Instructor's Manual

to accompany

Elements of Modern Algebra, *Eighth Edition*

Linda Gilbert and the late Jimmie Gilbert

University of South Carolina Upstate

Spartanburg, South Carolina

Contents

Preface

This manual provides answers for the computational exercises and a few of the exercises requiring proofs in **Elements of Modern Algebra,** *Eighth Edition,* by Linda Gilbert and the late Jimmie Gilbert. These exercises are listed in the table of contents. In constructing proof of exercises, we have freely utilized prior results, including those results stated in preceding problems.

My sincere thanks go to Danielle Hallock and Lauren Crosby for their careful management of the production of this manual and to Eric Howe for his excellent work on the accuracy checking of all the answers.

Linda Gilbert

Answers to Selected Exercises

Section 1.1

1. True **2**. True **3**. False **4**. True **5**. True **6.** False **7**. True

8. True **9**. False **10**. False

Exercises 1.1

1. **a**. $A = \{x \mid x \text{ is a nonnegative even integer less than } 12\}$ **b**. $\left\{x \mid x^2 = 1\right\}$
c. $A = \{x \mid x \text{ is a negative integer}\}$ **d**. $\left\{x \mid x = n^2 \text{ for } n \in \mathbf{Z}^+\right\}$

2. **a**. False **b**. True **c**. False **d**. False **e**. False **f**. True

3. **a**. True **b**. True **c**. True **d**. True **e**. True **f**. False
g. True **h**. True **i**. False **j**. False **k**. False **l**. True

4. **a**. False **b**. True **c**. True **d**. False **e**. True **f**. False
g. False **h**. True **i**. False **j**. False **k**. False **l**. False

5. **a**. $\{0, 1, 2, 3, 4, 5, 6, 8, 10\}$ **b**. $\{2, 3, 5\}$ **c**. $\{0, 2, 4, 6, 7, 8, 9, 10\}$ **d**. $\{2\}$
e. $\varnothing$ **f**. A **g**. $\{0, 2, 3, 4, 5\}$ **h**. $\{6, 8, 10\}$ **i**. $\{1, 3, 5\}$
j. $\{6, 8, 10\}$ **k**. $\{1, 2, 3, 5\}$ **l**. C **m**. $\{3, 5\}$ **n**. $\{1\}$

6. **a**. A **b**. A **c**. $\varnothing$ **d**. U **e**. A **f**. $\varnothing$ **g**. A **h**. U **i**. U
j. A **k**. U **l**. $\varnothing$ **m**. A **n**. $\varnothing$

7. **a**. $\{\varnothing, A\}$ **b**. $\{\varnothing, \{0\}, \{1\}, A\}$
c. $\{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}$
d. $\{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, A\}$
e. $\{\varnothing, \{1\}, \{\{1\}\}, A\}$ **f**. $\{\varnothing, A\}$ **g**. $\{\varnothing, A\}$ **h**. $\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, A\}$

8. **a**. Two possible partitions are:
$X_1 = \{x \mid x \text{ is a negative integer}\}$ and $X_2 = \{x \mid x \text{ is a nonnegative integer}\}$,
or
$X_1 = \{x \mid x \text{ is a negative integer}\}, X_2 = \{x \mid x \text{ is a positive integer}\}, X_3 = \{0\}$.

b. One possible partition is $X_1 = \{a, b\}$ and $X_2 = \{c, d\}$. Another possible partition is $X_1 = \{a\}, X_2 = \{b, c\}, X_3 = \{d\}$.

c. One partition is $X_1 = \{1, 5, 9\}$ and $X_2 = \{11, 15\}$. Another partition is $X_1 = \{1, 15\}, X_2 = \{11\}$ and $X_3 = \{5, 9\}$.

d. One possible partition is $X_1 = \{x \mid x = a + b\,i$, where a is a positive real number, b is a real number$\}$ and $X_2 = \{x \mid x = a + b\,i$, where a is a nonpositive real number, b is a real number$\}$. Another possible partition is $X_1 = \{x \mid x = a$, where a is a real number$\}$, $X_2 = \{x \mid x = b\,i$, where b is a nonzero real number$\}$ and $X_3 = \{x \mid x = a + b\,i$, where a and b are both nonzero real numbers$\}$.

9. **a**. $X_1 = \{1\}, X_2 = \{2\}, X_3 = \{3\}$;
$X_1 = \{1\}, X_2 = \{2, 3\}$;
$X_1 = \{2\}, X_2 = \{1, 3\}$;
$X_1 = \{3\}, X_2 = \{1, 2\}$

