# Introduction to Probability

## Detailed Solutions to Exercises

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# Preface

This collection of solutions is for reference for the instructors who use our book. The authors firmly believe that the best way to master new material is via problem solving. Having all the detailed solutions readily available would undermine this process. Hence, we ask that instructors not distribute this document to the students in their courses.

The authors welcome comments and corrections to the solutions. A list of corrections and clarifications to the textbook is updated regularly at the website `https://www.math.wisc.edu/asv/`

# Solutions to Chapter 1

**1.1.** One sample space is

$$\Omega = \{1, \ldots, 6\} \times \{1, \ldots, 6\} = \{(i, j) : i, j \in \{1, \ldots, 6\}\},$$

where we view order as mattering. Note that $\#\Omega = 6^2 = 36$. Since all outcomes are equally likely, we take $P(\omega) = \frac{1}{36}$ for each $\omega \in \Omega$. The event $A$ is

$$A = \left\{ \begin{array}{ccccc} (1,2), & (1,3), & (1,4), & (1,5), & (1,6) \\ & (2,3), & (2,4), & (2,5), & (2,6) \\ & & (3,4), & (3,5), & (3,6) \\ & & & (4,5), & (4,6) \\ & & & & (5,6) \end{array} \right\} = \{(i,j) : i, j \in \{1,2,3,4,5,6\}, i < j\},$$

and

$$P(A) = \frac{\#A}{\#\Omega} = \frac{15}{36}.$$

One way to count the number of elements in $A$ without explicitly writing them out is to note that for a first roll of $i \in \{1, 2, 3, 4, 5\}$, there are only $6 - i$ allowable rolls for the second. Hence,

$$\#A = \sum_{i=1}^{5} (6 - i) = 5 + 4 + 3 + 2 + 1 = 15.$$

**1.2.** (a) Since Bob has to choose exactly two options, $\Omega$ consists of the 2-element subsets of the set $\{\texttt{cereal}, \texttt{eggs}, \texttt{fruit}\}$:

$$\Omega = \{\{\texttt{cereal}, \texttt{eggs}\}, \{\texttt{cereal}, \texttt{fruit}\}, \{\texttt{eggs}, \texttt{fruit}\}\}$$

The items in Bob's breakfast do not come in any particular order, hence the outcomes are sets instead of ordered pairs.

(b) The two outcomes in the event $A$ are $\{\texttt{cereal}, \texttt{eggs}\}$ and $\{\texttt{cereal}, \texttt{fruit}\}$. In symbols,

$$A = \{\text{Bob's breakfast includes cereal}\} = \{\{\texttt{cereal}, \texttt{eggs}\}, \{\texttt{cereal}, \texttt{fruit}\}\}.$$

**1.3.** (a) This is a Cartesian product where the first factor covers the outcome of the coin flip ($\{H, T\}$ or $\{0, 1\}$, depending on how you want to encode heads and tails) and the second factor represents the outcome of the die. Hence

$$\Omega = \{0, 1\} \times \{1, 2, \ldots, 6\} = \{(i, j) : i = 0 \text{ or } 1 \text{ and } j \in \{1, 2, \ldots, 6\}\}.$$

(b) Now we need a larger Cartesian product space because the outcome has to contain the coin flip and die roll of each person. Let $c_i$ be the outcome of the coin flip of person $i$, and let $d_i$ be the outcome of the die roll of person $i$. Index $i$ runs from 1 to 10 (one index value for each person). Each $c_i \in \{0, 1\}$ and each $d_i \in \{1, 2, \ldots, 6\}$. Here are various ways of writing down the sample space:

$$\begin{aligned}\Omega &= (\{0, 1\} \times \{1, 2, \ldots, 6\})^{10} \\ &= \{(c_1, d_1, c_2, d_2, \ldots, c_{10}, d_{10}) : \text{each } c_i \in \{0, 1\} \text{ and each } d_i \in \{1, 2, \ldots, 6\}\} \\ &= \{(c_i, d_i)_{1 \le i \le 10} : \text{each } c_i \in \{0, 1\} \text{ and each } d_i \in \{1, 2, \ldots, 6\}\}.\end{aligned}$$

The last formula illustrates the use of indexing to shorten the writing of the 20-tuple of all outcomes. The number of elements is $\#\Omega = 2^{10} \cdot 6^{10} = 12^{10} =$ 61,917,364,224.

