Preface

This Instructor's Solutions Manual contains solutions for essentially all of the exercises in the text that are intended to be done by hand. Solutions to Matlab exercises are not included. The Student's Solutions Manual that accompanies this text contains solutions for only selected odd-numbered exercises, including those exercises whose answers appear in the answer key. The solutions that appear in the students' manual are identical to those provided in this manual, and generally provide a more detailed solution than is available in the answer key. Although no pattern is strictly adhered to throughout the student manual, the solutions provided there are primarily to the computational exercises, whereas solutions that involve proof are generally not included. None of the solutions to the supplementary end-of-chapter exercises are included in the student manual.

Contents

Chapter 1

Matrices and Systems of Equations

1.1 Introduction to Matrices and Systems of Linear Equations

1. Linear.

2. Nonlinear.

3. Linear.

4. Nonlinear.

5. Nonlinear.

6. Linear.

7. $$\begin{aligned} x_1 + 3x_2 &= 7 & 1 + 3 \cdot 2 &= 7 \\ 4x_1 - x_2 &= 2 & 4 \cdot 1 - 2 &= 2 \end{aligned}$$

8. $$\begin{aligned} 6x_1 - x_2 + x_3 &= 14 & 6 \cdot 2 - (-1) + 1 &= 14 \\ x_1 + 2x_2 + 4x_3 &= 4 & 2 + 2 \cdot (-1) + 4 \cdot 1 &= 4 \end{aligned}$$

9. $$\begin{aligned} x_1 + x_2 &= 0 & 1 + (-1) &= 0 \\ 3x_1 + 4x_2 &= -1 & 3 \cdot 1 + 4 \cdot (-1) &= -1 \\ -x_1 + 2x_2 &= -3 & -1 + 2 \cdot (-1) &= -3 \end{aligned}$$

10. $$\begin{aligned} 3x_2 &= 9, & 3 \cdot 3 &= 9 \\ 4x_1 &= 8, & 4 \cdot 2 &= 8 \end{aligned}$$

11. Unique solution.

12. No Solution

13. Infinitely many solutions.

14. No solution.

15. (a) The planes do not intersect; that is, the planes are parallel.
    (b) The planes intersect in a line or the planes are coincident.

16. The planes intersect in the line $x = (1-t)/2, y = 2, z = t$.

17. The planes intersect in the line $x = 4 - 3t, y = 2t - 1, z = t$.

18. Coincident planes.

19. $A = \begin{bmatrix} 2 & 1 & 6 \\ 4 & 3 & 8 \end{bmatrix}$.

20. $C = \begin{bmatrix} 1 & 2 & 7 & 1 \\ 2 & 2 & 4 & 3 \end{bmatrix}$.

21. $Q = \begin{bmatrix} 1 & 4 & -3 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$.

22. $\begin{array}{rrrcl} x_1 & +2x_2 & +7x_3 & = & 1 \\ 2x_1 & +2x_2 & +4x_3 & = & 3 \end{array}$

23. $\begin{array}{rcrcr} 2x_1 & + & x_2 & = & 6 \\ 4x_1 & + & 3x_2 & = & 8 \end{array}$; $\begin{array}{rcrcr} x_1 & + & 4x_2 & = & -3 \\ 2x_1 & + & x_2 & = & 1 \\ 3x_1 & + & 2x_2 & = & 1 \end{array}$

24. $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 3 \end{bmatrix}$.

25. $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 0 & -1 & 1 \end{bmatrix}$.

26. $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 5 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 2 & 5 & 1 & 5 \\ 1 & 1 & 1 & 3 \end{bmatrix}$.

27. $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & -1 \\ -1 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 2 & 6 \\ 3 & 4 & -1 & 5 \\ -1 & 1 & 1 & 2 \end{bmatrix}$.

28. $A = \begin{bmatrix} 1 & 1 & -3 \\ 1 & 2 & -5 \\ -1 & -3 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & -3 & -1 \\ 1 & 2 & -5 & -2 \\ -1 & -3 & 7 & 3 \end{bmatrix}$.

29. $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 2 \\ 1 & -1 & 3 & 2 \end{bmatrix}$.

