Instructors' Solution Manual
Introduction to Quantum Mechanics, 3rd ed.

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Preface

These are our own solutions to the problems in *Introduction to Quantum Mechanics, 3rd ed.* We have made every effort to insure that they are clear and correct, but errors are bound to occur, and for this we apologize in advance. We would like to thank the many people who pointed out mistakes in the solution manual for the second edition, and encourage anyone who finds defects in this one to alert us (griffith@reed.edu or schroetd@reed.edu). We especially thank Kenny Scott, Alain Thys, and Sergei Walter, who found many errors in the 2nd edition solution manual. We maintain a list of errata on the web page (http://academic.reed.edu/physics/faculty/griffiths.html), and incorporate corrections in the manual itself from time to time. We also thank our students for many useful suggestions, and Neelaksh Sadhoo, who did much of the typesetting for the second edition.

David Griffiths and Darrell Schroeter

Chapter 1

The Wave Function

Problem 1.1

(a)

$$\langle j\rangle^2 = 21^2 = \boxed{441.}$$

$$\langle j^2\rangle = \frac{1}{N}\sum j^2 N(j) = \frac{1}{14}\left[(14^2) + (15^2) + 3(16^2) + 2(22^2) + 2(24^2) + 5(25^2)\right]$$
$$= \frac{1}{14}(196 + 225 + 768 + 968 + 1152 + 3125) = \frac{6434}{14} = \boxed{459.571.}$$

(b)

j	$\Delta j = j - \langle j\rangle$
14	$14 - 21 = -7$
15	$15 - 21 = -6$
16	$16 - 21 = -5$
22	$22 - 21 = 1$
24	$24 - 21 = 3$
25	$25 - 21 = 4$

$$\sigma^2 = \frac{1}{N}\sum(\Delta j)^2 N(j) = \frac{1}{14}\left[(-7)^2 + (-6)^2 + (-5)^2\cdot 3 + (1)^2\cdot 2 + (3)^2\cdot 2 + (4)^2\cdot 5\right]$$
$$= \frac{1}{14}(49 + 36 + 75 + 2 + 18 + 80) = \frac{260}{14} = \boxed{18.571.}$$

$$\sigma = \sqrt{18.571} = \boxed{4.309.}$$

(c)

$$\langle j^2\rangle - \langle j\rangle^2 = 459.571 - 441 = 18.571. \quad \text{[Agrees with (b).]}$$

Problem 1.2

(a)

$$\langle x^2\rangle = \int_0^h x^2 \frac{1}{2\sqrt{hx}}dx = \frac{1}{2\sqrt{h}}\left(\frac{2}{5}x^{5/2}\right)\Big|_0^h = \frac{h^2}{5}.$$

$$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{h^2}{5} - \left(\frac{h}{3}\right)^2 = \frac{4}{45}h^2 \;\Rightarrow\; \sigma = \boxed{\frac{2h}{3\sqrt{5}} = 0.2981h.}$$

(b)

$$P = 1 - \int_{x_-}^{x_+} \frac{1}{2\sqrt{hx}}dx = 1 - \frac{1}{2\sqrt{h}}(2\sqrt{x})\Big|_{x_-}^{x_+} = 1 - \frac{1}{\sqrt{h}}\left(\sqrt{x_+} - \sqrt{x_-}\right).$$

$$x_+ \equiv \langle x\rangle + \sigma = 0.3333h + 0.2981h = 0.6315h; \quad x_- \equiv \langle x\rangle - \sigma = 0.3333h - 0.2981h = 0.0352h.$$

$$P = 1 - \sqrt{0.6315} + \sqrt{0.0352} = \boxed{0.393.}$$

Problem 1.3

(a)

$$1 = \int_{-\infty}^{\infty} Ae^{-\lambda(x-a)^2}dx. \qquad \text{Let } u \equiv x-a,\ du = dx,\ u: -\infty \to \infty.$$

$$1 = A\int_{-\infty}^{\infty} e^{-\lambda u^2}du = A\sqrt{\frac{\pi}{\lambda}} \quad\Rightarrow\quad \boxed{A = \sqrt{\frac{\lambda}{\pi}}.}$$