b. $X_1 = \{1\}, X_2 = \{2\}, X_3 = \{3\}, X_4 = \{4\}$;
$X_1 = \{1\}, X_2 = \{2\}, X_3 = \{3, 4\}$; $X_1 = \{1\}, X_2 = \{3\}, X_3 = \{2, 4\}$;
$X_1 = \{1\}, X_2 = \{4\}, X_3 = \{2, 3\}$; $X_1 = \{2\}, X_2 = \{3\}, X_3 = \{1, 4\}$;
$X_1 = \{2\}, X_2 = \{4\}, X_3 = \{1, 3\}$; $X_1 = \{3\}, X_2 = \{4\}, X_3 = \{1, 2\}$;
$X_1 = \{1, 2\}, X_2 = \{3, 4\}$; $X_1 = \{1, 3\}, X_2 = \{2, 4\}$;
$X_1 = \{1, 4\}, X_2 = \{2, 3\}$; $X_1 = \{1\}, X_2 = \{2, 3, 4\}$;
$X_1 = \{2\}, X_2 = \{1, 3, 4\}$; $X_1 = \{3\}, X_2 = \{1, 2, 4\}$;
$X_1 = \{4\}, X_2 = \{1, 2, 3\}$.

10. **a**. 2^n **b**. $\dfrac{n!}{k!\,(n-k)!}$

11. **a**. $A \subseteq B$ **b**. $B' \subseteq A$ or $A \cup B = U$ **c**. $B \subseteq A$
d. $A \cap B = \varnothing$ or $A \subseteq B'$ **e**. $A = B = U$ **f**. $A' \subseteq B$ or $A \cup B = U$
g. $A = U$ **h**. $A = U$

36. Let $A = \{a, b\}, B = \{b\}$ and $C = \{a\}$. Then $A \cup B = A = A \cup C$ but $B \neq C$.

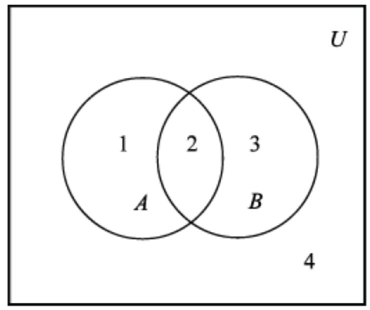
37. Let $A = \{a\}, B = \{a, b\}$ and $C = \{a, c\}$. Then $A \cap B = \{a\} = A \cap C$ but $B \neq C$.

38. Let $A = \{a, b\}$ and $B = \{a, c\}$. Then $A \cup B = \{a, b, c\}$ and $\{a, b, c\} \in \mathcal{P}(A \cup B)$ but $\{a, b, c\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

40. Let $A = \{a, b\}$ and $B = \{b\}$ Then $A - B = \{a\}$ and $\varnothing \in \mathcal{P}(A - B)$ but $\varnothing \notin \mathcal{P}(A) - \mathcal{P}(B)$.

41. $(A \cap B') \cup (A' \cap B) = (A \cup B) \cap (A' \cup B')$

42. **a.**



$A \cup B$: Regions 1,2,3

$A \cap B$: Region 2

$(A \cup B) - (A \cap B)$: Regions 1,3

$A + B$: Regions 1,3

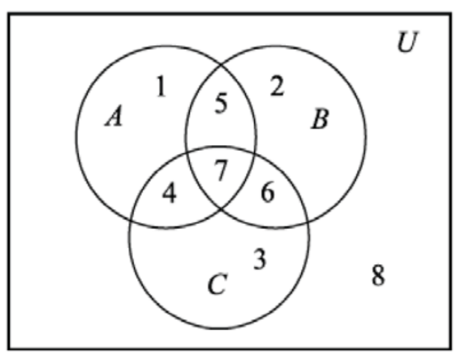
$A - B$: Region 1

$B - A$: Region 3

$(A - B) \cup (B - A)$: Regions 1,3

Each of $A + B$ and $(A - B) \cup (B - A)$ consists of Regions 1,3.

b.



A : Regions 1,4,5,7

$B + C$: Regions 2,3,4,5

$A + (B + C)$: Regions 1,2,3,7

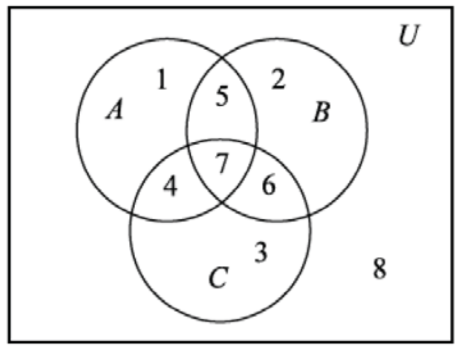
$A + B$: Regions 1,2,4,6

C : Regions 3,4,6,7

$(A + B) + C$: Regions 1,2,3,7

Each of $A + (B + C)$ and $(A + B) + C$ consists of Regions 1,2,3,7.

c.



$$A: \text{ Regions 1,4,5,7} \qquad A \cap B: \text{ Regions 5,7}$$

$$B + C: \text{ Regions 2,3,4,5} \qquad A \cap C: \text{ Regions 4,7}$$

$$A \cap (B + C): \text{ Regions 4,5} \qquad (A \cap B) + (A \cap C): \text{ Regions 4,5}$$

Each of $A \cap (B + C)$ and $(A \cap B) + (A \cap C)$ consists of Regions 4,5.

