Solutions Manual to accompany

AN INTRODUCTION TO MECHANICS

2nd edition

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corrected November 2014

KLEPPNER / KOLENKOW

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NOTE

Corrected solutions through November 2014 are collected in the Appendix.

CONTENTS

1 VECTORS AND KINEMATICS

1.1 *Vector algebra 1*

$\mathbf{A} = (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}) \quad \mathbf{B} = (5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

(a) $\mathbf{A} + \mathbf{B} = (2 + 5)\hat{\mathbf{i}} + (-3 + 1)\hat{\mathbf{j}} + (7 + 2)\hat{\mathbf{k}} = 7\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 9\hat{\mathbf{k}}$

(b) $\mathbf{A} - \mathbf{B} = (2 - 5)\hat{\mathbf{i}} + (-3 - 1)\hat{\mathbf{j}}(7 - 2)\hat{\mathbf{k}} = -3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$

(c) $\mathbf{A} \cdot \mathbf{B} = (2)(5) + (-3)(1) + (7)(2) = 21$

(d) $$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -3 & 7 \\ 5 & 1 & 2 \end{vmatrix}$$

$$= -13\hat{\mathbf{i}} + 31\hat{\mathbf{j}} + 17\hat{\mathbf{k}}$$

1.2 *Vector algebra 2*

$\mathbf{A} = (3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) \quad \mathbf{B} = (6\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$

(a) $A^2 = \mathbf{A} \cdot \mathbf{A} = 3^2 + (-2)^2 + 5^2 = 38$

(b) $B^2 = \mathbf{B} \cdot \mathbf{B} = 6^2 + (-7)^2 + 4^2 = 101$

(c) $(\mathbf{A} \cdot \mathbf{B})^2 = [(3)(6) + (-2)(-7) + (5)(4)]^2 = [18 + 14 + 20]^2 = 52^2 = 2704$

1.3 *Cosine and sine by vector algebra*

$\mathbf{A} = (3\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) \quad \mathbf{B} = (-2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$

(a)

$$\mathbf{A}\cdot\mathbf{B} = A\,B\cos(\mathbf{A},\mathbf{B})$$

$$\cos(\mathbf{A},\mathbf{B}) = \frac{\mathbf{A}\cdot\mathbf{B}}{A\,B} = \frac{(-6+1+1)}{\sqrt{(9+1+1)}\sqrt{4+1+1)}} = \frac{-4}{\sqrt{11}\sqrt{6}} \approx 0.492$$

(b) *method 1:*

$$|\mathbf{A}\times\mathbf{B}| = A\,B\sin(\mathbf{A},\mathbf{B})$$

$$\sin(\mathbf{A},\mathbf{B}) = \frac{|\mathbf{A}\times\mathbf{B}|}{A\,B}$$

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 1 & 1 \\ -2 & 1 & 1 \end{vmatrix} = (1-1)\hat{\mathbf{i}} - (3+2)\hat{\mathbf{j}} + (3+2)\hat{\mathbf{k}} = -5\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$$

$$|\mathbf{A}\times\mathbf{B}| = \sqrt{5^2+5^2} = 5\sqrt{2}$$

$$\sin(\mathbf{A},\mathbf{B}) = \frac{|\mathbf{A}\times\mathbf{B}|}{A\,B} = \frac{5\sqrt{2}}{\sqrt{11}\sqrt{6}} \approx 0.870$$

(c) *method 2 (simpler) – use:*

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin(\mathbf{A},\mathbf{B}) = \sqrt{1-\cos^2(\mathbf{A},\mathbf{B})} = \sqrt{1-(0.492)^2} \quad \text{from (a)} \approx 0.871$$

1.4 *Direction cosines*

Note that here α, β, γ stand for direction cosines, not for the angles shown in the figure:
$\theta_x = \cos^{-1}\alpha$,
$\theta_y = \cos^{-1}\beta$,
$\theta_z = \cos^{-1}\gamma$.

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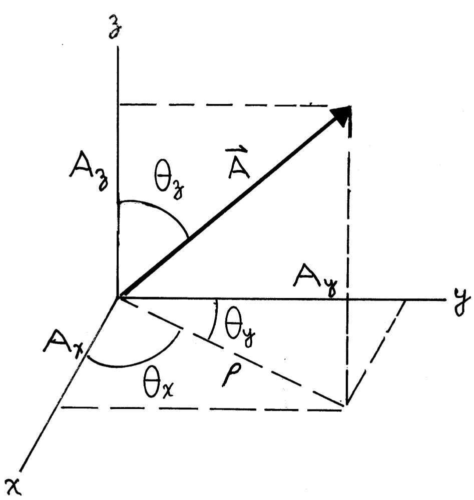


$$\begin{aligned}
\mathbf{A} &= A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}} \\
A_x &= \mathbf{A}\cdot\hat{\mathbf{i}} \;= A\;\cos(\mathbf{A},\hat{\mathbf{i}}) \equiv A\,\alpha \\
\alpha &= \cos(\mathbf{A},\hat{\mathbf{i}}) \;= \cos\theta_x.
\end{aligned}$$

Similarly,

$$\begin{aligned}
A_y &= A\;\cos(\mathbf{A},\hat{\mathbf{j}}) \;\equiv A\,\beta \\
\beta &= \cos(\mathbf{A},\hat{\mathbf{j}}) \;= \cos\theta_y \\
A_z &= A\;\cos(\mathbf{A},\hat{\mathbf{k}}) \;\equiv A\,\gamma \\
\gamma &= \cos(\mathbf{A},\hat{\mathbf{k}}) \;= \cos\theta_z
\end{aligned}$$

