INTRODUCTION
TO
LINEAR
ALGEBRA
Fourth Edition

MANUAL FOR INSTRUCTORS

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Problem Set 1.1, page 8

1 The combinations give (a) a line in $\mathbf{R}^3$ (b) a plane in $\mathbf{R}^3$ (c) all of $\mathbf{R}^3$.

2 $\boldsymbol{v}+\boldsymbol{w}=(2,3)$ and $\boldsymbol{v}-\boldsymbol{w}=(6,-1)$ will be the diagonals of the parallelogram with $\boldsymbol{v}$ and $\boldsymbol{w}$ as two sides going out from $(0,0)$.

3 This problem gives the diagonals $\boldsymbol{v}+\boldsymbol{w}$ and $\boldsymbol{v}-\boldsymbol{w}$ of the parallelogram and asks for the sides: The opposite of Problem 2. In this example $\boldsymbol{v}=(3,3)$ and $\boldsymbol{w}=(2,-2)$.

4 $3\boldsymbol{v}+\boldsymbol{w}=(7,5)$ and $c\boldsymbol{v}+d\boldsymbol{w}=(2c+d,c+2d)$.

5 $\boldsymbol{u}+\boldsymbol{v}=(-2,3,1)$ and $\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}=(0,0,0)$ and $2\boldsymbol{u}+2\boldsymbol{v}+\boldsymbol{w}=($ add first answers$)=(-2,3,1)$. The vectors $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ are in the same plane because a combination gives $(0,0,0)$. Stated another way: $\boldsymbol{u}=-\boldsymbol{v}-\boldsymbol{w}$ is in the plane of $\boldsymbol{v}$ and $\boldsymbol{w}$.

6 The components of every $c\boldsymbol{v}+d\boldsymbol{w}$ add to zero. $c=3$ and $d=9$ give $(3,3,-6)$.

7 The nine combinations $c(2,1)+d(0,1)$ with $c=0,1,2$ and $d=(0,1,2)$ will lie on a lattice. If we took all whole numbers c and d, the lattice would lie over the whole plane.

8 The other diagonal is $\boldsymbol{v}-\boldsymbol{w}$ (or else $\boldsymbol{w}-\boldsymbol{v}$). Adding diagonals gives $2\boldsymbol{v}$ (or $2\boldsymbol{w}$).

9 The fourth corner can be $(4,4)$ or $(4,0)$ or $(-2,2)$. Three possible parallelograms!

10 $\boldsymbol{i}-\boldsymbol{j}=(1,1,0)$ is in the base (x-y plane). $\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}=(1,1,1)$ is the opposite corner from $(0,0,0)$. Points in the cube have $0\le x\le 1$, $0\le y\le 1$, $0\le z\le 1$.

11 Four more corners $(1,1,0),(1,0,1),(0,1,1),(1,1,1)$. The center point is $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$. Centers of faces are $(\frac{1}{2},\frac{1}{2},0),(\frac{1}{2},\frac{1}{2},1)$ and $(0,\frac{1}{2},\frac{1}{2}),(1,\frac{1}{2},\frac{1}{2})$ and $(\frac{1}{2},0,\frac{1}{2}),(\frac{1}{2},1,\frac{1}{2})$.

12 A four-dimensional cube has $2^4=16$ corners and $2\cdot 4=8$ three-dimensional faces and 24 two-dimensional faces and 32 edges in Worked Example **2.4 A**.

13 Sum $=$ zero vector. Sum $=-2{:}00$ vector $=8{:}00$ vector. 2:00 is 30° from horizontal $=(\cos\frac{\pi}{6},\sin\frac{\pi}{6})=(\sqrt{3}/2,1/2)$.

14 Moving the origin to 6:00 adds $\boldsymbol{j}=(0,1)$ to every vector. So the sum of twelve vectors changes from $\mathbf{0}$ to $12\boldsymbol{j}=(0,12)$.

15 The point $\dfrac{3}{4}\boldsymbol{v}+\dfrac{1}{4}\boldsymbol{w}$ is three-fourths of the way to $\boldsymbol{v}$ starting from $\boldsymbol{w}$. The vector $\dfrac{1}{4}\boldsymbol{v}+\dfrac{1}{4}\boldsymbol{w}$ is halfway to $\boldsymbol{u}=\dfrac{1}{2}\boldsymbol{v}+\dfrac{1}{2}\boldsymbol{w}$. The vector $\boldsymbol{v}+\boldsymbol{w}$ is $2\boldsymbol{u}$ (the far corner of the parallelogram).

16 All combinations with $c+d=1$ are on the line that passes through $\boldsymbol{v}$ and $\boldsymbol{w}$. The point $\boldsymbol{V}=-\boldsymbol{v}+2\boldsymbol{w}$ is on that line but it is beyond $\boldsymbol{w}$.

17 All vectors $c\boldsymbol{v}+c\boldsymbol{w}$ are on the line passing through $(0,0)$ and $\boldsymbol{u}=\frac{1}{2}\boldsymbol{v}+\frac{1}{2}\boldsymbol{w}$. That line continues out beyond $\boldsymbol{v}+\boldsymbol{w}$ and back beyond $(0,0)$. With $c\ge 0$, half of this line is removed, leaving a *ray* that starts at $(0,0)$.

18 The combinations $c\boldsymbol{v}+d\boldsymbol{w}$ with $0\le c\le 1$ and $0\le d\le 1$ *fill the parallelogram* with sides $\boldsymbol{v}$ and $\boldsymbol{w}$. For example, if $\boldsymbol{v}=(1,0)$ and $\boldsymbol{w}=(0,1)$ then $c\boldsymbol{v}+d\boldsymbol{w}$ fills the unit square.

19 With $c\ge 0$ and $d\ge 0$ we get the infinite "cone" or "wedge" between $\boldsymbol{v}$ and $\boldsymbol{w}$. For example, if $\boldsymbol{v}=(1,0)$ and $\boldsymbol{w}=(0,1)$, then the cone is the whole quadrant $x\ge 0$, $y\ge 0$. *Question*: What if $\boldsymbol{w}=-\boldsymbol{v}$? The cone opens to a half-space.

20 (a) $\frac{1}{3}\boldsymbol{u} + \frac{1}{3}\boldsymbol{v} + \frac{1}{3}\boldsymbol{w}$ is the center of the triangle between $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$; $\frac{1}{2}\boldsymbol{u} + \frac{1}{2}\boldsymbol{w}$ lies between $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) To fill the triangle keep $c \geq 0$, $d \geq 0$, $e \geq 0$, and $c + d + e = \mathbf{1}$.

21 The sum is $(\boldsymbol{v} - \boldsymbol{u}) + (\boldsymbol{w} - \boldsymbol{v}) + (\boldsymbol{u} - \boldsymbol{w}) =$ **zero vector**. Those three sides of a triangle are in the same plane!

22 The vector $\frac{1}{2}(\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w})$ is *outside* the pyramid because $c + d + e = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} > 1$.

23 *All* vectors are combinations of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ as drawn (not in the same plane). Start by seeing that $c\boldsymbol{u} + d\boldsymbol{v}$ fills a plane, then adding $e\boldsymbol{w}$ fills all of $\mathbf{R}^3$.

24 The combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$ fill one plane. The combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ fill another plane. Those planes meet in a *line*: *only the vectors* $c\boldsymbol{v}$ are in both planes.

