf{u}_1+0\mathbf{u}_2 = 0\begin{bmatrix} 2 \\ -3 \end{bmatrix}+0\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $1\mathbf{u}_1+0\mathbf{u}_2 = 1\begin{bmatrix} 2 \\ -3 \end{bmatrix}+0\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, $0\mathbf{u}_1+1\mathbf{u}_2 = 0\begin{bmatrix} 2 \\ -3 \end{bmatrix}+1\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

   (b) $0\mathbf{u}_1+0\mathbf{u}_2 = 0\begin{bmatrix} 6 \\ 1 \\ 4 \end{bmatrix}+0\begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $1\mathbf{u}_1+0\mathbf{u}_2 = 1\begin{bmatrix} 6 \\ 1 \\ 4 \end{bmatrix}+0\begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 4 \end{bmatrix}$, $0\mathbf{u}_1+1\mathbf{u}_2 = 0\begin{bmatrix} 6 \\ 1 \\ 4 \end{bmatrix}+1\begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix}$

2. Set $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 = \mathbf{b} \Rightarrow x_1\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \quad \Rightarrow$ $\begin{bmatrix} x_1 \\ 2x_1 + 4x_2 \\ -2x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$. From the first equation, $x_1 = -1$. Then the second equation is $2(-1) + 4x_2 = 2 \Rightarrow x_2 = 1$. The third equation is now $-2(-1) + 3(1) = 5 \Rightarrow 5 = 5$. So $\mathbf{b}$ is in the span of $\{\mathbf{u}_1, \mathbf{u}_2\}$, with $(-1)\mathbf{u}_1 + (1)\mathbf{u}_2 = \mathbf{b}$.

3. (a) $A = \begin{bmatrix} 7 & -2 & -2 \\ -1 & 7 & 4 \\ 3 & -1 & -2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 6 \\ 11 \\ 1 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 4 & -3 & -1 & 5 \\ 3 & 12 & 6 & 0 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$

4. (a) Row-reduce to echelon form:

$$\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \overset{(1/2)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 2 & 3 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

   There is not a row of zeros, so every choice of $\mathbf{b}$ is in the span of the columns of the given matrix and, therefore, the columns of the matrix span $\mathbf{R}^2$.

   (b) Row-reduce to echelon form:

$$\begin{bmatrix} 4 & 1 \\ 1 & -3 \end{bmatrix} \overset{(-1/4)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 4 & 1 \\ 0 & -\frac{13}{4} \end{bmatrix}$$

   Since there is not a row of zeros, every choice of $\mathbf{b}$ is in the span of the columns of the given matrix, and therefore the columns of the matrix span $\mathbf{R}^2$.

5. (a) Row-reduce to echelon form:

$$\begin{bmatrix} 1 & 3 & -1 \\ -1 & -2 & 3 \\ 0 & 2 & 5 \end{bmatrix} \underset{\sim}{R_1+R_2 \to R_2} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix} \underset{\sim}{-2R_2+R_3 \to R_3} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

There is not a row of zeros, so every choice of $\mathbf{b}$ is in the span of the columns of the given matrix and, therefore, the columns of the matrix span $\mathbf{R}^3$.

(b) Row-reduce to echelon form:

$$\begin{bmatrix} 2 & 0 & 6 \\ 1 & -2 & 1 \\ -1 & 4 & 1 \end{bmatrix} \underset{\sim}{\substack{(-1/2)R_1+R_2 \to R_2 \\ (1/2)R_1+R_3 \to R_3}} \begin{bmatrix} 2 & 0 & 6 \\ 0 & -2 & -2 \\ 0 & 4 & 4 \end{bmatrix} \underset{\sim}{2R_2+R_3 \to R_3} \begin{bmatrix} 2 & 0 & 6 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Because there is a row of zeros, there exists a vector $\mathbf{b}$ that is not in the span of the columns of the matrix and, therefore, the columns of the matrix do not span $\mathbf{R}^3$.

6. (a) False. If the vectors span $\mathbf{R}^3$, then vectors have three components, and cannot span $\mathbf{R}^2$.

(b) True. Every vector $\mathbf{b}$ in $\mathbf{R}^2$ can be written as

$$\begin{aligned} \mathbf{b} &= x_1\mathbf{u}_1 + x_2\mathbf{u}_2 \\ &= \frac{x_1}{2}(2\mathbf{u}_1) + \frac{x_2}{3}(3\mathbf{u}_2) \end{aligned}$$

which shows that $\{2\mathbf{u}_1, 3\mathbf{u}_2\}$ spans $\mathbf{R}^2$.

(c) True. Every vector $\mathbf{b}$ in $\mathbf{R}^3$ can be written as $\mathbf{b} = x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3$. So $A\mathbf{x} = \mathbf{b}$ has the solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

(d) True. Every vector $\mathbf{b}$ in $\mathbf{R}^2$ can be written as $\mathbf{b} = x_1\mathbf{u}_1+x_2\mathbf{u}_2 = x_1\mathbf{u}_1+x_2\mathbf{u}_2+0\mathbf{u}_3$, so $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ spans $\mathbf{R}^2$.

2.2 Span

1. $0\mathbf{u}_1 + 0\mathbf{u}_2 = 0\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 0\begin{bmatrix} 9 \\ 15 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $1\mathbf{u}_1 + 0\mathbf{u}_2 = 1\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 0\begin{bmatrix} 9 \\ 15 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, $0\mathbf{u}_1 + 1\mathbf{u}_2 = 0\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 1\begin{bmatrix} 9 \\ 15 \end{bmatrix} = \begin{bmatrix} 9 \\ 15 \end{bmatrix}$

2. $0\mathbf{u}_1 + 0\mathbf{u}_2 = 0\begin{bmatrix} -2 \\ 7 \end{bmatrix} + 0\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $1\mathbf{u}_1 + 0\mathbf{u}_2 = 1\begin{bmatrix} -2 \\ 7 \end{bmatrix} + 0\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$, $0\mathbf{u}_1 + 1\mathbf{u}_2 = 0\begin{bmatrix} -2 \\ 7 \end{bmatrix} + 1\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

3. $0\mathbf{u}_1 + 0\mathbf{u}_2 = 0\begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $1\mathbf{u}_1 + 0\mathbf{u}_2 = 1\begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$, $0\mathbf{u}_1 + 1\mathbf{u}_2 = 0\begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$

4. $0\mathbf{u}_1 + 0\mathbf{u}_2 + 0\mathbf{u}_3 = 0\begin{bmatrix} 0 \\ 5 \\ -2 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + 0\begin{bmatrix} -6 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $1\mathbf{u}_1 + 0\mathbf{u}_2 + 0\mathbf{u}_3 = 1\begin{bmatrix} 0 \\ 5 \\ -2 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + 0\begin{bmatrix} -6 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ -2 \end{bmatrix}$, $0\mathbf{u}_1 + 1\mathbf{u}_2 + 0\mathbf{u}_3 = 0\begin{bmatrix} 0 \\ 5 \\ -2 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + 0\begin{bmatrix} -6 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$

5. $0\mathbf{u}_1 + 0\mathbf{u}_2 + 0\mathbf{u}_3 = 0\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} + 0\begin{bmatrix} -4 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $1\mathbf{u}_1 + 0\mathbf{u}_2 + 0\mathbf{u}_3 = 1\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} + 0\begin{bmatrix} -4 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $0\mathbf{u}_1 + 1\mathbf{u}_2 + 0\mathbf{u}_3 = 0\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} + 0\begin{bmatrix} -4 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$

6. $0\mathbf{u}_1 + 0\mathbf{u}_2 + 0\mathbf{u}_3 = 0\begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 8 \\ -5 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 12 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $1\mathbf{u}_1 + 0\mathbf{u}_2 + 0\mathbf{u}_3 = 1\begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 8 \\ -5 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 12 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}$, $0\mathbf{u}_1 + 1\mathbf{u}_2 + 0\mathbf{u}_3 = 0\begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 8 \\ -5 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 12 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ -5 \\ 2 \end{bmatrix}$

7. Set $x_1\mathbf{a}_1 = \mathbf{b} \Rightarrow x_1\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 9 \\ -15 \end{bmatrix} \Rightarrow \begin{bmatrix} 3x_1 \\ 5x_1 \end{bmatrix} = \begin{bmatrix} 9 \\ -15 \end{bmatrix}$.
From the first component, $x_1 = 3$, but from the second component $x_1 = -3$. Thus $\mathbf{b}$ is not in the span of $\mathbf{a}_1$.

8. Set $x_1\mathbf{a}_1 = \mathbf{b} \Rightarrow x_1\begin{bmatrix} 10 \\ -15 \end{bmatrix} = \begin{bmatrix} -30 \\ 45 \end{bmatrix} \Rightarrow \begin{bmatrix} 10x_1 \\ -15x_1 \end{bmatrix} = \begin{bmatrix} -30 \\ 45 \end{bmatrix}$.
From the first component, $x_1 = -3$, and from the second component $x_1 = 3$. Thus $\mathbf{b} = -3\mathbf{a}_1$,and $\mathbf{b}$ is in the span of $\mathbf{a}_1$.

9. Set $x_1\mathbf{a}_1 = \mathbf{b} \Rightarrow x_1\begin{bmatrix} 4 \\ -2 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 4x_1 \\ -2x_1 \\ 10x_1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix}$.
From the first and second components, $x_1 = \frac{1}{2}$, but from the third component $x_1 = -\frac{1}{2}$. Thus $\mathbf{b}$ is not in the span of $\mathbf{a}_1$.

10. Set $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \Rightarrow x_1\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ -3 \\ 6 \end{bmatrix} = \begin{bmatrix} -6 \\ 9 \\ 2 \end{bmatrix} \Rightarrow$
$\begin{bmatrix} -x_1 - 2x_2 \\ 3x_1 - 3x_2 \\ -x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} -6 \\ 9 \\ 2 \end{bmatrix}$. We obtain 3 equations and row-reduce the associated augmented matrix

to determine if there are solutions.

$$\begin{bmatrix} -1 & -2 & -6 \\ 3 & -3 & 9 \\ -1 & 6 & 2 \end{bmatrix} \overset{\substack{3R_1+R_2\to R_2 \\ -R_1+R_3\to R_3}}{\sim} \begin{bmatrix} -1 & -2 & -6 \\ 0 & -9 & -9 \\ 0 & 8 & 8 \end{bmatrix}$$

$$\overset{(8/9)R_2+R_3\to R_3}{\sim} \begin{bmatrix} -1 & -2 & -6 \\ 0 & -9 & -9 \\ 0 & 0 & 0 \end{bmatrix}$$

From the second row, $-9x_2 = -9 \Rightarrow x_2 = 1$.From row 1, $-x_1 - 2(1) = -6 \Rightarrow x_1 = 4$. We conclude $\mathbf{b}$ is in the span of $\mathbf{a}_1$ and $\mathbf{a}_2$, with $\mathbf{b} = 4\mathbf{a}_1 + \mathbf{a}_2$.

11. Set $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \Rightarrow x_1 \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 8 \\ -7 \end{bmatrix} = \begin{bmatrix} -10 \\ -8 \\ 7 \end{bmatrix} \Rightarrow$
$\begin{bmatrix} -x_1 + 2x_2 \\ 4x_1 + 8x_2 \\ -3x_1 - 7x_2 \end{bmatrix} = \begin{bmatrix} -10 \\ -8 \\ 7 \end{bmatrix}$. We obtain 3 equations and row-reduce the associated augmented matrix to determine if there are solutions.

$$\begin{bmatrix} -1 & 2 & -10 \\ 4 & 8 & -8 \\ -3 & -7 & 7 \end{bmatrix} \overset{\substack{4R_1+R_2\to R_2 \\ -3R_1+R_3\to R_3}}{\sim} \begin{bmatrix} -1 & 2 & -10 \\ 0 & 16 & -48 \\ 0 & -13 & 37 \end{bmatrix}$$

$$\overset{(13/16)R_2+R_3\to R_3}{\sim} \begin{bmatrix} -1 & 2 & -10 \\ 0 & 16 & -48 \\ 0 & 0 & -2 \end{bmatrix}$$

From the third row, $0 = -2$,and hence there are no solutions. We conclude that there do not exist x_1 and x_2 such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$, and therefore $\mathbf{b}$ is not in the span of $\mathbf{a}_1$ and $\mathbf{a}_2$.

12. Set $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \Rightarrow x_1 \begin{bmatrix} 3 \\ 1 \\ -2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 1 \\ 5 \end{bmatrix} \Rightarrow$
$\begin{bmatrix} 3x_1 - 4x_2 \\ x_1 + 2x_2 \\ -2x_1 + 3x_2 \\ -x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 1 \\ 5 \end{bmatrix}$. We obtain 4 equations and row-reduce the associated augmented matrix to determine if there are solutions.

$$\begin{bmatrix} 3 & -4 & 0 \\ 1 & 2 & 10 \\ -2 & 3 & 1 \\ -1 & 3 & 5 \end{bmatrix} \overset{\substack{(-1/3)R_1+R_2\to R_2 \\ (2/3)R_1+R_3\Rightarrow R_3 \\ (1/3)R_1+R_4\to R_4}}{\sim} \begin{bmatrix} 3 & -4 & 0 \\ 0 & \frac{10}{3} & 10 \\ 0 & \frac{1}{3} & 1 \\ 0 & \frac{5}{3} & 5 \end{bmatrix}$$

$$\overset{\substack{(-1/10)R_2+R_4\to R_3 \\ (-1/2)R_3+R_4\to R_4}}{\sim} \begin{bmatrix} 3 & -4 & 0 \\ 0 & \frac{10}{3} & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From the second row, $\frac{10}{3}x_2 = 10 \Rightarrow x_2 = 3$.From row 1, $3x_1 - 4(3) = 0 \Rightarrow x_1 = 4$. We conclude $\mathbf{b}$ is in the span of $\mathbf{a}_1$ and $\mathbf{a}_2$, with $\mathbf{b} = 4\mathbf{a}_1 + 3\mathbf{a}_2$.

13. $A = \begin{bmatrix} 2 & 8 & -4 \\ -1 & -3 & 5 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -10 \\ 4 \end{bmatrix}$

14. $A = \begin{bmatrix} -2 & 5 & -10 \\ 1 & -2 & 3 \\ 7 & -17 & 34 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ -1 \\ -16 \end{bmatrix}$

15. $A = \begin{bmatrix} 1 & -1 & -3 & -1 \\ -2 & 2 & 6 & 2 \\ -3 & -3 & 10 & 0 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix}$

16. $A = \begin{bmatrix} -5 & 9 \\ 3 & -5 \\ 1 & -2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 13 \\ -9 \\ -2 \end{bmatrix}$

17. $x_1 \begin{bmatrix} 5 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 7 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \end{bmatrix}$

18. $x_1 \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 4 \\ -13 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

19. $x_1 \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -5 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 7 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \\ 2 \end{bmatrix}$

20. $x_1 \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -9 \\ 4 \\ -7 \end{bmatrix} = \begin{bmatrix} 11 \\ 9 \\ 2 \end{bmatrix}$

21. Row-reduce to echelon form:

$$\begin{bmatrix} 15 & -6 \\ -5 & 2 \end{bmatrix} \overset{(1/3)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 15 & -6 \\ 0 & 0 \end{bmatrix}$$

Since there is a row of zeros, there exists a vector $\mathbf{b}$ which is not in the span of the columns of A, and therefore the columns of A do not span $\mathbf{R}^2$.

22. Row-reduce to echelon form:

$$\begin{bmatrix} 4 & -12 \\ 2 & 6 \end{bmatrix} \overset{(-1/2)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 4 & -12 \\ 0 & 12 \end{bmatrix}$$

Since there is not a row of zeros, every choice of $\mathbf{b}$ is in the span of the columns of A, and therefore the columns of A span $\mathbf{R}^2$.

23. Row-reduce to echelon form:

$$\begin{bmatrix} 2 & 1 & 0 \\ 6 & -3 & -1 \end{bmatrix} \overset{-3R_1+R_2\to R_2}{\sim} \begin{bmatrix} 2 & 1 & 0 \\ 0 & -6 & -1 \end{bmatrix}$$

Since there is not a row of zeros, every choice of $\mathbf{b}$ is in the span of the columns of A, and therefore the columns of A span $\mathbf{R}^2$.

