Chapter 1 Topics in Linear Algebra

Chapter 1 in the main text covers a few key topics in linear algebra that are often not treated in standard undergraduate courses in mathematics for economists. Some topics are especially important in understanding parts of the later chapters devoted to second-order conditions for the maximum or minimum of a function of several variables.

The chapter begins in Section 1.1 with a brief review of elementary linear algebra for easy reference.

An important topic concerns linear independence, discussed in Section 1.2. A set of vectors is linearly dependent iff at least one of them can be expressed as a linear combination of the others. Otherwise the set is linearly independent. More customary definitions of linear dependence and independence then follow and are shown to be equivalent to these possibly more intuitive definitions. Actually, experience suggests that some students have difficulties in understanding these definitions, so it is important to be quite precise and explain them carefully. The diagrams illustrating the difference between linear dependence and independence for sets of vectors in ^{2} and ^{3} helps the students in getting an intuitive understanding of the concepts.

Next, Section 1.3 defines the rank of a matrix as the maximum number of linearly independent columns (or zero, if all columns are zero). It follows that a square matrix of order n has rank n iff its determinant is nonzero. Furthermore, the minors of a matrix are defined, and it is argued that the rank is equal to the order of the largest nonzero minor. This implies that the rank of a matrix is equal to the rank of its transpose, so the rank is also equal to the maximum number of linearly independent rows. However, in most cases the most efficient way of finding the rank of a matrix is to apply elementary operations.

Linear systems of equations receive more extensive discussion in Section 1.4. One key result is Theorem 1.4.1, which says that a necessary and sufficient condition for a linear system of equations to have a solution is that the rank of the coefficient matrix does not increase when the vector of right-hand sides of the equations is added as an extra column. Of course, this is rather obviously equivalent to the requirement that the right-hand side vector *can* be expressed as a linear combination of the columns of the coefficient matrix—i.e., that there is a solution to the system of equations. The next result is Theorem 1.4.2, stating that if the rank k of the coefficient matrix is less than the number of equations m, and if any solution exists at all, then $m - k$ of the equations are superfluous because any solution of the remaining k equations will automatically solve all m equations. Also, if the rank k of the coefficient matrix is less than the number of unknowns n, and if there is any solution to the system of equations, then there are $n-k$ degrees of freedom because there exist $n-k$ unknowns whose values can be chosen arbitrarily. The remaining k variables will then be uniquely determined.

Section 1.5 defines eigenvalues and eigenvectors. It is shown that the standard definition implies that the eigenvalues are the roots of a so-called characteristic (polynomial) equation.

Diagonal matrices have many advantages, of course. Section 1.6 turns to the question of when an $n \times n$ matrix $\mathbf{A}$ can be diagonalized in the sense that $\mathbf{A} = \mathbf{P}^{-1}\mathbf{D}\mathbf{P}$ for some matrix $\mathbf{P}$ and diagonal matrix $\mathbf{D}$. Theorem 1.6.1 claims that this is possible *if and only if* $\mathbf{A}$ has n linearly independent eigenvectors—a remarkably, perhaps deceptively, simple and powerful result. Where n linearly independent eigenvectors can be found, they can be used as the columns of the matrix $\mathbf{P}$, which is immensely useful. In fact, one can prove that if $\mathbf{A}$ is known to have n distinct eigenvalues, there will always exist a linearly independent set of n eigenvectors, so that the matrix is diagonalizable. However, an $n \times n$ matrix can be diagonalizable even if it does not have n distinct eigenvalues. For example, the identity matrix has $\lambda = 1$ as the only eigenvalue, with the three standard unit vectors as eigenvectors.

The "spectral" Theorem 1.6.2 extends to symmetric $n \times n$ matrices the result shown for symmetric 2×2 matrices in Section 1.5—namely, that they have only real eigenvalues. Moreover, eigenvectors associated with distinct eigenvalues must be orthogonal. It follows that the matrix **P** of eigenvectors considered can be made orthogonal (i.e. $\mathbf{P}' = \mathbf{P}^{-1}$) by rescaling the eigenvectors so that each has length 1.