43. **a.** $A + A = (A \cup A) - (A \cap A) = A - A = A \cap A' = \varnothing$

b. $A + \varnothing = (A \cup \varnothing) - (A \cap \varnothing) = A - \varnothing = A \cap \varnothing' = A$

Section 1.2

1. False **2.** False **3.** False **4.** False **5.** False **6.** True **7.** True

8. False **9.** True

Exercises 1.2

1. **a.** $\{(a, 0), (a, 1), (b, 0), (b, 1)\}$ **b.** $\{(0, a), (0, b), (1, a), (1, b)\}$

c. $\{(2, 2), (4, 2), (6, 2), (8, 2)\}$

d. $\{(-1, 1), (-1, 5), (-1, 9), (1, 1), (1, 5), (1, 9)\}$

e. $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

2. **a.** Domain $= \mathbf{E}$, Codomain $= \mathbf{Z}$, Range $= \mathbf{Z}$

b. Domain $= \mathbf{E}$, Codomain $= \mathbf{Z}$, Range $= \mathbf{E}$

c. Domain $= \mathbf{E}$, Codomain $= \mathbf{Z}$,
Range $= \{y \mid y \text{ is a nonnegative even integer}\} = (\mathbf{Z}^+ \cap \mathbf{E}) \cup \{0\}$

d. Domain $= \mathbf{E}$, Codomain $= \mathbf{Z}$, Range $= \mathbf{Z} - \mathbf{E}$

3. **a.** $f(S) = \{1, 3, 5, \ldots\} = \mathbf{Z}^+ - \mathbf{E}$, $f^{-1}(T) = \{-4, -3, -1, 1, 3, 4\}$

b. $f(S) = \{1, 5, 9\}$, $f^{-1}(T) = \mathbf{Z}$ **c.** $f(S) = \{0, 1, 4\}$, $f^{-1}(T) = \varnothing$

d. $f(S) = \{0, 2, 14\}$, $f^{-1}(T) = \mathbf{Z}^+ \cup \{0, -1, -2\}$

4. **a.** The mapping f is not onto, since there is no $x \in \mathbf{Z}$ such that $f(x) = 1$. It is one-to-one.

b. The mapping f is not onto, since there is no $x \in \mathbf{Z}$ such that $f(x) = 1$. It is one-to-one.

c. The mapping f is onto and one-to-one.

d. The mapping f is one-to-one. It is not onto, since there is no $x \in \mathbf{Z}$ such that $f(x) = 2$.

e. The mapping f is not onto, since there is no $x \in \mathbf{Z}$ such that $f(x) = -1$. It is not one-to-one, since $f(1) = f(-1)$ and $1 \neq -1$.

f. We have $f(3) = f(2) = 0$, so f is not one-to-one. Since $f(x)$ is always even, there is no $x \in \mathbf{Z}$ such that $f(x) = 1$, and f is not onto.

g. The mapping f is not onto, since there is no $x \in \mathbf{Z}$ such that $f(x) = 3$. It is one-to-one.

h. The mapping f is not onto, since there is no $x \in \mathbf{Z}$ such that $f(x) = 1$. Neither is f one-to-one since $f(0) = f(1)$ and $0 \neq 1$.

i. The mapping f is onto. It is not one-to-one, since $f(9) = f(4)$ and $9 \neq 4$.

j. The mapping f is not onto, since there is no $x \in \mathbf{Z}$ such that $f(x) = 4$. It is one-to-one.

5. **a.** The mapping is onto and one-to-one.

b. The mapping is onto and one-to-one.

c. The mapping is onto and one-to-one.

d. The mapping is onto and one-to-one.

e. The mapping is not onto, since there is no $x \in \mathbf{R}$ such that $f(x) = -1$. It is not one-to-one, since $f(1) = f(-1)$ and $1 \neq -1$.

f. The mapping is not onto, since there is no $x \in \mathbf{R}$ such that $f(x) = 1$. It is not one-to-one, since $f(0) = f(1) = 0$ and $0 \neq 1$.

6. **a.** The mapping f is onto and one-to-one.

b. The mapping f is one-to-one. Since there is no $x \in \mathbf{E}$ such that $f(x) = 2$, the mapping is not onto.

7. **a.** The mapping f is onto. The mapping f is not one-to-one, since $f(1) = f(-1)$ and $1 \neq -1$.

b. The mapping f is not onto, since there is no $x \in \mathbf{Z}^+$ such that $f(x) = -1$. The mapping f is one-to-one.

c. The mapping f is onto and one-to-one.

d. The mapping f is onto. The mapping f is not one-to-one, since $f(1) = f(-1)$ and $1 \neq -1$.

8. **a.** The mapping f is not onto, since there is no $x \in \mathbf{Z}$ such that $|x+4| = -1$. The mapping f is not one-to-one, since $f(1) = f(-9) = 5$ but $1 \neq -9$.

b. The mapping f is not onto, since there is no $x \in \mathbf{Z}^+$ such that $|x+4| = 1$. The mapping f is one-to-one.