25 (a) For a line, choose $\boldsymbol{u} = \boldsymbol{v} = \boldsymbol{w} =$ any nonzero vector (b) For a plane, choose $\boldsymbol{u}$ and $\boldsymbol{v}$ in different directions. A combination like $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$ is in the same plane.

26 Two equations come from the two components: $c + 3d = 14$ and $2c + d = 8$. *The solution is* $c = 2$ *and* $d = 4$. Then $2(1,2) + 4(3,1) = (14,8)$.

27 The combinations of $\boldsymbol{i} = (1,0,0)$ and $\boldsymbol{i} + \boldsymbol{j} = (1,1,0)$ fill the xy plane in xyz space.

28 There are **6** unknown numbers $v_1, v_2, v_3, w_1, w_2, w_3$. The six equations come from the components of $\boldsymbol{v} + \boldsymbol{w} = (4,5,6)$ and $\boldsymbol{v} - \boldsymbol{w} = (2,5,8)$. Add to find $2\boldsymbol{v} = (6,10,14)$ so $\boldsymbol{v} = (3,5,7)$ and $\boldsymbol{w} = (1,0,-1)$.

29 Two combinations out of infinitely many that produce $\boldsymbol{b} = (0,1)$ are $-2\boldsymbol{u} + \boldsymbol{v}$ and $\frac{1}{2}\boldsymbol{w} - \frac{1}{2}\boldsymbol{v}$. **No**, three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in the x-y plane could fail to produce $\boldsymbol{b}$ if all three lie on a line that does not contain $\boldsymbol{b}$. *Yes*, if one combination produces $\boldsymbol{b}$ then two (and infinitely many) combinations will produce $\boldsymbol{b}$. This is true even if $\boldsymbol{u} = \mathbf{0}$; the combinations can have different $c\boldsymbol{u}$.

30 The combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ fill the plane *unless* $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *lie on the same line through* $(0,0)$. Four vectors whose combinations fill 4-dimensional space: one example is the "standard basis" $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, and $(0,0,0,1)$.

31 The equations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} = \boldsymbol{b}$ are

$$\begin{array}{rrrl} 2c & -d & & = 1 \\ -c & +2d & -e & = 0 \\ & -d & +2e & = 0 \end{array} \qquad \begin{array}{l} \text{So } d = 2e \\ \text{then } c = 3e \\ \text{then } 4e = 1 \end{array} \qquad \begin{array}{l} c = 3/4 \\ d = 2/4 \\ e = 1/4 \end{array}$$

Problem Set 1.2, page 19

1 $\boldsymbol{u} \cdot \boldsymbol{v} = -1.8 + 3.2 = 1.4$, $\boldsymbol{u} \cdot \boldsymbol{w} = -4.8 + 4.8 = 0$, $\boldsymbol{v} \cdot \boldsymbol{w} = 24 + 24 = 48 = \boldsymbol{w} \cdot \boldsymbol{v}$.

2 $\|\boldsymbol{u}\| = 1$ and $\|\boldsymbol{v}\| = 5$ and $\|\boldsymbol{w}\| = 10$. Then $1.4 < (1)(5)$ and $48 < (5)(10)$, confirming the Schwarz inequality.

3 Unit vectors $\boldsymbol{v}/\|\boldsymbol{v}\| = (\frac{3}{5}, \frac{4}{5}) = (.6, .8)$ and $\boldsymbol{w}/\|\boldsymbol{w}\| = (\frac{4}{5}, \frac{3}{5}) = (.8, .6)$. The cosine of θ is $\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \cdot \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = \frac{24}{25}$. The vectors $\boldsymbol{w}, \boldsymbol{u}, -\boldsymbol{w}$ make $0°, 90°, 180°$ angles with $\boldsymbol{w}$.

4 (a) $\boldsymbol{v} \cdot (-\boldsymbol{v}) = -1$ (b) $(\boldsymbol{v} + \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{w} = 1 + (\quad) - (\quad) - 1 = 0$ so $\theta = 90°$ (notice $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$) (c) $(\boldsymbol{v} - 2\boldsymbol{w}) \cdot (\boldsymbol{v} + 2\boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} - 4\boldsymbol{w} \cdot \boldsymbol{w} = 1 - 4 = -3$.

5 $\boldsymbol{u}_1 = \boldsymbol{v}/\|\boldsymbol{v}\| = (3,1)/\sqrt{10}$ and $\boldsymbol{u}_2 = \boldsymbol{w}/\|\boldsymbol{w}\| = (2,1,2)/3$. $\boldsymbol{U}_1 = (1,-3)/\sqrt{10}$ is perpendicular to $\boldsymbol{u}_1$ (and so is $(-1,3)/\sqrt{10}$). $\boldsymbol{U}_2$ could be $(1,-2,0)/\sqrt{5}$: There is a whole plane of vectors perpendicular to $\boldsymbol{u}_2$, and a whole circle of unit vectors in that plane.

6 All vectors $\boldsymbol{w} = (c, 2c)$ are perpendicular to $\boldsymbol{v}$. All vectors (x, y, z) with $x+y+z = 0$ lie on a *plane*. All vectors perpendicular to $(1,1,1)$ and $(1,2,3)$ lie on a *line*.

7 (a) $\cos\theta = \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = 1/(2)(1)$ so $\theta = 60°$ or $\pi/3$ radians (b) $\cos\theta = 0$ so $\theta = 90°$ or $\pi/2$ radians (c) $\cos\theta = 2/(2)(2) = 1/2$ so $\theta = 60°$ or $\pi/3$ (d) $\cos\theta = -1/\sqrt{2}$ so $\theta = 135°$ or $3\pi/4$.

8 (a) False: $\boldsymbol{v}$ and $\boldsymbol{w}$ are any vectors in the plane perpendicular to $\boldsymbol{u}$ (b) True: $\boldsymbol{u}\cdot(\boldsymbol{v} + 2\boldsymbol{w}) = \boldsymbol{u}\cdot\boldsymbol{v} + 2\boldsymbol{u}\cdot\boldsymbol{w} = 0$ (c) True, $\|\boldsymbol{u} - \boldsymbol{v}\|^2 = (\boldsymbol{u} - \boldsymbol{v})\cdot(\boldsymbol{u} - \boldsymbol{v})$ splits into $\boldsymbol{u}\cdot\boldsymbol{u} + \boldsymbol{v}\cdot\boldsymbol{v} = \mathbf{2}$ when $\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{u} = 0$.

9 If $v_2w_2/v_1w_1 = -1$ then $v_2w_2 = -v_1w_1$ or $v_1w_1 + v_2w_2 = \boldsymbol{v}\cdot\boldsymbol{w} = 0$: perpendicular!

10 Slopes $2/1$ and $-1/2$ multiply to give -1: then $\boldsymbol{v}\cdot\boldsymbol{w} = 0$ and the vectors (the directions) are perpendicular.

11 $\boldsymbol{v}\cdot\boldsymbol{w} < 0$ means angle $> 90°$; these $\boldsymbol{w}$'s fill half of 3-dimensional space.

12 $(1,1)$ perpendicular to $(1,5) - c(1,1)$ if $6 - 2c = 0$ or $c = 3$; $\boldsymbol{v}\cdot(\boldsymbol{w} - c\boldsymbol{v}) = 0$ if $c = \boldsymbol{v}\cdot\boldsymbol{w}/\boldsymbol{v}\cdot\boldsymbol{v}$. Subtracting $c\boldsymbol{v}$ is the key to perpendicular vectors.