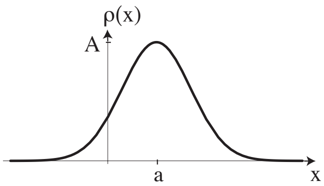
(b)

$$\begin{aligned}\langle x\rangle &= A\int_{-\infty}^{\infty} xe^{-\lambda(x-a)^2}dx = A\int_{-\infty}^{\infty}(u+a)e^{-\lambda u^2}du \\ &= A\left[\int_{-\infty}^{\infty} ue^{-\lambda u^2}du + a\int_{-\infty}^{\infty} e^{-\lambda u^2}du\right] = A\left(0 + a\sqrt{\frac{\pi}{\lambda}}\right) = \boxed{a.}\end{aligned}$$

$$\begin{aligned}\langle x^2\rangle &= A\int_{-\infty}^{\infty} x^2 e^{-\lambda(x-a)^2}dx \\ &= A\left\{\int_{-\infty}^{\infty} u^2 e^{-\lambda u^2}du + 2a\int_{-\infty}^{\infty} ue^{-\lambda u^2}du + a^2\int_{-\infty}^{\infty} e^{-\lambda u^2}du\right\} \\ &= A\left[\frac{1}{2\lambda}\sqrt{\frac{\pi}{\lambda}} + 0 + a^2\sqrt{\frac{\pi}{\lambda}}\right] = \boxed{a^2 + \frac{1}{2\lambda}.}\end{aligned}$$

$$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = a^2 + \frac{1}{2\lambda} - a^2 = \frac{1}{2\lambda}; \qquad \boxed{\sigma = \frac{1}{\sqrt{2\lambda}}.}$$

(c)

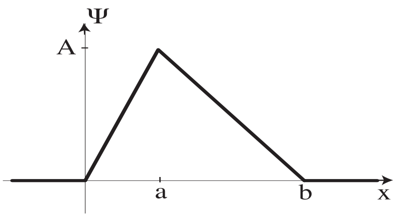


Problem 1.4

(a)

$$1 = \frac{|A|^2}{a^2}\int_0^a x^2 dx + \frac{|A|^2}{(b-a)^2}\int_a^b (b-x)^2 dx = |A|^2 \left\{ \frac{1}{a^2}\left(\frac{x^3}{3}\right)\Big|_0^a + \frac{1}{(b-a)^2}\left(-\frac{(b-x)^3}{3}\right)\Big|_a^b \right\}$$
$$= |A|^2\left[\frac{a}{3} + \frac{b-a}{3}\right] = |A|^2\frac{b}{3} \Rightarrow \boxed{A = \sqrt{\frac{3}{b}}.}$$

(b)



(c) At $\boxed{x = a.}$

(d)

$$P = \int_0^a |\Psi|^2 dx = \frac{|A|^2}{a^2}\int_0^a x^2 dx = |A|^2\frac{a}{3} = \boxed{\frac{a}{b}.} \begin{cases} P = 1 & \text{if } b = a, \checkmark \\ P = 1/2 & \text{if } b = 2a. \checkmark \end{cases}$$

(e)

$$\langle x \rangle = \int x|\Psi|^2 dx = |A|^2\left\{\frac{1}{a^2}\int_0^a x^3 dx + \frac{1}{(b-a)^2}\int_a^b x(b-x)^2 dx\right\}$$
$$= \frac{3}{b}\left\{\frac{1}{a^2}\left(\frac{x^4}{4}\right)\Big|_0^a + \frac{1}{(b-a)^2}\left(b^2\frac{x^2}{2} - 2b\frac{x^3}{3} + \frac{x^4}{4}\right)\Big|_a^b\right\}$$
$$= \frac{3}{4b(b-a)^2}\left[a^2(b-a)^2 + 2b^4 - 8b^4/3 + b^4 - 2a^2b^2 + 8a^3b/3 - a^4\right]$$
$$= \frac{3}{4b(b-a)^2}\left(\frac{b^4}{3} - a^2b^2 + \frac{2}{3}a^3b\right) = \frac{1}{4(b-a)^2}(b^3 - 3a^2b + 2a^3) = \boxed{\frac{2a+b}{4}.}$$

Problem 1.5

(a)

$$1 = \int |\Psi|^2 dx = 2|A|^2 \int_0^\infty e^{-2\lambda x} dx = 2|A|^2 \left(\frac{e^{-2\lambda x}}{-2\lambda} \right)\bigg|_0^\infty = \frac{|A|^2}{\lambda}; \quad \boxed{A = \sqrt{\lambda}.}$$