30. Elementary operations on equations: $E_2 - 2E_1$.

Reduced system of equations: $\begin{array}{rcr} 2x_1 + 3x_2 & = & 6 \\ -7x_2 & = & -5 \end{array}$.

Elementary row operations: $R_2 - 2R_1$.

Reduced augmented matrix: $\begin{bmatrix} 2 & 3 & 6 \\ 0 & -7 & -5 \end{bmatrix}$.

31. Elementary operations on equations: $E_2 - E_1,\ E_3 + 2E_1$.

Reduced system of equations: $\begin{array}{rcr} x_1 + 2x_2 - x_3 & = & 1 \\ -x_2 + 3x_3 & = & 1 \\ 5x_2 - 2x_3 & = & 6 \end{array}$.

Elementary row operations: $R_2 - R_1,\ R_3 + 2R_1$.

Reduced augmented matrix: $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 5 & -2 & 6 \end{bmatrix}$.

32. Elementary operations on equations: $E_1 \leftrightarrow E_2,\ E_3 - 2E_1$.

Reduced system of equations: $\begin{array}{rcr} x_1 - x_2 + 2x_3 & = & 1 \\ x_2 + x_3 & = & 4 \\ 3x_2 - 5x_3 & = & 4 \end{array}$.

Elementary row operations: $R_1 \leftrightarrow R_2,\ R_3 - 2R_1$.

Reduced augmented matrix: $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 3 & -5 & 4 \end{bmatrix}$.

33. Elementary operations on equations: $E_2 - E_1,\ E_3 - 3E_1$.

Reduced system of equations: $\begin{array}{rcr} x_1 + x_2 & = & 9 \\ -2x_2 & = & -2 \\ -2x_2 & = & -21 \end{array}$.

Elementary row operations: $R_2 - R_1,\ R_3 - 3R_1$.

Reduced augmented matrix: $\begin{bmatrix} 1 & 1 & 9 \\ 0 & -2 & -2 \\ 0 & -2 & -21 \end{bmatrix}$.

34. Elementary operations on equations: $E_2 + E_1,\ E_3 + 2E_1$.

Reduced system of equations: $$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 1 \\ 2x_2 &= 4 \\ 3x_2 + 3x_3 - 3x_4 &= 4 \end{aligned}.$$

Elementary row operations: $R_2 + R_1,\ R_3 + 2R_1$.

Reduced augmented matrix: $$\begin{bmatrix} 1 & 1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 0 & 4 \\ 0 & 3 & 3 & -3 & 4 \end{bmatrix}.$$

35. Elementary operations on equations: $E_2 \leftrightarrow E_1,\ E_3 + E_1$.

Reduced system of equations: $$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 1 \\ x_2 + x_3 - x_4 &= 3 \\ 3x_2 + 6x_3 &= 1 \end{aligned}.$$

Elementary row operations: $R_2 \leftrightarrow R_1,\ R_3 + R_1$.

Reduced augmented matrix: $$\begin{bmatrix} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & 3 & 6 & 0 & 1 \end{bmatrix}.$$

36. Elementary operations on equations: $E_2 - E_1,\ E_3 - 3E_1$.

Reduced system of equations: $$\begin{aligned} x_1 + x_2 &= 0 \\ -2x_2 &= 0 \\ -2x_2 &= 0 \end{aligned}.$$

Elementary row operations: $R_2 - R_1,\ R_3 - 3R_1$.

Reduced augmented matrix: $$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & -2 & 0 \end{bmatrix}.$$

37. (b) In each case, the graph of the resulting equation is a line.