(c) If nobody rolled a five, then each die outcome $d_i$ comes from the set $\{1, 2, 3, 4, 6\}$ that has 5 elements. Hence the number of these outcomes is $2^{10} \cdot 5^{10} = 10^{10}$. To get the number of outcomes where at least 1 person rolls a five, subtract the number of outcomes where no one rolls a 5 from the total: $12^{10} - 10^{10} =$ 51,917,364,224.

**1.4.** (a) This is an example of sampling with replacement, where order matters. Thus, the sample space is

$$\Omega = \{\omega = (x_1, x_2, x_3) : x_i \in \{\text{states in the U.S.}\}\}.$$

In other words, each sample point is a 3-tuple or ordered triple of U.S. states.

The problem statement contains the assumption that every day each state is equally likely to be chosen. Since $\#\Omega = 50^3 =$125,000, each sample point $\omega$ has equal probability $P\{\omega\} = 50^{-3} = \frac{1}{125{,}000}$. This specifies the probability measure completely because then the probability of any event $A$ comes from the formula $P(A) = \frac{\#A}{125{,}000}$.

(b) The 3-tuple (Wisconsin, Minnesota, Florida) is a particular outcome, and hence as explained above,

$$P((\text{Wisconsin, Minnesota, Florida})) = \tfrac{1}{50^3}.$$

(c) The number of ways to have Wisconsin come on Monday and Tuesday, but not Wednesday is $1 \cdot 1 \cdot 49$, with similar expressions for the other combinations. Since there is only 1 way for Wisconsin to come each of the three days, we see the total number of positive outcomes is

$$1 \cdot 1 \cdot 49 + 1 \cdot 49 \cdot 1 + 49 \cdot 1 \cdot 1 + 1 = 3 \cdot 49 + 1 = 148.$$

Thus

$$\begin{aligned}&P(\text{Wisconsin's flag hung at least two of the three days}) \\ &\quad = \frac{3 \cdot 49 + 1}{50^3} = \frac{37}{31250} = 0.001184.\end{aligned}$$

**1.5.** (a) There are two natural sample spaces we can choose, depending upon whether or not we want to let order matter.

If we let the order of the numbers matter, then we may choose

$$\Omega_1 = \{(x_1, \dots, x_5) : x_i \in \{1, \dots, 40\}, x_i \neq x_j \text{ if } i \neq j\},$$

the set of ordered 5-tuples of distinct elements from the set $\{1, 2, 3, \dots, 40\}$. In this case $\#\Omega_1 = 40 \cdot 39 \cdot 38 \cdot 37 \cdot 36$ and $P_1(\omega) = \frac{1}{\#\Omega_1}$ for each $\omega \in \Omega_1$.

If we do not let order matter, then we take

$$\Omega_2 = \{\{x_1, \dots, x_5\} : x_i \in \{1, 2, 3, \dots, 40\}, x_i \neq x_j \text{ if } i \neq j\},$$

the set of 5-element subsets of the set $\{1, 2, 3, \dots, 40\}$. In this case $\#\Omega_2 = \binom{40}{5}$ and $P_2(\omega) = \frac{1}{\#\Omega_2}$ for each $\omega \in \Omega_2$.

(b) The correct calculation for this question depends on which sample space was chosen in part (a).

When order matters, we imagine filling the positions of the 5-tuple with three even and two odd numbers. There are $\binom{5}{3}$ ways to choose the positions of the three even numbers. The remaining two positions are for the two odd numbers. We fill these positions in order, separately for the even and odd numbers. There are $20 \cdot 19 \cdot 18$ ways to choose the even numbers and $20 \cdot 19$ ways to choose the odd numbers. This gives

$$P(\text{exactly three numbers are even}) = \frac{\binom{5}{3} \cdot 20 \cdot 19 \cdot 18 \cdot 20 \cdot 19}{40 \cdot 39 \cdot 38 \cdot 37 \cdot 36} = \frac{475}{1443}.$$

When order does not matter, we choose sets. There are $\binom{20}{3}$ ways to choose a set of three even numbers between 1 and 40, and $\binom{20}{2}$ ways to choose a set of two odd numbers. Therefore, the probability can be computed as

$$P(\text{exactly three numbers are even}) = \frac{\binom{20}{3} \cdot \binom{20}{2}}{\binom{40}{5}} = \frac{475}{1443}.$$