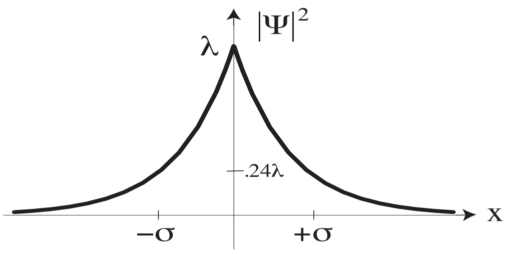
(b)

$$\langle x \rangle = \int x|\Psi|^2 dx = |A|^2 \int_{-\infty}^{\infty} x e^{-2\lambda|x|} dx = \boxed{0.} \qquad \text{[Odd integrand.]}$$

$$\langle x^2 \rangle = 2|A|^2 \int_0^\infty x^2 e^{-2\lambda x} dx = 2\lambda \left[\frac{2}{(2\lambda)^3} \right] = \boxed{\frac{1}{2\lambda^2}.}$$

(c)

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2\lambda^2}; \qquad \boxed{\sigma = \frac{1}{\sqrt{2}\lambda}.} \qquad |\Psi(\pm\sigma)|^2 = |A|^2 e^{-2\lambda\sigma} = \lambda e^{-2\lambda/\sqrt{2}\lambda} = \lambda e^{-\sqrt{2}} = 0.2431\lambda.$$



Probability outside:

$$2\int_\sigma^\infty |\Psi|^2 dx = 2|A|^2 \int_\sigma^\infty e^{-2\lambda x} dx = 2\lambda \left(\frac{e^{-2\lambda x}}{-2\lambda} \right)\bigg|_\sigma^\infty = e^{-2\lambda\sigma} = \boxed{e^{-\sqrt{2}} = 0.2431.}$$

Problem 1.6

For integration by parts, the differentiation has to be with respect to the *integration* variable – in this case the differentiation is with respect to t, but the integration variable is x. It's true that

$$\frac{\partial}{\partial t}(x|\Psi|^2) = \frac{\partial x}{\partial t}|\Psi|^2 + x\frac{\partial}{\partial t}|\Psi|^2 = x\frac{\partial}{\partial t}|\Psi|^2,$$

but this does *not* allow us to perform the integration:

$$\int_a^b x\frac{\partial}{\partial t}|\Psi|^2 dx = \int_a^b \frac{\partial}{\partial t}(x|\Psi|^2)dx \neq (x|\Psi|^2)\big|_a^b.$$

Problem 1.7

From Eq. 1.33, $\frac{d\langle p\rangle}{dt} = -i\hbar\int \frac{\partial}{\partial t}\left(\Psi^*\frac{\partial\Psi}{\partial x}\right)dx$. But, noting that $\frac{\partial^2\Psi}{\partial x\partial t} = \frac{\partial^2\Psi}{\partial t\partial x}$ and using Eqs. 1.23-1.24:

$$\frac{\partial}{\partial t}\left(\Psi^*\frac{\partial\Psi}{\partial x}\right) = \frac{\partial\Psi^*}{\partial t}\frac{\partial\Psi}{\partial x} + \Psi^*\frac{\partial}{\partial x}\left(\frac{\partial\Psi}{\partial t}\right) = \left[-\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*\right]\frac{\partial\Psi}{\partial x} + \Psi^*\frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi\right]$$
$$= \frac{i\hbar}{2m}\left[\Psi^*\frac{\partial^3\Psi}{\partial x^3} - \frac{\partial^2\Psi^*}{\partial x^2}\frac{\partial\Psi}{\partial x}\right] + \frac{i}{\hbar}\left[V\Psi^*\frac{\partial\Psi}{\partial x} - \Psi^*\frac{\partial}{\partial x}(V\Psi)\right]$$

The first term integrates to zero, using integration by parts twice, and the second term can be simplified to $V\Psi^*\frac{\partial\Psi}{\partial x} - \Psi^*V\frac{\partial\Psi}{\partial x} - \Psi^*\frac{\partial V}{\partial x}\Psi = -|\Psi|^2\frac{\partial V}{\partial x}$. So

$$\frac{d\langle p\rangle}{dt} = -i\hbar\left(\frac{i}{\hbar}\right)\int -|\Psi|^2\frac{\partial V}{\partial x}dx = \langle -\frac{\partial V}{\partial x}\rangle. \qquad \text{QED}$$

Problem 1.8

Suppose Ψ satisfies the Schrödinger equation *without* V_0: $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$. We want to find the solution Ψ_0 *with* V_0: $i\hbar\frac{\partial\Psi_0}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi_0}{\partial x^2} + (V+V_0)\Psi_0$.

Claim: $\Psi_0 = \Psi e^{-iV_0t/\hbar}$.