13 The plane perpendicular to $(1,0,1)$ contains all vectors $(c, d, -c)$. In that plane, $\boldsymbol{v} = (1,0,-1)$ and $\boldsymbol{w} = (0,1,0)$ are perpendicular.

14 One possibility among many: $\boldsymbol{u} = (1,-1,0,0)$, $\boldsymbol{v} = (0,0,1,-1)$, $\boldsymbol{w} = (1,1,-1,-1)$ and $(1,1,1,1)$ are perpendicular to each other. "We can rotate those $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in their 3D hyperplane."

15 $\frac{1}{2}(x + y) = (2 + 8)/2 = 5$; $\cos\theta = 2\sqrt{16}/\sqrt{10}\sqrt{10} = 8/10$.

16 $\|\boldsymbol{v}\|^2 = 1 + 1 + \cdots + 1 = 9$ so $\|\boldsymbol{v}\| = 3$; $\boldsymbol{u} = \boldsymbol{v}/3 = (\frac{1}{3}, \ldots, \frac{1}{3})$ is a unit vector in 9D; $\boldsymbol{w} = (1,-1,0,\ldots,0)/\sqrt{2}$ is a unit vector in the 8D hyperplane perpendicular to $\boldsymbol{v}$.

17 $\cos\alpha = 1/\sqrt{2}$, $\cos\beta = 0$, $\cos\gamma = -1/\sqrt{2}$. For any vector $\boldsymbol{v}$, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = (v_1^2 + v_2^2 + v_3^2)/\|\boldsymbol{v}\|^2 = 1$.

18 $\|\boldsymbol{v}\|^2 = 4^2 + 2^2 = 20$ and $\|\boldsymbol{w}\|^2 = (-1)^2 + 2^2 = 5$. Pythagoras is $\|(3,4)\|^2 = 25 = 20 + 5$.

19 Start from the rules (1), (2), (3) for $\boldsymbol{v}\cdot\boldsymbol{w} = \boldsymbol{w}\cdot\boldsymbol{v}$ and $\boldsymbol{u}\cdot(\boldsymbol{v}+\boldsymbol{w})$ and $(c\boldsymbol{v})\cdot\boldsymbol{w}$. Use rule (2) for $(\boldsymbol{v}+\boldsymbol{w})\cdot(\boldsymbol{v}+\boldsymbol{w}) = (\boldsymbol{v}+\boldsymbol{w})\cdot\boldsymbol{v} + (\boldsymbol{v}+\boldsymbol{w})\cdot\boldsymbol{w}$. By rule (1) this is $\boldsymbol{v}\cdot(\boldsymbol{v}+\boldsymbol{w}) + \boldsymbol{w}\cdot(\boldsymbol{v}+\boldsymbol{w})$. Rule (2) again gives $\boldsymbol{v}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{v} + \boldsymbol{w}\cdot\boldsymbol{w} = \boldsymbol{v}\cdot\boldsymbol{v} + 2\boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{w}$. Notice $\boldsymbol{v}\cdot\boldsymbol{w} = \boldsymbol{w}\cdot\boldsymbol{v}$! The main point is to be free to open up parentheses.

20 We know that $(\boldsymbol{v} - \boldsymbol{w})\cdot(\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{v}\cdot\boldsymbol{v} - 2\boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{w}$. The Law of Cosines writes $\|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos\theta$ for $\boldsymbol{v}\cdot\boldsymbol{w}$. When $\theta < 90°$ this $\boldsymbol{v}\cdot\boldsymbol{w}$ is positive, so in this case $\boldsymbol{v}\cdot\boldsymbol{v} + \boldsymbol{w}\cdot\boldsymbol{w}$ is larger than $\|\boldsymbol{v} - \boldsymbol{w}\|^2$.

21 $2\boldsymbol{v}\cdot\boldsymbol{w} \le 2\|\boldsymbol{v}\|\|\boldsymbol{w}\|$ leads to $\|\boldsymbol{v}+\boldsymbol{w}\|^2 = \boldsymbol{v}\cdot\boldsymbol{v} + 2\boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{w} \le \|\boldsymbol{v}\|^2 + 2\|\boldsymbol{v}\|\|\boldsymbol{w}\| + \|\boldsymbol{w}\|^2$. This is $(\|\boldsymbol{v}\| + \|\boldsymbol{w}\|)^2$. Taking square roots gives $\|\boldsymbol{v} + \boldsymbol{w}\| \le \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$.

22 $v_1^2w_1^2 + 2v_1w_1v_2w_2 + v_2^2w_2^2 \le v_1^2w_1^2 + v_1^2w_2^2 + v_2^2w_1^2 + v_2^2w_2^2$ is true (cancel 4 terms) because the difference is $v_1^2w_2^2 + v_2^2w_1^2 - 2v_1w_1v_2w_2$ which is $(v_1w_2 - v_2w_1)^2 \ge 0$.

23 $\cos\beta = w_1/\|\boldsymbol{w}\|$ and $\sin\beta = w_2/\|\boldsymbol{w}\|$. Then $\cos(\beta - a) = \cos\beta\cos\alpha + \sin\beta\sin\alpha = v_1w_1/\|\boldsymbol{v}\|\|\boldsymbol{w}\| + v_2w_2/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\|$. This is $\cos\theta$ because $\beta - \alpha = \theta$.

24 Example 6 gives $|u_1||U_1| \le \frac{1}{2}(u_1^2 + U_1^2)$ and $|u_2||U_2| \le \frac{1}{2}(u_2^2 + U_2^2)$. The whole line becomes $.96 \le (.6)(.8) + (.8)(.6) \le \frac{1}{2}(.6^2 + .8^2) + \frac{1}{2}(.8^2 + .6^2) = 1$. True: $.96 < 1$.

25 The cosine of θ is $x/\sqrt{x^2 + y^2}$, near side over hypotenuse. Then $|\cos\theta|^2$ is not greater than 1: $x^2/(x^2 + y^2) \le 1$.

26 The vectors $\boldsymbol{w} = (x, y)$ with $(1,2)\cdot\boldsymbol{w} = x + 2y = 5$ lie on a line in the xy plane. The shortest $\boldsymbol{w}$ on that line is $(1,2)$. (The Schwarz inequality $\|\boldsymbol{w}\| \ge \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\| = \sqrt{5}$ is an equality when $\cos\theta = 0$ and $\boldsymbol{w} = (1,2)$ and $\|\boldsymbol{w}\| = \sqrt{5}$.)

27 The length $\|\boldsymbol{v} - \boldsymbol{w}\|$ is between 2 and 8 (triangle inequality when $\|\boldsymbol{v}\| = 5$ and $\|\boldsymbol{w}\| = 3$). The dot product $\boldsymbol{v}\cdot\boldsymbol{w}$ is between -15 and 15 by the Schwarz inequality.

28 Three vectors in the plane could make angles greater than 90° with each other: for example $(1,0), (-1,4), (-1,-4)$. Four vectors could *not* do this (360° total angle). How many can do this in $\mathbf{R}^3$ or $\mathbf{R}^n$? Ben Harris and Greg Marks showed me that the answer is $n + 1$. The vectors from the center of a regular simplex in $\mathbf{R}^n$ to its $n + 1$ vertices all have negative dot products. If $n+2$ vectors in $\mathbf{R}^n$ had negative dot products, project them onto the plane orthogonal to the last one. Now you have $n + 1$ vectors in $\mathbf{R}^{n-1}$ with negative dot products. Keep going to 4 vectors in $\mathbf{R}^2$: no way!