Using these results,

$$\begin{aligned}
A^2 &= A_x^2 + A_y^2 + A_z^2 \\
&= A^2\,(\alpha^2 + \beta^2 + \gamma^2)
\end{aligned}$$

from which it follows that

$\alpha^2 + \beta^2 + \gamma^2 = 1$

Another way to see this is

$$A^2 = \rho^2 + A_z^2 \;= A_x^2 + A_y^2 + A_z^2 \;= A^2\,(\alpha^2 + \beta^2 + \gamma^2)$$

and it follows as before that

$\alpha^2 + \beta^2 + \gamma^2 = 1.$

1.5 *Perpendicular vectors*

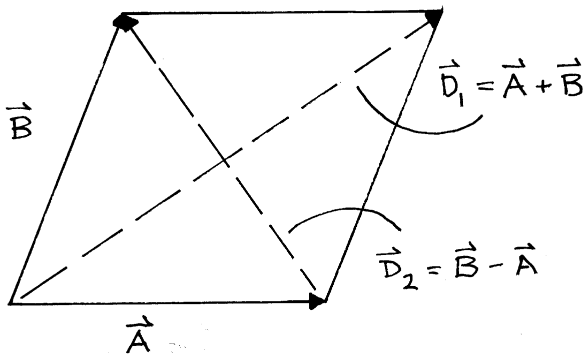
Given $|\mathbf{A}-\mathbf{B}| = |\mathbf{A}+\mathbf{B}|$ with $\mathbf{A}$ and $\mathbf{B}$ nonzero. Evaluate the magnitudes by squaring.

$$\begin{gathered}
A^2 - 2\,\mathbf{A}\cdot\mathbf{B} + B^2 = A^2 + 2\,\mathbf{A}\cdot\mathbf{B} + B^2 \\
-2\,\mathbf{A}\cdot\mathbf{B} = +2\,\mathbf{A}\cdot\mathbf{B}. \\
\mathbf{A}\cdot\mathbf{B} = 0
\end{gathered}$$

and it follows that $\mathbf{A} \perp \mathbf{B}$.

1.6 *Diagonals of a parallelogram*

The parallelogram is equilateral, so $A = B$.

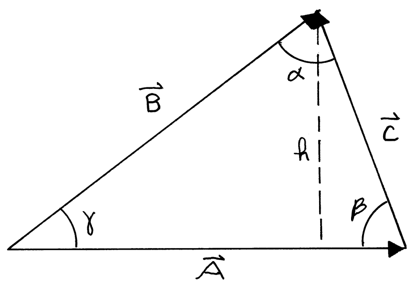


$$\mathbf{D_1} = \mathbf{A} + \mathbf{B}$$
$$\mathbf{D_2} = \mathbf{B} - \mathbf{A}$$
$$\mathbf{D_1} \cdot \mathbf{D_2} = (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{B} - \mathbf{A}) = A^2 - B^2 = 0.$$

Hence $\mathbf{D_1} \cdot \mathbf{D_2} = \mathbf{0}$ and it follows that $\mathbf{D_1} \perp \mathbf{D_2}$.

1.7 *Law of sines*

The area $\mathcal{A}$ of the triangle is



$$\mathcal{A} = \frac{1}{2}A\,h = \frac{1}{2}A\,B\,\sin\gamma = \frac{1}{2}\,|\mathbf{A} \times \mathbf{B}|$$

Similarly,

$$\mathcal{A} = \frac{1}{2}|\mathbf{B} \times \mathbf{C}| = \frac{1}{2}BC\sin\alpha$$
$$\mathcal{A} = \frac{1}{2}|\mathbf{C} \times \mathbf{A}| = \frac{1}{2}AC\sin\beta.$$

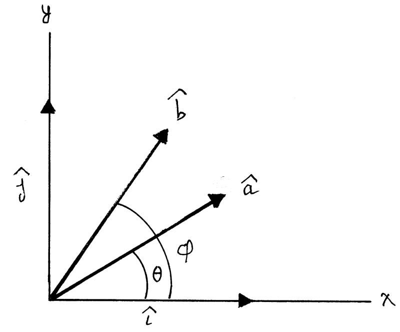
Hence $AB\ \sin\gamma = BC\ \sin\alpha = AC\ \sin\beta$, from which it follows

$$\frac{\sin\gamma}{C} = \frac{\sin\alpha}{A} = \frac{\sin\beta}{B}$$

Introducing the cross product makes the notation convenient, and emphasizes the relation between the cross product and the area of the triangle, but it is not essential for the proof.

1.8 *Vector proof of a trigonometric identity*

Given two unit vectors $\hat{\mathbf{a}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$ and $\hat{\mathbf{b}} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}}$, with $a = 1, b = 1$. First evaluate their scalar product using components:



$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= ab\ \cos\theta\cos\phi + ab\ \sin\theta\sin\phi\\ &= \cos\theta\cos\phi + \sin\theta\sin\phi\end{aligned}$$

then evaluate their scalar product geometrically.

$$\mathbf{a}\cdot\mathbf{b} = ab\ \cos(\mathbf{a},\mathbf{b})\ = ab\cos(\phi-\theta) = \cos(\phi-\theta)$$

Equating the two results,

$$\cos(\phi-\theta) = \cos\phi\cos\theta + \sin\phi\sin\theta$$

1.9 *Perpendicular unit vector*

Given $\mathbf{A} = (\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}})$ and $\mathbf{B} = (\mathbf{2}\,\hat{\mathbf{i}}+\hat{\mathbf{j}}-\mathbf{3}\,\hat{\mathbf{k}})$, find $\mathbf{C}$ such that $\mathbf{A}\cdot\mathbf{C} = \mathbf{0}$ and $\mathbf{B}\cdot\mathbf{C} = \mathbf{0}$.

$$\begin{aligned}\mathbf{C} &= C_x\,\hat{\mathbf{i}} + C_y\,\hat{\mathbf{j}} + C_z\,\hat{\mathbf{k}}\\ &= C_x(\hat{\mathbf{i}} + (C_y/C_x)\,\hat{\mathbf{j}} + (C_z/C_x)\,\hat{\mathbf{k}})\\ \mathbf{A}\cdot\mathbf{C} &= C_x(1 + (C_y/C_x) - (C_z/C_x)) = 0\\ \mathbf{B}\cdot\mathbf{C} &= C_x(2 + (C_y/C_x) - 3(C_z/C_x)) = 0\end{aligned}$$

We have two equations for the two unknowns (C_y/C_x) and (C_z/C_x).

$$\begin{aligned}1 + (C_y/C_x) - (C_z/C_x) &= 0\\ 2 + (C_y/C_x) - 3(C_z/C_x) &= 0.\end{aligned}$$

The solutions are $(C_y/C_x) = -\frac{1}{2}$ and $(C_z/C_x) = \frac{1}{2}$, so that $\mathbf{C} = \mathbf{C_x}(\hat{\mathbf{i}} - \frac{\mathbf{1}}{\mathbf{2}}\hat{\mathbf{j}} + \frac{\mathbf{1}}{\mathbf{2}}\hat{\mathbf{k}})$. To evaluate C_x, apply the condition that $\mathbf{C}$ is a unit vector.