38. Now if $a_{11} = 0$ we easily obtain the equivalent system

$$\begin{aligned} a_{21}x_1 + a_{22}x_2 &= b_2 \\ a_{12}x_2 &= b_1 \end{aligned}$$

Thus we may suppose that $a_{11} \neq 0$. Then :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned} \quad \left\{ \begin{array}{c} E_2 - (a_{21}/a_{11})E_1 \\ \Longrightarrow \end{array} \right\}$$

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 & = & b_1 \\ ((-a_{21}/a_{11})a_{12} + a_{22})x_2 & = & (-a_{21}/a_{11})b_1 + b_2 \end{array} \left\{ \begin{array}{c} a_{11}E_2 \\ \Longrightarrow \end{array} \right\}$$

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 & = & b_1 \\ (a_{11}a_{22} - a_{12}a_{21})x_2 & = & -a_{21}b_1 + a_{11}b_2 \end{array}$$

Each of a_{11} and $(a_{11}a_{22} - a_{12}a_{21})$ is non-zero. ■

39. Let

$$\mathcal{A} = \left\{ \begin{array}{c} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{array} \right\}$$

and let

$$\mathcal{B} = \left\{ \begin{array}{c} a_{11}x_1 + a_{12}x_2 = b_1 \\ ca_{21}x_1 + ca_{22}x_2 = cb_2 \end{array} \right\}$$

Suppose that $x_1 = s_1$, $x_2 = s_2$ is a solution to $\mathcal{A}$. Then $a_{11}s_1 + a_{12}s_2 = b_1$, and $a_{21}s_1 + a_{22}s_2 = b_2$. But this means that $ca_{21}s_1 + ca_{22}s_2 = cb_2$ and so $x_1 = s_1$, $x_2 = s_2$ is also a solution to $\mathcal{B}$. Now suppose that $x_1 = t_1$, $x_2 = t_2$ is a solution to $\mathcal{B}$. Then $a_{11}t_1 + a_{12}t_2 = b_1$ and $ca_{21}t_1 + ca_{22}t_2 = cb_2$. Since $c \neq 0$, $a_{21}x_1 + a_{22}x_2 = b_2$. ■

40. Let

$$\mathcal{A} = \left\{ \begin{array}{rcl} a_{11}x_1 + a_{12}x_2 & = & b_1 \\ a_{21}x_1 + a_{22}x_2 & = & b_2 \end{array} \right\}$$

and let

$$\mathcal{B} = \left\{ \begin{array}{r} a_{11}x_1 + a_{12}x_2 = b_1 \\ (a_{21} + ca_{11})x_1 + (a_{22} + ca_{12})x_2 = b_2 + cb_1 \end{array} . \right\}$$

Let $x_1 = s_1$ and $x_2 = s_2$ be a solution to $\mathcal{A}$. Then $a_{11}s_1 + a_{12}s_2 = b_1$ and $a_{21}s_1 + a_{22}s_2 = b_2$ so $a_{11}s_1 + a_{12}s_2 = b_1$ and $(a_{21} + ca_{11})s_1 + (a_{22} + ca_{12})s_2 = b_2 + cb_1$ as required. Now if $x_1 = t_1$ and $x_2 = t_2$ is a solution to $\mathcal{B}$ then $a_{11}t_1 + a_{12}t_2 = b_1$ and $(a_{21} + ca_{11})t_1 + (a_{22} + ca_{12})t_2 = b_2 + cb_1$, so $a_{11}t_1 + a_{12}t_2 = b_1$ and $a_{21}t_1 + a_{12}t_2 = b_2$ as required. ■

41. The proof is very similar to that of 45 and 46.

42. By adding the two equations we obtain: $2x_1^2 - 2x_1 = 4$. Then $x_1 = 2$ or $x_1 = -1$ and substituting these values in the second equation we find that there are three solutions: $x_1 = -1, x_2 = 0$; $x_1 = 2, x_2 = \sqrt{3},$; $x_1 = 2, x_2 = -\sqrt{3}$.

1.2 Echelon Form and Gauss-Jordan Elimination

1. The matrix is in echelon form. The row operation $R_2 - 2R_1$ transforms the matrix to reduced echelon form $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

2. Echelon form. $R_2 - 2R_1$ yields reduced row echelon form $\begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 3 \end{bmatrix}$.

3. Not in echelon form. $(1/2)R_1,\ R_2 - 4R_1,\ (-1/5)R_2$ yields echelon form $\begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 2/5 \end{bmatrix}$.