Proof: $i\hbar\frac{\partial\Psi_0}{\partial t} = i\hbar\frac{\partial\Psi}{\partial t}e^{-iV_0t/\hbar} + i\hbar\Psi\left(-\frac{iV_0}{\hbar}\right)e^{-iV_0t/\hbar} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi\right]e^{-iV_0t/\hbar} + V_0\Psi e^{-iV_0t/\hbar}$
$= -\frac{\hbar^2}{2m}\frac{\partial^2\Psi_0}{\partial x^2} + (V+V_0)\Psi_0.$ QED

This has *no* effect on the expectation value of a dynamical variable, since the extra phase factor, being independent of x, cancels out in Eq. 1.36.

Problem 1.9

(a)

$$1 = 2|A|^2\int_0^\infty e^{-2amx^2/\hbar}dx = 2|A|^2\frac{1}{2}\sqrt{\frac{\pi}{(2am/\hbar)}} = |A|^2\sqrt{\frac{\pi\hbar}{2am}}; \quad \boxed{A = \left(\frac{2am}{\pi\hbar}\right)^{1/4}.}$$

(b)

$$\frac{\partial\Psi}{\partial t} = -ia\Psi; \quad \frac{\partial\Psi}{\partial x} = -\frac{2amx}{\hbar}\Psi; \quad \frac{\partial^2\Psi}{\partial x^2} = -\frac{2am}{\hbar}\left(\Psi + x\frac{\partial\Psi}{\partial x}\right) = -\frac{2am}{\hbar}\left(1 - \frac{2amx^2}{\hbar}\right)\Psi.$$

Plug these into the Schrödinger equation, $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$:

$$V\Psi = i\hbar(-ia)\Psi + \frac{\hbar^2}{2m}\left(-\frac{2am}{\hbar}\right)\left(1 - \frac{2amx^2}{\hbar}\right)\Psi$$
$$= \left[\hbar a - \hbar a\left(1 - \frac{2amx^2}{\hbar}\right)\right]\Psi = 2a^2mx^2\Psi, \quad \text{so} \quad \boxed{V(x) = 2ma^2x^2.}$$

(c)

$$\langle x\rangle = \int_{-\infty}^{\infty} x|\Psi|^2 dx = \boxed{0.} \qquad \text{[Odd integrand.]}$$

$$\langle x^2\rangle = 2|A|^2\int_0^{\infty} x^2 e^{-2amx^2/\hbar}dx = 2|A|^2\frac{1}{2^2(2am/\hbar)}\sqrt{\frac{\pi\hbar}{2am}} = \boxed{\frac{\hbar}{4am}.}$$

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \boxed{0.}$$

$$\begin{aligned}\langle p^2\rangle &= \int \Psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2\Psi dx = -\hbar^2\int\Psi^*\frac{\partial^2\Psi}{\partial x^2}dx \\ &= -\hbar^2\int\Psi^*\left[-\frac{2am}{\hbar}\left(1-\frac{2amx^2}{\hbar}\right)\Psi\right]dx = 2am\hbar\left\{\int|\Psi|^2dx - \frac{2am}{\hbar}\int x^2|\Psi|^2dx\right\} \\ &= 2am\hbar\left(1-\frac{2am}{\hbar}\langle x^2\rangle\right) = 2am\hbar\left(1-\frac{2am}{\hbar}\frac{\hbar}{4am}\right) = 2am\hbar\left(\frac{1}{2}\right) = \boxed{am\hbar.}\end{aligned}$$

(d)

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{\hbar}{4am} \Longrightarrow \boxed{\sigma_x = \sqrt{\frac{\hbar}{4am}}}; \quad \sigma_p^2 = \langle p^2\rangle - \langle p\rangle^2 = am\hbar \Longrightarrow \boxed{\sigma_p = \sqrt{am\hbar}.}$$

$\sigma_x\sigma_p = \sqrt{\frac{\hbar}{4am}}\sqrt{am\hbar} = \frac{\hbar}{2}$. This *is* (just barely) consistent with the uncertainty principle.

Problem 1.10

From Math Tables: $\pi = 3.141592653589793238462643\cdots$

(a)

$P(0)=0$	$P(1)=2/25$	$P(2)=3/25$	$P(3)=5/25$	$P(4)=3/25$
$P(5)=3/25$	$P(6)=3/25$	$P(7)=1/25$	$P(8)=2/25$	$P(9)=3/25$

In general, $P(j) = \frac{N(j)}{N}$.