We go on to consider quadratic forms, and whether they are definite or not. Section 1.7 gives the basic definitions and results, starting with the case of two variables. Testing for the definiteness of a quadratic form is actually quite hard in general. That is one reason for carefully studying the 2 × 2 case first. Theorem 1.7.1 gives the general result, but for a full proof we refer to the literature. (Many texts get the semidefinite case wrong, we might add.) Theorem 1.7.2 gives the elegant and easy to remember tests for definiteness and semidefiniteness based on eigenvalues. The necessity part of each test is very easy to prove. But proving sufficiency relies on being able to diagonalize the matrix, and even then the proof is quite subtle.

Section 1.8 deals with quadratic forms subject to linear constraints. (One cannot drop the assumption in the main Theorem 1.8.1 that the first m columns in the matrix (b_{ij}) are linearly independent. To see why, consider $Q = x_1^2 + x_2^2 - x_3^2$, which is positive definite subject to $x_3 = 0$, but (5) fails. This is overlooked in many texts.)

Finally, Section 1.9 treats partitioned matrices and their inverses, which often arise in econometrics.

Problem 1-01 Prove that the vectors $\begin{pmatrix}1\\0\\1\end{pmatrix}$, $\begin{pmatrix}2\\1\\0\end{pmatrix}$, and $\begin{pmatrix}0\\1\\1\end{pmatrix}$ are linearly independent.

Problem 1-02

For which value of t are the three vectors $\begin{pmatrix}1\\-1\\1\end{pmatrix}$, $\begin{pmatrix}1\\1\\-1\end{pmatrix}$, and $\begin{pmatrix}3\\1\\t\end{pmatrix}$ linearly independent?

Problem 1-03

For which value of t are the three vectors $\begin{pmatrix}1\\8\\1\end{pmatrix}$, $\begin{pmatrix}t\\-2\\1\end{pmatrix}$, and $\begin{pmatrix}0\\4\\1\end{pmatrix}$ linearly dependent?

Problem 1-04

Given three linearly independent vectors **a**, **b**, and **c** in $\mathbb{R}^n$.

(a) Are $\mathbf{a} - 2\mathbf{b}$, $\mathbf{b} - \mathbf{c}$ and $\mathbf{a} - 2\mathbf{c}$ linearly dependent?

(b) Let $\mathbf{d} = 4\mathbf{a} - \mathbf{b} - \mathbf{c}$. Is it possible to find numbers x, y and z such that

$$x(\mathbf{a} - \mathbf{b}) + y(\mathbf{b} - \mathbf{c}) + z(\mathbf{a} - \mathbf{c}) = \mathbf{d}\ ?$$

Problem 1-05

Determine the ranks of the following matrices for all values of t:

(a) $\begin{pmatrix} t & 2 \\ -1 & -2 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 4 & 1+t \\ 5 & 4+t & 1+t \\ t-1 & t-1 & 0 \end{pmatrix}$

Problem 1-06

Discuss the rank of the matrix $\mathbf{A}_t = \begin{pmatrix} t & 0 & 0 & 1 \\ 0 & 2 & t & 3 \\ 1 & -2 & t & 0 \\ 2t & 1 & 0 & 3 \end{pmatrix}$.

Problem 1-07

(a) Find the rank of $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 1 & 1 & 2 & 0 \\ 0 & -1 & 2 & -3 \\ 1 & 0 & -2 & 0 \end{pmatrix}$.

(b) For what values of x, y and z are the three vectors $(x, 1, 0, 1)$, $(2, y, -1, 0)$ and $(0, 2, 2x, z)$ linearly independent?

Problem 1-08

(a) Consider the matrix $\mathbf{D}(s) = \begin{pmatrix} 1 & 2s & 1 & 1 \\ -2 & 1 & -2 & 3s \\ 1 & 1-s & -1 & 5 \\ -1 & 2 & s & -3 \end{pmatrix}$.

Find a necessary and sufficient condition for $\mathbf{D}(s)$ to have rank 4. What is the rank if $s = 1$?