**1.6.** We give two solutions, first with an ordered sample, and then without order.

(a) Label the three green balls 1, 2, and 3, and label the yellow balls 4, 5, 6, and 7. We imagine picking the balls in order, and hence take

$$\Omega = \{(i, j) : i, j \in \{1, 2, \dots, 7\}, i \neq j\},$$

the set of ordered pairs of distinct elements from the set $\{1, 2, \dots, 7\}$. The event of two different colored balls is,

$$A = \{(i, j) : (i \in \{1, 2, 3\} \text{ and } j \in \{4, \dots, 7\}) \textbf{ or } (i \in \{4, \dots, 7\} \text{ and } j \in \{1, 2, 3\})\}.$$

(b) We have $\#\Omega = 7 \cdot 6 = 42$ and $\#A = 3 \cdot 4 + 4 \cdot 3 = 24$. Thus,

$$P(A) = \frac{24}{42} = \frac{4}{7}.$$

Alternatively, we could have chosen a sample space in which order does not matter. In this case the size of the sample space is $\binom{7}{2}$. There are $\binom{3}{1}$ ways to choose one of the green balls and $\binom{4}{1}$ ways to choose one yellow ball. Hence, the probability is computed as

$$P(A) = \frac{\binom{3}{1}\binom{4}{1}}{\binom{7}{2}} = \frac{4}{7}.$$

**1.7.** (a) Label the balls 1 through 7, with the green balls labeled 1, 2 and 3, and the yellow balls labeled 4, 5, 6 and 7. Let

$$\Omega = \{(i, j, k) : i, j, k \in \{1, 2, \dots, 7\}, i \neq j, j \neq k, i \neq k\},$$

which captures the idea that order matters for this problem. Note that $\#\Omega = 7 \cdot 6 \cdot 5$. There are exactly

$$3 \cdot 4 \cdot 2 = 24$$

ways to choose first a green ball, then a yellow ball, and then a green ball. Thus the desired probability is

$$P(\text{green, yellow, green}) = \frac{24}{7 \cdot 6 \cdot 5} = \frac{4}{35}.$$

(b) We can use the same reasoning as in the previous part, by accounting for all the different orders in which the colors can come:

$$\begin{aligned} P(\text{2 greens and one yellow}) &= P(\text{green, green, yellow}) \\ &\quad P(\text{green, yellow, green}) + P(\text{yellow, green, green}) \\ &= \frac{3 \cdot 2 \cdot 4 + 3 \cdot 4 \cdot 2 + 4 \cdot 3 \cdot 2}{7 \cdot 6 \cdot 5} = \frac{72}{210} = \frac{12}{35}. \end{aligned}$$

Alternatively, since this question does not require ordering the sample of balls, we can take

$$\Omega = \{\{i, j, k\} : i, j, k \in \{1, 2, \dots, 7\}, i \neq j, j \neq k, i \neq k\},$$

the set of 3-element subsets of the set $\{1, 2, \dots, 7\}$. Now $\#\Omega = \binom{7}{3}$. There are $\binom{3}{2}$ ways to choose 2 green balls from the 3 green balls, and $\binom{4}{1}$ ways to choose one yellow ball from the 4 yellow balls. So the desired probability is

$$P(\text{2 greens and one yellow}) = \frac{\binom{3}{2} \cdot \binom{4}{1}}{\binom{7}{3}} = \frac{12}{35}.$$

**1.8.** (a) Label the letters from 1 to 14 so that the first 5 are `E`s, the next 4 are `A`s, the next 3 are `N`s and the last 2 are `B`s.

Our $\Omega$ consists of (ordered) sequences of four distinct elements:

$$\Omega = \{(a_1, a_2, a_3, a_4) : a_i \neq a_j, a_i \in \{1, 2, \dots, 14\}\}.$$

The size of $\Omega$ is $14 \cdot 13 \cdot 12 \cdot 11 = 24024$. (Because we can choose $a_1$ 14 different ways, then $a_2$ 13 different ways and so on.)

The event $C$ consists of sequences $(a_1, a_2, a_3, a_4)$ consisting of two numbers between 1 and 5, one between 6 and 9 and one between 10 and 12. We can count these by constructing such a sequence step-by-step: we first choose the positions of the two `E`s: we can do that $\binom{4}{2} = 6$ ways. Then we choose a first `E` out of the 5 choices and place it to the first chosen position. Then we choose the second `E` out of the remaining 4 and place it to the second (remaining) chosen position. Then we choose the `A` out of the 4 choices, and its position (there are 2 possibilities left), Finally we choose the letter `N` out of the 3 choices and place it in the remaining position (we only have one possibility here). In each step the number of choices did not depend on the previous choices so we can just multiply the numbers together to get $6 \cdot 5 \cdot 4 \cdot 4 \cdot 2 \cdot 3 \cdot 1 = 2880$.