4. Not in echelon form. $R_1 \leftrightarrow R_2$ yields echelon form $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

5. Not in echelon form.
   $R_1 \leftrightarrow R_2,\ (1/2)R_1,\ (1/2)R_2$ yields the echelon form $\begin{bmatrix} 1 & 0 & 1/2 & 2 \\ 0 & 0 & 1 & 3/2 \end{bmatrix}$.

6. Not in echelon form.
   $(1/2)R_1$ yields the echelon form $\begin{bmatrix} 1 & 0 & 3/2 & 1/2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$.

7. Not in echelon form. $R_2 - 4R_3,\ R_1 - 2R_3,\ R_1 - 3R_2$ yields the reduced echelon form $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

8. Not in echelon form. $(1/2)R_1,\ (-1/3)R_3$ yields the echelon form $\begin{bmatrix} 1 & -1/2 & 3/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

9. Not in echelon form. $(1/2)R_2$ yields the echelon form $\begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & -1 & -3/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

10. Not in echelon form $-R_1,\ (1/2)R_2$ yields the echelon form $\begin{bmatrix} 1 & -4 & 3 & -4 & -6 \\ 0 & 1 & 1/2 & -3/2 & -3/2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$.

11. $x_1 = 0$, $x_2 = 0$.

12. The system is inconsistent.

13. $x_1 = -2 + 5x_3$, $x_2 = 1 - 3x_3$, x_3 is arbitrary.

14. $x_1 = 1 - 2x_3$, $x_2 = 0$.

15. $x_1 = 0$, $x_2 = 0$, $x_3 = 0$.

16. $x_1 = 0$, $x_2 = 0$, $x_3 = 0$.

17. $x_1 = x_3 = x_4 = 0$, x_2 is arbitrary.

18. The system is inconsistent.

19. The system is inconsistent.

20. $x_1 = 3x_4 - 5x_5 - 2$, $x_2 = x_4 + x_5 - 2$, $x_3 = -2x_4 - x_5 + 2$, x_4 and x_5 are arbitrary.

21. $x_1 = -1 - (1/2)x_2 + (1/2)x_4$, $x_3 = 1 - x_4$, x_2 and x_4 arbitrary, $x_5 = 0$.

22. $x_1 = (5 + 3x_2)/2$, x_2 arbitrary.

23. The system is inconsistent.

24. $x_1 = x_3$, $x_2 = -3 + 2x_3$, x_3 arbitrary.

25. $x_1 = 2 - x_2$, x_2 arbitrary.

26. $x_1 = 10 + x_2$, x_2 arbitrary, $x_3 = -6$.

27. $x_1 = 2 - x_2 + x_3$, x_2 and x_3 arbitrary.

28. $x_1 = 2x_3$, $x_2 = 1$, x_3 arbitrary.

29. $x_1 = 3 - 2x_3$, $x_2 = -2 + 3x_3$, x_3 arbitrary.

30. $x_1 = -3x_4 - 6x_5$, $x_2 = 1 + 3x_4 + 7x_5$, $x_3 = -2x_4 - 5x_5$, x_4 and x_5 arbitrary.

31. $x_1 = 3 - (7x_4 - 16x_5)/2$, $x_2 = (x_4 + 2x_5)/2$, $x_3 = -2 + (5x_4 - 12x_5)/2$, x_4 and x_5 arbitrary.

32. $x_1 = 2$, $x_2 = -1$.

33. The system is inconsistent.

34. $x_1 = 1 - 2x_2$, x_2 arbitrary.

35. The system is inconsistent.

36. $$\begin{array}{rcr} x_1 + 2x_2 & = & -3 \\ ax_1 - 2x_2 & = & 5 \end{array} \left\{ \begin{array}{c} E_1 + E_2 \\ \Longrightarrow \end{array} \right\} \begin{array}{rcr} x_1 + 2x_2 & = & -3 \\ (a+1)x_1 & = & 2 \end{array}$$

$\boxed{\text{Hence if } a = -1 \text{ there is no solution.}}$

37. $$\begin{array}{rcr} x_1 + 3x_2 & = & 4 \\ 2x_1 + 6x_2 & = & a \end{array} \left\{ \begin{array}{c} E_2 - 2E_1 \\ \Longrightarrow \end{array} \right\} \begin{array}{rcr} x_1 + 3x_2 & = & 4 \\ 0 & = & a - 8 \end{array}$$