29 For a specific example, pick $\boldsymbol{v} = (1, 2, -3)$ and then $\boldsymbol{w} = (-3, 1, 2)$. In this example $\cos\theta = \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = -7/\sqrt{14}\sqrt{14} = -1/2$ and $\theta = 120°$. This always happens when $x + y + z = 0$:

$$\boldsymbol{v}\cdot\boldsymbol{w} = xz + xy + yz = \frac{1}{2}(x + y + z)^2 - \frac{1}{2}(x^2 + y^2 + z^2)$$

This is the same as $\boldsymbol{v}\cdot\boldsymbol{w} = 0 - \dfrac{1}{2}\ \|\boldsymbol{v}\|\|\boldsymbol{w}\|$. Then $\cos\theta = \dfrac{1}{2}$.

30 Wikipedia gives this proof of geometric mean $G = \sqrt[3]{xyz} \le$ arithmetic mean $A = (x + y + z)/3$. First there is equality in case $x = y = z$. Otherwise A is somewhere between the three positive numbers, say for example $z < A < y$.

Use the known inequality $g \le a$ for the *two* positive numbers x and $y + z - A$. Their mean $a = \frac{1}{2}(x + y + z - A)$ is $\frac{1}{2}(3A - A) =$ same as A! So $a \ge g$ says that $A^3 \ge g^2A = x(y + z - A)A$. But $(y + z - A)A = (y - A)(A - z) + yz > yz$. Substitute to find $A^3 > xyz = G^3$ as we wanted to prove. Not easy!

There are many proofs of $G = (x_1x_2\cdots x_n)^{1/n} \le A = (x_1 + x_2 + \cdots + x_n)/n$. In calculus you are maximizing G on the plane $x_1 + x_2 + \cdots + x_n = n$. The maximum occurs when all x's are equal.

31 The columns of the 4 by 4 "Hadamard matrix" (times $\frac{1}{2}$) are perpendicular unit vectors:

$$\frac{1}{2}H = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

32 The commands $V =$ **randn** $(3, 30)$; $D =$ **sqrt** (**diag** $(V' * V)$); $U = V\backslash D$; will give 30 random unit vectors in the columns of U. Then $u' * U$ is a row matrix of 30 dot products whose average absolute value may be close to $2/\pi$.

Problem Set 1.3, page 29

1 $2\boldsymbol{s}_1 + 3\boldsymbol{s}_2 + 4\boldsymbol{s}_3 = (2,5,9)$. The same vector $\boldsymbol{b}$ comes from S times $\boldsymbol{x} = (2,3,4)$:

$$\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}(\text{row } 1)\cdot\boldsymbol{x}\\(\text{row } 2)\cdot\boldsymbol{x}\\(\text{row } 2)\cdot\boldsymbol{x}\end{bmatrix} = \begin{bmatrix}2\\5\\9\end{bmatrix}.$$

2 The solutions are $y_1 = 1$, $y_2 = 0$, $y_3 = 0$ (right side = column 1) and $y_1 = 1$, $y_2 = 3$, $y_3 = 5$. That second example illustrates that the first n odd numbers add to n^2.

3 $$\begin{aligned} y_1 \quad\quad\quad &= B_1\\ y_1 + y_2 \quad &= B_2\\ y_1 + y_2 + y_3 &= B_3\end{aligned} \quad\text{gives}\quad \begin{aligned} y_1 &= B_1\\ y_2 &= -B_1 + B_2\\ y_3 &= \quad -B_2 + B_3\end{aligned} = \begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}B_1\\B_2\\B_3\end{bmatrix}$$

The inverse of $S = \begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}$ is $A = \begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}$: independent columns in A and S!

4 The combination $0\boldsymbol{w}_1 + 0\boldsymbol{w}_2 + 0\boldsymbol{w}_3$ always gives the zero vector, but this problem looks for other *zero* combinations (then the vectors are *dependent*, they lie in a plane): $\boldsymbol{w}_2 = (\boldsymbol{w}_1 + \boldsymbol{w}_3)/2$ so one combination that gives zero is $\frac{1}{2}\boldsymbol{w}_1 - \boldsymbol{w}_2 + \frac{1}{2}\boldsymbol{w}_3$.

5 The rows of the 3 by 3 matrix in Problem 4 must also be *dependent*: $\boldsymbol{r}_2 = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_3)$. The column and row combinations that produce $\mathbf{0}$ are the same: this is unusual.

6 $c = 3$ $\begin{bmatrix}1&3&5\\1&2&4\\1&1&\mathbf{3}\end{bmatrix}$ has column 3 = 2 (column 1) + column 2

$c = -1$ $\begin{bmatrix}1&0&\mathbf{-1}\\1&1&0\\0&1&1\end{bmatrix}$ has column 3 = $-$ column 1 + column 2

$c = 0$ $\begin{bmatrix}\mathbf{0}&\mathbf{0}&\mathbf{0}\\2&1&5\\3&3&6\end{bmatrix}$ has column 3 = 3 (column 1) $-$ column 2

7 All three rows are perpendicular to the solution $\boldsymbol{x}$ (the three equations $\boldsymbol{r}_1\cdot\boldsymbol{x} = 0$ and $\boldsymbol{r}_2\cdot\boldsymbol{x} = 0$ and $\boldsymbol{r}_3\cdot\boldsymbol{x} = 0$ tell us this). Then the whole plane of the rows is perpendicular to $\boldsymbol{x}$ (the plane is also perpendicular to all multiples $c\boldsymbol{x}$).

8 $$\begin{aligned} x_1 - 0 &= b_1\\ x_2 - x_1 &= b_2\\ x_3 - x_2 &= b_3\\ x_4 - x_3 &= b_4\end{aligned} \qquad \begin{aligned} x_1 &= b_1\\ x_2 &= b_1 + b_2\\ x_3 &= b_1 + b_2 + b_3\\ x_4 &= b_1 + b_2 + b_3 + b_4\end{aligned} = \begin{bmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{bmatrix}\begin{bmatrix}b_1\\b_2\\b_3\\b_4\end{bmatrix} = A^{-1}\boldsymbol{b}$$

9 The cyclic difference matrix C has a line of solutions (in 4 dimensions) to $C\boldsymbol{x} = \mathbf{0}$:

$$\begin{bmatrix}1&0&0&-1\\-1&1&0&0\\0&-1&1&0\\0&0&-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \text{ when } \boldsymbol{x} = \begin{bmatrix}c\\c\\c\\c\end{bmatrix} = \text{any constant vector.}$$

10 $\begin{aligned} z_2 - z_1 &= b_1 \\ z_3 - z_2 &= b_2 \\ 0 - z_3 &= b_3 \end{aligned} \qquad \begin{aligned} z_1 &= -b_1 - b_2 - b_3 \\ z_2 &= \quad\;\; -b_2 - b_3 \\ z_3 &= \qquad\quad\;\; -b_3 \end{aligned} \qquad = \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \Delta^{-1}\boldsymbol{b}$

11 The forward differences of the squares are $(t+1)^2 - t^2 = t^2 + 2t + 1 - t^2 = 2t + 1$. Differences of the nth power are $(t+1)^n - t^n = t^n - t^n + nt^{n-1} + \cdots$. The leading term is the derivative nt^{n-1}. The binomial theorem gives all the terms of $(t+1)^n$.