9. **a.** The mapping f is not onto, since there is no $x \in \mathbf{Z}^+$ such that $2^x = 3$. The mapping f is one-to-one.

b. The mapping f is not onto, since there is no $x \in \mathbf{Z}^+ \cap \mathbf{E}$ such that $2^x = 6$. The mapping f is one-to-one.

10. **a.** Let $f : \mathbf{E} \to \mathbf{E}$ where $f(x) = x$. **b.** Let $f : \mathbf{E} \to \mathbf{E}$ where $f(x) = 2x$.

c. Let $f : \mathbf{E} \to \mathbf{E}$ where $f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is a multiple of 4} \\ x & \text{if } x \text{ is not a multiple of 4.} \end{cases}$

d. Let $f : \mathbf{E} \to \mathbf{E}$ where $f(x) = x^2$.

11. **a.** For arbitrary $a \in \mathbf{Z}, 2a$ is even and $f(2a) = \frac{2a}{2} = a$. Thus f is onto. But f is not one-to-one, since $f(1) = f(-1) = 0$.

b. The mapping f is not onto, since there is no x in $\mathbf{Z}$ such that $f(x) = 1$. The mapping f is not one-to-one, since $f(0) = f(2) = 0$.

c. For arbitrary a in $\mathbf{Z}, 2a-1$ is odd, and therefore

$$f(2a-1) = \frac{(2a-1)+1}{2} = a.$$

Thus f is onto. But f is not one-to-one, since $f(2) = 5$ and also $f(9) = 5$.

d. For arbitrary a in $\mathbf{Z}$, $2a$ is even and $f(2a) = \frac{2a}{2} = a$. Thus f is onto. But f is not one-to-one, since $f(4) = 2$ and $f(7) = 2$.

e. The mapping f is not onto, because there is no x in $\mathbf{Z}$ such that $f(x) = 4$. Since $f(2) = 6$ and $f(3) = 6$, then f is not one-to-one.

f. The mapping f is not onto, since there is no x in $\mathbf{Z}$ such that $f(x) = 1$. Suppose that $f(a_1) = f(a_2)$. It can be seen from the definition of f that the image of an even integer is always an odd integer, and also that the image of an odd integer is always an even integer. Therefore, $f(a_1) = f(a_2)$ requires that either both a_1 and a_2 are even, or both a_1 and a_2 are odd. If both a_1 and a_2 are even,

$$f(a_1) = f(a_2) \Rightarrow 2a_1 - 1 = 2a_2 - 1 \Rightarrow 2a_1 = 2a_2 \Rightarrow a_1 = a_2.$$

If both a_1 and a_2 are odd,

$$f(a_1) = f(a_2) \Rightarrow 2a_1 = 2a_2 \Rightarrow a_1 = a_2.$$

Hence, $f(a_1) = f(a_2)$ always implies $a_1 = a_2$ and f is one-to-one.

12. **a.** The mapping f is not onto, because there is no $x \in \mathbf{R} - \{0\}$ such that $f(x) = 1$. If $a_1, a_2 \in \mathbf{R} - \{0\}$,

$$\begin{aligned} f(a_1) = f(a_2) \Rightarrow \quad & \frac{a_1 - 1}{a_1} = \frac{a_2 - 1}{a_2} \\ \Rightarrow \quad & a_2(a_1 - 1) = a_1(a_2 - 1) \\ \Rightarrow \quad & a_2 a_1 - a_2 = a_1 a_2 - a_1 \\ \Rightarrow \quad & -a_2 = -a_1 \\ \Rightarrow \quad & a_2 = a_1. \end{aligned}$$

Thus f is one-to-one.

b. The mapping f is not onto, because there is no $x \in \mathbf{R} - \{0\}$ such that $f(x) = 2$. If $a_1, a_2 \in \mathbf{R} - \{0\}$,

$$\begin{aligned} f(a_1) = f(a_2) \Rightarrow \quad & \frac{2a_1 - 1}{a_1} = \frac{2a_2 - 1}{a_2} \\ \Rightarrow \quad & 2 - \frac{1}{a_1} = 2 - \frac{1}{a_2} \\ \Rightarrow \quad & -\frac{1}{a_1} = -\frac{1}{a_2} \\ \Rightarrow \quad & a_1 = a_2. \end{aligned}$$

Thus f is one-to-one.

c. The mapping f is not onto, since there is no $x \in \mathbf{R} - \{0\}$ such that $f(x) = 0$. It is not one-to-one, since $f(2) = \frac{2}{5}$ and $f\left(\frac{1}{2}\right) = \frac{2}{5}$.

d. The mapping f is not onto, since there is no $x \in \mathbf{R} - \{0\}$ such that $f(x) = 1$. Since $f(1) = f(3) = \frac{1}{2}$, then f is not one-to-one.