The probability of $C$ is

$$P(C) = \frac{\#C}{\#\Omega} = \frac{2880}{24024} = \frac{120}{1001}.$$

(b) As before, we label the letters from 1 to 14 so that the first 5 are `E`s, the next 4 are `A`s, the next 3 are `N`s and the last 2 are `B`s. Our $\Omega$ is the set of unordered samples of size 4, or in other words: all subsets of $\{1, 2, \dots, 14\}$ of size 4:

$$\Omega = \{\{a_1, a_2, a_3, a_4\} : a_i \neq a_j, a_i \in \{1, 2, \dots, 14\}\}.$$

The size of $\Omega$ is $\binom{14}{4} = 1001$.

The event $C$ is that $\{a_1, a_2, a_3, a_4\}$ has two numbers between 1 and 5, one between 6 and 9 and one between 10 and 12. The number of ways we can choose such a set is $\binom{5}{2}\binom{4}{1}\binom{3}{1} = 120$. (Because we can choose the two `E`s out of 5 possibilities, the single `A` out of 4 possibilities and the single `N` out of 3 possibilities.)

This gives

$$P(C) = \frac{\#C}{\#\Omega} = \frac{120}{1001},$$

the same as in part (a).

**1.9.** We model the point at which the stick is broken as being chosen uniformly at random along the length of the stick, which we take to be $L$ (in some arbitrary units). Thus, $\Omega = [0, L]$. The event we care about is $A = \{\omega \in \Omega : \omega \leq L/5 \text{ or } \omega \geq 4L/5\}$. Hence, since the two events are mutually exclusive,

$$P(A) = P\{\omega \in [0, L] : \omega \leq L/5\} + P\{\omega \in [0, L] : \omega \geq 4L/5\} = \frac{L/5}{L} + \frac{L/5}{L} = \frac{2}{5}.$$

**1.10.** (a) Since the outcome of the experiment is the number of times we roll the die (as in Example 1.16), we take

$$\Omega = \{\infty, 1, 2, 3, \dots\}.$$

Element $k$ in $\Omega$ means that it took $k$ rolls to see the first four. Element $\infty$ means that four never appeared.

Next we deduce the probability measure $P$ on $\Omega$. Since $\Omega$ is a discrete sample space (countably infinite), $P$ is determined by giving the probabilities of all the individual sample points.

For an integer $k \geq 1$, we have

$$P(k) = P\{\text{needed } k \text{ rolls}\} = P\{\text{no fours in the first } k-1 \text{ rolls, then a } 4\}.$$

Each roll has 6 outcomes so the total number of outcomes from $k$ rolls is $6^k$. Each roll can fail to be a four in 5 ways. Hence by taking the ratio of the number of favorable outcomes over the total number of outcomes,

$$P(k) = P\{\text{no fours in the first } k-1 \text{ rolls, then a } 4\} = \frac{5^{k-1} \cdot 1}{6^k} = \left(\frac{5}{6}\right)^{k-1} \frac{1}{6}.$$

To complete the specification of the measure $P$, we find the value $P(\infty)$. Since the outcomes are mutually exclusive,

$$\begin{aligned}
1 = P(\Omega) &= P(\infty) + \sum_{k=1}^{\infty} P(k) \\
&= P(\infty) + \sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^{k-1} \frac{1}{6} \\
\text{(reindex)} \qquad &= P(\infty) + \frac{1}{6} \sum_{j=0}^{\infty} \left(\frac{5}{6}\right)^{j} \\
\text{(geometric series)} \qquad &= P(\infty) + \frac{1}{6} \cdot \frac{1}{1 - 5/6} \\
&= P(\infty) + 1.
\end{aligned}$$

Thus, $P(\infty) = 0$.

(b) We already deduced above that

$$P(\text{the number four never appears}) = P(\infty) = 0.$$

Here is an alternative solution.

$$P(\text{the number four never appears}) \le P(\text{no fours in the first } n \text{ rolls}) = \left(\frac{5}{6}\right)^n.$$

Since $(\frac{5}{6})^n \to 0$ as $n \to \infty$ and the inequality holds for any $n$, the probability on the left must be zero.