24. Row-reduce to echelon form:

$$\begin{bmatrix} 1 & 0 & 5 \\ -2 & 2 & 7 \end{bmatrix} \overset{2R_1+R_2\to R_2}{\sim} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & 17 \end{bmatrix}$$

Since there is not a row of zeros, every choice of $\mathbf{b}$ is in the span of A, and therefore the columns of A span $\mathbf{R}^2$.

25. Row-reduce to echelon form:

$$\begin{bmatrix} 3 & 1 & 0 \\ 5 & -2 & -1 \\ 4 & -4 & -3 \end{bmatrix} \underset{\substack{(-5/3)R_1+R_2\to R_2 \\ (-4/3)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 3 & 1 & 0 \\ 0 & -\frac{11}{3} & -1 \\ 0 & -\frac{16}{3} & -3 \end{bmatrix}$$

$$\underset{(-16/11)R_2+R_3\to R_3}{\sim} \begin{bmatrix} 3 & 1 & 0 \\ 0 & -\frac{11}{3} & -1 \\ 0 & 0 & -\frac{17}{11} \end{bmatrix}$$

Since there is not a row of zeros, every choice of $\mathbf{b}$ is in the span of the columns of A, and therefore the columns of A span $\mathbf{R}^3$.

26. Row-reduce to echelon form:

$$\begin{bmatrix} 1 & 2 & 8 \\ -2 & 3 & 7 \\ 3 & -1 & 1 \end{bmatrix} \underset{\substack{2R_1+R_2\to R_2 \\ -3R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 1 & 2 & 8 \\ 0 & 7 & 23 \\ 0 & -7 & -23 \end{bmatrix}$$

$$\underset{R_2+R_3\to R_3}{\sim} \begin{bmatrix} 1 & 2 & 8 \\ 0 & 7 & 23 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a row of zeros, there exists a vector $\mathbf{b}$ which is not in the span of A, and therefore the columns of A do not span $\mathbf{R}^3$.

27. Row-reduce to echelon form:

$$\begin{bmatrix} 2 & 1 & -3 & 5 \\ 1 & 4 & 2 & 6 \\ 0 & 3 & 3 & 3 \end{bmatrix} \underset{(-1/2)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 2 & 1 & -3 & 5 \\ 0 & \frac{7}{2} & \frac{7}{2} & \frac{7}{2} \\ 0 & 3 & 3 & 3 \end{bmatrix}$$

$$\underset{(-6/7)R_2+R_3\to R_3}{\sim} \begin{bmatrix} 2 & 1 & -3 & 5 \\ 0 & \frac{7}{2} & \frac{7}{2} & \frac{7}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is a row of zeros, there exists a vector $\mathbf{b}$ which is not in the span of the columns of A, and therefore the columns of A do not span $\mathbf{R}^3$.

28. Row-reduce to echelon form:

$$\begin{bmatrix} -4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 8 \\ 5 & -1 & 1 & -4 \end{bmatrix} \underset{R_2\leftrightarrow R_3}{\sim} \begin{bmatrix} -4 & -7 & 1 & 2 \\ 5 & -1 & 1 & -4 \\ 0 & 0 & 3 & 8 \end{bmatrix}$$

$$\underset{(5/4)R_1+R_2\to R_2}{\sim} \begin{bmatrix} -4 & -7 & 1 & 2 \\ 0 & -\frac{39}{4} & \frac{9}{4} & -\frac{3}{2} \\ 0 & 0 & 3 & 8 \end{bmatrix}$$

Since there is not a row of zeros, every choice of $\mathbf{b}$ is in the span of A, and therefore the columns of A span $\mathbf{R}^3$.

29. Row-reduce A to echelon form:

$$\begin{bmatrix} 3 & -4 \\ 4 & 2 \end{bmatrix} \underset{(-4/3)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 3 & -4 \\ 0 & \frac{22}{3} \end{bmatrix}$$

Since there is not a row of zeros, for every choice of $\mathbf{b}$ there is a solution of $A\mathbf{x} = \mathbf{b}$.

30. Row-reduce A to echelon form:

$$\begin{bmatrix} -9 & 21 \\ 6 & -14 \end{bmatrix} \overset{(2/3)R_1+R_2\to R_2}{\sim} \begin{bmatrix} -9 & 21 \\ 0 & 0 \end{bmatrix}$$

Since there is a row of zeros, there is a choice of $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has no solution.

31. Since the number of columns, $m = 2$, is less than $n = 3$, the columns of A do not span $\mathbf{R}^3$, and by Theorem 2.9, there is a choice of $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has no solution.

32. Row-reduce A to echelon form.

$$\begin{bmatrix} 1 & -1 & 2 \\ -2 & 3 & -1 \\ 1 & 0 & 5 \end{bmatrix} \underset{-R_1+R_3\to R_3}{\overset{2R_1+R_2\to R_2}{\sim}} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\overset{-R_2+R_3\to R_3}{\sim} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a row of zeros, there is a choice of $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has no solution.

33. Row-reduce A to echelon form:

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -1 & -1 \\ 5 & -4 & -3 \end{bmatrix} \underset{(5/3)R_1+R_3\to R_3}{\overset{(1/3)R_1+R_2\to R_2}{\sim}} \begin{bmatrix} -3 & 2 & 1 \\ 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & -\frac{2}{3} & -\frac{4}{3} \end{bmatrix}$$

$$\overset{(-2)R_2+R_3\to R_3}{\sim} \begin{bmatrix} -3 & 2 & 1 \\ 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a row of zeros, there is a choice of $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has no solution.

34. Since the number of columns, $m = 3$, is less than $n = 4$, the columns of A do not span $\mathbf{R}^4$, and by Theorem 2.11, there is a choice of $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has no solution.

35. $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not in span$\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix} \right\}$, since span$\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix} \right\} = $ span$\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$ and $\mathbf{b} \neq c \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ for any scalar c.

36. $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not in span$\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \end{bmatrix} \right\}$, since span$\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \end{bmatrix} \right\} = $ span$\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$ and $\mathbf{b} \neq c \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ for any scalar c.

37. $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in span$\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$, since $c_1 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has no solutions.

38. $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in span$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} \right\}$, since $c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has no solutions.

39. $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in span$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix} \right\}$, because span$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix} \right\} = $ span$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and $\mathbf{b} \neq c \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ for any scalar c.

40. $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in span $\left\{ \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 15 \\ -10 \end{bmatrix} \right\}$, because span $\left\{ \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 15 \\ -10 \end{bmatrix} \right\}$ $=$ span $\left\{ \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right\}$ and $\mathbf{b} \neq c \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ for any scalar c.

41. $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in span $\left\{ \begin{bmatrix} 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -5 \\ 10 \end{bmatrix}, \begin{bmatrix} 7 \\ -14 \end{bmatrix} \right\}$, because span $\left\{ \begin{bmatrix} 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -5 \\ 10 \end{bmatrix}, \begin{bmatrix} 7 \\ -14 \end{bmatrix} \right\} =$ span $\left\{ \begin{bmatrix} 2 \\ -4 \end{bmatrix} \right\}$ and $\mathbf{b} \neq c \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ for any scalar c.

42. $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in span $\left\{ \begin{bmatrix} 4 \\ 10 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -6 \\ -15 \end{bmatrix} \right\}$, because span $\left\{ \begin{bmatrix} 4 \\ 10 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -6 \\ -15 \end{bmatrix} \right\} =$ span $\left\{ \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$ and $\mathbf{b} \neq c \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ for any scalar c.

43. $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in span $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} \right\}$, because $c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has no solutions.

44. $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in span $\left\{ \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} \right\}$, because $c_1 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has no solutions.

45. $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in span $\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 7 \end{bmatrix} \right\}$, because $c_1 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 4 \\ 7 \end{bmatrix} =$ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has no solutions.

46. $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$, because $c_1 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} =$ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has no solutions.

47. $h \neq 3$,since when $h = 3$ the vectors $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ are parallel and do not span $\mathbf{R}^2$.

48. $h \neq \frac{12}{5}$, since when $h = \frac{12}{5}$ the vectors $\begin{bmatrix} -3 \\ \frac{12}{5} \end{bmatrix}$ and $\begin{bmatrix} 5 \\ -4 \end{bmatrix}$ are parallel and do not span $\mathbf{R}^2$.

49. $h \neq 4$. This value for h was determined by row-reducing

$$\begin{bmatrix} 2 & h & 1 \\ 4 & 8 & 2 \\ 5 & 10 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & h & 1 \\ 0 & 8-2h & 0 \\ 0 & 0 & \frac{7}{2} \end{bmatrix}$$

Then $c_1 \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} h \\ 8 \\ 10 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ has a solution provided $h \neq 4$.

50. $h \neq -27$. This value for h was determined by row-reducing

$$\begin{bmatrix} -1 & 4 & 1 \\ h & -2 & -3 \\ 7 & 5 & 2 \end{bmatrix} \sim \begin{bmatrix} -1 & 4 & 1 \\ 0 & 33 & 9 \\ 0 & 0 & -\frac{1}{11}h - \frac{27}{11} \end{bmatrix}$$

Then $c_1\begin{bmatrix}-1\\h\\7\end{bmatrix}+c_2\begin{bmatrix}4\\-2\\5\end{bmatrix}+c_3\begin{bmatrix}1\\-3\\2\end{bmatrix}=\begin{bmatrix}x\\y\\z\end{bmatrix}$ has a solution provided $h\neq -27$.

51. $\mathbf{u}_1=(1,0,0)$, $\mathbf{u}_2=(0,1,0)$, $\mathbf{u}_3=(0,0,1)$, $\mathbf{u}_4=(1,1,1)$

52. $\mathbf{u}_1=(1,0,0,0)$, $\mathbf{u}_2=(0,1,0,0)$, $\mathbf{u}_3=(0,0,1,0)$, $\mathbf{u}_4=(0,0,0,1)$

53. $\mathbf{u}_1=(1,0,0)$, $\mathbf{u}_2=(2,0,0)$, $\mathbf{u}_3=(3,0,0)$, $\mathbf{u}_4=(4,0,0)$

54. $\mathbf{u}_1=(1,0,0,0)$, $\mathbf{u}_2=(2,0,0,0)$, $\mathbf{u}_3=(3,0,0,0)$, $\mathbf{u}_4=(4,0,0,0)$

55. $\mathbf{u}_1=(1,0,0)$, $\mathbf{u}_2=(0,1,0)$

56. $\mathbf{u}_1=(0,1,0,0)$, $\mathbf{u}_2=(0,0,1,0)$, $\mathbf{u}_3=(0,0,0,1)$

57. $\mathbf{u}_1=(1,-1,0)$, $\mathbf{u}_2=(1,0,-1)$

58. $\mathbf{u}_1=(1,-1,0,0)$, $\mathbf{u}_2=(1,0,-1,0)$, $\mathbf{u}_3=(1,0,0,-1)$

59. (a) True, by Theorem 2.9.
    (b) False, the zero vector can be included with any set of vectors which already span $\mathbf{R}^n$.

60. (a) False, since every column of A may be a zero column.
    (b) False, by Example 5.

61. (a) False. Consider $A=[1]$.
    (b) True, by Theorem 2.11.

62. (a) True, the span of a set of vectors can only increase (with respect to set containment) when adding a vector to the set.
    (b) False. Consider $\mathbf{u}_1=(0,0,0)$, $\mathbf{u}_2=(1,0,0)$, $\mathbf{u}_3=(0,1,0)$, and $\mathbf{u}_4=(0,0,1)$.

63. (a) False. Consider $\mathbf{u}_1=(0,0,0)$, $\mathbf{u}_2=(1,0,0)$, $\mathbf{u}_3=(0,1,0)$, and $\mathbf{u}_4=(0,0,1)$.
    (b) True. The span of $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ will be a subset of the span of $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}$.

64. (a) True. span $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}\subseteq$ span $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}$ is always true. If a vector $\mathbf{w}\in$ span $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}$, then since $\mathbf{u}_4$ is a linear combination of $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$, we can express $\mathbf{w}$ as a linear combination of just the vectors $\mathbf{u}_1,\mathbf{u}_2$, and $\mathbf{u}_3$. Hence $\mathbf{w}$ is in span $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$, and we have span $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}\subseteq$ span $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$.
    (b) False. If $\mathbf{u}_4$ is a linear combination of $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ then span $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}=$ span $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$. (See problem 61, and the solutions to problems 43 and 45 for examples.)

65. (a) False. Consider $\mathbf{u}_1=(1,0,0,0)$, $\mathbf{u}_2=(0,1,0,0)$, $\mathbf{u}_3=(0,0,1,0)$, and $\mathbf{u}_4=(0,0,0,1)$.
    (b) True. Since $\mathbf{u}_4\in$ span $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}$, but $\mathbf{u}_4\notin$ span $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$.

66. (a) True, because $c_1\mathbf{0}+c_2\mathbf{u}_1+c_3\mathbf{u}_2+c_4\mathbf{u}_3=c_2\mathbf{u}_1+c_3\mathbf{u}_2+c_4\mathbf{u}_3$, span $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}=$ span $\{\mathbf{0},\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$.
    (b) False, because span $\{\mathbf{u}_1,\mathbf{u}_2\}=$ span $\{\mathbf{u}_1\}\notin\mathbf{R}^2$, and $\begin{bmatrix}1\\0\end{bmatrix}\notin$ span $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$.

67. (a) Cannot possibly span $\mathbf{R}^3$, since $m=1<n=3$.
    (b) Cannot possibly span $\mathbf{R}^3$, since $m=2<n=3$.
    (c) Can possibly span $\mathbf{R}^3$. For example, $\mathbf{u}_1=(1,0,0)$, $\mathbf{u}_2=(0,1,0)$, $\mathbf{u}_3=(0,0,1)$.
    (d) Can possibly span $\mathbf{R}^3$. For example, $\mathbf{u}_1=(1,0,0)$, $\mathbf{u}_2=(0,1,0)$, $\mathbf{u}_3=(0,0,1)$, $\mathbf{u}_4=(0,0,0)$.

68. (a) Cannot possibly span $\mathbf{R}^3$, since $m=1<n=3$.
    (b) Cannot possibly span $\mathbf{R}^3$, since $m=1<n=3$.

(c) Can possibly span $\mathbf{R}^3$. For example, $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (0, 1, 0)$, $\mathbf{u}_3 = (0, 0, 1)$.

(d) Can possibly span $\mathbf{R}^3$. For example, $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (0, 1, 0)$, $\mathbf{u}_3 = (0, 0, 1)$, $\mathbf{u}_4 = (0, 0, 0)$.

69. Let $\mathbf{w} \in \operatorname{span}\{\mathbf{u}\}$, then $\mathbf{w} = x_1\mathbf{u} = \left(\frac{x_1}{c}\right)(c\mathbf{u})$, so $\mathbf{w} \in \operatorname{span}\{c\mathbf{u}\}$ and thus $\operatorname{span}\{\mathbf{u}\} \subseteq \operatorname{span}\{c\mathbf{u}\}$. Now let $\mathbf{w} \in \operatorname{span}\{c\mathbf{u}\}$,then $\mathbf{w} = x_1(c\mathbf{u}) = (x_1 c)(\mathbf{u})$, so $\mathbf{w} \in \operatorname{span}\{\mathbf{u}\}$ and thus $\operatorname{span}\{c\mathbf{u}\} \subseteq \operatorname{span}\{\mathbf{u}\}$. Together, we conclude $\operatorname{span}\{\mathbf{u}\} = \operatorname{span}\{c\mathbf{u}\}$.

70. Let $\mathbf{w} \in \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\}$, then $\mathbf{w} = x_1\mathbf{u}_1 + x_2\mathbf{u}_2 = \left(\frac{x_1}{c_1}\right)(c_1\mathbf{u}_1) + \left(\frac{x_2}{c_2}\right)(c_2\mathbf{u}_2)$, so $\mathbf{w} \in \operatorname{span}\{c_1\mathbf{u}_1, c_2\mathbf{u}_2\}$ and thus $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\} \subseteq \operatorname{span}\{c_1\mathbf{u}_1, c_2\mathbf{u}_2\}$. Now let $\mathbf{w} \in \operatorname{span}\{c_1\mathbf{u}_1, c_2\mathbf{u}_2\}$, then $\mathbf{w} = x_1(c_1\mathbf{u}_1) + x_2(c_2\mathbf{u}_2) = (x_1c_1)(\mathbf{u}_1) + (x_2c_2)(\mathbf{u}_2)$,so $\mathbf{w} \in \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\}$and thus $\operatorname{span}\{c_1\mathbf{u}_1, c_2\mathbf{u}_2\} \subseteq \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Together, we conclude $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\} = \operatorname{span}\{c_1\mathbf{u}_1, c_2\mathbf{u}_2\}$.