13. **a.** The mapping f is onto, since for every $(y, x) \in B = \mathbf{Z} \times \mathbf{Z}$ there exists an $(x, y) \in A = \mathbf{Z} \times \mathbf{Z}$ such that $f(x, y) = (y, x)$. To show that f is one-to-one, we assume $(a, b) \in A = \mathbf{Z} \times \mathbf{Z}$ and $(c, d) \in A = \mathbf{Z} \times \mathbf{Z}$ and

$$f(a, b) = f(c, d)$$

or

$$(b, a) = (d, c).$$

This means $b = d$ and $a = c$ and

$$(a, b) = (c, d).$$

b. For any $x \in \mathbf{Z}, (x, 0) \in A$ and $f(x, 0) = x$. Thus f is onto. Since $f(2, 3) = f(4, 1) = 5$, f is not one-to-one.

c. Since for every $x \in B = \mathbf{Z}$ there exists an $(x, y) \in A = \mathbf{Z} \times \mathbf{Z}$ such that $f(x, y) = x$, the mapping f is onto. However, f is not one-to-one, since $f(1, 0) = f(1, 1)$ and $(1, 0) \neq (1, 1)$.

d. The mapping f is one-to-one since $f(a_1) = f(a_2) \Rightarrow (a_1, 1) = (a_2, 1) \Rightarrow a_1 = a_2$. Since there is no $x \in \mathbf{Z}$ such that $f(x) = (0, 0)$, then f is not onto.

e. The mapping f is not onto, since there is no $(x, y) \in \mathbf{Z} \times \mathbf{Z}$ such that $f(x, y) = 2$. The mapping f is not one-to-one, since $f(2, 0) = f(2, 1) = 4$ and $(2, 0) \neq (2, 1)$.

f. The mapping f is not onto, since there is no $(x, y) \in \mathbf{Z} \times \mathbf{Z}$ such that $f(x, y) = 3$. The mapping is not one-to-one, since $f(1, 0) = f(-1, 0) = 1$ and $(1, 0) \neq (-1, 0)$.

g. The mapping f is not onto, since there is no (x, y) in $\mathbf{Z}^+ \times \mathbf{Z}^+$ such that $f(x, y) = \frac{x}{y} = 0$. The mapping f is not one-to-one, since $f(2, 1) = f(4, 2) = 2$.

h. The mapping f is not onto, since there is no (x, y) in $\mathbf{R} \times \mathbf{R}$ such that $f(x, y) = 2^{x+y} = 0$. The mapping f is not one-to-one, since $f(1, 0) = f(0, 1) = 2^1$.

14. **a**. The mapping f is obviously onto.

b. The mapping f is not one-to-one, since $f(0) = f(2) = 1$.

c. Let both x_1 and x_2 be even. Then $x_1 + x_2$ is even and $f(x_1 + x_2) = 1 = 1 \cdot 1 = f(x_1) f(x_2)$. Let both x_1 and x_2 be odd. Then $x_1 + x_2$ is even and $f(x_1 + x_2) = 1 = (-1)(-1) = f(x_1) f(x_2)$. Finally, if one of x_1, x_2 is even and the other is odd, then $x_1 + x_2$ is odd and $f(x_1 + x_2) = -1 = (1)(-1) = f(x_1) f(x_2)$. Thus it is true that $f(x_1 + x_2) = f(x_1) f(x_2)$.

d. Let both x_1 and x_2 be odd. Then $x_1 x_2$ is odd and $f(x_1 x_2) = -1 \neq (-1)(-1) = f(x_1) f(x_2)$.

15. **a**. The mapping f is not onto, since there is no $a \in A$ such that $f(a) = 9 \in B$. It is not one-to-one, since $f(-2) = f(2)$ and $-2 \neq 2$.

b. $f^{-1}(f(S)) = f^{-1}(\{1, 4\}) = \{-2, 1, 2\} \neq S$

c. With $T = \{4, 9\}$, $f^{-1}(T) = \{-2, 2\}$, and $f\left(f^{-1}(T)\right) = f(\{-2, 2\}) = \{4\} \neq T$.

16. **a**. $g(S) = \{2, 4\}$, $g^{-1}(g(S)) = \{2, 3, 4, 7\}$

b. $g^{-1}(T) = \{9, 6, 11\}$, $g\left(g^{-1}(T)\right) = T$

17. **a**. $f(S) = \{-1, 2, 3\}$, $f^{-1}(f(S)) = S$

b. $f^{-1}(T) = \{0\}$, $f\left(f^{-1}(T)\right) = \{-1\}$

18. **a**. $(f \circ g)(x) = \begin{cases} 2x & \text{if } x \text{ is even} \\ 2(2x-1) & \text{if } x \text{ is odd} \end{cases}$ **b**. $(f \circ g)(x) = 2x^3$

c. $(f \circ g)(x) = \begin{cases} \dfrac{x+|x|}{2} & \text{if } x \text{ is even} \\ |x| - x & \text{if } x \text{ is odd} \end{cases}$ **d**. $(f \circ g)(x) = x$

e. $(f \circ g)(x) = (x - |x|)^2$

19. **a**. $(g \circ f)(x) = 2x$ **b**. $(g \circ f)(x) = 8x^3$ **c**. $(g \circ f)(x) = \dfrac{x+|x|}{2}$

d. $(g \circ f)(x) = \begin{cases} \frac{x}{2} - 1, \text{ if } x = 4k, \text{ for } k \text{ an integer} \\ x, \text{ otherwise} \end{cases}$ **e**. $(g \circ f)(x) = 0$