12 Centered difference matrices of *even* size seem to be invertible. Look at eqns. 1 and 4:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad \begin{matrix} \text{First} \\ \text{solve} \\ x_2 = b_1 \\ -x_3 = b_4 \end{matrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -b_2 - b_4 \\ b_1 \\ -b_4 \\ b_1 + b_3 \end{bmatrix}$$

13 *Odd size*: The five centered difference equations lead to $b_1 + b_3 + b_5 = 0$.

$$\begin{aligned} x_2 \qquad &= b_1 \\ x_3 - x_1 &= b_2 \\ x_4 - x_2 &= b_3 \\ x_5 - x_3 &= b_4 \\ -x_4 &= b_5 \end{aligned}$$

Add equations $1, 3, 5$
The left side of the sum is zero
The right side is $b_1 + b_3 + b_5$
There cannot be a solution unless $b_1 + b_3 + b_5 = 0$.

14 An example is $(a, b) = (3, 6)$ and $(c, d) = (1, 2)$. The ratios a/c and b/d are equal. Then $ad = bc$. Then (when you divide by bd) the ratios a/b and c/d are equal!

Problem Set 2.1, page 40

1 The columns are $\boldsymbol{i} = (1, 0, 0)$ and $\boldsymbol{j} = (0, 1, 0)$ and $\boldsymbol{k} = (0, 0, 1)$ and $\boldsymbol{b} = (2, 3, 4) = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$.

2 The planes are the same: $2x = 4$ is $x = 2$, $3y = 9$ is $y = 3$, and $4z = 16$ is $z = 4$. The solution is the same point $\boldsymbol{X} = \boldsymbol{x}$. The columns are changed; but same combination.

3 The solution is not changed! The second plane and row 2 of the matrix and all columns of the matrix (vectors in the column picture) are changed.

4 If $z = 2$ then $x + y = 0$ and $x - y = z$ give the point $(1, -1, 2)$. If $z = 0$ then $x + y = 6$ and $x - y = 4$ produce $(5, 1, 0)$. Halfway between those is $(3, 0, 1)$.

5 If x, y, z satisfy the first two equations they also satisfy the third equation. The line $\mathbf{L}$ of solutions contains $\boldsymbol{v} = (1, 1, 0)$ and $\boldsymbol{w} = (\frac{1}{2}, 1, \frac{1}{2})$ and $\boldsymbol{u} = \frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$ and all combinations $c\boldsymbol{v} + d\boldsymbol{w}$ with $c + d = 1$.

6 Equation 1 + equation 2 − equation 3 is now $0 = -4$. Line misses plane; *no solution*.

7 Column 3 = Column 1 makes the matrix singular. Solutions $(x, y, z) = (1, 1, 0)$ or $(0, 1, 1)$ and you can add any multiple of $(-1, 0, 1)$; $\boldsymbol{b} = (4, 6, c)$ needs $c = 10$ for solvability (then $\boldsymbol{b}$ lies in the plane of the columns).

8 Four planes in 4-dimensional space normally meet at a *point*. The solution to $A\boldsymbol{x} = (3, 3, 3, 2)$ is $\boldsymbol{x} = (0, 0, 1, 2)$ if A has columns $(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1)$. The equations are $x + y + z + t = 3$, $y + z + t = 3$, $z + t = 3$, $t = 2$.

9 (a) $A\boldsymbol{x} = (18, 5, 0)$ and (b) $A\boldsymbol{x} = (3, 4, 5, 5)$.

10 Multiplying as linear combinations of the columns gives the same $A\boldsymbol{x}$. By rows or by columns: **9** separate multiplications for 3 by 3.

11 $A\boldsymbol{x}$ equals $(14, 22)$ and $(0, 0)$ and $(9, 7)$.

12 $A\boldsymbol{x}$ equals (z, y, x) and $(0, 0, 0)$ and $(3, 3, 6)$.

13 (a) $\boldsymbol{x}$ has n components and $A\boldsymbol{x}$ has m components (b) Planes from each equation in $A\boldsymbol{x} = \boldsymbol{b}$ are in n-dimensional space, but the columns are in m-dimensional space.

14 $2x + 3y + z + 5t = 8$ is $A\boldsymbol{x} = \boldsymbol{b}$ with the 1 by 4 matrix $A = [2 \;\; 3 \;\; 1 \;\; 5]$. The solutions $\boldsymbol{x}$ fill a 3D "plane" in 4 dimensions. It could be called a *hyperplane*.

15 (a) $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

16 $90°$ rotation from $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $180°$ rotation from $R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$.

17 $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ produces (y, z, x) and $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ recovers (x, y, z). Q is the inverse of P.

18 $E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ subtract the first component from the second.

19 $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $E\boldsymbol{v} = (3, 4, 8)$ and $E^{-1}E\boldsymbol{v}$ recovers $(3, 4, 5)$.

20 $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ projects onto the x-axis and $P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ projects onto the y-axis. $\boldsymbol{v} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ has $P_1\boldsymbol{v} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ and $P_2 P_1 \boldsymbol{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

21 $R = \dfrac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$ rotates all vectors by $45°$. The columns of R are the results from rotating $(1, 0)$ and $(0, 1)$!

22 The dot product $A\boldsymbol{x} = [1 \;\; 4 \;\; 5]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (1 \text{ by } 3)(3 \text{ by } 1)$ is zero for points (x, y, z) on a plane in three dimensions. The columns of A are one-dimensional vectors.

23 $A = [1 \;\; 2 \;\; ; \;\; 3 \;\; 4]$ and $\boldsymbol{x} = [5 \;\; -2]'$ and $\boldsymbol{b} = [1 \;\; 7]'$. $\boldsymbol{r} = \boldsymbol{b} - A * \boldsymbol{x}$ prints as zero.

24 $A * \boldsymbol{v} = [3 \;\; 4 \;\; 5]'$ and $\boldsymbol{v}' * \boldsymbol{v} = 50$. But $\boldsymbol{v} * A$ gives an error message from 3 by 1 times 3 by 3.

25 **ones**$(4, 4) *$ **ones**$(4, 1) = [4 \;\; 4 \;\; 4 \;\; 4]'$; $B * \boldsymbol{w} = [10 \;\; 10 \;\; 10 \;\; 10]'$.

26 The row picture has two lines meeting at the solution $(4, 2)$. The column picture will have $4(1, 1) + 2(-2, 1) = 4(\text{column } 1) + 2(\text{column } 2) = \text{right side } (0, 6)$.

27 The row picture shows **2 planes** in **3-dimensional space**. The column picture is in **2-dimensional space**. The solutions normally lie on a ***line***.

28 The row picture shows four *lines* in the 2D plane. The column picture is in *four*-dimensional space. No solution unless the right side is a combination of *the two columns*.

29 $\boldsymbol{u}_2 = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$ and $\boldsymbol{u}_3 = \begin{bmatrix} .65 \\ .35 \end{bmatrix}$. The components add to 1. They are always positive. $\boldsymbol{u}_7, \boldsymbol{v}_7, \boldsymbol{w}_7$ are all close to $(.6, .4)$. Their components still add to 1.

30 $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}\begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix} =$ *steady state* $\boldsymbol{s}$. No change when multiplied by $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$.

31 $M = \begin{bmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 5+u & 5-u+v & 5-v \\ 5-u-v & 5 & 5+u+v \\ 5+v & 5+u-v & 5-u \end{bmatrix}$; $M_3(1,1,1) = (15,15,15)$; $M_4(1,1,1,1) = (34,34,34,34)$ because $1+2+\cdots+16 = 136$ which is $4(34)$.