$$\begin{aligned}C^2 &= \frac{3}{2}C_x^2\ = 1\\ C_x &= \pm\sqrt{(2/3)}\\ \hat{\mathbf{C}} &= \pm\sqrt{(2/3)}\,(\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}} + \frac{1}{2}\hat{\mathbf{k}})\end{aligned}$$

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which can be written

$$\hat{\mathbf{C}} = \pm\frac{1}{\sqrt{6}}(2\,\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})$$

Geometrically, $\mathbf{C}$ can be perpendicular to both $\mathbf{A}$ and $\mathbf{B}$ only if $\mathbf{C}$ is perpendicular to the plane determined by $\mathbf{A}$ and $\mathbf{B}$. From the standpoint of vector algebra, this implies that $\mathbf{C} \propto \mathbf{A} \times \mathbf{B}$. To prove this, evaluate $\mathbf{A} \times \mathbf{B}$.

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & -1 \\ 2 & 1 & -3 \end{vmatrix} \\ &= -2\,\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}} \\ &\propto \mathbf{C}.\end{aligned}$$

1.10 *Perpendicular unit vectors*

Given $\mathbf{A} = \mathbf{3\hat{i}} + \mathbf{4\hat{j}} - \mathbf{4\hat{k}}$, find a unit vector $\hat{\mathbf{B}}$ perpendicular to $\mathbf{A}$.

(a)

$$\begin{aligned}\mathbf{B} &= B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} \ = B_x[\hat{\mathbf{i}} + (B_y/B_x)\hat{\mathbf{j}}] \\ \mathbf{A}\cdot\mathbf{B} &= B_x[3 + 4(B_y/B_x)] \ = 0 \\ B_y/B_x &= -3/4 \\ \mathbf{B} &= B_x[\hat{\mathbf{i}} - \frac{3}{4}\,\hat{\mathbf{j}}]\end{aligned}$$

To evaluate B_x, note that $\mathbf{B}$ is a unit vector, $B^2 = 1$.

$$1 = B_x^2\left[(1)^2 + \left(\frac{3}{4}\right)^2\right] \ = \left(\frac{25}{16}\right)B_x^2$$

which gives

$$\begin{aligned}B_x &= \pm(4/5) \\ \hat{\mathbf{B}} &= \pm(4/5)(\hat{\mathbf{i}} - (3/4)\hat{\mathbf{j}}) \ = \pm\frac{1}{5}(4\,\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}})\end{aligned}$$

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(b)

$$\begin{aligned}
\mathbf{C} &= C_x\,\hat{\mathbf{i}} + C_y\,\hat{\mathbf{j}} + C_z\,\hat{\mathbf{k}} \\
&= C_x[\hat{\mathbf{i}} + (C_y/C_x)\,\hat{\mathbf{j}} + (C_z/C_x)\,\hat{\mathbf{k}}]
\end{aligned}$$

$$\mathbf{A}\cdot\mathbf{C} = 0 \;\Rightarrow\; C_x[3 + 4(C_y/C_x) - 4(C_z/C_x)] = 0$$

$$\mathbf{B}\cdot\mathbf{C} = 0 \;\Rightarrow\; \frac{1}{5}C_x[4 - 3(C_y/C_x)] = 0$$

$$C_y/C_x = 4/3 \quad C_z/C_x = 25/12$$

To make $\mathbf{C}$ a unit vector,

$$C^2 = C_x^2\left[(1)^2 + \left(\frac{4}{3}\right)^2 + \left(\frac{25}{12}\right)^2\right] = 1$$

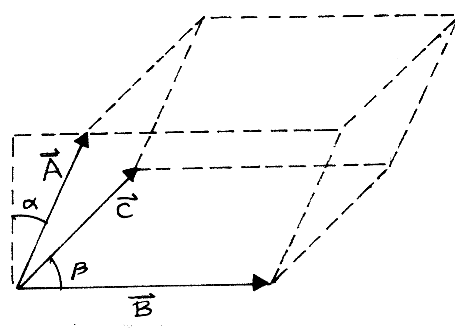
$$C_x \approx \pm 0.348$$

(c) The vector $\mathbf{B}\times\mathbf{C}$ is perpendicular (normal) to the plane defined by $\mathbf{B}$ and $\mathbf{C}$, so we want to prove

$$\mathbf{A} \propto \mathbf{B}\times\mathbf{C}$$

$$\begin{aligned}
\mathbf{B}\times\mathbf{C} &= C_x\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{4}{5} & -\frac{3}{5} & 0 \\ 1 & \frac{4}{3} & \frac{25}{12}\end{vmatrix} \\
&= C_x\left[-\left(\frac{75}{60}\right)\hat{\mathbf{i}} - \left(\frac{100}{60}\right)\hat{\mathbf{j}} + \left(\frac{25}{15}\right)\hat{\mathbf{k}}\right] \\
&= \left(\frac{5}{12}\right)C_x(-3\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}} + 4\,\hat{\mathbf{k}}) \;\propto \mathbf{A}.
\end{aligned}$$

1.11 *Volume of a parallelepiped*



With reference to the sketch, the height is $A\cos\alpha$, so the frontal area is $AB\cos\alpha$. The depth is $C\sin\beta$, so the volume V is

$$V = (AB\cos\alpha)(C\sin\beta) = (A\cos\alpha)(BC\sin\beta) = \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$$

The same approach can be used starting with a different face.

$$V = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) \quad V = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$$

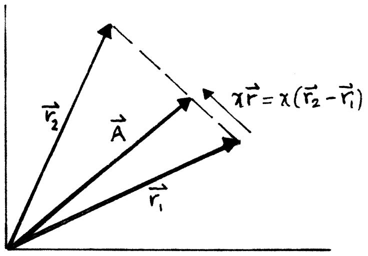
Note that $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are arbitrary vectors. This proves the vector identity