20. n^m **21**. $n!$ **22**. $n(n-1)(n-2)\cdots(n-m+1) = \dfrac{n!}{(n-m)!}$

28. Let $f : A \to B$, where A and B are nonempty.

Assume first that $f\left(f^{-1}(T)\right) = T$ for every subset T of B. For an arbitrary element b of B, let $T = \{b\}$. The equality $f\left(f^{-1}(\{b\})\right) = \{b\}$ implies that $f^{-1}(\{b\})$ is not empty. For any $x \in f^{-1}(\{b\})$, we have $f(x) = b$. Thus f is onto.

Assume now that f is onto. For an arbitrary $y \in f\left(f^{-1}(T)\right)$, we have

$$\begin{aligned} y \in f\left(f^{-1}(T)\right) &\Rightarrow y = f(x) \text{ for some } x \in f^{-1}(T) \\ &\Rightarrow y = f(x) \text{ for some } f(x) \in T \\ &\Rightarrow y \in T. \end{aligned}$$

Thus $f\left(f^{-1}(T)\right) \subseteq T$. For an arbitrary $t \in T$, there exists $x \in A$ such that $f(x) = t$, since f is onto. Now

$$\begin{aligned} f(x) = t \in T &\Rightarrow x \in f^{-1}(T) \\ &\Rightarrow f(x) \in f\left(f^{-1}(T)\right) \\ &\Rightarrow t \in f\left(f^{-1}(T)\right). \end{aligned}$$

Thus $T \subseteq f\left(f^{-1}(T)\right)$, and we have proved that $f\left(f^{-1}(T)\right) = T$ for an arbitrary subset T of B.

Section 1.3

1. False **2**. True **3**. False **4**. False **5**. False **6**. False

Exercises 1.3

1. **a**. The mapping $f \circ g$ is not onto, since there is no $x \in \mathbf{Z}$ such that $(f \circ g)(x) = 1$. It is not one-to-one, since $(f \circ g)(1) = (f \circ g)(-1)$ and $1 \neq -1$.

b. The mapping $f \circ g$ is not onto, since there is no $x \in \mathbf{Z}$ such that $(f \circ g)(x) = 0$. The mapping $f \circ g$ is one-to-one.

c. The mapping $f \circ g$ is not onto, since there is no $x \in \mathbf{Z}$ such that $(f \circ g)(x) = 1$. The mapping $f \circ g$ is one-to-one.

d. The mapping $f \circ g$ is not onto, since there is no $x \in \mathbf{Z}$ such that $(f \circ g)(x) = 1$. The mapping $f \circ g$ is one-to-one.

e. The mapping $f \circ g$ is not onto, since there is no $x \in \mathbf{Z}$ such that $(f \circ g)(x) = 1$. It is not one-to-one, since $(f \circ g)(-2) = (f \circ g)(0)$ and $-2 \neq 0$.

f. The mapping $f \circ g$ is both onto and one-to-one.

g. The mapping $f \circ g$ is not onto, since there is no $x \in \mathbf{Z}$ such that $(f \circ g)(x) = -1$. It is not one-to-one, since $(f \circ g)(1) = (f \circ g)(2)$ and $1 \neq 2$.

2. **a**. The mapping $g \circ f$ is not onto, since there is no $x \in \mathbf{Z}$ such that $(g \circ f)(x) = -1$. It is not one-to-one since $(g \circ f)(0) = (g \circ f)(2)$ and $0 \neq 2$.

b. The mapping $g \circ f$ is not onto, since there is no $x \in \mathbf{Z}$ such that $(g \circ f)(x) = 1$. The mapping $g \circ f$ is one-to-one.

c. The mapping $g \circ f$ is not onto, since there is no $x \in \mathbf{Z}$ such that $(g \circ f)(x) = 1$. The mapping $g \circ f$ is one-to-one.

d. The mapping $g \circ f$ is not onto, since there is no $x \in \mathbf{Z}$ such that $(g \circ f)(x) = 1$. The mapping $g \circ f$ is one-to-one.

e. The mapping $g \circ f$ is not onto, since there is no $x \in \mathbf{Z}$ such that $(g \circ f)(x) = -1$. It is not one-to-one, since $(g \circ f)(-1) = (g \circ f)(-2)$ and $-1 \neq -2$.

f. The mapping $g \circ f$ is not onto, since there is no $x \in \mathbf{Z}$ such that $(g \circ f)(x) = 0$. The mapping $g \circ f$ is not one-to-one, since $(g \circ f)(1) = (g \circ f)(4)$ and $1 \neq 4$.

g. The mapping $g \circ f$ is not onto, since there is no $x \in \mathbf{Z}$ such that $(g \circ f)(x) = 1$. It is not one-to-one, since $(g \circ f)(0) = (g \circ f)(1)$ and $0 \neq 1$.