71. We may let $S_1 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ and $S_2 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m, \mathbf{u}_{m+1}, \ldots \mathbf{u}_n\}$ where $m \le n$. Let $\mathbf{w} \in \operatorname{span}(S_1)$, then

$$\begin{aligned}\mathbf{w} &= x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_m\mathbf{u}_m \\ &= x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_m\mathbf{u}_m + 0\mathbf{u}_{m+1} + \cdots + 0\mathbf{u}_n\end{aligned}$$

and thus $\mathbf{w} \in \operatorname{span}(S_2)$. We conclude that $\operatorname{span}(S_1) \subseteq \operatorname{span}(S_2)$.

72. Let $\mathbf{b} \in \mathbf{R}^2$,then $\mathbf{b} = x_1\mathbf{u}_1 + x_2\mathbf{u}_2$ for some scalars x_1 and x_2 because $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\} = \mathbf{R}^2$. We can rewrite $\mathbf{b} = \frac{x_1+x_2}{2}(\mathbf{u}_1 + \mathbf{u}_2) + \frac{x_1-x_2}{2}(\mathbf{u}_1 - \mathbf{u}_2)$, thus $\mathbf{b} \in \operatorname{span}\{\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2\}$. Since $\mathbf{b}$ was arbitrary, $\operatorname{span}\{\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2\} = \mathbf{R}^2$.

73. Let $\mathbf{b} \in \mathbf{R}^3$,then $\mathbf{b} = x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3$ for some scalars x_1, x_2, and x_3 because $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \mathbf{R}^3$. We can rewrite $\mathbf{b} = \frac{x_1+x_2-x_3}{2}(\mathbf{u}_1 + \mathbf{u}_2) + \frac{x_1-x_2+x_3}{2}(\mathbf{u}_1 + \mathbf{u}_3) + \frac{-x_1+x_2+x_3}{2}(\mathbf{u}_2 + \mathbf{u}_3)$,thus $\mathbf{b} \in \operatorname{span}\{\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_3, \mathbf{u}_2 + \mathbf{u}_3\}$. Since $\mathbf{b}$ was arbitrary, $\operatorname{span}\{\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_3, \mathbf{u}_2 + \mathbf{u}_3\} = \mathbf{R}^3$.

74. If $\mathbf{b}$ is in $\operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$, then by Theorem 2.11 the linear system corresponding to the augmented matrix

$$[\mathbf{u}_1 \quad \cdots \quad \mathbf{u}_m \quad \mathbf{b}]$$

has at least one solution. Since $m > n$, this system has more variables than equations. Hence the echelon form of the system will have free variables, and since the system is consistent this implies that it has infinitely many solutions.

75. Let $A = [\mathbf{u}_1 \cdots \mathbf{u}_m]$ and suppose $A \sim B$, where B is in echelon form. Since $m < n$, the last row of B must consist of zeros. Form B_1 by appending to B the vector $\mathbf{e} = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$, so that $B_1 = [B \quad \mathbf{e}]$. If B_1 is viewed as an augmented matrix, then the bottom row corresponds to the equation $0 = 1$, so the corresponding linear system is inconsistent. Now reverse the row operations used to transform A to B, and apply these to B_1. Then the resulting matrix will have the form $[A \quad \mathbf{e}']$. This implies that $\mathbf{e}'$ is not in the span of the columns of A, as required.

76. $[(a) \Rightarrow (b)]$ Since $\mathbf{b} \in \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ there exists scalars x_1, x_2, ... , x_m such that $\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots x_m\mathbf{a}_m$, which is statement (b).
$[(b) \Rightarrow (c)]$ The linear system corresponding to $[\ \mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_m \quad \mathbf{b}\ ]$ can be expressed by the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots x_m\mathbf{a}_m = \mathbf{b}$. By (b), $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots x_m\mathbf{a}_m = \mathbf{b}$ has a solution, hence we conclude that linear system corresponding to $[\ \mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_m \quad \mathbf{b}\ ]$ has a solution.
$[(c) \Rightarrow (d)]$ $A\mathbf{x} = \mathbf{b}$ has a solution provided the augmented matrix $[\ A \quad \mathbf{b}\ ]$ has a solution. In terms of the columns of A, this is true if the augmented matrix $[\ \mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_m \quad \mathbf{b}\ ]$ has a solution. This is what (c) implies, hence $A\mathbf{x} = \mathbf{b}$ has a solution.
$[(d) \Rightarrow (a)]$ If $A\mathbf{x} = \mathbf{b}$ has a solution, then $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots x_m\mathbf{a}_m = \mathbf{b}$ where $A = [\ \mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_m\ ]$ and $\mathbf{x} = (x_1, x_2, \ldots, x_m)$. Thus $\mathbf{b} \in \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$.

77. True. Using a computer algebra system, the row-reduced echelon form of the matrix with the given vectors as columns does not have any zero rows. Hence the vectors span $\mathbf{R}^3$.

78. False. Using a computer algebra system, the row-reduced echelon form of the matrix with the given vectors as columns does have a zero row. Hence the vectors do not span $\mathbf{R}^3$.

79. False. Using a computer algebra system, the row-reduced echelon form of the matrix with the given vectors as columns does have a zero row. Hence the vectors do not span $\mathbf{R}^4$.

80. True. Using a computer algebra system, the row-reduced echelon form of the matrix with the given vectors as columns does not have any zero rows. Hence the vectors span $\mathbf{R}^4$.

2.3 Practice Problems

Section 2.3

1. (a) Consider $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 = \mathbf{0}$, and solve using the corresponding augmented matrix:

$$\begin{bmatrix} 2 & 4 & 0 \\ -3 & 1 & 0 \end{bmatrix} \underset{\sim}{\overset{(3/2)R_1+R_2\to R_2}{}} \begin{bmatrix} 2 & 4 & 0 \\ 0 & 7 & 0 \end{bmatrix}$$

The only solution is the trivial solution, so the vectors are linearly independent.

(b) Consider $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 = \mathbf{0}$, and solve using the corresponding augmented matrix:

$$\begin{bmatrix} 6 & -2 & 0 \\ 1 & 3 & 0 \\ 4 & -3 & 0 \end{bmatrix} \underset{\sim}{\overset{\substack{(-1/6)R_1+R_2\to R_2 \\ (-2/3)R_1+R_3\to R_3}}{}} \begin{bmatrix} 6 & -2 & 0 \\ 0 & \frac{10}{3} & 0 \\ 0 & -\frac{5}{3} & 0 \end{bmatrix}$$

$$\underset{\sim}{\overset{(1/2)R_2+R_3\to R_3}{}} \begin{bmatrix} 6 & -2 & 0 \\ 0 & \frac{10}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The only solution is the trivial solution, so the vectors are linearly independent.

2. (a) We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & 5 & 0 \\ 3 & -4 & 0 \end{bmatrix} \underset{\sim}{\overset{-3R_1+R_2\to R_2}{}} \begin{bmatrix} 1 & 5 & 0 \\ 0 & -19 & 0 \end{bmatrix}$$

The only solution is the trivial solution, so the columns of the matrix are linearly independent.

(b) We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & -2 & 4 & 0 \\ -3 & 7 & 2 & 0 \end{bmatrix} \underset{\sim}{\overset{\substack{-2R_1+R_2\to R_2 \\ 3R_1+R_3\to R_3}}{}} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 7 & 11 & 0 \end{bmatrix}$$

$$\underset{\sim}{\overset{(7/2)R_2+R_3\to R_3}{}} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

There is only the trivial solution; the columns of the matrix are linearly independent.

3. (a) We solve the homogeneous equation using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 8 & 4 & 0 \end{bmatrix} \underset{\sim}{\overset{-2R_2+R_3\to R_3}{}} \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Because there exist nontrivial solutions, the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

(b) We solve the homogeneous equation using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 0 \\ -2 & 2 & 1 & 0 & 0 \end{bmatrix} \underset{\substack{R_1+R_2\to R_2 \\ 2R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 2 & -1 & 2 & 0 \end{bmatrix}$$

$$\underset{2R_2+R_3\to R_3}{\sim} \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & -3 & 6 & 0 \end{bmatrix}$$

Because there exist nontrivial solutions, the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

4. (a) False, because $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is linearly independent in $\mathbf{R}^3$ but does not span $\mathbf{R}^3$.

(b) True, by the Unifying Theorem.

(c) True. Because $\mathbf{u}_1 - 4\mathbf{u}_2 = 4\mathbf{u}_2 - 4\mathbf{u}_2 = \mathbf{0}$, $\{\mathbf{u}_1, \mathbf{u}_2\}$ is linearly dependent.

(d) False. Suppose $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then the columns of A are linearly dependent, and $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ has no solutions.

2.3 Linear Independence

1. Consider $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{0}$, and solve using the corresponding augmented matrix:

$$\begin{bmatrix} 3 & -1 & 0 \\ -2 & -4 & 0 \end{bmatrix} \underset{(2/3)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 3 & -1 & 0 \\ 0 & -\frac{14}{3} & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the vectors are linearly independent.

2. Consider $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{0}$, and solve using the corresponding augmented matrix:

$$\begin{bmatrix} 6 & -4 & 0 \\ -15 & -10 & 0 \end{bmatrix} \underset{(5/2)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 6 & -4 & 0 \\ 0 & -20 & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the vectors are linearly independent.

3. Consider $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{0}$, and solve using the corresponding augmented matrix:

$$\begin{bmatrix} 7 & 5 & 0 \\ 1 & -3 & 0 \\ -13 & 2 & 0 \end{bmatrix} \underset{\substack{(-1/7)R_1+R_2\to R_2 \\ (13/7)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 7 & 5 & 0 \\ 0 & -\frac{26}{7} & 0 \\ 0 & \frac{79}{7} & 0 \end{bmatrix}$$

$$\underset{(79/26)R_2+R_3\to R_3}{\sim} \begin{bmatrix} 7 & 5 & 0 \\ 0 & -\frac{26}{7} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the vectors are linearly independent.

4. Consider $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$, and solve using the corresponding augmented matrix:

$$\begin{bmatrix} -4 & -2 & -8 & 0 \\ 0 & -1 & 2 & 0 \\ -3 & 5 & -19 & 0 \end{bmatrix} \underset{(-3/4)R_1+R_3\to R_3}{\sim} \begin{bmatrix} -4 & -2 & -8 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & \frac{13}{2} & -13 & 0 \end{bmatrix}$$

$$\underset{(13/2)R_2+R_3\to R_3}{\sim} \begin{bmatrix} -4 & -2 & -8 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there exist nontrivial solutions, the vectors are not linearly independent.

5. Consider $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$, and solve using the corresponding augmented matrix:

$$\begin{bmatrix} 3 & 0 & 2 & 0 \\ -1 & 4 & 4 & 0 \\ 2 & 1 & 7 & 0 \end{bmatrix} \underset{\sim}{\substack{(1/3)R_1+R_2\to R_2 \\ (-2/3)R_1+R_3\to R_3}} \begin{bmatrix} 3 & 0 & 2 & 0 \\ 0 & 4 & \frac{14}{3} & 0 \\ 0 & 1 & \frac{17}{3} & 0 \end{bmatrix}$$
$$\underset{\sim}{(-1/4)R_2+R_3\to R_3} \begin{bmatrix} 3 & 0 & 2 & 0 \\ 0 & 4 & \frac{14}{3} & 0 \\ 0 & 0 & \frac{9}{2} & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the vectors are linearly independent.

6. Consider $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$, and solve using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & 4 & -1 & 0 \\ 8 & -2 & 2 & 0 \\ 3 & 5 & 0 & 0 \\ 3 & -5 & 1 & 0 \end{bmatrix} \underset{\sim}{\substack{-8R_1+R_2\to R_2 \\ -3R_1+R_3\to R_3 \\ -3R_1+R_4\to R_4}} \begin{bmatrix} 1 & 4 & -1 & 0 \\ 0 & -34 & 10 & 0 \\ 0 & -7 & 3 & 0 \\ 0 & -17 & 4 & 0 \end{bmatrix}$$
$$\underset{\sim}{\substack{(-7/34)R_2+R_3\to R_3 \\ (-1/2)R_2+R_4\to R_4}} \begin{bmatrix} 1 & 4 & -1 & 0 \\ 0 & -34 & 10 & 0 \\ 0 & 0 & \frac{16}{17} & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$
$$\underset{\sim}{(17/16)R_3+R_4\to R_4} \begin{bmatrix} 1 & 4 & -1 & 0 \\ 0 & -34 & 10 & 0 \\ 0 & 0 & \frac{16}{17} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the vectors are linearly independent.

7. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 15 & -6 & 0 \\ -5 & 2 & 0 \end{bmatrix} \underset{\sim}{(2/3)R_1+R_2\to R_2} \begin{bmatrix} 15 & -6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there exist nontrivial solutions, the columns of A are not linearly independent.

8. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 4 & -12 & 0 \\ 2 & 6 & 0 \end{bmatrix} \underset{\sim}{(-1/2)R_1+R_2\to R_2} \begin{bmatrix} 4 & -12 & 0 \\ 0 & 12 & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the columns of A are linearly independent.

9. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 5 & -7 & 0 \end{bmatrix} \underset{\sim}{\substack{2R_1+R_2\to R_2 \\ -5R_1+R_3\to R_3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -7 & 0 \end{bmatrix}$$
$$\underset{\sim}{(7/2)R_2+R_3\to R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

There is only the trivial solution, the columns of A are linearly independent.

10. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ -4 & 5 & -5 & 0 \\ -1 & 2 & 1 & 0 \end{bmatrix} \underset{\substack{4R_1+R_2\to R_2 \\ R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix}$$

$$\underset{-R_2+R_3\to R_3}{\sim} \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are trivial solutions, the columns of A are linearly dependent.

11. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 3 & 1 & 0 & 0 \\ 5 & -2 & -1 & 0 \\ 4 & -4 & -3 & 0 \end{bmatrix} \underset{\substack{(-5/3)R_1+R_2\to R_2 \\ (-4/3)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & -\frac{11}{3} & -1 & 0 \\ 0 & -\frac{16}{3} & -3 & 0 \end{bmatrix}$$

$$\underset{(-16/11)R_2+R_3\to R_3}{\sim} \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & -\frac{11}{3} & -1 & 0 \\ 0 & 0 & -\frac{17}{11} & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the columns of A are linearly independent.

12. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} -4 & -7 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 5 & -1 & 1 & 0 \\ 8 & 2 & -4 & 0 \end{bmatrix} \underset{R_2\leftrightarrow R_4}{\sim} \begin{bmatrix} -4 & -7 & 1 & 0 \\ 8 & 2 & -4 & 0 \\ 5 & -1 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$\underset{\substack{(5/4)R_1+R_3\to R_3 \\ 2R_1+R_2\Rightarrow R_2}}{\sim} \begin{bmatrix} -4 & -7 & 1 & 0 \\ 0 & -12 & -2 & 0 \\ 0 & -\frac{39}{4} & \frac{9}{4} & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$\underset{(-13/16)R_2+R_3\to R_3}{\sim} \begin{bmatrix} -4 & -7 & 1 & 0 \\ 0 & -12 & -2 & 0 \\ 0 & 0 & \frac{31}{8} & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$\underset{(-24/31)R_3+R_4\to R_4}{\sim} \begin{bmatrix} -4 & -7 & 1 & 0 \\ 0 & -12 & -2 & 0 \\ 0 & 0 & \frac{31}{8} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the columns of A are linearly independent.