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$$

1.12 *Constructing a vector to a point*

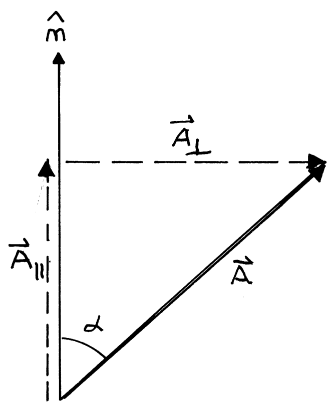
Applying vector addition to the lower triangle in the sketch,

$$\mathbf{A} = \mathbf{r}_1 + x(\mathbf{r}_2 - \mathbf{r}_1)$$
$$= (1 - x)\mathbf{r}_1 + x\mathbf{r}_2$$



1.13 *Expressing one vector in terms of another*

We will express vector $\mathbf{A}$ in terms of a unit vector $\hat{\mathbf{n}}$. As shown in the sketch, we can write $\mathbf{A}$ as the vector sum of a vector $\mathbf{A}_{\|}$ parallel to $\hat{\mathbf{n}}$ and a vector $\mathbf{A}_{\perp}$ perpendicular to $\hat{\mathbf{n}}$, so that $\mathbf{A} = \mathbf{A}_{\|} + \mathbf{A}_{\perp}$.



$$|\mathbf{A}_{\|}| = A\cos\alpha$$

The direction of $\mathbf{A}_{\|}$ is along $\hat{\mathbf{n}}$, so it follows that

$$\mathbf{A}_{\|} = (\mathbf{A}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}.$$
$$|\mathbf{A}_{\perp}| = A\sin\alpha = |\hat{\mathbf{n}}\times\mathbf{A}|$$

The direction of $(\hat{\mathbf{n}}\times\mathbf{A})$ is into the paper, so taking its cross product with $\hat{\mathbf{n}}$ gives a vector $(\hat{\mathbf{n}}\times\mathbf{A})\times\hat{\mathbf{n}}$ along $\mathbf{A}_{\perp}$ and with the correct magnitude. Hence

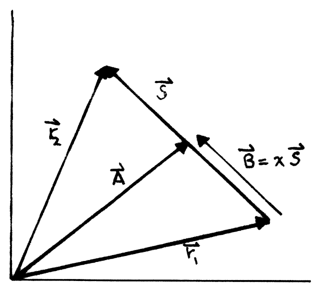
$$\mathbf{A} = (\mathbf{A}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + (\hat{\mathbf{n}}\times\mathbf{A})\times\hat{\mathbf{n}}$$

1.14 *Two points*

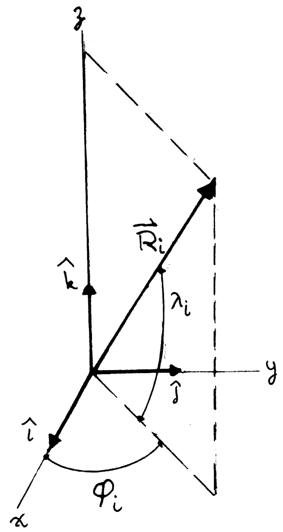
$$\mathbf{S} = \mathbf{r}_2 - \mathbf{r}_1 \qquad \mathbf{B} = x\mathbf{S} \qquad \mathbf{A} = \mathbf{r}_1 + \mathbf{B}$$

$x = 0$ at $t = 0$; $x = 1$ at $t = T$
so that $x = t/T$, linear in t

$$\mathbf{A} = \mathbf{r}_1 + x\mathbf{S} = \mathbf{r}_1 + \frac{t}{T}(\mathbf{r}_2 - \mathbf{r}_1) = \left(1 - \frac{t}{T}\right)\mathbf{r}_1 + \frac{t}{T}\mathbf{r}_2$$



1.15 *Great circle*



Consider vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ from the center of a sphere of radius R to points on the surface. To avoid complications, the sketch shows the geometry of a generic vector $\mathbf{R_i}$ (i = 1 or 2) making angles λ_i and ϕ_i. The magnitude of $\mathbf{R_i}$ is R, so $R_1 = R_2 = R$. The coordinates of a point on the surface are

$$\mathbf{R}_i = R\cos\lambda_i\cos\phi_i\,\hat{\mathbf{i}} + R\cos\lambda_i\sin\phi_i\,\hat{\mathbf{j}} + R\sin\lambda_i\,\hat{\mathbf{k}}$$

The angle between two points can be found using the dot product.

$$\theta(1,2) = \arccos\left(\frac{\mathbf{R_1}\cdot\mathbf{R_2}}{R_1R_2}\right) = \arccos\left(\frac{\mathbf{R_1}\cdot\mathbf{R_2}}{R^2}\right)$$

Note that $\theta(1,2)$ is in radians.

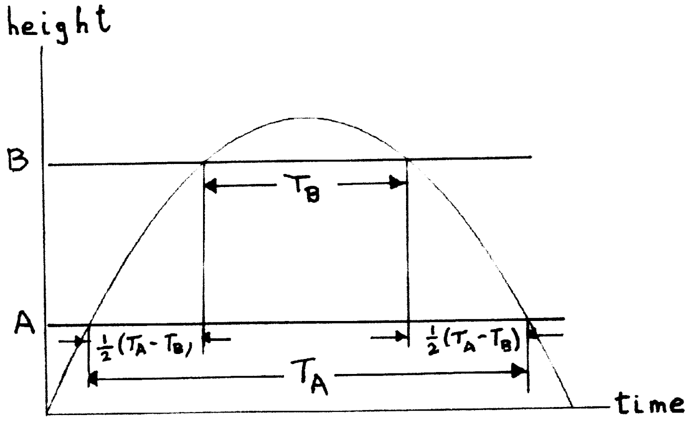
The great circle distance between $\mathbf{R_1}$ and $\mathbf{R_2}$ is $S = R\theta(1,2)$.

$$\mathbf{R_1}\cdot\mathbf{R_2} = R^2(\cos\lambda_1\cos\phi_1\cos\lambda_2\cos\phi_2 + \cos\lambda_1\sin\phi_1\cos\lambda_2\sin\phi_2 + \sin\lambda_1\sin\lambda_2)$$

Hence

$$\begin{aligned}
S &= R\,\theta(1,2)\\
&= R\arccos\left[\cos\lambda_1\cos\lambda_2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2) + \sin\lambda_1\sin\lambda_2\right]\\
&= R\arccos\left\{\frac{1}{2}\cos(\lambda_1+\lambda_2)\left[\cos(\phi_1-\phi_2)-1\right] + \frac{1}{2}\cos(\lambda_1-\lambda_2)\left[\cos(\phi_1-\phi_2)+1\right]\right\}
\end{aligned}$$