$\boxed{\text{Thus, if } a \neq 8 \text{ there is no solution.}}$

38. $\begin{array}{rcl} 2x_1 + 4x_2 & = & a \\ 3x_1 + 6x_2 & = & 5 \end{array} \left\{ \begin{array}{c} E_2 - (3/2)E_1 \\ \Longrightarrow \end{array} \right\} \begin{array}{rcr} 2x_1 + 4x_2 & = & a \\ 0 & = & 5 - (3/2)a \end{array}$

$\boxed{\text{Thus, if } a \neq 10/3 \text{ there is no solution.}}$

39. $\begin{array}{rcl} 3x_1 + ax_2 & = & 3 \\ ax_1 + 3x_2 & = & 5 \end{array} \left\{ \begin{array}{c} E_2 - (a/3)E_1 \\ \Longrightarrow \end{array} \right\} \begin{array}{rcr} 3x_1 + ax_2 & = & 3 \\ (a^2/3 - 3)x_2 & = & 5 - a \end{array}$

$\boxed{\text{Thus, if } a = \pm 3 \text{ there is no solution.}}$

40. $\begin{array}{rcl} x_1 + ax_2 & = & 6 \\ ax_1 + 2ax_2 & = & 4 \end{array} \left\{ \begin{array}{c} E_2 - aE_1 \\ \Longrightarrow \end{array} \right\} \begin{array}{rcr} x_1 + ax_2 & = & 6 \\ (2a - a^2)x_2 & = & 4 - 6a \end{array}$

41. $\cos\alpha = 1/2$ and $\sin\beta = 1/2$, so $\alpha = \pi/3$ or $\alpha = 5\pi/3$ and $\beta = \pi/6$ or $\beta = 5\pi/6$.

42. $\cos^2\alpha = 3/4$ and $\sin^2\beta = 1/2$. The choices for α are $\pi/6$, $5\pi/6$, $7\pi/6$, and $11\pi/6$. The choices for β are $\pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$.

43. $x_1 = 1 - 2x_3$, $x_2 = 2 + x_3, x_3$ arbitrary. (a) $x_3 = 1/2$. (b) In order for $x_1 \geq 0, x_2 \geq 0$, we must have $-2 \leq x_3 \leq 1/2$; for a given x_1 and $x_2, y = -6 - 7x_3$, so the minimum value is $y = 8$ at $x_3 = -2$. (c) The minimum value is 20.

44. $\begin{bmatrix} 1 & d \\ c & b \end{bmatrix} \left\{ \begin{array}{c} R_2 - cR_1 \\ \Longrightarrow \end{array} \right\} \begin{bmatrix} 1 & d \\ 0 & b - cd \end{bmatrix} \left\{ \begin{array}{c} R_1 - (d/(b - cd))R_2 \\ (\text{ recall } b - cd \neq 0 \\ \Longrightarrow \end{array} \right\}$

$\begin{bmatrix} 1 & 0 \\ 0 & b - cd \end{bmatrix} \left\{ \begin{array}{c} 1/(b - cd)R_2 \\ \Longrightarrow \end{array} \right\} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

45. $\begin{bmatrix} 1 & x & x \\ 0 & 1 & x \end{bmatrix}, \begin{bmatrix} 1 & x & x \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & x & x \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & x \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

46. (a) $\begin{bmatrix} 1 & x \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & x \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$

(b) $\begin{bmatrix} 1 & x & x \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & x & x \\ 0 & 1 & x \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & x & x \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & x & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$

$\begin{bmatrix} 0 & 1 & x \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

(c) $\begin{bmatrix} 1 & x & x & x \\ 0 & 1 & x & x \\ 0 & 0 & 1 & x \end{bmatrix}$, $\begin{bmatrix} 1 & x & x & x \\ 0 & 1 & x & x \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & x & x & x \\ 0 & 1 & x & x \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

$\begin{bmatrix} 1 & x & x & x \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & x & x & x \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & x & x & x \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

$\begin{bmatrix} 1 & x & x & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & x & x \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & x & x \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