13. We solve the homogeneous equation using the corresponding augmented matrix:

$$\begin{bmatrix} -3 & 5 & 0 \\ 4 & 1 & 0 \end{bmatrix} \underset{(4/3)R_1+R_2\to R_2}{\sim} \begin{bmatrix} -3 & 5 & 0 \\ 0 & \frac{23}{3} & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

14. We solve the homogeneous equation using the corresponding augmented matrix:

$$\begin{bmatrix} 12 & 10 & 0 \\ 6 & 5 & 0 \end{bmatrix} \underset{(-1/2)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 12 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there exist nontrivial solutions, the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

15. We solve the homogeneous equation using the corresponding augmented matrix:

$$\begin{bmatrix} 8 & 1 & 0 \\ 0 & -1 & 0 \\ -3 & 2 & 0 \end{bmatrix} \underset{\sim}{\scriptstyle (3/8)R_1+R_3\to R_3} \begin{bmatrix} 8 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & \frac{19}{8} & 0 \end{bmatrix}$$

$$\underset{\sim}{\scriptstyle (19/8)R_2+R_3\to R_3} \begin{bmatrix} 8 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

16. We solve the homogeneous equation using the corresponding augmented matrix:

$$\begin{bmatrix} -3 & 2 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 5 & -4 & -3 & 0 \end{bmatrix} \underset{\sim}{\substack{(1/3)R_1+R_2\to R_2 \\ (5/3)R_1+R_3\to R_3}} \begin{bmatrix} -3 & 2 & 1 & 0 \\ 0 & -\frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & -\frac{2}{3} & -\frac{4}{3} & 0 \end{bmatrix}$$

$$\underset{\sim}{\scriptstyle -2R_2+R_3\to R_3} \begin{bmatrix} -3 & 2 & 1 & 0 \\ 0 & -\frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there exist nontrivial solutions, the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

17. We solve the homogeneous equation using the corresponding augmented matrix:

$$\begin{bmatrix} -1 & 3 & 1 & 0 \\ 4 & -3 & -1 & 0 \\ 3 & 0 & 5 & 0 \end{bmatrix} \underset{\sim}{\substack{4R_1+R_2\to R_2 \\ 3R_1+R_3\to R_3}} \begin{bmatrix} -1 & 3 & 1 & 0 \\ 0 & 9 & 3 & 0 \\ 0 & 9 & 8 & 0 \end{bmatrix}$$

$$\underset{\sim}{\scriptstyle -R_2+R_3\to R_3} \begin{bmatrix} -1 & 3 & 1 & 0 \\ 0 & 9 & 3 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

18. We solve the homogeneous equation using the corresponding augmented matrix:

$$\begin{bmatrix} 2 & -3 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ -5 & 3 & -9 & 0 \\ 3 & 0 & 9 & 0 \end{bmatrix} \underset{\sim}{\substack{(5/2)R_1+R_3\to R_3 \\ (-3/2)R_1+R_4\to R_4}} \begin{bmatrix} 2 & -3 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -\frac{9}{2} & -9 & 0 \\ 0 & \frac{9}{2} & 9 & 0 \end{bmatrix}$$

$$\underset{\sim}{\substack{(9/2)R_2+R_3\to R_3 \\ (-9/2)R_2+R_3\to R_3}} \begin{bmatrix} 2 & -3 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there exist nontrivial solutions, the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

19. Linearly dependent. Notice that $\mathbf{u} = 2\mathbf{v}$, so $\mathbf{u} - 2\mathbf{v} = \mathbf{0}$.

20. Linearly independent. The vectors are not scalar multiples of each other.

21. Linearly dependent. Apply Theorem 2.14.

22. Linearly independent. The vectors are not scalar multiples of each other.

23. Linearly dependent. Any collection of vectors containing the zero vector must be linearly dependent.

24. Linearly dependent. Since $\mathbf{u} = \mathbf{v}$, $\mathbf{u} - \mathbf{v} = \mathbf{0}$.

25. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 6 & 1 & 0 \\ 2 & 7 & 0 \\ -5 & 0 & 0 \end{bmatrix} \underset{\sim}{\substack{(-1/3)R_1+R_2 \to R_2 \\ (5/6)R_1+R_3 \to R_3}} \begin{bmatrix} 6 & 1 & 0 \\ 0 & \frac{20}{3} & 0 \\ 0 & \frac{5}{6} & 0 \end{bmatrix}$$
$$\underset{\sim}{(-1/8)R_2+R_3 \to R_3} \begin{bmatrix} 6 & 1 & 0 \\ 0 & \frac{20}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the columns of the matrix are linearly independent. By Theorem 2.15, none of the vectors is in the span of the other vectors.

26. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 7 & 1 & 3 & 0 \\ -1 & 6 & 0 & 0 \end{bmatrix} \underset{\sim}{\substack{(-7/2)R_1+R_2 \to R_2 \\ (1/2)R_1+R_3 \to R_3}} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -\frac{5}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{13}{2} & \frac{1}{2} & 0 \end{bmatrix}$$
$$\underset{\sim}{(13/5)R_2+R_3 \to R_3} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -\frac{5}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{4}{5} & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the columns of the matrix are linearly independent. By Theorem 2.15, none of the vectors is in the span of the other vectors.

27. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 4 & 3 & -5 & 0 \\ -1 & 5 & 7 & 0 \\ 3 & -2 & -7 & 0 \end{bmatrix} \underset{\sim}{\substack{(1/4)R_1+R_2 \to R_2 \\ (-3/4)R_1+R_3 \to R_3}} \begin{bmatrix} 4 & 3 & -5 & 0 \\ 0 & \frac{23}{4} & \frac{23}{4} & 0 \\ 0 & -\frac{17}{4} & -\frac{13}{4} & 0 \end{bmatrix}$$
$$\underset{\sim}{(17/23)R_2+R_3 \to R_3} \begin{bmatrix} 4 & 3 & -5 & 0 \\ 0 & \frac{23}{4} & \frac{23}{4} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the columns of the matrix are linearly independent. By Theorem 2.15, none of the vectors is in the span of the other vectors.

28. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 7 & 3 & 1 & 0 \\ 8 & 5 & -2 & 0 \\ 4 & 2 & 0 & 0 \end{bmatrix} \underset{\sim}{\substack{(-7)R_1+R_2 \to R_2 \\ (-8)R_1+R_3 \to R_3 \\ (-4)R_1+R_3 \to R_3}} \begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 10 & -20 & 0 \\ 0 & 13 & -26 & 0 \\ 0 & 6 & -12 & 0 \end{bmatrix}$$
$$\underset{\sim}{\substack{(-13/10)R_2+R_3 \to R_3 \\ (-3/5)R_2+R_4 \to R_4}} \begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 10 & -20 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there exist nontrivial solutions, the columns of the matrix are linearly dependent. By Theorem 2.15, one of the vectors is in the span of the other vectors.

29. We row–reduce to echelon form:

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \underset{\sim}{-(1/2)R_1+R_2 \to R_2} \begin{bmatrix} 2 & -1 \\ 0 & \frac{1}{2} \end{bmatrix}$$

Because the echelon form has a pivot in every row, by Theorem 2.9 $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$ in $\mathbf{R}^2$.

30. We row–reduce to echelon form:

$$\begin{bmatrix} 4 & 1 \\ -8 & 2 \end{bmatrix} \underset{\sim}{\overset{2R_1+R_2\to R_2}{}} \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$$

Because the echelon form has a pivot in every row, by Theorem 2.9 $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$ in $\mathbf{R}^2$.

31. We row–reduce to echelon form:

$$\begin{bmatrix} 6 & -9 \\ -4 & 6 \end{bmatrix} \underset{\sim}{\overset{(2/3)R_1+R_2\to R_2}{}} \begin{bmatrix} 6 & -9 \\ 0 & 0 \end{bmatrix}$$

Because the echelon form does not have a pivot in every row, by Theorem 2.9 $A\mathbf{x} = \mathbf{b}$ does not have a solution for all $\mathbf{b}$ in $\mathbf{R}^2$.

32. We row–reduce to echelon form:

$$\begin{bmatrix} 1 & -2 \\ 2 & 7 \end{bmatrix} \underset{\sim}{\overset{-2R_1+R_2\to R_2}{}} \begin{bmatrix} 1 & -2 \\ 0 & 11 \end{bmatrix}$$

Because the echelon form has a pivot in every row, by Theorem 2.9 $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$ in $\mathbf{R}^2$.

33. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -3 & 4 & 5 & 0 \end{bmatrix} \underset{\sim}{\overset{\substack{(-1/2)R_1+R_2\to R_2 \\ (3/2)R_1+R_3\to R_3}}{}} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & \frac{5}{2} & 5 & 0 \end{bmatrix}$$
$$\underset{\sim}{\overset{-5R_2+R_3\to R_3}{}} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there exist nontrivial solutions, the columns of the matrix are linearly dependent. By The Unifying Theorem, $A\mathbf{x} = \mathbf{b}$ does not have a unique solution for all $\mathbf{b}$ in $\mathbf{R}^3$.

34. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 3 & 4 & 7 & 0 \\ 7 & -1 & 6 & 0 \\ -2 & 0 & 2 & 0 \end{bmatrix} \underset{\sim}{\overset{\substack{(-7/3)R_1+R_2\to R_2 \\ (2/3)R_1+R_3\to R_3}}{}} \begin{bmatrix} 3 & 4 & 7 & 0 \\ 0 & -\frac{31}{3} & -\frac{31}{3} & 0 \\ 0 & \frac{8}{3} & \frac{20}{3} & 0 \end{bmatrix}$$
$$\underset{\sim}{\overset{(8/31)R_2+R_3\to R_3}{}} \begin{bmatrix} 3 & 4 & 7 & 0 \\ 0 & -\frac{31}{3} & -\frac{31}{3} & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the columns of the matrix are linearly independent. By The Unifying Theorem, $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$ in $\mathbf{R}^3$.

35. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 3 & -2 & 1 & 0 \\ -4 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \end{bmatrix} \underset{\sim}{\overset{\substack{(4/3)R_1+R_2\to R_2 \\ (5/3)R_1+R_3\to R_3}}{}} \begin{bmatrix} 3 & -2 & 1 & 0 \\ 0 & -\frac{5}{3} & \frac{4}{3} & 0 \\ 0 & -\frac{10}{3} & \frac{8}{3} & 0 \end{bmatrix}$$
$$\underset{\sim}{\overset{-2R_2+R_3\to R_3}{}} \begin{bmatrix} 3 & -2 & 1 & 0 \\ 0 & -\frac{5}{3} & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there exist nontrivial solutions, the columns of the matrix are linearly dependent. By The Unifying Theorem, $A\mathbf{x} = \mathbf{b}$ does not have a unique solution for all $\mathbf{b}$ in $\mathbf{R}^3$.

36. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & -3 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 4 & 7 & 0 \end{bmatrix} \underset{\sim}{-2R_1+R_3 \to R_3} \begin{bmatrix} 1 & -3 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 10 & 11 & 0 \end{bmatrix}$$
$$\underset{\sim}{-10R_2+R_3 \to R_3} \begin{bmatrix} 1 & -3 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the columns of the matrix are linearly independent. By The Unifying Theorem, $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$ in $\mathbf{R}^3$.

37. $\mathbf{u} = (1,0,0,0)$, $\mathbf{v} = (0,1,0,0)$, $\mathbf{w} = (1,1,0,0)$

38. $\mathbf{u} = (1,0,0,0,0)$, $\mathbf{v} = (0,1,0,0,0)$, $\mathbf{w} = (0,0,1,0,0)$

39. $\mathbf{u} = (1,0)$, $\mathbf{v} = (2,0)$, $\mathbf{w} = (3,0)$

40. $\mathbf{u} = (1,0)$, $\mathbf{v} = (0,1)$, $\mathbf{w} = (1,1)$

41. $\mathbf{u} = (1,0,0)$, $\mathbf{v} = (0,1,0)$, $\mathbf{w} = (1,1,0)$

42. $\mathbf{u} = (1,0,0)$, $\mathbf{v} = (0,1,0)$, $\mathbf{w} = (0,0,1), \mathbf{x} = (0,0,0)$. The collection is linearly dependent, and $\mathbf{x}$ is a *trivial* linear combination of the other vectors, so Theorem 2.15 is not violated.

43. (a) False. For example, $\mathbf{u} = (1,0)$ and $\mathbf{v} = (2,0)$ are linearly dependent but do not span $\mathbf{R}^2$.

    (b) False. For example, $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ spans $\mathbf{R}^2$, but is not linearly independent.

44. (a) True, by Theorem 2.14.

    (b) False. For example, $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}$ does not span $\mathbf{R}^2$.

45. (a) False. For example, $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and has a pivot in every row, but the columns of A are not linearly independent.

    (b) True. If every column has a pivot, then $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, and therefore the columns of A are linearly independent.

46. (a) False. If $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$, then $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, but the columns of A are linearly dependent.

    (b) False. For example, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has linearly dependent columns, and the columns of A do not span $\mathbf{R}^2$.

47. (a) False. For example, $A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 0 & 0 \end{bmatrix}$ has more rows than columns but the columns are linearly dependent.

    (b) False. For example, $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ has more columns than rows, but the columns are linearly dependent. (Theorem 2.14 can also be applied here to show that no matrix with more columns than rows can have linearly independent columns.)

48. (a) False. $A\mathbf{x} = \mathbf{0}$ corresponds to $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$, and by linear independence, each $x_i = 0$.
    (b) False. For example, if $A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $A\mathbf{x} = \mathbf{b}$ has no solution.

49. (a) False. Consider for example $\mathbf{u}_4 = \mathbf{0}$.
    (b) True. If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly dependent, then $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 = \mathbf{0}$ with at least one of the $x_i \neq 0$. Since $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 = \mathbf{0} \Rightarrow x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 + 0\mathbf{u}_4 = \mathbf{0}$, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly dependent.

50. (a) True. Consider $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 = \mathbf{0}$. If one of the $x_i \neq 0$, then $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 + 0\mathbf{u}_4 = \mathbf{0}$ would imply that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly dependent, a contradiction. Hence each $x_i = 0$, and $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent.
    (b) False. Consider $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (0, 1, 0)$, $\mathbf{u}_3 = (0, 0, 1), \mathbf{u}_4 = (0, 0, 0)$.

51. (a) False. If $\mathbf{u}_4 = x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3$, then $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 - \mathbf{u}_4 = \mathbf{0}$, and since the coefficient of $\mathbf{u}_4$ is -1, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly dependent.
    (b) True. If $\mathbf{u}_4 = x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3$, then $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 - \mathbf{u}_4 = \mathbf{0}$, and since the coefficient of $\mathbf{u}_4$ is -1, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly dependent.

52. (a) False. Consider $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (1, 0, 0)$, $\mathbf{u}_3 = (1, 0, 0), \mathbf{u}_4 = (0, 1, 0)$.
    (b) False. Consider $\mathbf{u}_1 = (1, 0, 0, 0)$, $\mathbf{u}_2 = (0, 1, 0, 0)$, $\mathbf{u}_3 = (0, 0, 1, 0), \mathbf{u}_4 = (0, 0, 0, 1)$.

53. (a), (b), and (c). For example, consider $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (1, 0, 0)$, and $\mathbf{u}_3 = (1, 0, 0)$. (d) cannot be linearly independent, by Theorem 2.14.

54. Only (c), since to span $\mathbf{R}^3$ we need at least 3 vectors, and to be linearly independent in $\mathbf{R}^3$ we can have at most 3 vectors.

55. Consider $x_1(c_1\mathbf{u}_1) + x_2(c_2\mathbf{u}_2) + x_3(c_3\mathbf{u}_3) = \mathbf{0}$. Then $(x_1c_1)\,\mathbf{u}_1 + (x_2c_2)\,\mathbf{u}_2 + (x_3c_3)\,\mathbf{u}_3 = \mathbf{0}$, and since $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent, $x_1c_1 = 0$, $x_2c_2 = 0$, and $x_3c_3 = 0$. Since each $c_i \neq 0$, we must have each $x_i = 0$. Hence, $\{c_1\mathbf{u}_1, c_2\mathbf{u}_2, c_3\mathbf{u}_3\}$ is linearly independent.

56. Consider $x_1(\mathbf{u}+\mathbf{v}) + x_2(\mathbf{u}-\mathbf{v}) = \mathbf{0}$. This implies $(x_1 + x_2)\mathbf{u} + (x_1 - x_2)\mathbf{v} = \mathbf{0}$. Since $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent, $x_1 + x_2 = 0$ and $x_1 - x_2 = 0$. Solving this system, we obtain $x_1 = 0$ and $x_2 = 0$. Thus $\{\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}\}$ is linearly independent.