1.16 *Measuring g*



The motion is free fall with uniform acceleration, so the trajectory is a parabola, as shown in the sketch. Take the initial conditions at T=0 to be $z = z_A$ and $v = v_A$. The height z is then

$$z = z_A + v_A T - \frac{1}{2} g T^2$$

The height is again z_A when $T = T_A$.

$$z_A = z_A + v_A T_A - \frac{1}{2} g T_A^2$$

so that

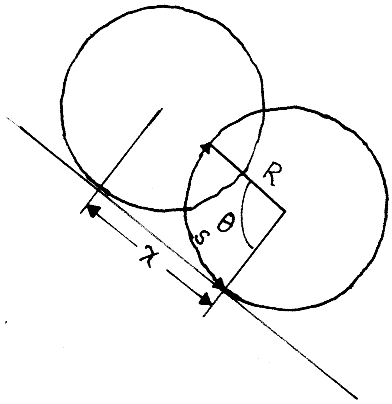
$$0 = v_A T_A - \frac{1}{2} g T_A^2 \;\Rightarrow\; v_A = \frac{1}{2} g T_A$$

By the symmetry of the trajectory, the body reaches height z_B for the second time at $T = \frac{1}{2}(T_A + T_B)$.

$$\begin{aligned}
h &= z_B - z_A \\
&= \left[z_A + \frac{1}{2} v_A (T_A + T_B) - \frac{1}{2} g [\frac{1}{2}(T_A + T_B)]^2 \right] - \left[z_A + v_A T_A - \frac{1}{2} g T_A^2 \right] \\
&= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) g T_A (T_A + T_B) - \frac{1}{8} g (T_A + T_B)^2 \\
&= \frac{1}{8} g (T_A^2 - T_B^2) \\
g &= \frac{8h}{T_A^2 - T_B^2}
\end{aligned}$$

1.17 *Rolling drum*

The drum rolls without slipping, so that when it has rotated through an angle θ, it advances down the plane by a distance x equal to the arc length $s = R\theta$ laid down.



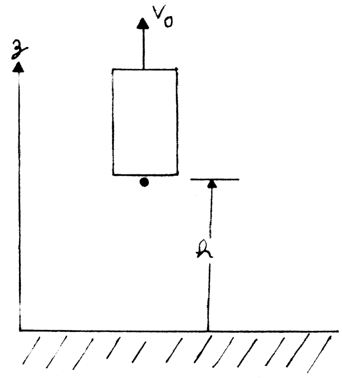
$$x = R\theta$$

$$a = \ddot{x} = R\ddot{\theta} = R\alpha$$

so that

$$\alpha = \frac{a}{R}$$

1.18 *Elevator and falling marble*



Starting at $t = 0$, the elevator moves upward with uniform speed v_0, so its height above the ground at time t is $z = v_0 t$.

At time T_1, $h = v_0 T_1$, so that $T_1 = h/v_0$. At the instant T_1 when the marble is released, the marble is at height h and has an instantaneous speed v_0. Its height z at a later time t is then

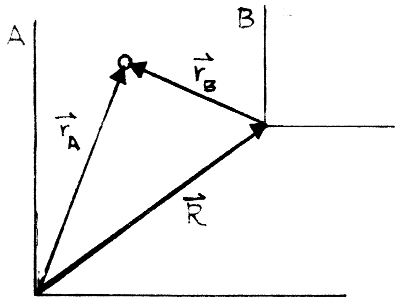
$$z = h + v_0(t - T_1) - \frac{1}{2}g(t - T_1)^2$$

The marble hits the ground $h = 0$ at time $t = T_2$.

$$\begin{aligned}
0 &= h + v_0(T_2 - T_1) - \frac{1}{2}g(T_2 - T_1)^2 \\
&= h + \frac{h}{T_1}(T_2 - T_1) - \frac{1}{2}g(T_2 - T_1)^2 \\
&= h\frac{T_2}{T_1} - \frac{1}{2}g(T_2 - T_1)^2 \\
h &= \frac{1}{2}\frac{T_1}{T_2}g(T_2 - T_1)^2
\end{aligned}$$

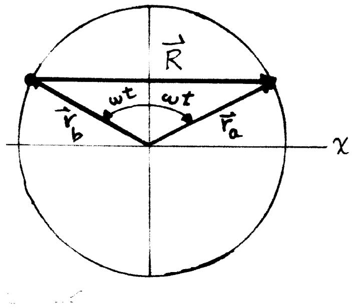
1.19 *Relative velocity*

(a)



$$\mathbf{r_A} = \mathbf{r_B} + \mathbf{R}$$
$$\dot{\mathbf{r_A}} = \dot{\mathbf{r_B}} + \dot{\mathbf{R}}$$
$$\mathbf{v_B} = \mathbf{v_A} - \dot{\mathbf{R}}$$

(b)



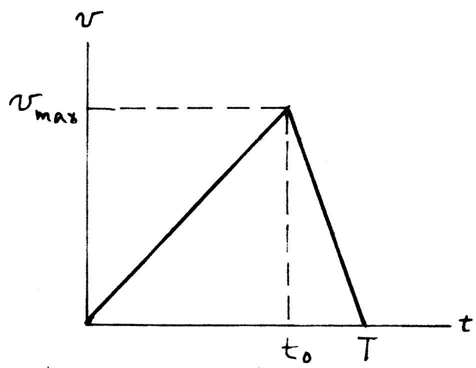
$$\mathbf{R} = 2l \sin(\omega t)\,\hat{\mathbf{i}}$$
$$\dot{\mathbf{R}} = 2l\omega \cos(\omega t)\,\hat{\mathbf{i}}$$

From the result of part (a)

$$\mathbf{v_a} = \mathbf{v_b} + 2l\omega \cos(\omega t)\,\hat{\mathbf{i}}$$

1.20 *Sportscar*

With reference to the sketch, the distance D traveled is the area under the plot of speed vs. time. The goal is to minimize the time while keeping D constant. This involves accelerating with maximum acceleration a_a for time t_0 and then braking with maximum (negative) acceleration a_b to bring the car to rest.