3. $f(x) = x^2, g(x) = -x$

4. Let $A = \{0, 1\}, B = \{-2, 1, 2\}, C = \{1, 4\}$. Let $g : A \to B$ be defined by $g(x) = x+1$ and $f : B \to C$ be defined by $f(x) = x^2$. Then g is not onto, since $-2 \notin g(A)$. The mapping f is onto. Also $f \circ g$ is onto, since $(f \circ g)(0) = f(1) = 1$ and $(f \circ g)(1) = f(2) = 4$.

5. Let f and g be defined as in Problem 1f. Then f is not one-to-one, g is one-to-one, and $f \circ g$ is one-to-one.

6. **a**. Let $f : \mathbf{Z} \to \mathbf{Z}$ and $g : \mathbf{Z} \to \mathbf{Z}$ be defined by

$$f(x) = x, \quad g(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ x & \text{if } x \text{ is odd.} \end{cases}$$

The mapping f is one-to-one and the mapping g is onto, but the composition $f \circ g = g$ is not one-to-one, since $(f \circ g)(1) = (f \circ g)(2)$ and $1 \neq 2$.

b. Let $f : \mathbf{Z} \to \mathbf{Z}$ and $g : \mathbf{Z} \to \mathbf{Z}$ be defined by $f(x) = x^3$ and $g(x) = x$. The mapping f is one-to-one, the mapping g is onto, but the mapping $f \circ g$ given by $(f \circ g)(x) = x^3$ is not onto, since there is no $x \in \mathbf{Z}$ such that $(f \circ g)(x) = 2$.

7. **a.** Let $f : \mathbf{Z} \to \mathbf{Z}$ and $g : \mathbf{Z} \to \mathbf{Z}$ be defined by

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ x & \text{if } x \text{ is odd} \end{cases} \qquad g(x) = x.$$

The mapping f is onto and the mapping g is one-to-one, but the composition $f \circ g = f$ is not one-to-one, since $(f \circ g)(1) = (f \circ g)(2)$ and $1 \neq 2$.

b. Let $f : \mathbf{Z} \to \mathbf{Z}$ and $g : \mathbf{Z} \to \mathbf{Z}$ be defined by $f(x) = x$ and $g(x) = x^3$. The mapping f is onto, the mapping g is one-to-one, but the mapping $f \circ g$ given by $(f \circ g)(x) = x^3$ is not onto, since there is no $x \in \mathbf{Z}$ such that $(f \circ g)(x) = 2$.

9. **a.** Let $f(x) = x, g(x) = x^2$, and $h(x) = |x|$, for all $x \in \mathbf{Z}$.

b. Let $f(x) = x^2, g(x) = x$ and $h(x) = -x$, for all $x \in \mathbf{Z}$.

12. To prove that f is one-to-one, suppose $f(a_1) = f(a_2)$, for a_1 and a_2 in A. Since $g \circ f$ is onto, there exist α_1 and α_2 in A such that

$$a_1 = (g \circ f)(\alpha_1) \quad \text{and} \quad a_2 = (g \circ f)(\alpha_2).$$

Then $f((g \circ f)(\alpha_1)) = f((g \circ f)(\alpha_2))$, since $f(a_1) = f(a_2)$, or

$$(f \circ g)(f(\alpha_1)) = (f \circ g)(f(\alpha_2)).$$

This implies that

$$f(\alpha_1) = f(\alpha_2)$$

since $f \circ g$ is one-to-one. Since g is a mapping, then

$$g(f(\alpha_1)) = g(f(\alpha_2)).$$

Thus

$$(g \circ f)(\alpha_1) = (g \circ f)(\alpha_2)$$

and

$$a_1 = a_2.$$

Therefore f is one-to-one.

To show that f is onto, let $b \in B$. Then $g(b) \in A$ and therefore $g(b) = (g \circ f)(a)$ for some $a \in A$ since $g \circ f$ is onto. It follows then that

$$(f \circ g)(b) = (f \circ g)(f(a)).$$

Since $f \circ g$ is one-to-one, we have

$$b = f(a),$$

and f is onto.

Section 1.4

1. False **2**. True **3**. True **4**. False **5**. True **6**. True **7**. True

8. True **9**. True

Exercises 1.4

1. **a**. The set B is not closed, since $-1 \in B$ and $-1 * -1 = 1 \notin B$.

b. The set B is not closed, since $1 \in B$ and $2 \in B$ but $1 * 2 = 1 - 2 = -1 \notin B$.

c. The set B is closed.

d. The set B is closed.

e. The set B is not closed, since $1 \in B$ and $1 * 1 = 0 \notin B$.

f. The set B is closed.

g. The set B is closed.

h. The set B is closed.