57. Consider $x_1(\mathbf{u}_1 + \mathbf{u}_2) + x_2(\mathbf{u}_1 + \mathbf{u}_3) + x_3\,(\mathbf{u}_2 + \mathbf{u}_3) = \mathbf{0}$. This implies $(x_1 + x_2)\mathbf{u}_1 + (x_1 + x_3)\mathbf{u}_2 + (x_2 + x_3)\,\mathbf{u}_3 = \mathbf{0}$. Since $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent, $x_1 + x_2 = 0$, $x_1 + x_3 = 0$, and $x_2 + x_3 = 0$. Solving this system, we obtain $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$. Thus $\{\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_3, \mathbf{u}_2 + \mathbf{u}_3\}$ is linearly independent.

58. We can, by re-indexing, consider the non-empty subset as $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ where $1 \leq n \leq m$. Let $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_n\mathbf{u}_n = \mathbf{0}$, then $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_n\mathbf{u}_n + 0\mathbf{u}_{n+1} + \cdots + 0\mathbf{u}_m = \mathbf{0}$. Since $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n, \mathbf{u}_{n+1}, \ldots, \mathbf{u}_m\}$ is linearly independent, every $x_i = 0$, $1 \leq i \leq n$. Therefore, $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is linearly independent.

59. Suppose $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is linearly dependent set, and we add vectors to form a new set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n, \ldots \mathbf{u}_m\}$. There exist x_i with a least one $x_i \neq 0$ such that $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_n\mathbf{u}_n = \mathbf{0}$. Thus $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_n\mathbf{u}_n + 0\mathbf{u}_{n+1} + \cdots + 0\mathbf{u}_m = \mathbf{0}$, and so $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n, \ldots \mathbf{u}_m\}$ is linearly dependent.

60. Since $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent, there exists scalars x_1, x_2, x_3 such that $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$, and at least one $x_i \neq 0$. If $x_3 = 0$, then $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{0}$ with either x_1 or x_2 nonzero, contradicting $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent. Hence $x_3 \neq 0$, and we may write then $\mathbf{w} = (-x_1/x_3)\mathbf{u} + (-x_2/x_3)\mathbf{v}$, and therefore $\mathbf{w}$ is in the span of $\{\mathbf{u}, \mathbf{v}\}$.

61. $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent if and only if there exist scalars x_1 and x_2, not both zero, such that $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{0}$. If $x_1 \neq 0$, then $\mathbf{u} = (-x_2/x_1)\,\mathbf{v} = c\mathbf{v}$. If $x_2 \neq 0$, then $\mathbf{v} = (-x_1/x_2)\,\mathbf{u} = c\mathbf{u}$.

62. Let $\mathbf{u}_i$ be the vector in the i^{th} nonzero row of A. Suppose the pivot in row i occurs in column k_i. Let r be the number of pivots, and consider $x_1\mathbf{u}_1 + \cdots x_r\mathbf{u}_r = \mathbf{0}$. Since A is in echelon form, the k_1 component of $\mathbf{u}_i$ for $i \geq 2$ must be 0. Hence when we equate the k_1 component of $x_1\mathbf{u}_1 + \cdots x_r\mathbf{u}_r = \mathbf{0}$ we obtain $x_1 = 0$. Applying the same argument to the k_2 component now with the equation $x_2\mathbf{u}_2 + \cdots x_r\mathbf{u}_r = \mathbf{0}$ we conclude that $x_2 = 0$. Continuing in this way we see that $x_i = 0$ for all i, and hence the nonzero rows of A are linearly independent.

63. Suppose $A = [\ \mathbf{a}_1 \quad \mathbf{a}_2 \quad \ldots \quad \mathbf{a}_m\ ]$, $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_m)$. Then we have $\mathbf{x} - \mathbf{y} = (x_1 - y_1, x_2 - y_2, \ldots, x_m - y_m)$, and thus

$$\begin{aligned} A(\mathbf{x} - \mathbf{y}) &= (x_1 - y_1)\,\mathbf{a}_1 + (x_2 - y_2)\,\mathbf{a}_2 + \cdots + (x_m - y_m)\,\mathbf{a}_m \\ &= (x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_m\mathbf{a}_m) - (y_1\mathbf{a}_1 + y_2\mathbf{a}_2 + \cdots + y_m\mathbf{a}_m) \\ &= A\mathbf{x} - A\mathbf{y} \end{aligned}$$

64. Since $\mathbf{u}_1 \neq \mathbf{0}$ and $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ is linearly dependent, there exists a smallest index r such that $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$ is linearly independent but $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r, \mathbf{u}_{r+1}\}$ is linearly dependent. Consider $x_1\mathbf{u}_1 + \cdots + x_r\mathbf{u}_r + x_{r+1}\mathbf{u}_{r+1} = \mathbf{0}$. Since $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r, \mathbf{u}_{r+1}\}$ is linearly dependent, at least one of the $x_i \neq 0$. If $x_{r+1} = 0$, then $x_1\mathbf{u}_1 + \cdots + x_r\mathbf{u}_r = \mathbf{0}$, which implies that $x_i = 0$ for all $i \leq r$ since $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$ is linearly independent. But this contradicts that some $x_i \neq 0$, and so we must have $x_{r+1} \neq 0$. Thus we may write $\mathbf{u}_{r+1} = (-x_1/x_{r+1})\,\mathbf{u}_1 + \cdots + (-x_r/x_{r+1})\,\mathbf{u}_r$. We select those subscripts i with $x_i \neq 0$ (there must be at least one, otherwise $\mathbf{u}_{r+1} = \mathbf{0}$, a contradiction), and rewrite $\mathbf{u}_{r+1} = (-x_{k_1}/x_{r+1})\,\mathbf{u}_{k_1} + \cdots + \left(-x_{k_p}/x_{r+1}\right)\mathbf{u}_{k_p}$. We now have a vector $\mathbf{u}_{r+1}$ written as a linear combination of a subset of the remaining vectors, with nonzero coefficients. Since $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$ is linearly independent, this subset of vectors $\left\{\mathbf{u}_{k_1}, \mathbf{u}_{k_2}, \ldots, \mathbf{u}_{k_p}\right\}$ is also linearly independent (see exercise 56). Finally, these coefficients are unique, since if $(-x_{k_1}/x_{r+1})\,\mathbf{u}_{k_1} + \cdots + \left(-x_{k_p}/x_{r+1}\right)\mathbf{u}_{k_p} = y_1\mathbf{u}_{k_1} + \cdots + y_p\mathbf{u}_{k_p}$, then $(y_1 - x_{k_1}/x_{r+1})\,\mathbf{u}_{k_1} + \cdots + \left(y_p - x_{k_p}/x_{r+1}\right)\mathbf{u}_{k_p} = \mathbf{0}$, and by linear independence of $\left\{\mathbf{u}_{k_1}, \mathbf{u}_{k_2}, \ldots, \mathbf{u}_{k_p}\right\}$, each $y_i - x_{k_i}/x_{r+1} = 0$, and thus $y_i = x_{k_i}/x_{r+1}$.

65. Using a computer algebra system, the vectors are linearly independent.

66. Using a computer algebra system, the vectors are linearly dependent.

67. Using a computer algebra system, the vectors are linearly independent.

68. Using a computer algebra system, the vectors are linearly dependent.

69. We row–reduce to using computer software to obtain

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ -5 & 3 & 1 & 2 \\ -1 & 2 & -2 & 1 \\ 1 & -2 & 0 & -3 \\ 3 & 1 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, because $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, we conclude that the vectors are linearly dependent.

70. We row–reduce to using computer software to obtain

$$\begin{bmatrix} 4 & 2 & -3 & 0 \\ 2 & 3 & 2 & 2 \\ -1 & 1 & 1 & -1 \\ 5 & -1 & 1 & 3 \\ 2 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, because $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, we conclude that the vectors are linearly independent.

71. Using a computer algebra system, $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$ in $\mathbf{R}^3$.

72. Using a computer algebra system, $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$ in $\mathbf{R}^3$.

73. Using a computer algebra system, $A\mathbf{x} = \mathbf{b}$ does not have a unique solution for all $\mathbf{b}$ in $\mathbf{R}^4$.

74. Using a computer algebra system, $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$ in $\mathbf{R}^4$.

Chapter 2 Supplementary Exercises

1. $\mathbf{u}+\mathbf{v} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$;

   $3\mathbf{w} = 3\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ -15 \\ 21 \end{bmatrix}$

2. $\mathbf{v}-\mathbf{w} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}$;

   $-4\mathbf{u} = -4\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 12 \\ -8 \end{bmatrix}$

3. $2\mathbf{w}+3\mathbf{v} = 2\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} + 3\begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 17 \end{bmatrix}$;

   $2\mathbf{u}-5\mathbf{w} = 2\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} - 5\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 \\ 19 \\ -31 \end{bmatrix}$

4. $3\mathbf{v}+2\mathbf{u} = 3\begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \\ 7 \end{bmatrix}$;

   $-2\mathbf{u}+4\mathbf{w} = -2\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ -14 \\ 24 \end{bmatrix}$

5. $2\mathbf{u}+\mathbf{v}+3\mathbf{w} = 2\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ -17 \\ 26 \end{bmatrix}$;

   $\mathbf{u}-3\mathbf{v}+2\mathbf{w} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} - 3\begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} 9 \\ -25 \\ 13 \end{bmatrix}$

6. $\mathbf{u}-2\mathbf{v}+4\mathbf{w} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} 9 \\ -31 \\ 28 \end{bmatrix}$;

   $-3\mathbf{u}+\mathbf{v}-2\mathbf{w} = -3\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} -7 \\ 23 \\ -19 \end{bmatrix}$

7. $$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & 1 \\ -3x_1 & + & 4x_2 & = & -5 \\ 2x_1 & + & x_2 & = & 7 \end{array}$$

8. $$\begin{array}{rcrcr} x_1 & + & x_2 & = & 4 \\ -5x_1 & - & 3x_2 & = & -8 \\ 7x_1 & + & 2x_2 & = & -2 \end{array}$$

9. $0\begin{bmatrix}1\\-3\\2\end{bmatrix}+0\begin{bmatrix}-2\\4\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$; $1\begin{bmatrix}1\\-3\\2\end{bmatrix}+0\begin{bmatrix}-2\\4\\1\end{bmatrix}=\begin{bmatrix}1\\-3\\2\end{bmatrix}$;
$0\begin{bmatrix}1\\-3\\2\end{bmatrix}+1\begin{bmatrix}-2\\4\\1\end{bmatrix}=\begin{bmatrix}-2\\4\\1\end{bmatrix}$

10. $0\begin{bmatrix}1\\-5\\7\end{bmatrix}+0\begin{bmatrix}-2\\4\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$; $1\begin{bmatrix}1\\-5\\7\end{bmatrix}+0\begin{bmatrix}-2\\4\\1\end{bmatrix}=\begin{bmatrix}1\\-5\\7\end{bmatrix}$;
$0\begin{bmatrix}1\\-5\\7\end{bmatrix}+1\begin{bmatrix}-2\\4\\1\end{bmatrix}=\begin{bmatrix}-2\\4\\1\end{bmatrix}$

11. $x_1\mathbf{u}+x_2\mathbf{v}=\mathbf{w} \quad\Leftrightarrow\quad x_1\begin{bmatrix}1\\-3\\2\end{bmatrix}+x_2\begin{bmatrix}-2\\4\\1\end{bmatrix}=\begin{bmatrix}1\\-5\\7\end{bmatrix} \quad\Leftrightarrow\quad \begin{bmatrix}x_1-2x_2\\-3x_1+4x_2\\2x_1+x_2\end{bmatrix}$
$=\begin{bmatrix}1\\-5\\7\end{bmatrix}$ $\Leftrightarrow$ the augmented matrix $\begin{bmatrix}1&-2&1\\-3&4&-5\\2&1&7\end{bmatrix}$ has a solution:

$$\begin{bmatrix}1&-2&1\\-3&4&-5\\2&1&7\end{bmatrix}\underset{\sim}{\overset{3R_1+R_2\to R_2}{-2R_1+R_3\to R_3}}\begin{bmatrix}1&-2&1\\0&-2&-2\\0&5&5\end{bmatrix}\underset{\sim}{(5/2)R_2+R_3\to R_3}\begin{bmatrix}1&-2&1\\0&-2&-2\\0&0&0\end{bmatrix}$$

Because a solution exists, $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

12. $x_1\mathbf{w}+x_2\mathbf{u}=\mathbf{v} \quad\Leftrightarrow\quad x_1\begin{bmatrix}1\\-5\\7\end{bmatrix}+x_2\begin{bmatrix}1\\-3\\2\end{bmatrix}=\begin{bmatrix}-2\\4\\1\end{bmatrix} \quad\Leftrightarrow$
$\begin{bmatrix}x_1+x_2\\-5x_1-3x_2\\7x_1+2x_2\end{bmatrix}=\begin{bmatrix}-2\\4\\1\end{bmatrix}$ $\Leftrightarrow$ the augmented matrix $\begin{bmatrix}1&1&-2\\-5&-3&4\\7&2&1\end{bmatrix}$ has a solution:

$$\begin{bmatrix}1&1&-2\\-5&-3&4\\7&2&1\end{bmatrix}\underset{\sim}{\overset{5R_1+R_2\to R_2}{-7R_1+R_3\to R_3}}\begin{bmatrix}1&1&-2\\0&2&-6\\0&-5&15\end{bmatrix}\underset{\sim}{(5/2)R_2+R_3\to R_3}\begin{bmatrix}1&1&-2\\0&2&-6\\0&0&0\end{bmatrix}$$

Because a solution exists, $\mathbf{v}$ is a linear combination of $\mathbf{w}$ and $\mathbf{u}$.

13. Because $\mathbf{w}$ is in the span of $\mathbf{u}$ and $\mathbf{v}$, by Exercise 11, $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is linearly dependent.

14. Because $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is linearly dependent, by Exercise 13, span $\{\mathbf{u},\mathbf{v},\mathbf{w}\}\neq\mathbf{R}^3$.

15. $x_1\begin{bmatrix}4\\1\end{bmatrix}+x_2\begin{bmatrix}13\\-7\end{bmatrix}+x_3\begin{bmatrix}-1\\4\end{bmatrix}=\begin{bmatrix}-7\\12\end{bmatrix}$

16. $x_1\begin{bmatrix}3\\-1\\-3\end{bmatrix}+x_2\begin{bmatrix}-2\\5\\0\end{bmatrix}+x_3\begin{bmatrix}-1\\0\\10\end{bmatrix}+x_4\begin{bmatrix}2\\1\\-3\end{bmatrix}=\begin{bmatrix}0\\-7\\2\end{bmatrix}$

17. $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-1\\0\\0\end{bmatrix}+s_1\begin{bmatrix}2\\3\\1\end{bmatrix}$

18. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} -7 \\ 0 \\ 1 \end{bmatrix}$

19. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} -5 \\ 0 \\ 8 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

20. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} 0 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

21. $2\begin{bmatrix} -3 \\ a \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} b \\ 4 \end{bmatrix} = \begin{bmatrix} -2b-7 \\ 2a-8 \end{bmatrix}$, so we have the equations $-2b-7=-2$, and $2a-8=5$. We solve these and obtain $a = \frac{13}{2}$ and $b = -\frac{5}{2}$.

22. $-\begin{bmatrix} a \\ 1 \\ -2 \end{bmatrix} + 3\begin{bmatrix} 3 \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} 9-a \\ 3b-1 \\ 2 \end{bmatrix}$, so we have the equations $9-a=1$, $3b-1=-4$, and $2=c$. We solve these and obtain $a=8$, $b=-1$, and $c=2$.