(b) Determine the number of degrees of freedom for the equation system

$$
\begin{aligned}
x + 2y + z + w &= 0 \\
-2x + y - 2z + 3w &= 0 \\
x \qquad - z + 5w &= 0 \\
-x + 2y + z - 3w &= 0
\end{aligned}
$$

Problem 1-09

(a) Consider the 3×5 equation system

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{15}x_5 &= c_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{25}x_5 &= c_2 \qquad (*) \\
a_{31}x_1 + a_{32}x_2 + \cdots + a_{35}x_5 &= c_3
\end{aligned}
$$

where the coefficient matrix has rank 3 and $x_1, \ldots, x_5$ are the unknowns. Does $(*)$ always have a solution? And if so, how many degrees of freedom are there?

(b) Add the fourth equation $a_{41}x_1 + \cdots + a_{45}x_5 + a_{46}x_6 = c_4$ to system $(*)$, where x_6 is an additional unknown. Describe possible solutions, including the degrees of freedom, in the new system. (Explicit solutions are not required.)

Problem 1-10

(a) Let the matrix $\mathbf{A}$ be defined by $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$. Compute $\mathbf{A}^2$ and $\mathbf{A}^3$.

(b) Find the eigenvalues of $\mathbf{A}$ and corresponding eigenvectors.

(c) Let $\mathbf{P} = \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}$. Compute $\mathbf{P}^{-1}$, and show that $\mathbf{A} = \mathbf{P} \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{P}^{-1}$.

Problem 1-11

Verify that the matrix $\mathbf{B} = \begin{pmatrix} 5 & 4 & -2 \\ 4 & 5 & -2 \\ -2 & -2 & 2 \end{pmatrix}$ has the eigenvector $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ and find the associated eigenvalue.

Problem 1-12

(a) Find the eigenvalues of the matrix $\mathbf{A}_a = \begin{pmatrix} 2a & 0 & 0 \\ 0 & 0 & -a \\ 2-a & 1 & 2 \end{pmatrix}$, $a \leq 1$

(b) Find corresponding eigenvectors in the case $a = 1$.

Problem 1-13

Let $\mathbf{A}$ be the matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

(a) Find the eigenvalues and a set of corresponding eigenvectors of $\mathbf{A}$.

(b) Let $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ be a sequence of vectors given by

$$\mathbf{x}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t \quad \text{for } t = 0, 1, 2, \ldots$$

Show that $\mathbf{x}_0$ can be written as a linear combination of eigenvectors of $\mathbf{A}$, and then find $\mathbf{x}_t$ for all t.

Problem 1-14

(a) Find the eigenvalues and the eigenvectors of $\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

(b) Find the eigenvalues of $\mathbf{A}^2$.

Problem 1-15

Find the eigenvalues of $\mathbf{A} = \begin{pmatrix} a & 1 & 1 \\ 0 & a & 4 \\ 0 & 1 & a \end{pmatrix}$.

Problem 1-16

(a) Let $\mathbf{A} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ where a, b, and c are different from 0. Find $\mathbf{A}^{-1}$.

(b) Let $\mathbf{B}$ any 3×3 matrix whose column vectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ are mutually orthogonal and different from the zero vector. Put $\mathbf{C} = \mathbf{B}'\mathbf{B}$ and show that $\mathbf{C}$ is a diagonal matrix.

(c) Find $\mathbf{B}^{-1}$ expressed in terms of $\mathbf{C} = \mathbf{B}'\mathbf{B}$ and $\mathbf{B}$.

(d) Prove that the columns of $\mathbf{P} = \begin{pmatrix} 1 & -8 & 4 \\ -8 & 1 & 4 \\ 4 & 4 & 7 \end{pmatrix}$ are mutually orthogonal. Find $\mathbf{P}^{-1}$.

Problem 1-17

Consider the matrix $\mathbf{A} = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$.

(a) Show that the characteristic polynomial of $\mathbf{A}$ can be written as $(4 - \lambda)(\lambda^2 + a\lambda + b)$ for suitable constants a and b. Find the eigenvalues of $\mathbf{A}$.

(b) Show that $\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$, $\begin{pmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ -1/\sqrt{6} \end{pmatrix}$, and $\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix}$ are eigenvectors of $\mathbf{A}$.

Let $\mathbf{C}$ be the matrix with the three vectors from part (b) as columns.

(c) Show that $\mathbf{C}\mathbf{C}' = \mathbf{I}_3$ (the identity matrix of order 3), and use this to find the inverse of $\mathbf{C}$. Compute $\mathbf{C}^{-1}\mathbf{A}\mathbf{C}$. (This will be a diagonal matrix.)