32 A is singular when its third column $\boldsymbol{w}$ is a combination $c\boldsymbol{u} + d\boldsymbol{v}$ of the first columns. A typical column picture has $\boldsymbol{b}$ outside the plane of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$. A typical row picture has the intersection line of two planes parallel to the third plane. *Then no solution.*

33 $\boldsymbol{w} = (5,7)$ is $5\boldsymbol{u} + 7\boldsymbol{v}$. Then $A\boldsymbol{w}$ equals 5 times $A\boldsymbol{u}$ plus 7 times $A\boldsymbol{v}$.

34 $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ has the solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 8 \\ 6 \end{bmatrix}$.

35 $\boldsymbol{x} = (1,\ldots,1)$ gives $S\boldsymbol{x} =$ sum of each row $= 1+\cdots+9 = 45$ for Sudoku matrices. 6 row orders $(1,2,3)$, $(1,3,2)$, $(2,1,3)$, $(2,3,1)$, $(3,1,2)$, $(3,2,1)$ are in Section 2.7. The same 6 permutations of *blocks* of rows produce Sudoku matrices, so $6^4 = 1296$ orders of the 9 rows all stay Sudoku. (And also 1296 permutations of the 9 columns.)

Problem Set 2.2, page 51

1 Multiply by $\ell_{21} = \frac{10}{2} = 5$ and subtract to find $2x+3y = 14$ and $-6y = 6$. The pivots to circle are 2 and -6.

2 $-6y = 6$ gives $y = -1$. Then $2x + 3y = 1$ gives $x = 2$. Multiplying the right side $(1, 11)$ by 4 will multiply the solution by 4 to give the new solution $(x, y) = (8, -4)$.

3 Subtract $-\frac{1}{2}$ (or add $\frac{1}{2}$) times equation 1. The new second equation is $3y = 3$. Then $y = 1$ and $x = 5$. If the right side changes sign, so does the solution: $(x, y) = (-5, -1)$.

4 Subtract $\ell = \frac{c}{a}$ times equation 1. The new second pivot multiplying y is $d - (cb/a)$ or $(ad - bc)/a$. Then $y = (ag - cf)/(ad - bc)$.

5 $6x + 4y$ is 2 times $3x + 2y$. There is no solution unless the right side is $2 \cdot 10 = 20$. Then all the points on the line $3x + 2y = 10$ are solutions, including $(0, 5)$ and $(4, -1)$. (The two lines in the row picture are the same line, containing all solutions).

6 Singular system if $b = 4$, because $4x + 8y$ is 2 times $2x + 4y$. Then $g = 32$ makes the lines become the *same*: infinitely many solutions like $(8, 0)$ and $(0, 4)$.

7 If $a = 2$ elimination must fail (two parallel lines in the row picture). The equations have no solution. With $a = 0$, elimination will stop for a row exchange. Then $3y = -3$ gives $y = -1$ and $4x + 6y = 6$ gives $x = 3$.

8 If $k = 3$ elimination must fail: no solution. If $k = -3$, elimination gives $0 = 0$ in equation 2: infinitely many solutions. If $k = 0$ a row exchange is needed: one solution.

9 On the left side, $6x - 4y$ is 2 times $(3x - 2y)$. Therefore we need $b_2 = 2b_1$ on the right side. Then there will be infinitely many solutions (two parallel lines become one single line).

10 The equation $y = 1$ comes from elimination (subtract $x + y = 5$ from $x + 2y = 6$). Then $x = 4$ and $5x - 4y = c = 16$.

11 (a) Another solution is $\frac{1}{2}(x + X, y + Y, z + Z)$. (b) If 25 planes meet at two points, they meet along the whole line through those two points.

12 Elimination leads to an upper triangular system; then comes back substitution.

$$\begin{aligned} 2x + 3y + z &= 8 \\ y + 3z &= 4 \\ 8z &= 8 \end{aligned} \quad \text{gives} \quad \begin{aligned} x &= 2 \\ y &= 1 \\ z &= 1 \end{aligned}$$

If a zero is at the start of row 2 or 3, that avoids a row operation.

13

$$\begin{aligned} 2x - 3y \phantom{{}+ z} &= 3 \\ 4x - 5y + z &= 7 \\ 2x - y - 3z &= 5 \end{aligned} \quad \text{gives} \quad \begin{aligned} 2x - 3y &= 3 \\ y + z &= 1 \\ 2y + 3z &= 2 \end{aligned} \quad \text{and} \quad \begin{aligned} 2x - 3y &= 3 \\ y + z &= 1 \\ -5z &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} x &= 3 \\ y &= 1 \\ z &= 0 \end{aligned}$$

Subtract $2 \times$ row 1 from row 2, subtract $1 \times$ row 1 from row 3, subtract $2 \times$ row 2 from row 3

14 Subtract 2 times row 1 from row 2 to reach $(d-10)y - z = 2$. Equation (3) is $y - z = 3$. If $d = 10$ exchange rows 2 and 3. If $d = 11$ the system becomes singular.

15 The second pivot position will contain $-2 - b$. If $b = -2$ we exchange with row 3. If $b = -1$ (singular case) the second equation is $-y - z = 0$. A solution is $(1, 1, -1)$.

16 (a) **Example of 2 exchanges**

$$\begin{aligned} 0x + 0y + 2z &= 4 \\ x + 2y + 2z &= 5 \\ 0x + 3y + 4z &= 6 \end{aligned}$$

(exchange 1 and 2, then 2 and 3)

(b) **Exchange but then break down**

$$\begin{aligned} 0x + 3y + 4z &= 4 \\ x + 2y + 2z &= 5 \\ 0x + 3y + 4z &= 6 \end{aligned}$$

(rows 1 and 3 are not consistent)

17 If row 1 = row 2, then row 2 is zero after the first step; exchange the zero row with row 3 and there is no *third* pivot. If column 2 = column 1, then column 2 has no pivot.

18 *Example* $x + 2y + 3z = 0$, $4x + 8y + 12z = 0$, $5x + 10y + 15z = 0$ has 9 different coefficients but rows 2 and 3 become $0 = 0$: infinitely many solutions.

19 Row 2 becomes $3y - 4z = 5$, then row 3 becomes $(q + 4)z = t - 5$. If $q = -4$ the system is singular—no third pivot. Then if $t = 5$ the third equation is $0 = 0$. Choosing $z = 1$ the equation $3y - 4z = 5$ gives $y = 3$ and equation 1 gives $x = -9$.

20 Singular if row 3 is a combination of rows 1 and 2. From the end view, the three planes form a triangle. This happens if rows $1 + 2 =$ row 3 on the left side but not the right side: $x + y + z = 0$, $x - 2y - z = 1$, $2x - y = 4$. No parallel planes but still no solution.

21 (a) Pivots $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}$ in the equations $2x + y = 0$, $\frac{3}{2}y + z = 0$, $\frac{4}{3}z + t = 0$, $\frac{5}{4}t = 5$ after elimination. Back substitution gives $t = 4$, $z = -3$, $y = 2$, $x = -1$. (b) If the off-diagonal entries change from $+1$ to -1, the pivots are the same. The solution is $(1, 2, 3, 4)$ instead of $(-1, 2, -3, 4)$.