23. $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \quad \Leftrightarrow \quad x_1\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -11 \\ 10 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} x_1+2x_2 \\ -2x_1+3x_2 \\ 4x_1-x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -11 \\ 10 \end{bmatrix} \Leftrightarrow$ the augmented matrix $\begin{bmatrix} 1 & 2 & -1 \\ -2 & 3 & -11 \\ 4 & -1 & 10 \end{bmatrix}$ yields a solution.

$$\begin{bmatrix} 1 & 2 & -1 \\ -2 & 3 & -11 \\ 4 & -1 & 10 \end{bmatrix} \overset{\substack{2R_1+R_2\to R_2 \\ -4R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 7 & -13 \\ 0 & -9 & 14 \end{bmatrix}$$

$$\overset{(9/7)R_2+R_3\to R_3}{\sim} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 7 & -13 \\ 0 & 0 & -\frac{19}{7} \end{bmatrix}$$

From the third row, we have $0 = -\frac{19}{7}$, and hence the system does not have a solution. Hence $\mathbf{b}$ is not a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

24. $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b} \quad \Leftrightarrow \quad x_1\begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} -2 \\ 0 \\ 3 \\ -1 \end{bmatrix} =$
$\begin{bmatrix} -2 \\ -4 \\ 5 \\ 3 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} x_1-2x_3 \\ -3x_1+2x_2 \\ -3x_2+3x_3 \\ 2x_1+x_2-x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 5 \\ 3 \end{bmatrix} \Leftrightarrow$ the augmented matrix $\begin{bmatrix} 1 & 0 & -2 & -2 \\ -3 & 2 & 0 & -4 \\ 0 & -1 & 3 & 5 \\ 2 & 1 & -1 & 3 \end{bmatrix}$ yields a solution.

$$\begin{bmatrix} 1 & 0 & -2 & -2 \\ -3 & 2 & 0 & -4 \\ 0 & -1 & 3 & 5 \\ 2 & 1 & -1 & 3 \end{bmatrix} \overset{\substack{3R_1+R_2\to R_2 \\ -2R_1+R_4\to R_4}}{\sim} \begin{bmatrix} 1 & 0 & -2 & -2 \\ 0 & 2 & -6 & -10 \\ 0 & -1 & 3 & 5 \\ 0 & 1 & 3 & 7 \end{bmatrix}$$

$$\overset{\substack{(1/2)R_2+R_3\to R_3 \\ -(1/2)R_2+R_4\to R_4}}{\sim} \begin{bmatrix} 1 & 0 & -2 & -2 \\ 0 & 2 & -6 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 12 \end{bmatrix}$$

From row 4, $6x_3 = 12 \Rightarrow x_3 = 2$.From row 2, $2x_2 - 6(2) = -10 \Rightarrow x_2 = 1$. From row 1, $x_1 - 2(2) = -2 \Rightarrow x_1 = 2$. We conclude $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ with $\mathbf{b} = 2\mathbf{a}_1 + \mathbf{a}_2 + 2\mathbf{a}_3$.

25. $A = \begin{bmatrix} 2 & 3 & -8 & 1 \\ 6 & -1 & 4 & -2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$

26. $A = \begin{bmatrix} 3 & -1 & -7 \\ -4 & 5 & 0 \\ -8 & 2 & 6 \\ 1 & 3 & 9 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 2 \\ -4 \\ 3 \\ 7 \end{bmatrix}$

27. Set $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \Rightarrow x_1 \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 7 \end{bmatrix} \quad \Rightarrow$

$\begin{bmatrix} 3x_1 + x_2 \\ -x_1 + 4x_2 \\ -2x_1 + 5x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 7 \end{bmatrix}$.We obtain 3 equations and row-reduce the associated augmented matrix to determine if there are solutions.

$$\begin{bmatrix} 3 & 1 & -1 \\ -1 & 4 & 5 \\ -2 & 5 & 7 \end{bmatrix} \underset{\substack{(1/3)R_1+R_2\to R_2 \\ (2/3)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 3 & 1 & -1 \\ 0 & \frac{13}{3} & \frac{14}{3} \\ 0 & \frac{17}{3} & \frac{19}{3} \end{bmatrix}$$

$$\underset{-(17/13)R_2+R_3\to R_3}{\sim} \begin{bmatrix} 3 & 1 & -1 \\ 0 & \frac{13}{3} & \frac{14}{3} \\ 0 & 0 & \frac{3}{13} \end{bmatrix}$$

From the third row, $0 = \frac{3}{13}$,and hence there are no solutions. We conclude that there do not exist x_1 and x_2 such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$, and therefore $\mathbf{b}$ is not in the span of $\mathbf{a}_1$ and $\mathbf{a}_2$.

28. Set $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b} \Rightarrow x_1 \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ -7 \\ 1 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} x_1 - x_2 + 2x_3 \\ 3x_1 + 2x_2 + 2x_3 \\ x_1 + 3x_2 \\ 4x_2 - x_3 \end{bmatrix} =$

$\begin{bmatrix} -3 \\ 4 \\ -7 \\ 1 \end{bmatrix}$. We obtain 4 equations and row-reduce the associated augmented matrix to determine if there are solutions.

$$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 3 & 2 & 2 & 4 \\ 1 & 3 & 0 & -7 \\ 0 & 4 & -1 & 1 \end{bmatrix} \underset{\substack{-3R_1+R_2\to R_2 \\ -R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -4 & 13 \\ 0 & 4 & -2 & -4 \\ 0 & 4 & -1 & 1 \end{bmatrix}$$

$$\underset{\substack{(-4/5)R_2+R_4\to R_3 \\ (-4/5)R_3+R_4\to R_4}}{\sim} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -4 & 13 \\ 0 & 0 & \frac{6}{5} & -\frac{72}{5} \\ 0 & 0 & \frac{11}{5} & -\frac{47}{5} \end{bmatrix}$$

$$\underset{(-11/6)R_3+R_4\to R_4}{\sim} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -4 & 13 \\ 0 & 0 & \frac{6}{5} & -\frac{72}{5} \\ 0 & 0 & 0 & 17 \end{bmatrix}$$

From the third row, $0 = 17$,and hence there are no solutions. We conclude that there do not exist x_1, x_2, and x_3 such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$, and therefore $\mathbf{b}$ is not in the span of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

29. $\{\mathbf{a}_1\}$ does not span $\mathbf{R}^2$, by Theorem 2.9, because $m = 1 < 2 = n$.

30. Row-reduce to echelon form:

$$\begin{bmatrix} 6 & -2 \\ -9 & 3 \end{bmatrix} \overset{(3/2)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 6 & -2 \\ 0 & 0 \end{bmatrix}$$

Because there is a row of zeros, there exists a vector $\mathbf{b}$ which is not in the span of the columns of the matrix, and therefore $\{\mathbf{a}_1, \mathbf{a}_2\}$ does not span $\mathbf{R}^2$.

31. Row-reduce to echelon form:

$$\begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix} \overset{-2R_1+R_2\to R_2}{\sim} \begin{bmatrix} 1 & -3 \\ 0 & 11 \end{bmatrix}$$

Because there is not a row of zeros, every choice of $\mathbf{b}$ is in the span of the columns of the given matrix, and therefore $\{\mathbf{a}_1, \mathbf{a}_2\}$ spans $\mathbf{R}^2$.

32. Row-reduce to echelon form:

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & -3 & 4 \end{bmatrix} \overset{-3R_1+R_2\to R_2}{\sim} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

Because there is not a row of zeros, every choice of $\mathbf{b}$ is in the span of the columns of the given matrix, and therefore $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ spans $\mathbf{R}^2$.

33. $\{\mathbf{a}_1\}$ does not span $\mathbf{R}^3$, by Theorem 2.9, because $m = 1 < 3 = n$.

34. $\{\mathbf{a}_1, \mathbf{a}_2\}$ does not span $\mathbf{R}^3$, by Theorem 2.9, because $m = 2 < 3 = n$.

35. Row-reduce to echelon form:

$$\begin{bmatrix} 1 & -3 & 4 \\ 2 & -5 & 6 \\ 5 & 4 & 11 \end{bmatrix} \overset{\substack{-2R_1+R_2\to R_2 \\ -5R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & -2 \\ 0 & 19 & -9 \end{bmatrix} \overset{-19R_2+R_3\to R_3}{\sim} \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 29 \end{bmatrix}$$

Because there is not a row of zeros, every choice of $\mathbf{b}$ is in the span of the columns of the given matrix, and therefore $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ spans $\mathbf{R}^3$.

36. Row-reduce to echelon form:

$$\begin{bmatrix} 1 & -1 & 1 & -2 \\ -3 & 2 & -5 & 2 \\ 1 & -2 & -1 & -6 \end{bmatrix} \overset{\substack{3R_1+R_2\to R_2 \\ -R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & -1 & -2 & -4 \\ 0 & -1 & -2 & -4 \end{bmatrix} \overset{-R_2+R_3\to R_3}{\sim} \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & -1 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is a row of zeros, there exists a vector $\mathbf{b}$ which is not in the span of the columns of the matrix, and therefore $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ does not span $\mathbf{R}^3$.

37. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & -2 & 0 \\ -5 & 9 & 0 \end{bmatrix} \overset{5R_1+R_2\to R_2}{\sim} \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

Because the only solution is the trivial solution, the set of column vectors, $\{\mathbf{a}_1, \mathbf{a}_2\}$, is linearly independent.

38. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 9 & -6 & 0 \\ -6 & 4 & 0 \end{bmatrix} \overset{(2/3)R_1+R_2 \to R_2}{\sim} \begin{bmatrix} 9 & -6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Because there exist nontrivial solutions, the set of column vectors, $\{\mathbf{a}_1, \mathbf{a}_2\}$, is not linearly independent.

39. By Theorem 2.14, because $m = 3 > 2 = n$, the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is not linearly independent.

40. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & -2 & 0 \\ 6 & 3 & 0 \\ -2 & 0 & 0 \end{bmatrix} \overset{\substack{-6R_1+R_2 \to R_2 \\ 2R_1+R_3 \to R_3}}{\sim} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 15 & 0 \\ 0 & -4 & 0 \end{bmatrix}$$

$$\overset{(4/15)R_2+R_3 \to R_3}{\sim} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Because the only solution is the trivial solution, the set of column vectors, $\{\mathbf{a}_1, \mathbf{a}_2\}$, is linearly independent.

41. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & -2 & 0 \\ 4 & -8 & 0 \\ -5 & 10 & 0 \end{bmatrix} \overset{\substack{-4R_1+R_2 \to R_2 \\ 5R_1+R_3 \to R_3}}{\sim} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Because there exist nontrivial solutions, the set of column vectors, $\{\mathbf{a}_1, \mathbf{a}_2\}$, is not linearly independent.

42. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & -2 & 2 & 0 \\ -1 & 3 & -5 & 0 \\ 3 & 4 & 9 & 0 \end{bmatrix} \overset{\substack{R_1+R_2 \to R_2 \\ -3R_1+R_3 \to R_3}}{\sim} \begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 10 & 3 & 0 \end{bmatrix}$$

$$\overset{-10R_2+R_3 \to R_3}{\sim} \begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 33 & 0 \end{bmatrix}$$

Because the only solution is the trivial solution, the set of column vectors, $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, is linearly independent.

43. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 3 & -2 & 0 & 0 \\ 0 & 3 & 9 & 0 \\ 2 & -4 & -8 & 0 \end{bmatrix} \overset{(-2/3)R_1+R_3 \to R_3}{\sim} \begin{bmatrix} 3 & -2 & 0 & 0 \\ 0 & 3 & 9 & 0 \\ 0 & -\frac{8}{3} & -8 & 0 \end{bmatrix}$$

$$\overset{(8/9)R_2+R_3 \to R_3}{\sim} \begin{bmatrix} 3 & -2 & 0 & 0 \\ 0 & 3 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Because there exist nontrivial solutions, the set of column vectors, $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, is not linearly independent.

44. By Theorem 2.14, because $m = 4 > 3 = n$, the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ is not linearly independent.

Chapter 2

Euclidean Space

2.1 Vectors

Key Points in This Section

1. A *vector* is an ordered list of real numbers $u_1, u_2, \ldots, u_n$ written usually in column form:
$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \tag{2.1}$$
although sometimes also in row form:
$$\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \text{ or as } \mathbf{u} = (u_1, u_2, \ldots, u_n). \tag{2.2}$$
The individual entries in a vector are called the *components*. Two vectors are considered equal if their components are the same.

2. The set of all vectors with n components is denoted $\mathbf{R}^n$.

3. Vectors in $\mathbf{R}^n$ can be added component-wise:
$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} \tag{2.3}$$

4. A vector can be multiplied by a *scalar*, that is, a real number:

$$c \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix} \tag{2.4}$$

5. *Euclidean space* is the set $\mathbf{R}^n$ together with the operations of addition and scalar multiplication. The commutative, associative, and distributive laws from arithmetic hold for these vector operations (Theorem 2.3).

6. Given vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ in $\mathbf{R}^n$ and scalars $c_1, c_2, \ldots, c_m$, the sum:

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_m\mathbf{u}_m \tag{2.5}$$

   is called a *linear combination*.

7. Linear systems can be written as vector equations involving a linear combination where the scalars are the variables.

8. General solutions to linear systems can be expressed as a linear combination. This is called the *vector form* of the general solution.

9. Vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$ can be visualized as an arrow with a *tail* at the origin and a *tip* at the point whose coordinates are the components of the vector.

10. With this interpretation, the arithmetic of vectors can be restated:

    (a) **Addition:** The vector $\mathbf{u} + \mathbf{v}$ is the vector whose tip is the tip of $\mathbf{v}$ after translating $\mathbf{v}$ so that $\mathbf{v}$'s tail is the tip of $\mathbf{u}$.

    (b) **Scalar Multiplication:** The vector $c\mathbf{u}$ is in the direction of $\mathbf{u}$ if $c > 0$ and in the opposite direction if $c < 0$. The length of $c\mathbf{u}$ is $|c|$ times the length of $\mathbf{u}$.

    (c) **Subtraction:** The vector $\mathbf{u} - \mathbf{v}$ is the vector whose tail is the tip of $\mathbf{v}$ and whose tip is the tip of $\mathbf{u}$.

Teaching Suggestions

Students are introduced to the arithmetic of *vectors*, in particular the notions of *span* and *linear independence* in this chapter. These notions are central to the rest of the course, in particular for Chapters 4 and 7 where vector spaces and subspaces are studied. There are no new computation techniques introduced in this chapter, instead students learn to analyze what the solution set to a linear system means in terms of the associated vector equation.

Vectors are defined as ordered lists of real numbers and written as a column matrix. Most students will have probably used vectors before in a physics class, or possibly multivariable calculus. There, vectors usually have two or three components and are something that has both a length (magnitude), and a direction. It is good to acknowledge that those objects are in fact vectors as they are defined in linear algebra, but that the definition of vector in linear algebra is more general. For the time being, only vectors in $\mathbf{R}^n$ are considered. Vectors with more than three components can be confusing enough and deemed pointless to some students. Examples will help to familiarize your students to these concepts. Abstract vector spaces are defined and studied starting in Chapter 7.

Most students will not struggle with the simple arithmetic of adding vectors or multiplying by scalars. You should spend more time on emphasizing the translation between linear systems and vector equations as this helps to motivate the definition of matrix multiplication (Definition 2.10 in Section 2.2). Further examining the relationship between the consistency of a linear system and the ability to express a given vector as a linear combination will help to motivate the next two sections.

Starting in this chapter, you will have to decide how much time should be spent in class working through elementary row operations. Advanced classes can start to skip this step altogether, or at least begin to perform multiple steps at once, and after writing the augmented matrix on the board you can then write the solution and leave the steps as an exercise. In other classes you should plan on doing such a calculation occasionally but should not spend the majority of classtime on such calculations as this places too much emphasis on calculation rather than concepts. The amount of calculations to work out in class could be judged by quiz or homework performance.

Suggested Classroom Examples

2.1.1 Example. Let:

$$\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Compute the following and sketch the corresponding vectors.

1. $\mathbf{u} + \mathbf{v}$
2. $\mathbf{v} - \mathbf{w}$
3. $-2\mathbf{u}$
4. $\mathbf{u} + 2\mathbf{w}$

Solution.