(d) Let $\mathbf{D} = \mathrm{diag}(d_1, d_2, d_3)$ be a diagonal matrix, and let $\mathbf{B} = \mathbf{C}\mathbf{D}\mathbf{C}^{-1}$. Show that $\mathbf{B}^2 = \mathbf{C}\mathbf{D}^2\mathbf{C}^{-1}$, and that $\mathbf{B}^2 = \mathbf{A}$ for suitable values of d_1, d_2, and d_3.

Problem 1-18

Consider the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$

(a) Find the rank of $\mathbf{A}$, show that $(\mathbf{A}\mathbf{A}')^{-1}$ exists, and find this inverse.

(b) Compute the matrix $\mathbf{C} = \mathbf{A}'(\mathbf{A}\mathbf{A}')^{-1}$, and show that $\mathbf{A}\mathbf{C}\mathbf{b} = \mathbf{b}$ for every 2×1 matrix (2-dimensional column vector) $\mathbf{b}$.

(c) Use the results above to find a solution of the system of equations

$$x_1 + x_2 + x_3 = 1$$
$$x_1 + 2x_2 + 3x_3 = 1$$

(d) Consider in general a linear system of equations

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \quad \text{where } \mathbf{A} \text{ is an } m \times n \text{ matrix with } m \le n \qquad (*)$$

It can be shown that if $r(\mathbf{A}) = m$, then $r(\mathbf{A}\mathbf{A}') = m$. Why does this imply that $(\mathbf{A}\mathbf{A}')^{-1}$ exists? Put $\mathbf{C} = \mathbf{A}'(\mathbf{A}\mathbf{A}')^{-1}$, and show that if $\mathbf{v}$ is an arbitrary $m \times 1$ vector, then $\mathbf{A}\mathbf{C}\mathbf{v} = \mathbf{v}$. Use this to show that $\mathbf{x} = \mathbf{C}\mathbf{b}$ must be a solution of $(*)$.

Problem 1-19

Define the matrix $\mathbf{A}_a$ for all real numbers a by $\mathbf{A}_a = \begin{pmatrix} a & 0 & 1 \\ a & a & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(a) Compute the rank of $\mathbf{A}_a$ for all values of a.

(b) Find all eigenvalues and eigenvectors of $\mathbf{A}_0$. (NB! Here $a = 0$.) Show that eigenvectors corresponding to different eigenvalues are mutually orthogonal.

(c) When is the rank of the matrix product $\mathbf{A}_a \mathbf{A}_b$ equal to 3?

(d) (Difficult) Discuss the rank of the matrix product $\mathbf{A}_a \mathbf{A}_b$ for all values of a and b.

Problem 1-20

Given the matrices $\mathbf{A} = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}$ and $\mathbf{E} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(a) Find the eigenvalues and eigenvectors of $\mathbf{E}$.

(b) Find numbers p and q such that $\mathbf{A} = p\mathbf{I}_3 + q\mathbf{E}$.

(c) Show that if $\mathbf{x}_0$ is an eigenvector of $\mathbf{E}$, then $\mathbf{x}_0$ is also an eigenvector of $\mathbf{A}$.

(d) Find the eigenvalues of $\mathbf{A}$.

Problem 1-21

(a) Find the eigenvalues of the matrix $\mathbf{A} = \begin{pmatrix} -a^2b & 0 & ab \\ 0 & c & 0 \\ -ab & 0 & b \end{pmatrix}$.

(b) Let $\mathbf{H}$ be a 3×3 matrix with eigenvalues λ_1, λ_2 and λ_3, and let α be a number $= 0$. Show that $\alpha\lambda_1$, $\alpha\lambda_2$ and $\alpha\lambda_3$ are eigenvalues of the matrix $\mathbf{K} = \alpha\mathbf{H}$.

(c) Find the eigenvalues of $\mathbf{B} = \dfrac{1}{1-a^2}\begin{pmatrix} -4a^2 & 0 & 4a \\ 0 & 1-a^2 & 0 \\ -4a & 0 & 4 \end{pmatrix}$, $a = \pm 1$.