(b) *Most probable*: $\boxed{3.}$ *Median*: 13 are ≤ 4, 12 are ≥ 5, so median is $\boxed{4.}$

Average: $\langle j\rangle = \frac{1}{25}[0\cdot 0 + 1\cdot 2 + 2\cdot 3 + 3\cdot 5 + 4\cdot 3 + 5\cdot 3 + 6\cdot 3 + 7\cdot 1 + 8\cdot 2 + 9\cdot 3]$

$= \frac{1}{25}[0+2+6+15+12+15+18+7+16+27] = \frac{118}{25} = \boxed{4.72.}$

(c) $\langle j^2\rangle = \frac{1}{25}[0 + 1^2\cdot 2 + 2^2\cdot 3 + 3^2\cdot 5 + 4^2\cdot 3 + 5^2\cdot 3 + 6^2\cdot 3 + 7^2\cdot 1 + 8^2\cdot 2 + 9^2\cdot 3]$

$= \frac{1}{25}[0+2+12+45+48+75+108+49+128+243] = \frac{710}{25} = \boxed{28.4.}$

$\sigma^2 = \langle j^2\rangle - \langle j\rangle^2 = 28.4 - 4.72^2 = 28.4 - 22.2784 = 6.1216;\quad \sigma = \sqrt{6.1216} = \boxed{2.474.}$

Problem 1.11

(a)

$$\frac{1}{2}mv^2 + V = E \quad \rightarrow \quad \boxed{v(x) = \sqrt{\frac{2}{m}\left(E - V(x)\right)}.}$$

(b)

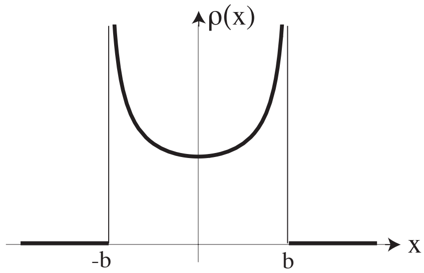
$$T = \int_a^b \frac{1}{\sqrt{\frac{2}{m}\left(E - \frac{1}{2}kx^2\right)}}\,dx = \sqrt{\frac{m}{k}}\int_a^b \frac{1}{\sqrt{(2E/k) - x^2}}\,dx.$$

Turning points: $v = 0 \Rightarrow E = V = \frac{1}{2}kb^2 \Rightarrow b = \sqrt{2E/k}$; $\quad a = -b$.

$$T = 2\sqrt{\frac{m}{k}}\int_0^b \frac{1}{\sqrt{b^2 - x^2}}\,dx = 2\sqrt{\frac{m}{k}}\sin^{-1}\left(\frac{x}{b}\right)\Big|_0^b = 2\sqrt{\frac{m}{k}}\sin^{-1}(1)$$
$$= 2\sqrt{\frac{m}{k}}\left(\frac{\pi}{2}\right) = \pi\sqrt{\frac{m}{k}}.$$

$$\rho(x) = \frac{1}{\pi\sqrt{\frac{m}{k}}\sqrt{\frac{2}{m}\left(E - \frac{1}{2}kx^2 1\right)}} = \boxed{\frac{1}{\pi\sqrt{b^2 - x^2}}.}$$

$$\int_a^b \rho(x)\,dx = \frac{2}{\pi}\int_0^b \frac{1}{\sqrt{b^2 - x^2}}\,dx = \frac{2}{\pi}\left(\frac{\pi}{2}\right) = 1. \checkmark$$



(c) $\boxed{\langle x \rangle = 0.}$

$$\langle x^2 \rangle = \frac{1}{\pi}\int_{-b}^b \frac{x^2}{\sqrt{b^2 - x^2}}\,dx = \frac{2}{\pi}\int_0^b \frac{x^2}{\sqrt{b^2 - x^2}}\,dx$$
$$= \frac{2}{\pi}\left[-\frac{x}{2}\sqrt{b^2 - x^2} + \frac{b^2}{2}\sin^{-1}\left(\frac{x}{b}\right)\right]\Big|_0^b = \frac{b^2}{\pi}\sin^{-1}(1) = \frac{b^2}{\pi}\frac{\pi}{2} = \frac{b^2}{2} = \boxed{\frac{E}{k}.}$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\langle x^2 \rangle} = \frac{b}{\sqrt{2}} = \boxed{\sqrt{\frac{E}{k}}.}$$

Problem 1.12

(a)

$$\rho(p)\,dp = \frac{dt}{T} = \frac{|dt/dp|\,dp}{T}$$

where dt is now the time it spends with momentum in the range dp (dt is intrinsically positive, but $dp/dt = F = -kx$ runs negative—hence the absolute value). Now

$$\frac{p^2}{2m} + \frac{1}{2}kx^2 = E \Rightarrow x = \pm\sqrt{\frac{2}{k}\left(E - \frac{p^2}{2m}\right)},$$

so

$$\rho(p) = \frac{1}{\pi\sqrt{\frac{m}{k}}\, k\sqrt{\frac{2}{k}\left(E - \frac{p^2}{2m}\right)}} = \boxed{\frac{1}{\pi\sqrt{2mE - p^2}}} = \frac{1}{\pi\sqrt{c^2 - p^2}},$$

where $c \equiv \sqrt{2mE}$. This is the same as $\rho(x)$ (Problem 1.11(b)), with c in place of b (and, of course, p in place of x).