22 The fifth pivot is $\frac{6}{5}$ for both matrices (1's or -1's off the diagonal). The nth pivot is $\frac{n+1}{n}$.

23 If ordinary elimination leads to $x + y = 1$ and $2y = 3$, the original second equation could be $2y + \ell(x + y) = 3 + \ell$ for any ℓ. Then ℓ will be the multiplier to reach $2y = 3$.

24 Elimination fails on $\begin{bmatrix} a & 2 \\ a & a \end{bmatrix}$ if $a = 2$ or $a = 0$.

25 $a = 2$ (equal columns), $a = 4$ (equal rows), $a = 0$ (zero column).

26 Solvable for $s = 10$ (add the two pairs of equations to get $a+b+c+d$ on the left sides, 12 and $2 + s$ on the right sides). The four equations for a, b, c, d are **singular**! Two solutions are $\begin{bmatrix} 1 & 3 \\ 1 & 7 \end{bmatrix}$ and $\begin{bmatrix} 0 & 4 \\ 2 & 6 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$.

27 Elimination leaves the diagonal matrix diag$(3, 2, 1)$ in $3x = 3, 2y = 2, z = 4$. Then $x = 1, y = 1, z = 4$.

28 $A(2, :) = A(2, :) - 3 * A(1, :)$ subtracts 3 times row 1 from row 2.

29 The average pivots for rand(3) *without* row exchanges were $\frac{1}{2}, 5, 10$ in one experiment—but pivots 2 and 3 can be arbitrarily large. Their averages are actually infinite ! *With row exchanges* in MATLAB's lu code, the averages .75 and .50 and .365 are much more stable (and should be predictable, also for randn with normal instead of uniform probability distribution).

30 If $A(5, 5)$ is 7 not 11, then the last pivot will be 0 not 4.

31 Row j of U is a combination of rows $1, \ldots, j$ of A. If $A\boldsymbol{x} = \mathbf{0}$ then $U\boldsymbol{x} = \mathbf{0}$ (not true if $\boldsymbol{b}$ replaces $\mathbf{0}$). U is the diagonal of A when A is *lower triangular.*

32 The question deals with 100 equations $A\boldsymbol{x} = \mathbf{0}$ when A is singular.

(a) Some linear combination of the 100 rows is **the row of 100 zeros**.

(b) Some linear combination of the 100 **columns** is **the column of zeros**.

(c) A very singular matrix has all ones: $A = \mathbf{eye}(100)$. A better example has 99 random rows (or the numbers $1^i, \ldots, 100^i$ in those rows). The 100th row could be the sum of the first 99 rows (or any other combination of those rows with no zeros).

(d) The row picture has 100 planes **meeting along a common line through 0**. The column picture has 100 vectors all in the same 99-dimensional hyperplane.

Problem Set 2.3, page 63

1 $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

2 $E_{32}E_{21}b = (1, -5, -35)$ but $E_{21}E_{32}b = (1, -5, 0)$. When E_{32} comes first, row 3 feels no effect from row 1.

3 $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ $M = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}$.

4 Elimination on column 4: $\boldsymbol{b} = \begin{bmatrix}1\\0\\0\end{bmatrix} \xrightarrow{E_{21}} \begin{bmatrix}1\\-4\\0\end{bmatrix} \xrightarrow{E_{31}} \begin{bmatrix}1\\-4\\2\end{bmatrix} \xrightarrow{E_{32}} \begin{bmatrix}1\\-4\\10\end{bmatrix}$. The original $A\boldsymbol{x} = \boldsymbol{b}$ has become $U\boldsymbol{x} = \boldsymbol{c} = (1, -4, 10)$. Then back substitution gives $\boldsymbol{z} = -5$, $y = \frac{1}{2}$, $\boldsymbol{x} = \frac{1}{2}$. This solves $A\boldsymbol{x} = (1, 0, 0)$.

5 Changing a_{33} from 7 to 11 will change the third pivot from 5 to 9. Changing a_{33} from 7 to 2 will change the pivot from 5 to *no pivot*.

6 Example: $\begin{bmatrix}2&3&7\\2&3&7\\2&3&7\end{bmatrix}\begin{bmatrix}1\\3\\-1\end{bmatrix} = \begin{bmatrix}4\\4\\4\end{bmatrix}$. If all columns are multiples of column 1, there is no second pivot.

7 To reverse E_{31}, **add** 7 times row **1** to row **3**. The inverse of the elimination matrix $E = \begin{bmatrix}1&0&0\\0&1&0\\\mathbf{-7}&0&1\end{bmatrix}$ is $E^{-1} = \begin{bmatrix}1&0&0\\0&1&0\\\mathbf{7}&0&1\end{bmatrix}$.

8 $M = \begin{bmatrix}a&b\\c&d\end{bmatrix}$ and $M^* = \begin{bmatrix}a&b\\c-\ell a&d-\ell b\end{bmatrix}$. $\det M^* = a(d-\ell b) - b(c-\ell a)$ reduces to $ad - bc$!

9 $M = \begin{bmatrix}1&0&0\\0&0&1\\-1&1&0\end{bmatrix}$. After the exchange, we need E_{31} (not E_{21}) to act on the new row 3.

10 $E_{13} = \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$; $\begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix}$; $E_{31}E_{13} = \begin{bmatrix}2&0&1\\0&1&0\\1&0&1\end{bmatrix}$. Test on the identity matrix!

11 An example with two negative pivots is $A = \begin{bmatrix}1&2&2\\1&1&2\\1&2&1\end{bmatrix}$. The diagonal entries can change sign during elimination.

12 The first product is $\begin{bmatrix}9&8&7\\6&5&4\\3&2&1\end{bmatrix}$ rows and also columns reversed. The second product is $\begin{bmatrix}1&2&3\\0&1&-2\\0&2&-3\end{bmatrix}$.

13 (a) E times the third column of B is the third column of EB. A column that starts at zero will stay at zero. (b) E could add row 2 to row 3 to change a zero row to a nonzero row.

14 E_{21} has $-\ell_{21} = \frac{1}{2}$, E_{32} has $-\ell_{32} = \frac{2}{3}$, E_{43} has $-\ell_{43} = \frac{3}{4}$. Otherwise the E's match I.

15 $a_{ij} = 2i - 3j$: $A = \begin{bmatrix}-1&-4&-7\\1&-2&-5\\3&\mathbf{0}&-3\end{bmatrix} \rightarrow \begin{bmatrix}-1&-4&-7\\0&-6&-12\\0&\mathbf{-12}&-24\end{bmatrix}$. The zero became -12, an example of *fill-in*. To remove that -12, choose $E_{32} = \begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}$.

16 (a) The ages of X and Y are x and y: $x - 2y = 0$ and $x + y = 33$; $x = 22$ and $y = 11$ (b) The line $y = mx + c$ contains $x = 2$, $y = 5$ and $x = 3$, $y = 7$ when $2m + c = 5$ and $3m + c = 7$. Then $m = 2$ is the slope.

17 The parabola $y = a + bx + cx^2$ goes through the 3 given points when $\begin{aligned} a + b + c &= 4 \\ a + 2b + 4c &= 8 \\ a + 3b + 9c &= 14 \end{aligned}$. Then $a = 2$, $b = 1$, and $c = 1$. This matrix with columns $(1,1,1)$, $(1,2,3)$, $(1,4,9)$ is a "Vandermonde matrix."

18 $EF = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$, $FE = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b+ac & c & 1 \end{bmatrix}$, $E^2 = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 2b & 0 & 1 \end{bmatrix}$, $F^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3c & 1 \end{bmatrix}$.