(d) Find a matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{B}\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \mathbf{D}$, where $\mathbf{B}$ is the matrix in (c). Then find a matrix $\mathbf{C}$ such that $\mathbf{C}^2 = \mathbf{B}$. (*Hint:* Find a diagonal matrix $\mathbf{E}$ such that $\mathbf{E}^2 = \mathbf{D}$, and then use the formula $\mathbf{P}\mathbf{E}^2\mathbf{P}^{-1} = \mathbf{P}\mathbf{E}\mathbf{P}^{-1}\mathbf{P}\mathbf{E}\mathbf{P}^{-1}$ to find $\mathbf{C}$ expressed in terms of $\mathbf{E}$ and $\mathbf{P}$.)

Problem 1-22

(a) Consider the matrix $\mathbf{C} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}$.

Find the characteristic polynoial $p(\lambda)$ of $\mathbf{C}$, and show that $(\lambda - 3)^2$ is a factor in $p(\lambda)$.

(b) Find the eigenvalues and eigenvectors of $\mathbf{C}$.

(c) For which values of x are the vectors $(x, 1, -1)$, $(1, x, 1)$, and $(x, 1, 3)$ linearly independent?

Problem 1-23

Let **A**, **B**, and **C** be $n \times n$ matrices, where **A** and **C** are invertible. Solve the following matrix equation for **X**: $\mathbf{CB} + \mathbf{CXA}^{-1} = \mathbf{A}^{-1}$.

Problem 1-24

(a) Let **A** be a symmetric $n \times n$ matrix with $|\mathbf{A}| = 0$, let **B** be a $1 \times n$ matrix, and let **X** be an $n \times 1$ matrix. Show that the expression

$$(\mathbf{X} + \tfrac{1}{2}\mathbf{A}^{-1}\mathbf{B}')'\mathbf{A}(\mathbf{X} + \tfrac{1}{2}\mathbf{A}^{-1}\mathbf{B}') - \tfrac{1}{4}\mathbf{B}\mathbf{A}^{-1}\mathbf{B}' \qquad (*)$$

is equal to $\mathbf{X}'\mathbf{AX} + \mathbf{BX}$.

(b) Suppose that **A** is symmetric and positive definite (i.e. $\mathbf{Y}'\mathbf{AY} > 0$ for all $n \times 1$ matrices $\mathbf{Y} = \mathbf{0}$). Using $(*)$, find the matrix **X** that minimizes the expression $\mathbf{X}'\mathbf{AX} + \mathbf{BX}$.

Problem 1-25

Investigate the (semi)definiteness of (a) $Q = 3x_1^2 - 8x_1x_2 + 8x_2$ (b) $Q = 25x_1^2 - 20x_1x_2 + 4x_2$.

Problem 1-26

(a) Write $Q(x_1, x_2, x_3) = 3x_1^2 + 2x_1x_2 + x_2 + 2x_2x_3 + 3x_3$ in matrix form with **A** symmetric.

(b) Determine the definiteness of $Q(x_1, x_2, x_3)$ by studying the signs of the (leading) principal minors of **A**.

(c) Confirm your result in (b) by finding the eigenvalues of **A**.

Problem 1-27

Classify the quadratic form $Q = 3x_1^2 + 4x_1x_2 + 2x_2^2 + 4x_2x_3 + x_3^2$ by computing the eigenvalues of the associated symmetric matrix. (Look for integer eigenvalues.)

Problem 1-28

Examine the definiteness of $x^2 - 2xy + xz - y^2 + 2z^2$ subject to $\begin{cases} x + y + z = 0 \\ 2x - 2y + z = 0 \end{cases}$

Problem 1-29

Find a necessary and sufficient condition for the quadratic form

$$\sum_{i=1}^{3}\sum_{j=1}^{3} L''_{ij}(x_1, x_2, x_3)h_i h_j$$

to be positive definite subject to $g'_1(x^*_1, x_2, x_3)h_1 + g_2(x_1, x_2, x_3)h_2 + g_3(x_1, x_2, x_3)h_3 = 0$, assuming that not all the three partials $g'_1(x^*_1, x_2, x_3), g_2(x_1, x_2, x_3), g_3(x_1, x_2, x_3)$ are 0.