1. $\mathbf{u} + \mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1+3 \\ 2+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$
2. $\mathbf{v} - \mathbf{w} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3-2 \\ 0-1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
3. $-2\mathbf{u} = 2\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2(-1) \\ 2(2) \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$
4. $\mathbf{u} + 2\mathbf{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+2(2) \\ 2+2(1) \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

The vectors are shown in Figure 2.1. $\diamondsuit$

2.1.2 Example. Find two different linear combinations of:

$$\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}$$

Solution. Using the scalars 2 and -1 we have:

$$2\mathbf{u} - \mathbf{v} = 2\begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 2(1)-0 \\ 2(-2)-4 \\ 2(6)--1 \end{bmatrix} = \begin{bmatrix} 2 \\ -8 \\ 13 \end{bmatrix} \tag{2.6}$$

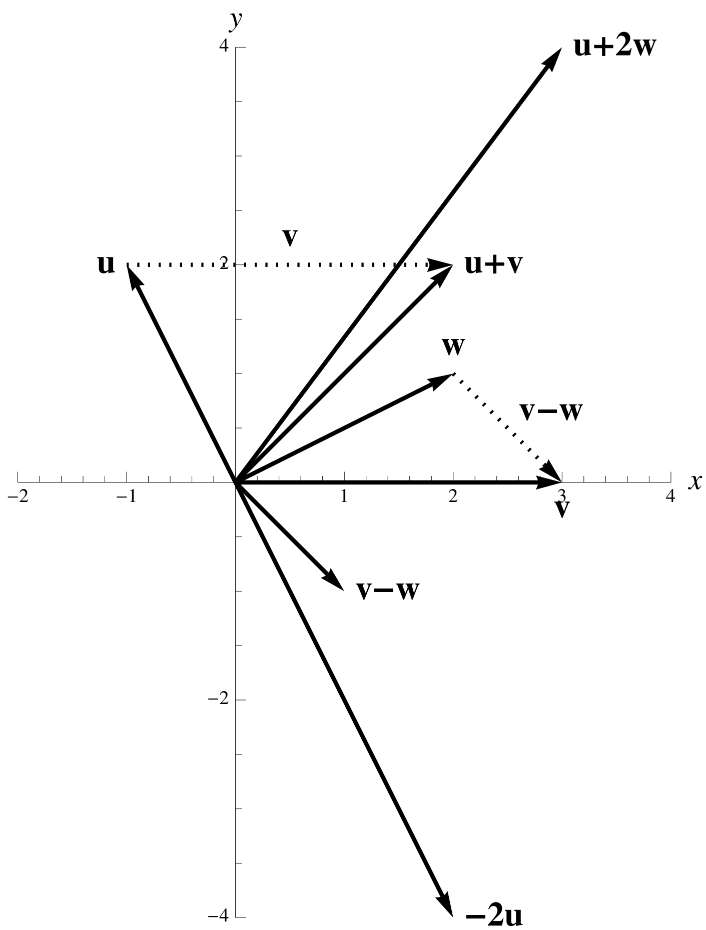


Figure 2.1: The vectors from Example 2.1.1. Translated vectors are shown as dotted.

Using the scalars 3 and 0 we have:

$$3\mathbf{u} + 0\mathbf{v} = 3\begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 3(1)+0 \\ 3(-2)+0 \\ 3(6)+0 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 18 \end{bmatrix} \tag{2.7}$$

$\diamondsuit$

2.1.3 Example. Express the given vector equation as a linear system.

1. $x_1 \begin{bmatrix} 3 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$

2. $x_1 \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 6 \\ -1 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}$

Solution.

1. $\begin{array}{rcrcr} 3x_1 & - & x_2 & = & -3 \\ -4x_1 & + & 7x_2 & = & 0 \end{array}$

2. $\begin{array}{rcrcrcr} 2x_1 & + & 6x_2 & + & x_3 & = & 4 \\ & & -x_2 & + & 3x_3 & = & 4 \\ -3x_1 & + & 3x_2 & & & = & 3 \end{array}$

$\diamondsuit$

2.1.4 Example. Express the given linear system as a vector equation.

1. $\begin{array}{rcrcr} x_1 & + & 3x_2 & = & 5 \\ -2x_1 & + & 3x_2 & = & -1 \end{array}$

2. $\begin{array}{rcrcrcr} 5x_1 & - & x_2 & + & 2x_3 & = & -4 \\ 2x_1 & & & - & x_3 & = & 3 \\ -x_1 & + & 3x_2 & + & 7x_3 & = & 0 \end{array}$

Solution.

1. $x_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$

2. $x_1 \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$

$\diamondsuit$

2.1.5 Example. Write the general solution to the following linear systems in vector form, that is, as a linear combination of vectors:

1. $\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & - & x_3 & = & -2 \\ 4x_1 & + & 5x_2 & - & 3x_3 & = & -2 \\ -x_1 & + & 3x_2 & + & 5x_3 & = & -8 \end{array}$

$$
2. \quad \begin{array}{rcrcrcrcrcr}
x_1 & + & 3x_2 & & & + & 3x_4 & - & 5x_5 & = & 5 \\
 & & & & x_3 & - & 6x_4 & + & 10x_5 & = & -7 \\
 & & & & & & 3x_4 & - & 4x_5 & = & 6
\end{array}
$$

Solution.

1. This system appears as the second system in Example 1.2.5. We found the general solution to be:

$$
\begin{array}{rclcrcr}
x_1 & = & 2+2s_1 & = & 2 & + & 2s_1 \\
x_2 & = & -2-s_1 & = & -2 & - & 1s_1 \\
x_3 & = & s_1 & = & 0 & + & 1s_1
\end{array} \tag{2.8}
$$

In vector form we have:

$$
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \tag{2.9}
$$

2. This system appears in Example 1.1.7. We found the general solution to be:

$$
\begin{array}{rclcrcrcr}
x_1 & = & -1-3s_1+s_2 & = & -1 & - & 3s_1 & + & 1s_2 \\
x_2 & = & s_1 & = & 0 & + & 1s_1 & + & 0s_2 \\
x_3 & = & 5+2s_2 & = & 5 & + & 0s_1 & + & 2s_2 \\
x_4 & = & 2+\frac{4}{3}s_2 & = & 2 & + & 0s_1 & + & \frac{4}{3}s_2 \\
x_5 & = & s_2 & = & 0 & + & 0s_1 & + & 1s_2
\end{array} \tag{2.10}
$$

In vector form we have:

$$
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 5 \\ 2 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 0 \\ 2 \\ \frac{4}{3} \\ 1 \end{bmatrix} \tag{2.11}
$$

$\diamondsuit$

2.1.6 Example. Express $\mathbf{b}$ as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$ if possible where:

$$
\mathbf{a}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
$$

Solution. We are seeking scalars x_1 and x_2 such that:

$$x_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{2.12}$$

This vector equation corresponds to the linear system:

$$\begin{array}{rcrcl} -x_1 & + & 3x_2 & = & 2 \\ 2x_1 & & & = & 1 \end{array} \tag{2.13}$$

Interchanging the roles of x_1 and x_2, this system is in triangular form so we can solve via back substitution starting with x_1. The second equation gives $x_1 = \frac{1}{2}$. Substituting this into the first equation, we have:

$$-\left(\frac{1}{2}\right) + 3x_2 = 2 \Rightarrow 3x_2 = 2 + \frac{1}{2} = \frac{5}{2} \tag{2.14}$$

Thus $x_2 = \frac{5}{6}$. This gives:

$$\frac{1}{2}\mathbf{a}_1 + \frac{5}{6}\mathbf{a} = \frac{1}{2}\begin{bmatrix} -1 \\ 2 \end{bmatrix} + \frac{5}{6}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} + \frac{5}{2} \\ 1 + 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \mathbf{b} \tag{2.15}$$

$\diamondsuit$

2.1.7 Example. Express $\mathbf{b}$ as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ if possible where:

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{a}_3 = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 9 \\ 7 \\ 1 \end{bmatrix}$$

Solution. We are seeking scalars x_1, x_2 and x_3 such that:

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 1 \end{bmatrix} \tag{2.16}$$

This vector equation corresponds to the linear system:

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 9 \\ & & x_2 & - & 3x_3 & = & 7 \\ x_1 & & & + & x_3 & = & 1 \end{array} \tag{2.17}$$

We apply elementary row operations to the augmented matrix to solve.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 0 & 1 & -3 & 7 \\ 1 & 0 & 1 & 1 \end{array}\right] \overset{-R_1+R_3 \Rightarrow R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 0 & 1 & -3 & 7 \\ 0 & -2 & 0 & -8 \end{array}\right]$$

$$\overset{R_2 \Leftrightarrow R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 0 & -2 & 0 & -8 \\ 0 & 1 & -3 & 7 \end{array}\right]$$

$$\overset{-\frac{1}{2}R_2 \Rightarrow R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 0 & 1 & 0 & 4 \\ 0 & 1 & -3 & 7 \end{array}\right]$$

$$\overset{\substack{-2R_2+R_1 \Rightarrow R_1 \\ -R_2+R_3 \Rightarrow R_3}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & -3 & 3 \end{array}\right]$$

$$\overset{-\frac{1}{3}R_3 \Rightarrow R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\overset{-R_3+R_1 \Rightarrow R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Therefore, the unique solution to the linear system in (2.17) is $x_1 = 2, x_2 = 4$ and $x_3 = -1$. This gives:

$$\begin{aligned} 2\mathbf{a}_1 + 4\mathbf{a}_2 - \mathbf{a}_3 &= 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2(1) + 4(2) - 1 \\ 2(0) + 4(1) - -3 \\ 2(1) + 4(0) - 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 1 \end{bmatrix} = \mathbf{b} \end{aligned} \tag{2.18}$$

$\diamondsuit$

2.1.8 Example. Express $\mathbf{b}$ as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ if possible where:

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{a}_3 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix}$$

Solution. We are seeking scalars x_1, x_2 and x_3 such that:

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix} \tag{2.19}$$

This vector equation corresponds to the linear system:

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & - & x_3 & = & 6 \\ & & x_2 & + & x_3 & = & 1 \\ x_1 & & & - & 3x_3 & = & 3 \end{array} \tag{2.20}$$

We apply elementary row operations to the augmented matrix to solve.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & -3 & 3 \end{array}\right] \overset{-R_1+R_3 \Rightarrow R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & -2 & -3 \end{array}\right]$$

$$\overset{\substack{-2R_2+R_1 \Rightarrow R_1 \\ 2R_2+R_3 \Rightarrow R_3}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

The third equation in the corresponding linear system is $0 = -1$ which has no solution. Hence it is not possible to express $\mathbf{b}$ as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. $\diamondsuit$

2.2 Span

Key Points in This Section

1. Given a set of vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ in $\mathbf{R}^n$, the set of all possible linear combinations:
$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_m\mathbf{u}_m \tag{2.21}$$
where $x_1, x_2, \ldots, x_m$ are real numbers, is called the *span* of the set and is denoted $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$.

2. A set of vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ *spans* $\mathbf{R}^n$ if $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} = \mathbf{R}^n$.

3. Determining whether or not a given vector $\mathbf{v}$ is in $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ gives rise to a vector equation and hence a linear system via the previous section. The augmented matrix of this linear system is:
$$\left[\begin{array}{cccc|c}\mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m & \mathbf{v}\end{array}\right] \tag{2.22}$$
This system has a solution if and only if $\mathbf{v}$ is in $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ (Theorem 2.6).

4. If $\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ are vectors in $\mathbf{R}^n$ and $\mathbf{u}$ is in $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$, then:
$$\operatorname{span}\{\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} \tag{2.23}$$
(Theorem 2.7).

5. Suppose $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ are vectors in $\mathbf{R}^n$ and let
$$A = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m\end{bmatrix} \sim B$$
where B is in echelon form. Then $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} = \mathbf{R}^n$ exactly when B has a pivot position in every row (Theorem 2.8).

6. If $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ are vectors in $\mathbf{R}^n$ and $m < n$, then
$$\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} \neq \mathbf{R}^n \tag{2.24}$$
(Theorem 2.9).

7. Matrix–vector multiplication is defined so that a linear system can be compactly expressed as $A\mathbf{x} = \mathbf{b}$. If:

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_m \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \tag{2.25}$$

where $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ are vectors in $\mathbf{R}^n$, then

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_m\mathbf{a}_m. \tag{2.26}$$

Notice that the number of columns of A must equal the number of components (rows) of $\mathbf{x}$.

8. Suppose $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ and $\mathbf{b}$ are vectors in $\mathbf{R}^n$. Then the following statements are equivalent.

   (a) $\mathbf{b}$ is in $\operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$.

   (b) The vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_m\mathbf{a}_m = \mathbf{b}$ has at least one solution.

   (c) The linear system corresponding to $\left[\begin{array}{cccc|c} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_m & \mathbf{b} \end{array}\right]$ has at least one solution.

   (d) The equation $A\mathbf{x} = \mathbf{b}$, with A and $\mathbf{x}$ as given in (2.25), has at least one solution.

   (Theorem 2.11)

Teaching Suggestions

There are no new computational techniques developed in this section, but students must learn to interpret what the calculations they have been performing in Chapter 1 mean in terms of vector equations. Examples with simple linear systems, as given in Example 2.2.1, are useful here to allow students to focus on the new concept and not bother with computations.

The definition of matrix-vector product as a linear combination of the columns of the matrix should be emphasized as it plays a major role in subsequent chapters. This comment is so important it warrants repetition: $A\mathbf{x}$

is a linear combination of the columns of A, the scalar being the components of $\mathbf{x}$.

You will need to make a decision now about how much rigor the course will include. For instance, the proof of Theorem 2.9 is mainly shown with an illustrative example whereas you might wish to give a general proof. The abstraction and notation necessary for such a proof might distract from the content. For classes where an introductory course on proof writing is a prerequisite, Exercises 69–76 make for good problems for students to try as they often involve writing definition, performing one or two lines of manipulation, and then interpretation.

A good problem to pose to the students to see if they understand span and as a lead into the next section is to ask them to give an example of a set of three vectors that do not span $\mathbf{R}^3$. They should stumble upon the notion of linear dependence on their own.

Suggested Classroom Examples

2.2.1 Example. Determine whether or not the given vector $\mathbf{b}$ is in the span of the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$.

1. $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

2. $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ -4 \\ -4 \end{bmatrix}$

3. $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}$

Solution.

1. We are looking for scalars x_1 and x_2 such that $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 = \mathbf{b}$. This translates to the linear system:

$$\begin{array}{rcrcr} x_1 & - & x_2 & = & 4 \\ 2x_1 & - & 3x_2 & = & -2 \end{array} \tag{2.27}$$

Subtracting 2 times the first equation from the second we have the triangular system:

$$\begin{array}{rcrcr} x_1 & - & x_2 & = & 4 \\ & & -x_2 & = & -10 \end{array} \tag{2.28}$$

We find $x_2 = 10$ and $x_1 = 4 + 10 = 14$. Therefore, $14\mathbf{u}_1 + 10\mathbf{u}_2 = \mathbf{b}$ and hence $\mathbf{b}$ is in $\text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

2. We are looking for scalars x_1 and x_2 such that $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 = \mathbf{b}$. This translates to the linear system:

$$\begin{array}{rcrcr} x_1 & & & = & 1 \\ & & 2x_2 & = & -4 \\ -2x_1 & + & x_2 & = & -4 \end{array} \tag{2.29}$$

In order for the first two equation to be satisfied, we need $x_1 = 1$ and $x_2 = -2$. This pair satisfies the third equation as well, and therefore it is the unique solution to the system. Therefore, $\mathbf{u}_1 - 2\mathbf{u}_2 = \mathbf{b}$, and hence $\mathbf{b}$ is in $\text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

3. This is very similar to the previous example. Now the linear system is:

$$\begin{array}{rcrcr} x_1 & & & = & 4 \\ & & 2x_2 & = & -1 \\ -2x_1 & + & x_2 & = & 4 \end{array} \tag{2.30}$$

In order for the first two equation to be satisfied, we need $x_1 = 4$ and $x_2 = -\frac{1}{2}$. This pair does not satisfy the third equation, and therefore $\mathbf{b}$ is not in $\text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

The difference between these last two examples is shown in Figure 2.2.