19 $PQ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. In the opposite order, two row exchanges give $QP = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, If M exchanges rows 2 and 3 then $M^2 = I$ (also $(-M)^2 = I$). There are many square roots of I: Any matrix $M = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ has $M^2 = I$ if $a^2 + bc = 1$.

20 (a) Each column of EB is E times a column of B (b) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{4} \\ \mathbf{1} & \mathbf{2} & \mathbf{4} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ \mathbf{2} & \mathbf{4} & \mathbf{8} \end{bmatrix}$. All rows of EB are *multiples* of $\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}$.

21 **No.** $E = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ give $EF = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ but $FE = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

22 (a) $\sum a_{3j} x_j$ (b) $a_{21} - a_{11}$ (c) $a_{21} - 2a_{11}$ (d) $(EA\boldsymbol{x})_1 = (A\boldsymbol{x})_1 = \sum a_{1j} x_j$.

23 $E(EA)$ subtracts 4 times row 1 from row 2 (EEA does the row operation twice). AE subtracts 2 times column 2 of A from column 1 (multiplication by E on the right side acts on **columns** instead of rows).

24 $\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 2 & 3 & \mathbf{1} \\ 4 & 1 & \mathbf{17} \end{bmatrix} \to \begin{bmatrix} 2 & 3 & \mathbf{1} \\ 0 & -5 & \mathbf{15} \end{bmatrix}$. The triangular system is $\begin{aligned} 2x_1 + 3x_2 &= 1 \\ -5x_2 &= 15 \end{aligned}$ Back substitution gives $x_1 = 5$ and $x_2 = -3$.

25 The last equation becomes $0 = 3$. If the original 6 is 3, then row 1 + row 2 = row 3.

26 (a) Add two columns $\boldsymbol{b}$ and $\boldsymbol{b}^*$ $\begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix} \to \boldsymbol{x} = \begin{bmatrix} -7 \\ 2 \end{bmatrix}$ and $\boldsymbol{x}^* = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$.

27 (a) No solution if $d = 0$ and $c \neq 0$ (b) Many solutions if $d = 0 = c$. No effect from a, b.

28 $A = AI = A(BC) = (AB)C = IC = C$. That middle equation is crucial.

29 $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ subtracts each row from the next row. The result $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$ still has multipliers $= 1$ in a 3 by 3 Pascal matrix. The product M of all elimination matrices is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$. This "alternating sign Pascal matrix" is on page 88.

30 Given positive integers with $ad - bc = 1$. Certainly $c < a$ and $b < d$ would be impossible. Also $c > a$ and $b > d$ would be impossible with integers. This leaves row 1 $<$ row 2 OR row 2 $<$ row 1. An example is $M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. Multiply by $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ to get $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, then multiply twice by $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ to get $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. This shows that $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

31 $E_{21} = \begin{bmatrix} 1 & & & \\ 1/2 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E_{32} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 2/3 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E_{43} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 3/4 & 1 \end{bmatrix}$,

$$E_{43}\,E_{32}\,E_{21} = \begin{bmatrix} 1 & & & \\ 1/2 & 1 & & \\ 1/3 & 2/3 & 1 & \\ 1/4 & 2/4 & 3/4 & 1 \end{bmatrix}$$

Problem Set 2.4, page 75

1 If all entries of A, B, C, D are 1, then $BA = 3$ **ones**(5) is 5 by 5; $AB = 5$ **ones**(3) is 3 by 3; $ABD = 15$ **ones**(3, 1) is 3 by 1. DBA and $A(B + C)$ are not defined.

2 (a) A (column 3 of B) (b) (Row 1 of A) B (c) (Row 3 of A)(column 4 of B) (d) (Row 1 of C)D(column 1 of E).

3 $AB + AC$ is the same as $A(B + C) = \begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix}$. (*Distributive law*).

4 $A(BC) = (AB)C$ by the *associative law*. In this example both answers are $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ from column 1 of AB and row 2 of C (multiply columns times rows).

5 (a) $A^2 = \begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix}$ and $A^n = \begin{bmatrix} 1 & nb \\ 0 & 1 \end{bmatrix}$. (b) $A^2 = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$ and $A^n = \begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix}$.

6 $(A + B)^2 = \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix} = A^2 + AB + BA + B^2$. But $A^2 + 2AB + B^2 = \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}$.

7 (a) True (b) False (c) True (d) False: usually $(AB)^2 \neq A^2B^2$.

8 The rows of DA are 3 (row 1 of A) and 5 (row 2 of A). Both rows of EA are row 2 of A. The columns of AD are 3 (column 1 of A) and 5 (column 2 of A). The first column of AE is zero, the second is column 1 of A + column 2 of A.

9 $AF = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$ and $E(AF)$ equals $(EA)F$ because matrix multiplication is *associative*.

10 $FA = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$ and then $E(FA) = \begin{bmatrix} a+c & b+d \\ a+2c & b+2d \end{bmatrix}$. $E(FA)$ is not the same as $F(EA)$ because multiplication is not commutative.

11 (a) $B = 4I$ (b) $B = 0$ (c) $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (d) Every row of B is $1, 0, 0$.

12 $AB = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = BA = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ gives $\boldsymbol{b} = \boldsymbol{c} = \boldsymbol{0}$. Then $AC = CA$ gives $\boldsymbol{a} = \boldsymbol{d}$. The only matrices that commute with B and C (and all other matrices) are multiples of I: $A = aI$.

13 $(A-B)^2 = (B-A)^2 = A(A-B) - B(A-B) = A^2 - AB - BA + B^2$. In a typical case (when $AB \neq BA$) the matrix $A^2 - 2AB + B^2$ is different from $(A-B)^2$.

14 (a) True (A^2 is only defined when A is square) (b) False (if A is m by n and B is n by m, then AB is m by m and BA is n by n). (c) True (d) False (take $B = 0$).

15 (a) mn (use every entry of A) (b) $mnp = p\times$part (a) (c) n^3 (n^2 dot products).

16 (a) Use only column 2 of B (b) Use only row 2 of A (c)–(d) Use row 2 of first A.

17 $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ has $a_{ij} = \min(i, j)$. $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ has $a_{ij} = (-1)^{i+j} =$ "alternating sign matrix". $A = \begin{bmatrix} 1/1 & 1/2 & 1/3 \\ 2/1 & 2/2 & 2/3 \\ 3/1 & 3/2 & 3/3 \end{bmatrix}$ has $a_{ij} = i/j$ (this will be an example of a *rank one matrix*).

18 Diagonal matrix, lower triangular, symmetric, all rows equal. Zero matrix fits all four.

19 (a) a_{11} (b) $\ell_{31} = a_{31}/a_{11}$ (c) $a_{32} - (\frac{a_{31}}{a_{11}})a_{12}$ (d) $a_{22} - (\frac{a_{21}}{a_{11}})a_{12}$.

20 $A^2 = \begin{bmatrix} 0&0&4&0 \\ 0&0&0&4 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0&0&0&8 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$, $A^4 =$ zero matrix for *strictly triangular* A.

Then $A\boldsymbol{v} = A\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2y \\ 2z \\ 2t \\ 0 \end{bmatrix}$, $A^2\boldsymbol{v} = \begin{bmatrix} 4z \\ 4t \\ 0 \\ 0 \end{bmatrix}$, $A^3\boldsymbol{v} = \begin{bmatrix} 8t \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $A^4\boldsymbol{v} = 0$.