**1.11.** The sample space $\Omega$ that represents the dartboard itself is a square of side length 20 inches. We can assume that the center of the board is at the origin. The event $A$, that the dart hits within 2 inches of the center, is then the subset of $\Omega$ described by $A = \{x : |x| \le 2\}$. Probability is now proportional to area, and so

$$P(A) = \frac{\text{area of } A}{\text{area of the board}} = \frac{\pi \cdot 2^2}{20^2} = \frac{\pi}{100}.$$

**1.12.** The sample space and probability measure for this experiment were described in the solution to Exercise 1.10: $P(k) = (\frac{5}{6})^{k-1}\frac{1}{6}$ for positive integers $k$.

(a) $P(\text{need at most 3 rolls}) = P(1) + P(2) + P(3) = \frac{1}{6}\big(1 + \frac{5}{6} + (\frac{5}{6})^2\big) = \frac{91}{216}$.

(b)

$$\begin{aligned}
P(\text{even number of rolls}) &= \sum_{m=1}^{\infty} P(2m) = \sum_{m=1}^{\infty} (\tfrac{5}{6})^{2m-1}\tfrac{1}{6} = \tfrac{1}{5} \sum_{m=1}^{\infty} (\tfrac{25}{36})^m \\
&= \tfrac{1}{5} \cdot \frac{\frac{25}{36}}{1 - \frac{25}{36}} = \tfrac{5}{11}.
\end{aligned}$$

**1.13.** (a) Imagine selecting one student uniformly at random from the school. Thus, $\Omega$ is the set of students and each outcome is equally likely. Let $W$ be the subset of $\Omega$ consisting of those students who wear a watch. Let $B$ be the subset of students who wear a bracelet. We are told that

$$P(W^c B^c) = 0.6, \quad P(W) = 0.25, \quad P(B) = 0.30.$$

We are asked for $P(W \cup B)$. By de Morgan (or a Venn Diagram) we have

$$P(W \cup B) = 1 - P((W \cup B)^c) = 1 - P(W^c B^c) = 1 - 0.6 = 0.4.$$

(b) We want $P(W \cap B)$. We have

$$P(W \cap B) = P(W) + P(B) - P(W \cup B) = 0.25 + 0.30 - 0.4 = 0.15.$$

**1.14.** From the inclusion-exclusion principle we get

$$P(A \cup B) = P(A) + P(B) - P(AB) = 0.4 + 0.7 - P(AB) = 1.1 - P(AB).$$

Rearranging this we get $P(AB) = 1.1 - P(A \cup B)$.

Since $P(A \cup B)$ is a probability, it is at most 1, so

$$P(AB) = 1.1 - P(A \cup B) \geq 1.1 - 1 = 0.1.$$

On the other hand, $B \subset A \cup B$ so $P(A \cup B) \geq P(B) = 0.7$ which gives

$$P(AB) = 1.1 - P(A \cup B) \leq 1.1 - 0.7 = 0.4.$$

Putting these together we get $0.1 \leq P(AB) \leq 0.4$.

**1.15.** (a) The event that one of the colors does not appear is $W \cup G \cup R$. If we use the inclusion-exclusion principle then

$$P(W \cup G \cup R) = P(W) + P(G) + P(R) - P(WG) - P(GR) - P(RW) + P(WGR).$$

We compute each term on the right-hand side. Note that the we can label the 4 balls so that we can differentiate between the 2 red balls. This way the three draws lead to equally likely outcomes, each with probability $\frac{1}{4^3}$.

We have

$$P(W) = P(\text{each pick is green or red}) = \frac{3^3}{4^3}$$

and similarly $P(G) = \frac{3^3}{4^3}$ and $P(R) = \frac{2^3}{4^3}$. Also:

$$P(WG) = P(\text{each pick is red}) = \frac{2^3}{4^3}$$

and similarly $P(GR) = \frac{1}{4^3}$ and $P(RW) = \frac{1}{4^3}$. Finally, $P(WGR) = 0$, since it is not possible to have none of the colors in the sample.

Putting everything together:

$$P(W \cup G \cup R) = \frac{1}{4^3}(3^3 + 3^3 + 2^3 - 2^3 - 1 - 1) = \frac{13}{16}.$$

(b) The complement of the event is $\{\text{all three colors appear}\}$. Let us count how many different ways we can get such an outcome. We have 2 choices to decide which red ball will show up, while there is only one possibility for the green and the white. Then there are $3! = 6$ different ways we can order the three colors. This gives $2 \cdot 6 = 12$ possibilities. Thus

$$P(\text{all three colors appear}) = \frac{12}{4^3} = \frac{3}{16}$$

from which

$$P(\text{one of the colors does not appear}) = 1 - P(\text{all three colors appear}) = \frac{13}{16}.$$