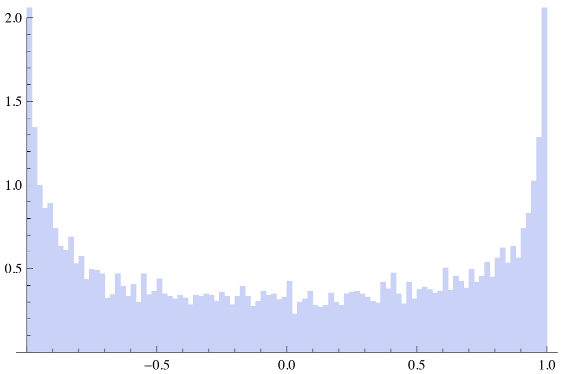
(b) From Problem 1.11(c), $\boxed{\langle p\rangle = 0,\ \langle p^2\rangle = \frac{c^2}{2},\ \sigma_p = \frac{c}{\sqrt{2}} = \sqrt{mE}\,.}$

(c) $\sigma_x\sigma_p = \sqrt{\frac{E}{k}}\sqrt{mE} = \boxed{\sqrt{\frac{m}{k}}\,E} = \frac{E}{\omega}$. If $E \geq \frac{1}{2}\hbar\omega$, then $\sigma_x\sigma_p \geq \frac{1}{2}\hbar$, which is precisely the Heisenberg uncertainty principle!

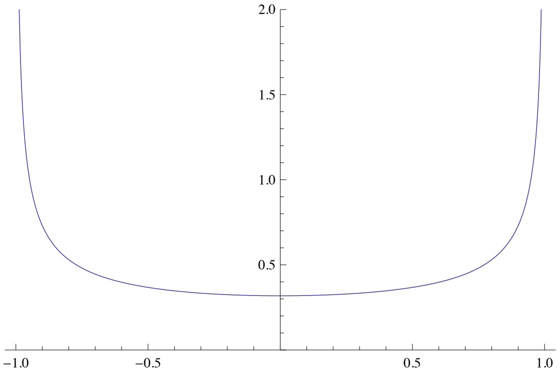
Problem 1.13

```
x[t_] := Cos[t]

snapshots = Table[x[π RandomReal[j]], {j, 10000}]
```

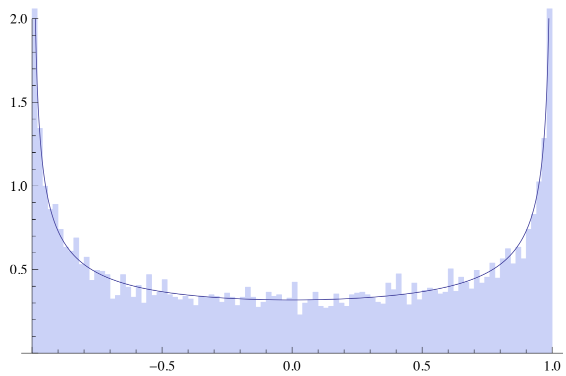
```
Histogram[snapshots, 100, "PDF", PlotRange → {0, 2}]
```



```
r[x_] := 1/(π Sqrt[1 - x^2])

Plot[ r[x], {x, -1, 1}, PlotRange -> {0, 2}]
```




```
Show[Histogram[snapshots, 100, "PDF", PlotRange → {0, 2}],
 Plot[ r[x], {x, -1, 1}, PlotRange -> {0, 2}]]
```



Problem 1.14

(a) $P_{ab}(t) = \int_a^b |\Psi(x,t)|^2 dx$, so $\frac{dP_{ab}}{dt} = \int_a^b \frac{\partial}{\partial t}|\Psi|^2 dx$. But (Eq. 1.25):

$$\frac{\partial|\Psi|^2}{\partial t} = \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\right] = -\frac{\partial}{\partial x}J(x,t).$$

$$\therefore \frac{dP_{ab}}{dt} = -\int_a^b \frac{\partial}{\partial x}J(x,t)dx = -\left[J(x,t)\right]\Big|_a^b = J(a,t) - J(b,t). \qquad \text{QED}$$