$\diamondsuit$

2.2.2 Example. Find a vector in $\mathbf{R}^3$ that is not in:

$$\text{span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \right\}.$$

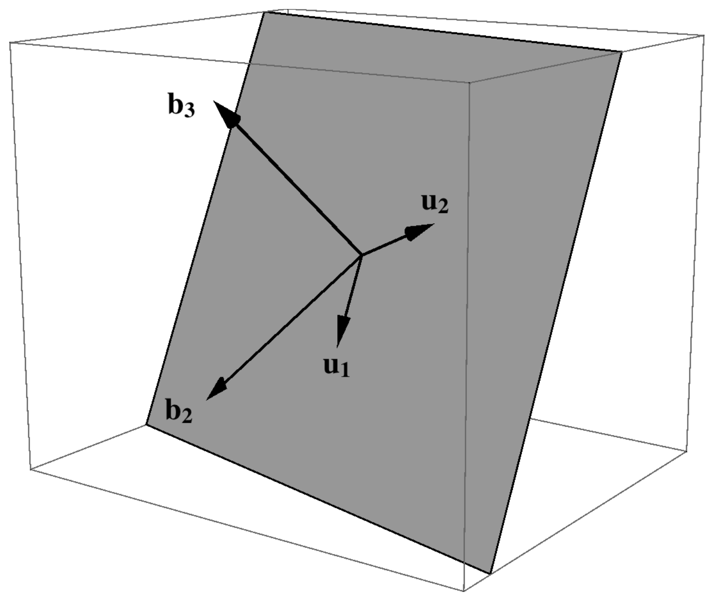


Figure 2.2: The vector $\mathbf{b}$ from part 2 of Example 2.2.1 is in span$\{\mathbf{u}_1, \mathbf{u}_2\}$ but the vector $\mathbf{b}$ from part 3 is not.

Solution. We being by applying elementary row operations on the matrix whose columns are the vectors in question adjoined with an arbitrary vector in $\mathbf{R}^3$ whose components are denoted a, b, c.

$$\left[\begin{array}{rr|c} 2 & 0 & a \\ -1 & 3 & b \\ 4 & -2 & c \end{array}\right] \overset{\frac{1}{2}R_1 \Rightarrow R_1}{\sim} \left[\begin{array}{rr|r} 1 & 0 & a/2 \\ -1 & 3 & b \\ 4 & -2 & c \end{array}\right]$$

$$\overset{\substack{R_1+R_2 \Rightarrow R_2 \\ -4R_1+R_3 \Rightarrow R_3}}{\sim} \left[\begin{array}{rr|r} 1 & 0 & a/2 \\ 0 & 3 & b+a/2 \\ 0 & -2 & c-2a \end{array}\right]$$

$$\overset{\frac{1}{3}R_2 \Rightarrow R_2}{\sim} \left[\begin{array}{rr|r} 1 & 0 & a/2 \\ 0 & 1 & \frac{1}{3}(b+a/2) \\ 0 & -2 & c-2a \end{array}\right]$$

$$\overset{2R_2+R_3 \Rightarrow R_3}{\sim} \left[\begin{array}{rr|r} 1 & 0 & a/2 \\ 0 & 1 & \frac{1}{3}(b+a/2) \\ 0 & 0 & c+\frac{2}{3}b-\frac{5}{3}a \end{array}\right]$$

We see that the corresponding linear system has a solution if and only if $c + \frac{2}{3}b - \frac{5}{3}a = 0$. Any vector whose components do not satisfy this equation will therefore not be in the span. For example, taking $a = 1$, we see that:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ is not in span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \right\}. \tag{2.31}$$

There are of course infinitely many other possibilities. $\diamondsuit$

2.2.3 Example. Determine if the following vectors span $\mathbf{R}^2$.

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Solution. Applying the row operation $-\frac{2}{3}R_1 + R_2 \Rightarrow R_2$ to $\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}$, we get the matrix $\left[\begin{smallmatrix} 3 & 5 \\ 0 & -1/9 \end{smallmatrix}\right]$. This matrix is in echelon form and every row has a pivot position. This shows that $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\} = \mathbf{R}^2$ (Theorem 2.8). $\diamondsuit$

2.2.4 Example. Determine if the following vectors span $\mathbf{R}^3$.

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Solution. No, two vectors can never span $\mathbf{R}^3$ (Theorem 2.9). ♢

2.2.5 Example. Give an example of a set of vectors that span $\mathbf{R}^n$.

Solution. Consider the vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ in $\mathbf{R}^n$ where every component of $\mathbf{e}_i$ is 0 except the i^{th} which is 1.

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{2.32}$$

These span $\mathbf{R}^n$ as we can always write:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots x_n\mathbf{e}_n. \tag{2.33}$$

♢

2.2.6 Example. Find all values of h such that the set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ spans $\mathbf{R}^3$.

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 0 \\ h \\ 1 \end{bmatrix}, \qquad \mathbf{u}_3 = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

Solution. We consider the matrix $\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}$. This matrix and the result of applying several elementary row operations are:

$$\begin{bmatrix} 1 & 0 & -3 \\ 2 & h & 2 \\ -3 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -8 \\ 0 & 0 & 8(h+1) \end{bmatrix}$$

The operations we applied are: $-2R_1+R_2 \Rightarrow R_2$, $3R_1+R_3 \Rightarrow R_3$, $R_2 \Leftrightarrow R_3$, $-hR_2+R_3 \Rightarrow R_3$. This matrix is in echelon form. Every row has a pivot position if and only if $h \neq -1$. Hence, $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \mathbf{R}^3$ if and only if $h \neq -1$ (Theorem 2.8). $\diamondsuit$

2.2.7 Example. Find A, $\mathbf{x}$, and $\mathbf{b}$ such that the equation $A\mathbf{x} = \mathbf{b}$ corresponds to the linear system:

$$\begin{array}{rcrcrcrcr} 2x_1 & + & 3x_2 & & & - & 5x_4 & = & -1 \\ 4x_1 & & & + & 7x_3 & + & x_4 & = & 4 \\ & & 3x_2 & - & x_3 & + & 2x_4 & = & 0 \end{array}$$

Solution. We have:

$$A = \begin{bmatrix} 2 & 3 & 0 & -5 \\ 4 & 0 & 7 & 1 \\ 0 & 3 & -1 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} \tag{2.34}$$

$\diamondsuit$

2.2.8 Example. Determine if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every choice of $\mathbf{b}$.

1. $A = \begin{bmatrix} 1 & 3 \\ -2 & -6 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$

Solution.

1. This is similar to Example 2.2.3. We adjoin the matrix A with a vector with components a and b and apply elementary row operations.

$$\left[\begin{array}{rr|r} 1 & 3 & a \\ -2 & -6 & b \end{array}\right] \overset{2R_1+R_2 \Rightarrow R_2}{\sim} \left[\begin{array}{rr|r} 1 & 3 & a \\ 0 & 0 & 2a+b \end{array}\right]$$

The equation corresponding to the bottom row does not have a solution unless $2a + b = 0$. Therefore, the system $A\mathbf{x} = \mathbf{b}$ does not have a solution for every $\mathbf{b}$ in $\mathbf{R}^2$. In other words, the columns of A do not span $\mathbf{R}^2$.

2. We adjoin the matrix A with a vector with components a, b, and c and apply elementary row operations.

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & a \\ 2 & 1 & 2 & b \\ 2 & 1 & 3 & c \end{array}\right] \underset{\sim}{\substack{-2R_1+R_2 \Rightarrow R_2 \\ -2R_1+R_3 \Rightarrow R_3}} \left[\begin{array}{ccc|c} 1 & 0 & -2 & a \\ 0 & 1 & 6 & b-2a \\ 0 & 1 & 7 & c-2a \end{array}\right]$$

$$\underset{\sim}{-R_2+R_3 \Rightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & -2 & a \\ 0 & 1 & 6 & b-2a \\ 0 & 0 & 1 & c-b \end{array}\right]$$

$$\underset{\sim}{\substack{2R_3+R_1 \Rightarrow R_1 \\ -6R_3+R_2 \Rightarrow R_2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & a-2b+2c \\ 0 & 1 & 0 & 7b-2a-6c \\ 0 & 0 & 1 & c-b \end{array}\right]$$

Therefore for any values for a, b, and c we can solve the linear system $A\mathbf{x} = \mathbf{b}$. The solution is:

$$\mathbf{x} = \begin{bmatrix} a-2b+2c \\ 7b-2a-6c \\ c-b \end{bmatrix} \tag{2.35}$$

Additionally, this shows that we can write:

$$(a-2b+2c)\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + (7b-2a-6c)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + (c-b)\begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{2.36}$$

This shows that the columns of A span $\mathbf{R}^3$.

$\diamondsuit$

2.3 Linear Independence

Key Points in This Section

1. A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ in $\mathbf{R}^n$ is *linearly independent* if whenever $c_1, c_2, \ldots, c_m$ are scalars such that:

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_m\mathbf{u}_m = \mathbf{0} \tag{2.37}$$

then necessarily $c_1 = c_2 = \cdots = c_m = 0$. In other words, the only solution to the vector equation:

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_m\mathbf{u}_m = \mathbf{0} \tag{2.38}$$

is the trivial solution $x_1 = x_2 = \cdots = x_m = 0$. Otherwise the set is said to be *linearly dependent*.

2. If a set of vectors in $\mathbf{R}^n$ contains $\mathbf{0}$, then the set is linearly dependent (Theorem 2.13).

3. The set $\{\mathbf{u}_1, \mathbf{u}_2\}$ is linearly dependent if and only if one of the vectors is a scalar multiple of the other.

4. If $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ are vectors in $\mathbf{R}^n$ and $m > n$, then $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ is linearly dependent (Theorem 2.14).

5. If $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ are vectors in $\mathbf{R}^n$, then $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ is linearly dependent if and only if one of the vectors is in the span of the others (Theorem 2.15).

6. Suppose $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ are vectors in $\mathbf{R}^n$ and let

$$A = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m \end{bmatrix} \sim B$$

where B is in echelon form. Then

   (a) $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} = \mathbf{R}^n$ exactly when B has a pivot position in every <u>row</u>, and

   (b) $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ is linearly independent exactly when B has a pivot position in every <u>column</u>.

   (Theorem 2.16)

7. A set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ in $\mathbf{R}^n$ is *linearly independent* if and only if the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ with coefficient matrix:

$$A = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m \end{bmatrix} \tag{2.39}$$

has a unique solution, the trivial solution $\mathbf{x} = \mathbf{0}$. (Theorem 2.17).

8. Given a linear system $A\mathbf{x} = \mathbf{b}$ is the *associated homogeneous linear system* $A\mathbf{x} = \mathbf{0}$.

9. Matrix–vector multiplication satisfies the distributive property:

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} \tag{2.40}$$

   (Theorem 2.18).

10. Let $\mathbf{x}_p$ be a (*particular*) solution to $A\mathbf{x} = \mathbf{b}$. Then the general solution to $A\mathbf{x} = \mathbf{b}$ is of the form $\mathbf{x}_g = \mathbf{x}_p + \mathbf{x}_h$ where $\mathbf{x}_h$ is a solution to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$ (Theorem 2.19).

11. Suppose $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ and $\mathbf{b}$ are vectors in $\mathbf{R}^n$. Then the following statements are equivalent.

    (a) The set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ is linearly independent.

    (b) The vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_m\mathbf{a}_m = \mathbf{b}$ has at most one solution.

    (c) The linear system corresponding to $\left[\begin{array}{cccc|c} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_m & \mathbf{b} \end{array}\right]$ has at most one solution.

    (d) The equation $A\mathbf{x} = \mathbf{b}$, with A and $\mathbf{x}$ as given in (2.25), has at most one solution.

    (Theorem 2.20)

12. **The Unifying Theorem – Version 1:** Let $\mathcal{S} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ be a set of vectors in $\mathbf{R}^n$ and let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$. Then the following are equivalent:

    (a) $\mathcal{S}$ spans $\mathbf{R}^n$.

    (b) $\mathcal{S}$ is linearly independent.

    (c) $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$ in $\mathbf{R}^n$.

    Notice the number of vectors in $\mathcal{S}$ is exactly n.

Teaching Suggestions

This section introduces the other central notion for a set of vectors, *linear independence*. This notion, along with span, play a key part in the remainder of the course. Students will need to be able to readily translate what these concepts are and how to test for them in terms of solutions of linear systems and echelon form of matrices. Contrasting Theorems 2.11 and 2.20 will help students to see the connections between span and linear independence.

One way to naturally introduce the concept of linear dependence is to ask the students to construct a set of three vectors in $\mathbf{R}^3$ that do not span. Students will realize that to do so, at least one of the vectors will need to be a linear combination of the others. In other words, they will need to construct a linearly dependent set. Along the way, they can probably start to convince themselves why The Unifying Theorem is true.

Linear independence can be a hard definition for students to grasp at first glance as it is defined as the absence of nontrivial solutions. Students can easily understand when something is there, but a definition based on the concept of something not being there can be difficult to comprehend unless they have seen such definitions before.

Again, if the class has some experience with writing proofs, Exercises 55–64 make for excellent problems and practice of proof writing techniques.

Suggested Classroom Examples

2.3.1 Example. Determine if the set of vectors is linearly independent.

1. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

2. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

3. $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

Solution.

1. Applying the row operation $-2R_1 + R_2 \Rightarrow R_2$ to $\left[\begin{smallmatrix} 1 & 3 \\ 2 & -1 \end{smallmatrix}\right]$, we get $\left[\begin{smallmatrix} 1 & 3 \\ 0 & -7 \end{smallmatrix}\right]$. This matrix is in echelon form and every column has a pivot position and so the set is linearly independent (Theorem 2.16).

2. Of course by Theorem 2.14 this set is linearly dependent but look at the homogeneous linear system to write an explicit linear dependence. The augmented matrix and its reduced echelon form are:

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 0 \\ -1 & 2 & 5 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

   The operations we applied are: $R_1 + R_2 \Rightarrow R_2$, $\frac{1}{4}R_2 \Rightarrow R_2$ and $-2R_2 + R_1 \Rightarrow R_1$. Therefore, the general solution to the vector equation:

$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \mathbf{0} \tag{2.41}$$

   is:

$$\begin{array}{rcl} x_1 & = & s_1 \\ x_2 & = & -2s_1 \\ x_3 & = & s_1 \end{array} \tag{2.42}$$

   Any nonzero value for s_1 gives a nontrivial solution, for example if $s_1 = 1$, then $x_1 = 1, x_2 = -2$ and $x_3 = 1$ is a nontrivial solution. This gives the linear dependence:

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \mathbf{0} \tag{2.43}$$

3. We consider the matrix has the given vectors as columns. This matrix and the result of the row operations $R_1 + R_2 \Rightarrow R_2$, $-R_1 + R_3 \Rightarrow R_3$ and $-2R_2 + R_3 \Rightarrow R_3$ are:

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & -1 \\ 1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

   The new matrix is in echelon form and the third column does not have a pivot position, hence the vectors are linearly dependent (Theorem 2.16).

$\diamondsuit$

2.3.2 Example. Determine if the columns of the given matrix are linearly independent.

$$A = \begin{bmatrix} 1 & 2 & -5 \\ -3 & 6 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$

Solution. We put the matrix into echelon form:

$$\begin{bmatrix} 1 & 2 & -5 \\ -3 & 6 & 0 \\ 0 & 1 & 4 \end{bmatrix} \quad \overset{\substack{3R_1+R_2 \Rightarrow R_2 \\ R_2 \Leftrightarrow R_3}}{\sim} \quad \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & -15 \end{bmatrix}$$

Since every column has a pivot position, the columns of A are linearly independent (Theorem 2.16). By The Unifying Theorem, this also implies that the columns of A span $\mathbf{R}^3$ and that the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbf{R}^3$. $\diamondsuit$

2.3.3 Example. Find the general solution and the solutions to the associated homogeneous system for:

$$\begin{array}{ccccccccccc} x_1 & + & 3x_2 & & & + & 3x_4 & - & 5x_5 & = & 5 \\ & & & & x_3 & - & 6x_4 & + & 10x_5 & = & -7 \\ & & & & & & 3x_4 & - & 4x_5 & = & 6 \end{array}$$

Solution. We solved this linear system in Example 1.1.7 and also considered it in Example 2.1.5. We found the general solution to be (2.11):

$$\mathbf{x}_g = \begin{bmatrix} -1 \\ 0 \\ 5 \\ 2 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 0 \\ 2 \\ \frac{4}{3} \\ 1 \end{bmatrix} \tag{2.44}$$

The solutions to the associated homogeneous linear system are:

$$\mathbf{x}_h = s_1 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 0 \\ 2 \\ \frac{4}{3} \\ 1 \end{bmatrix} \tag{2.45}$$

$\diamondsuit$

2.3.4 Example. Show that the columns of A are linearly independent:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

Solution. In the second part of Example 2.2.8, we showed that the linear system $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbf{R}^3$. (In fact, we actually showed that there is a unique solution.) Hence the columns of A span $\mathbf{R}^3$ and therefore by The Unifying Theorem, as the number of columns equals the number of rows, the columns of A are linearly independent. $\diamondsuit$