$$v_{max} = a_a t_0 = a_b(T - t_0)$$
$$t_0 = \frac{a_b T}{a_a + a_b}$$
$$D = \frac{1}{2} v_{max} T = \frac{1}{2} a_a t_0 T = \frac{1}{2}\left(\frac{a_a a_b}{a_a + a_b}\right) T^2$$
$$T = \sqrt{\frac{2D(a_a + a_b)}{a_a a_b}}$$
$$a_a = \frac{100\,\text{km/hr}}{3.5\,\text{s}} = \left(\frac{100\,\text{km}}{\text{hr}}\right)\left(\frac{1000\,\text{m}}{1\,\text{km}}\right)\left(\frac{1\,\text{hr}}{3600\,\text{s}}\right)\left(\frac{1}{3.5\,\text{s}}\right) \approx 7.94\text{m/s}^2$$
$$a_b = 0.7g = 0.7(9.80\,\text{m/s}^2) \approx 6.86\,\text{m/s}^2$$
$$T = \sqrt{\frac{(2000\,\text{m})(6.86 + 7.94)\,\text{m/s}^2}{(6.86\,\text{m/s}^2)(7.94\,\text{m/s}^2)}} \approx 23.5\text{s}$$

1.21 *Particle with constant radial velocity*

(a)

$$\mathbf{v} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} = (4.0\,\mathrm{m/s})\,\hat{\mathbf{r}} + (3.0\,\mathrm{m})(2.0\,\mathrm{rad/s})\,\hat{\boldsymbol{\theta}}$$

(Note that radians are dimensionless.)

$$\mathbf{v} = (4.0\,\hat{\mathbf{r}} + 6.0\,\hat{\boldsymbol{\theta}})\,\mathrm{m/s} \qquad \mathrm{v} = \sqrt{{v_r}^2 + {v_\theta}^2} = \sqrt{16.0 + 36.0} \approx 7.2\,\mathrm{m/s}$$

(b)

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\boldsymbol{\theta}}$$

$\ddot{r} = 0$ and $\ddot{\theta} = 0$

$$a_r = -r\dot{\theta}^2 = -(3.0\,\mathrm{m})(2.0\,\mathrm{rad/s})^2 = -12.0\,\mathrm{m/s^2}$$
$$a_\theta = 2\dot{r}\dot{\theta} = 2(4.0\,\mathrm{m/s})(2.0\,\mathrm{rad/s}) = 16.0\,\mathrm{m/s^2}$$
$$a = \sqrt{a_r^2 + a_\theta^2} = \sqrt{144.0 + 256.0} = 20.0\,\mathrm{m/s^2}$$

1.22 *Jerk*

Refer to the Appendix for a corrected solution.

1.23 *Smooth elevator ride*

(a) Let $a(t) \equiv$ *acceleration*

$$a(t) = \frac{1}{2}a_m[1 - \cos(2\pi t/T)] \quad 0 \le t \le T$$

$$a(t) = -\frac{1}{2}a_m[1 - \cos(2\pi t/T)] \quad T \le t \le 2T$$

Let $j(t) \equiv$ *jerk*

$$j(t) = \frac{da}{dt}$$

$$j(t) = a_m(\pi/T)\sin(2\pi t/T) \quad 0 \le t \le T$$

$$j(t) = -a_m(\pi/T)\sin(2\pi t/T) \quad T \le t \le 2T$$

Let $v(t) \equiv$ *speed*

$$v(t) = v(0) + \int_0^t a(t')dt' \quad 0 \le t \le T$$

$$= \frac{1}{2}a_m[t - (T/2\pi)\sin(2\pi t/T)]$$

$$v(t) = v(T) + \int_T^t a(t')dt' \quad T \le t \le 2T$$

$$= \frac{1}{2}a_m T - \frac{1}{2}a_m[(t - T) - (T/2\pi)\sin(2\pi t/T)]$$

$$= \frac{1}{2}a_m[(2T - t) + (T/2\pi)\sin 2\pi t/T]$$

The sketch (in color) shows the jerk $j(t)$ (red), the acceleration $a(t)$ (green), and the speed $v(t)$ (black) versus time t.

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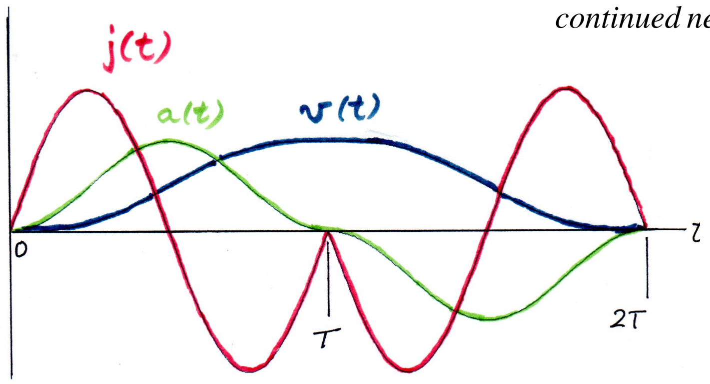

(b) The speed $v(t)$ is the area under the curve of $a(t)$. As the sketch indicates, $v(t)$ increases with time up to $t = T$, and then decreases. The maximum speed v_{max} therefore occurs at $t = T$, so that $v_{max} = v(T)$.

$$v_{max} = v(0) + \int_0^T a(t')dt' \;= \frac{1}{2}a_m \int_0^T [1 - \cos(2\pi t'/T)]dt'$$
$$= \frac{1}{2}a_m\, [t' - (T/2\pi)\sin(2\pi t'/T)]|_0^T \;= \frac{1}{2}a_m T$$

(c) For $t \ll T$, we can use the small angle approximation:

$$\sin\theta = [\theta - \frac{1}{3!}\theta^3 + \ldots]$$
$$v(t) = \int_0^t a(t')dt' \;= \frac{1}{2}a_m[t - (T/2\pi)\sin(2\pi t/T)]$$
$$= \frac{a_m}{2}\{t - (T/2\pi)[(2\pi t/T) - \frac{1}{3!}(2\pi t/T)^3 + \ldots\}$$
$$\approx \frac{a_m}{2}\{\frac{1}{3!}(2\pi/T)^2 t^3\} \;\approx a_m\left(\frac{\pi^2}{3}\right)\left(\frac{t^3}{T^2}\right)$$

(d) *direct method:*

Let the distance at time t be $x(t)$.