$\begin{bmatrix} 0 & 1 & x & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

47. $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \left\{ \begin{matrix} 2R_2 \\ \Longrightarrow \end{matrix} \right\} \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix} \left\{ \begin{matrix} R_2 - R_1 \\ \Longrightarrow \end{matrix} \right\} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

48. $\begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix} \left\{ \begin{matrix} R_2 - 3R_1 \\ \Longrightarrow \end{matrix} \right\} \begin{bmatrix} 1 & 4 \\ 0 & -5 \end{bmatrix} \left\{ \begin{matrix} R_1 + (2/5)R_2 \\ \Longrightarrow \end{matrix} \right\}$

$\begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} \left\{ \begin{matrix} (3/5)R_2 \\ \Longrightarrow \end{matrix} \right\} \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} \left\{ \begin{matrix} R_2 + 2R_1 \\ \Longrightarrow \end{matrix} \right\} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

49. $$\begin{aligned} 100x_1 + 10x_2 + x_3 &= 15(x_1 + x_2 + x_3) \\ 100x_3 + 10x_2 + x_1 &= 100x_1 + 10x_2 + x_3 + 396 \\ x_3 &= x_1 + x_2 + 1 \end{aligned}$$

$x_1 = 1$, $x_2 = 3$, and $x_3 = 5$, so $N = 135$.

50. $$\begin{aligned} a - b + c &= 6 \\ a + b + c &= 4 \\ 4a + 2b + c &= 9 \end{aligned}$$

$a = 2$, $b = -1$, $c = 3$. So $y = 2x^2 - x + 3$.

51. Let x_1, x_2, x_3 be the amounts initially held by players one, two and three, respectively. Also assume that player one loses the first game, player two loses the second game, and player three loses the third game. Then after three games, the amount of money held by each player is given by the following table

Player	**Amount of money**
1	$4x_1 - 4x_2 - 4x_3 = 24$
2	$-2x_1 + 6x_2 - 2x_3 = 24$
3	$-x_1 - x_2 + 7x_3 = 24$

Solving yields $x_1 = 39$, $x_2 = 21$, and $x_3 = 12$.

52. The resulting system of equations is

$$\begin{aligned} x_1 + x_2 + x_3 &= 34 \\ x_1 + x_2 &= 7 \\ x_2 + x_3 &= 22 \end{aligned}$$

The solution is $x_1 = 12$, $x_2 = -5$, $x_3 = 27$.

53. If x_1 is the number of adults, x_2 the number of students, and x_3 the number of children, then $x_1 + x_2 + x_3 = 79$, $6x_1 + 3x_2 + (1/2)x_3 = 207$, and for $j = 1, 2, 3$, x_j is an integer such that $0 \leq x_j \leq 79$. Following is a list of possiblities

Number of Adults	0	5	10	15	20	25	30
Number of Students	67	56	45	34	23	12	1
Number of Children	12	18	24	30	36	42	48

54. The resulting system of equations is

$$\begin{aligned} a + b + c + d &= 5 \\ b + 2c + 3d &= 5 \\ a + 2b + 4c + 8d &= 17 \\ b + 4c + 12d &= 21. \end{aligned}$$

The solution is $a = 3$, $b = 1$, $c = -1$, $d = 2$. So $p(x) = 3 + x - x^2 + 2x^3$.

55. By (7), $1 + 2 + 3 + \cdots + n = a_1 n + a_2 n^2$. Setting $n = 1$ and $n = 2$ gives

$$\begin{aligned} a_1 + a_2 &= 1 \\ 2a_1 + 4a_2 &= 3 \end{aligned}$$

The solution is $a_1 = a_2 = 1/2$, so $1 + 2 + 3 + \ldots + n = n(n + 1)/2$.

56. By (7), $1^2 + 2^2 + 3^2 + \cdots + n^2 = a_1 n + a_2 n^2 + a_3 n^3$. Setting $n = 1$, $n = 2$, $n = 3$, gives

$$\begin{aligned} a_1 + a_2 + a_3 &= 1 \\ 2a_1 + 4a_2 + 8a_3 &= 5 \\ 3a_1 + 9a_2 + 27a_3 &= 14 \end{aligned}$$

The solution is $a_1 = 1/6$, $a_2 = 1/2$ and $a_3 = 1/3$, so $1^2+2^2+3^2+\ldots+n^2 = n(n+1)(2n+1)/6$.