Probability is dimensionless, so J has the dimensions 1/time, and units $\boxed{\text{seconds}^{-1}.}$

(b) Here $\Psi(x,t) = f(x)e^{-iat}$, where $f(x) \equiv Ae^{-amx^2/\hbar}$, so $\Psi\frac{\partial\Psi^*}{\partial x} = fe^{-iat}\frac{df}{dx}e^{iat} = f\frac{df}{dx}$, and $\Psi^*\frac{\partial\Psi}{\partial x} = f\frac{df}{dx}$ too, so $\boxed{J(x,t) = 0.}$

Problem 1.15

Use Eqs. [1.23] and [1.24], and integration by parts:

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1^*\Psi_2\,dx &= \int_{-\infty}^{\infty}\frac{\partial}{\partial t}\left(\Psi_1^*\Psi_2\right)dx = \int_{-\infty}^{\infty}\left(\frac{\partial\Psi_1^*}{\partial t}\Psi_2+\Psi_1^*\frac{\partial\Psi_2}{\partial t}\right)dx \\ &= \int_{-\infty}^{\infty}\left[\left(\frac{-i\hbar}{2m}\frac{\partial^2\Psi_1^*}{\partial x^2}+\frac{i}{\hbar}V\Psi_1^*\right)\Psi_2+\Psi_1^*\left(\frac{i\hbar}{2m}\frac{\partial^2\Psi_2}{\partial x^2}-\frac{i}{\hbar}V\Psi_2\right)\right]dx \\ &= -\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\left(\frac{\partial^2\Psi_1^*}{\partial x^2}\Psi_2-\Psi_1^*\frac{\partial^2\Psi_2}{\partial x^2}\right)dx \\ &= -\frac{i\hbar}{2m}\left[\frac{\partial\Psi_1^*}{\partial x}\Psi_2\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{\partial\Psi_1^*}{\partial x}\frac{\partial\Psi_2}{\partial x}dx-\Psi_1^*\frac{\partial\Psi_2}{\partial x}\Big|_{-\infty}^{\infty}+\int_{-\infty}^{\infty}\frac{\partial\Psi_1^*}{\partial x}\frac{\partial\Psi_2}{\partial x}dx\right]=0.\ \text{QED}\end{aligned}$$

Problem 1.16

(a)

$$\begin{aligned}1 &= |A|^2\int_{-a}^{a}\left(a^2-x^2\right)^2dx = 2|A|^2\int_0^a\left(a^4-2a^2x^2+x^4\right)dx = 2|A|^2\left[a^4x-2a^2\frac{x^3}{3}+\frac{x^5}{5}\right]\Big|_0^a \\ &= 2|A|^2a^5\left(1-\frac{2}{3}+\frac{1}{5}\right)=\frac{16}{15}a^5|A|^2,\ \text{so}\ \boxed{A=\sqrt{\frac{15}{16a^5}}.}\end{aligned}$$

(b)

$$\langle x\rangle = \int_{-a}^{a}x|\Psi|^2\,dx = \boxed{0.}\quad \text{(Odd integrand.)}$$

(c)

$$\langle p\rangle = \frac{\hbar}{i}A^2\int_{-a}^{a}\left(a^2-x^2\right)\underbrace{\frac{d}{dx}\left(a^2-x^2\right)}_{-2x}dx = \boxed{0.}\quad \text{(Odd integrand.)}$$

Since we only know $\langle x\rangle$ at $t=0$ we cannot calculate $d\langle x\rangle/dt$ directly.

(d)

$$\begin{aligned}\langle x^2\rangle &= A^2\int_{-a}^{a}x^2\left(a^2-x^2\right)^2dx = 2A^2\int_0^a\left(a^4x^2-2a^2x^4+x^6\right)dx \\ &= 2\frac{15}{16a^5}\left[a^4\frac{x^3}{3}-2a^2\frac{x^5}{5}+\frac{x^7}{7}\right]\Big|_0^a = \frac{15}{8a^5}\left(a^7\right)\left(\frac{1}{3}-\frac{2}{5}+\frac{1}{7}\right) \\ &= \frac{\cancel{15}a^2}{8}\left(\frac{35-42+15}{\cancel{3}\cdot\cancel{5}\cdot 7}\right)=\frac{a^2}{8}\cdot\frac{8}{7}=\boxed{\frac{a^2}{7}.}\end{aligned}$$