$$x(t) = \int v(t')dt'$$

where

$$v(t) = \frac{1}{2}\int_0^t a(t')dt' \quad 0 \le t \le T$$
$$= \frac{a_m}{2}[t - (T/2\pi)\sin(2\pi t/T)] \quad 0 \le t \le T$$
$$v(t) = \int_0^T a(t')dt' + \int_T^t a(t')dt' \quad T \le t \le 2T$$
$$= \frac{a_m}{2}[T - t + T + (T/2\pi)\sin(2\pi t/T)] \quad T \le t \le 2T$$

(Note that $v(2T) = 0$.) Then

$$D = x(2T)$$
$$= \frac{a_m}{2}\int_0^T [t' - (T/2\pi)\sin(2\pi t'/T)]dt' + \frac{a_m}{2}\int_T^{2T} [2T - t' + (T/2\pi)\sin(2\pi t'/T)]dt'$$
$$= \frac{a_m}{2}T^2$$

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(e) *symmetry method:*

By symmetry, the distance from $x(0)$ to $x(T)$ and the distance from $x(T)$ to $x(2T)$ are equal. The distance from $x(0)$ to $x(T)$ is

$$\begin{aligned}
x(T) &= \int_0^T v(t')dt' \\
&= \frac{a_m}{2}\int_0^T [t - (T/2\pi)\sin(2\pi t'/T)]dt' \\
&= \frac{a_m}{2}\,[t'^2/2 + (T/2\pi)^2\cos(2\pi t'/T)]\Big|_0^T \;=\; \frac{a_m}{4}T^2
\end{aligned}$$

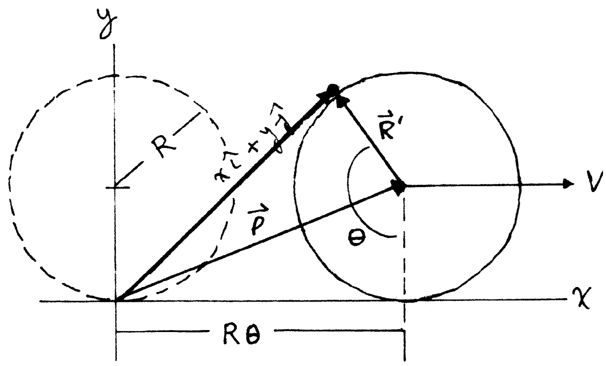
By symmetry

$$D = 2x(T) \;=\; \frac{1}{2}a_m T^2$$

as before.

1.24 *Rolling tire*

Let x, y be the coordinates of the pebble measured from the stationary origin. Let $\boldsymbol{\rho}$ be the vector from the stationary origin to the center of the rolling tire, and let $\mathbf{R}'$ be the vector from the center of the tire to the pebble.



$$\boldsymbol{\rho} = R\theta\,\hat{\mathbf{i}} + R\,\hat{\mathbf{j}}$$
$$\mathbf{R}' = -R\sin\theta\,\hat{\mathbf{i}} - R\cos\theta\,\hat{\mathbf{j}}$$

From the diagram, the vector from the origin to the pebble is

$$x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} = \boldsymbol{\rho} + \mathbf{R}' \;=\; R\theta\,\hat{\mathbf{i}} + R\,\hat{\mathbf{j}} - R\sin\theta\,\hat{\mathbf{i}} - R\cos\theta\,\hat{\mathbf{j}}$$
$$x = R\theta - R\sin\theta \quad \dot{x} = R\dot{\theta} - R\cos\theta\,\dot{\theta}$$
$$y = R - R\cos\theta \quad \dot{y} = R\sin\theta\,\dot{\theta}$$

The tire is rolling at constant speed without slipping: $\theta = \omega t = (V/R)t$.

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$\dot{x} = R\omega - R\omega\cos\theta \quad \ddot{x} = R\omega^2\sin\theta$

$\dot{y} = R\omega\sin\theta \quad \ddot{y} = R\omega^2\cos\theta$

Note that

$\ddot{x}\,\hat{\mathbf{i}} + \ddot{y}\,\hat{\mathbf{j}} = \ddot{\boldsymbol{\rho}} + \ddot{\mathbf{R}}' \;= \ddot{\mathbf{R}}'$

The pebble on the tire experiences an inward radial acceleration V^2/R, and from the results for $\ddot{x}$ and $\ddot{y}$

$$\begin{aligned}\sqrt{\ddot{x}^2 + \ddot{y}^2} &= R\omega^2 \\ &= \frac{V^2}{R}\end{aligned}$$

as expected.

This result shows that the acceleration measured in the stationary system is the same as measured in the system moving uniformly along with the tire.

1.25 *Spiraling particle*

(a)

$$r = \frac{\theta}{\pi} \qquad \theta = \frac{\alpha t^2}{2}$$

$$r = \frac{\alpha t^2}{2\pi}$$

$$\dot{r} = \frac{\alpha t}{\pi} \qquad \dot{\theta} = \alpha t$$

$$\ddot{r} = \frac{\alpha}{\pi} \qquad \ddot{\theta} = \alpha$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\boldsymbol{\theta}} \;= \left(\frac{\alpha}{\pi} - \frac{\alpha^3 t^4}{2\pi}\right)\hat{\mathbf{r}} + \left(\frac{5\alpha^2 t^2}{2\pi}\right)\hat{\boldsymbol{\theta}}$$

(b)

$$a_r = \frac{\alpha}{\pi} - \frac{\alpha^3 t^4}{2\pi} = 0 \text{ at time t'}$$

$$\frac{\alpha}{\pi} = \frac{\alpha^3 t'^4}{2\pi} \implies t'^2 = \frac{\sqrt{2}}{\alpha}$$

$$\theta(t') = \frac{\alpha t'^2}{2} = \frac{1}{\sqrt{2}} \text{ rad}$$

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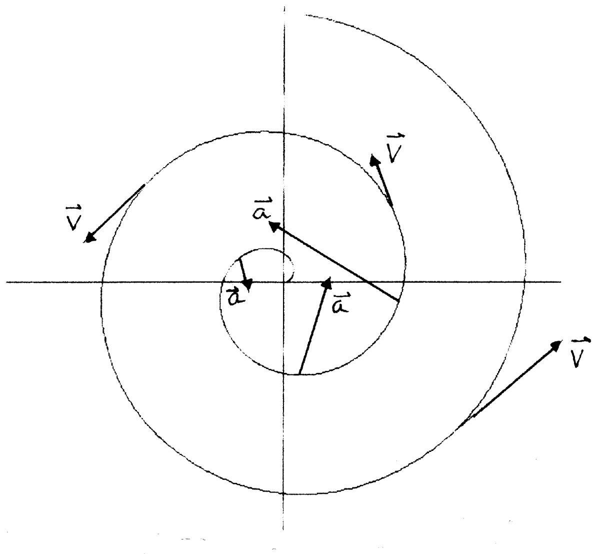
(c)