57. The system of equations obtained from (7) is

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 + a_5 &= 1 \\ 2a_1 + 4a_2 + 8a_3 + 16a_4 + 32a_5 &= 17 \\ 3a_1 + 9a_2 + 27a_3 + 81a_4 + 242a_5 &= 98 \\ 4a_1 + 16a_2 + 64a_3 + 256a_4 + 1024a_5 &= 354 \\ 5a_1 + 25a_2 + 125a_3 + 625a_4 + 3125a_5 &= 979 \end{aligned}$$

The solution is $a_1 = -1/30$, $a_2 = 0$, $a_3 = 1/3$, $a_4 = 1/2$, $a_5 = 1/5$. Therefore, $1^4 + 2^4 + 3^4 + \cdots + n^4 = n(n+1)(2n+1)(3n^2+3n-1)/30$.

58. $1^5 + 2^5 + 3^5 + \cdots + n^5 = n^2(n+1)^2(2n^2+2n-1)/12$.

1.3 Consistent Systems of Linear Equations

1. The augmented matrix reduces to $\begin{bmatrix} 1 & 1 & 0 & 5/6 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
   n = 3, r = 2, x_2 is independent.

2. The augmented matrix reduces to $\begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.
   n = 2, r = 2.

3. The augmented matrix reduces to $\begin{bmatrix} 1 & 0 & 4 & 0 & 13/2 \\ 0 & 1 & -1 & 0 & -3/2 \\ 0 & 0 & 0 & 1 & 1/2 \end{bmatrix}$.
   n = 4, r = 3, x_3 is independent.

4. The augmented matrix reduces to $\begin{bmatrix} 1 & 2 & 3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.
   n = 4, r = 2, x_2 and x_3 are independent.

5. $n = 2$ and $r \leq 2$ so $r = 0$, $n - r = 2$; $r = 1$, $n - r = 1$; $r = 2$, $n - r = 0$. There could be a unique solution.

6. $n = 4$ and $r \leq 3$ so $r = 0$, $n - r = 4$; $r = 1$, $n - r = 3$; $r = 2$, $n - r = 2$; $r = 3$, $n - r = 1$. By the corollary to Theorem 3, there are infinitely many solutions.

7. Infinitely many solutions.

8. Infinitely many solutions.

9. Infinitely many solutions, a unique solution or no solution.

10. Infinitely many solutions, a unique solution, or no solution.

11. A unique solution or infinitely many solutions.

12. Infinitely many solutions or a unique solution.

13. Infinitely many solutions.

14. Infinitely many solutions.

15. Infinitely many solutions or a unique solution.

16. Infinitely many solutions or a unique solution.

17. Infinitely many solutions.

18. Infinitely many solutions.

19. There are nontrivial solutions.

20. There are nontrivial solutions.

21. There is only the trivial solution.

22. There is only the trivial solution.

23. If $a = -1$ then when we reduce the augmented matrix we obtain a row of zeroes and hence infinitely many nontrivial solutions.

24. (a) Reduced row echelon form of the augmented matrix is $\begin{bmatrix} 1 & 0 & 2 & -2b_1 + 3b_2 \\ 0 & 1 & -1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_1 - 2b_2 \end{bmatrix}$.

    Hence, if $b_3 - b_1 - 2b_2 \neq 0$ then the system is inconsistent. Therefore, the system of equations is consistent if and only if $b_3 - b_1 - 2b_2 = 0$.

    (b) (i) The system is consistent. For example, a solution is $x_1 = -1$, $x_2 = 1$ and $x_3 = 1$. (ii) The system is inconsistent by part (a). (iii) The system is consistent. For example, a solution is $x_1 = 1$, $x_2 = 0$ and $x_3 = 1$.

25. (a) $B = \begin{bmatrix} * & x & x \\ 0 & * & x \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}$.