(e)

$$\langle p^2\rangle = -A^2\hbar^2\int_{-a}^{a}\left(a^2-x^2\right)\underbrace{\frac{d^2}{dx^2}\left(a^2-x^2\right)}_{-2}dx = 2A^2\hbar^2 2\int_0^a\left(a^2-x^2\right)dx$$

$$= 4\cdot\frac{15}{16a^5}\hbar^2\left(a^2x-\frac{x^3}{3}\right)\Big|_0^a = \frac{15\hbar^2}{4a^5}\left(a^3-\frac{a^3}{3}\right) = \frac{15\hbar^2}{4a^2}\cdot\frac{2}{3} = \boxed{\frac{5}{2}\frac{\hbar^2}{a^2}.}$$

(f)

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{1}{7}a^2} = \boxed{\frac{a}{\sqrt{7}}.}$$

(g)

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{5}{2}\frac{\hbar^2}{a^2}} = \boxed{\sqrt{\frac{5}{2}}\frac{\hbar}{a}.}$$

(h)

$$\sigma_x\sigma_p = \frac{a}{\sqrt{7}}\cdot\sqrt{\frac{5}{2}}\frac{\hbar}{a} = \sqrt{\frac{5}{14}}\hbar = \sqrt{\frac{10}{7}}\frac{\hbar}{2} > \frac{\hbar}{2}. \checkmark$$

Problem 1.17

(a) Eq. 1.24 now reads $\frac{\partial\Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V^*\Psi^*$, and Eq. 1.25 picks up an extra term:

$$\frac{\partial}{\partial t}|\Psi|^2 = \cdots + \frac{i}{\hbar}|\Psi|^2(V^*-V) = \cdots + \frac{i}{\hbar}|\Psi|^2(V_0+i\Gamma-V_0+i\Gamma) = \cdots - \frac{2\Gamma}{\hbar}|\Psi|^2,$$

and Eq. 1.27 becomes $\frac{dP}{dt} = -\frac{2\Gamma}{\hbar}\int_{-\infty}^{\infty}|\Psi|^2dx = -\frac{2\Gamma}{\hbar}P.$ QED

(b)

$$\frac{dP}{P} = -\frac{2\Gamma}{\hbar}dt \Longrightarrow \ln P = -\frac{2\Gamma}{\hbar}t + \text{constant} \Longrightarrow \boxed{P(t) = P(0)e^{-2\Gamma t/\hbar},} \text{ so } \boxed{\tau = \frac{\hbar}{2\Gamma}.}$$

Problem 1.18

$$\frac{h}{\sqrt{3mk_BT}} > d \;\Rightarrow\; T < \frac{h^2}{3mk_Bd^2}.$$

(a) Electrons ($m = 9.1\times10^{-31}$ kg):

$$T < \frac{(6.6\times10^{-34})^2}{3(9.1\times10^{-31})(1.4\times10^{-23})(3\times10^{-10})^2} = \boxed{1.3\times10^5\text{ K.}}$$

Silicon nuclei ($m = 28m_p = 28(1.7\times10^{-27}) = 4.8\times10^{-26}$ kg):

$$T < \frac{(6.6\times10^{-34})^2}{3(4.8\times10^{-26})(1.4\times10^{-23})(3\times10^{-10})^2} = \boxed{2.4\text{ K.}}$$

(b) $PV = Nk_BT$; volume occupied by one molecule $(N = 1,\ V = d^3) \Rightarrow d = (k_BT/P)^{1/3}$.

$$T < \frac{h^2}{3mk_B}\left(\frac{P}{k_BT}\right)^{2/3} \Rightarrow T^{5/3} < \frac{h^2}{3m}\frac{P^{2/3}}{k_B^{5/3}} \Rightarrow T < \frac{1}{k_B}\left(\frac{h^2}{3m}\right)^{3/5} P^{2/5}.$$

For helium $(m = 4m_p = 6.8 \times 10^{-27}$ kg) at 1 atm $= 1.0 \times 10^5$ N/m^2:

$$T < \frac{1}{(1.4 \times 10^{-23})}\left(\frac{(6.6 \times 10^{-34})^2}{3(6.8 \times 10^{-27})}\right)^{3/5} (1.0 \times 10^5)^{2/5} = \boxed{2.8\text{ K.}}$$

For atomic hydrogen $(m = m_p = 1.7 \times 10^{-27}$ kg) with $d = 0.01$ m:

$$T < \frac{(6.6 \times 10^{-34})^2}{3(1.7 \times 10^{-27})(1.4 \times 10^{-23})(10^{-2})^2} = \boxed{6.2 \times 10^{-14}\text{ K.}}$$

At 3 K it is definitely in the classical regime.