$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\boldsymbol{\theta}}$$

Using the expression for θ from part (a),

$$\mathbf{a} = \left(\frac{\alpha}{\pi}\right)[(1 - 2\theta^2)\,\hat{\mathbf{r}} + 5\theta\,\hat{\boldsymbol{\theta}}]$$

Setting $|a_r| = |a_\theta|$, then $|1 - 2\theta^2| = |5\theta|$

If $\theta < \frac{1}{\sqrt{2}}$, then $1 - 2\theta^2 = 5\theta$

Because $\theta \geq 0$, the only allowable root is

$$\theta = \frac{-5 + \sqrt{33}}{4} \approx 0.186 \text{ rad } \approx 10^\circ$$

If $\theta > \frac{1}{\sqrt{2}}$, then $2\theta^2 - 1 = 5\theta$

$$\theta = \frac{5 + \sqrt{33}}{4} \approx 2.69 \text{ rad } \approx 154^\circ$$

In the sketch, the velocity vectors are in scale to one another, as are the acceleration vectors.

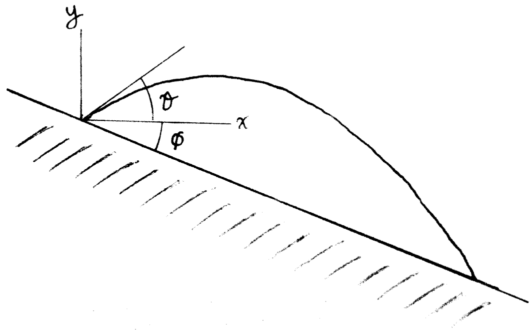
NOTE: The figure is an example of an *Archimedean spiral.* In polar coordinates, the equation of an Archimedean spiral is $r = A\,\theta$, where A is a constant. A fundamental property of an Archimedean spiral is that the radial spacing between adjacent turns is the same everywhere on the spiral. Consider a point (r, θ) on the spiral. The point on the adjacent turn along the same radial line thus has coordinates $(r', \theta + 2\pi)$. Then

$$\Delta r = r' - r = A\,(\theta + 2\pi) - A\,\theta)$$
$$= 2\pi A$$

a constant, the same at any point of the spiral.

1.26 *Range on a hill*

The trajectory of the rock is described by coordinates x and y, as shown in the sketch. Let the initial velocity of the rock be v_0 at angle θ.



$$x = (v_0 \cos\theta)\, t \qquad y = (v_0 \sin\theta)\, t - \frac{1}{2} g\, t^2$$

The locus of the hill is $y = -x \tan\phi$

Let the rock land on the hill at time t'.

$$t' = \frac{x}{v_0 \cos\theta}$$

The locus of the hill and the trajectory of the rock intersect at t'.

$$-x \tan\phi = x \tan\theta - \frac{1}{2}\left(\frac{g}{{v_0}^2}\right)\left(\frac{x^2}{\cos^2\theta}\right)$$

Solving for x,

$$x = \left(\frac{2v_0^2}{g}\right)\left[\cos\theta \sin\theta + (\cos^2\theta)\tan\phi\right] = \left(\frac{2v_0^2}{g}\right)\left[\frac{1}{2}\sin 2\theta + (\cos^2\theta)\tan\phi\right]$$

The condition for maximum range is $dx/d\theta = 0$. Note that ϕ is a constant.

$$\frac{dx}{d\theta} = 0 = \cos 2\theta - 2\sin\theta\cos\theta\tan\phi = \cos 2\theta - (\sin 2\theta)\tan\phi$$

$$\cot 2\theta = \tan\phi$$

$$\tan 2\theta = \tan\left(\frac{\pi}{2} - \phi\right)$$

$$\theta = \frac{\pi}{4} - \frac{\phi}{2} \quad \text{for maximum range}$$

The sketch is drawn for the case $\phi = 20°$ and $v_0 = 5.0$ m/s.

1.27 *Peaked roof*

Let the initial speed at $t = 0$ be v_0. A straightforward way to solve this problem is to write the equations of motion in a uniform gravitational field, as follows:

$$x = -h + v_{0x}t \qquad y = v_{0y}t - \frac{1}{2}gt^2$$
$$v_x = v_{0x} \qquad v_y = v_{0y} - gt$$

At time T, the ball is at the peak, where $y = h$ and $v_y = 0$.

$$0 = v_{0y} - gT \;\Rightarrow\; T = \frac{v_{0y}}{g}$$
$$h = v_0yT - \frac{1}{2}gT^2 \;=\; \frac{v_{0y}^2}{g} - \frac{1}{2}\frac{v_{0y}^2}{g}$$
$$v_{0y}^2 = 2gh$$

At time T, $x = 0$.

$$0 = -h + v_{0x}T \;\Rightarrow\; v_{0x} = \frac{h}{T} \;=\; \frac{\sqrt{gh}}{2}$$

We then have

$$v_0 = \sqrt{v_{0x}^2 + v_{0y}^2} \;=\; \sqrt{2 + \frac{1}{2}}\sqrt{gh} \;=\; \sqrt{\frac{5}{2}}\sqrt{gh}$$

A more physical approach is to note that the vertical speed needed to reach the peak is the same as the speed v_{0y} a mass acquires falling a distance h: $v_{0y} = \sqrt{2gh}$. The time T to fall that distance is $T = v_{0y}/g$. The horizontal distance traveled in the time T is

$$h = v_{0x}T \;=\; v_{0x}\left(\frac{v_{0y}}{g}\right) = v_{0x}\sqrt{\frac{2h}{g}}$$
$$v_{0x} = \sqrt{\frac{gh}{2}}$$

The initial speed v_0 is therefore

$$v_0 = \sqrt{v_{0x}^2 + v_{0y}^2} \;=\; \sqrt{2 + \frac{1}{2}}\sqrt{gh} \;=\; \sqrt{\frac{5}{2}}\sqrt{gh}$$