2. **a**. Not commutative, Not associative, No identity element

b. Not commutative, Associative, No identity element

c. Not commutative, Not associative, No identity element

d. Commutative, Not associative, No identity element

e. Commutative, Associative, No identity element

f. Not commutative, Not associative, No identity element

g. Commutative, Associative, 0 is an identity element. 0 is the only invertible element and its inverse is 0.

h. Commutative, Associative, -3 is an identity element. $-x - 6$ is the inverse of x.

i. Not commutative, Not associative, No identity element

j. Commutative, Not associative, No identity element

k. Not commutative, Not associative, No identity element

l. Commutative, Not associative, No identity element

m. Not commutative, Not associative, No identity element

n. Commutative, Not associative, No identity element

3. **a**. The binary operation $*$ is not commutative, since $B * C \neq C * B$.

b. There is no identity element.

4. **a.** The operation $*$ is commutative, since $x * y = y * x$ for all x, y in S.

b. A is an identity element.

c. The elements B and C are inverses of each other and A is its own inverse.

5. **a.** The binary operation $*$ is not commutative, since $D * A \neq A * D$.

b. C is an identity element.

c. The elements A and B are inverses of each other and C is its own inverse.

6. **a.** The binary operation $*$ is commutative.

b. D is an identity element.

c. D is the only invertible element and its inverse is D.

7. The set of nonzero integers is not closed with respect to division, since 1 and 2 are nonzero integers but $1 \div 2$ is not a nonzero integer.

8. The set of odd integers is not closed with respect to addition, since 1 is an odd integer but $1 + 1$ is not an odd integer.

10. **a.** The set of nonzero integers is not closed with respect to addition defined on $\mathbf{Z}$, since 1 and -1 are nonzero integers but $1 + (-1)$ is not a nonzero integer.

b. The set of nonzero integers is closed with respect to multiplication defined on $\mathbf{Z}$.

11. **a.** The set B is not closed with respect to addition defined on $\mathbf{Z}$, since $1 \in B, 8 \in B$ but $1 + 8 = 9 \notin B$.

b. The set B is closed with respect to multiplication defined on $\mathbf{Z}$.

12. **a.** The set $\mathbf{Q} - \{0\}$ is closed with respect to multiplication defined on $\mathbf{R}$.

b. The set $\mathbf{Q} - \{0\}$ is closed with respect to division defined on $\mathbf{R} - \{0\}$.

Section 1.5

1. True **2.** False **3.** False

Exercises 1.5

1. **a.** A right inverse does not exist, since f is not onto.

b. A right inverse does not exist, since f is not onto.

c. A right inverse $g : \mathbf{Z} \to \mathbf{Z}$ is defined by $g(x) = x - 2$.

d. A right inverse $g : \mathbf{Z} \to \mathbf{Z}$ is defined by $g(x) = 1 - x$.

e. A right inverse does not exist, since f is not onto.

f. A right inverse does not exist, since f is not onto.

g. A right inverse does not exist, since f is not onto.

h. A right inverse does not exist, since f is not onto.

i. A right inverse does not exist, since f is not onto.

j. A right inverse does not exist, since f is not onto.

k. A right inverse $g : \mathbf{Z} \to \mathbf{Z}$ is defined by $g(x) = \begin{cases} x & \text{if } x \text{ is even} \\ 2x+1 & \text{if } x \text{ is odd.} \end{cases}$

l. A right inverse does not exist, since f is not onto.

m. A right inverse $g : \mathbf{Z} \to \mathbf{Z}$ is defined by $g(x) = \begin{cases} 2x & \text{if } x \text{ is even} \\ x-2 & \text{if } x \text{ is odd.} \end{cases}$

n. A right inverse $g : \mathbf{Z} \to \mathbf{Z}$ is defined by $g(x) = \begin{cases} 2x-1 & \text{if } x \text{ is even} \\ x-1 & \text{if } x \text{ is odd.} \end{cases}$

2. **a**. A left inverse $g : \mathbf{Z} \to \mathbf{Z}$ is defined by $g(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ 1 & \text{if } x \text{ is odd.} \end{cases}$

b. A left inverse $g : \mathbf{Z} \to \mathbf{Z}$ is defined by $g(x) = \begin{cases} \frac{x}{3} & \text{if } x \text{ is a multiple of 3} \\ 0 & \text{if } x \text{ is not a multiple of 3.} \end{cases}$

c. A left inverse $g : \mathbf{Z} \to \mathbf{Z}$ is defined by $g(x) = x - 2$.

d. A left inverse $g : \mathbf{Z} \to \mathbf{Z}$ is defined by $g(x) = 1 - x$.

e. A left inverse $g : \mathbf{Z} \to \mathbf{Z}$ is defined by $g(x) = \begin{cases} y & \text{if } x = y^3 \text{ for some } y \in \mathbf{Z} \\ 0 & \text{if } x \neq y^3 \text{ for some } y \in \mathbf{Z}. \end{cases}$

f. A left inverse does not exist, since f is not one-to-one.

g. A left inverse $g : \mathbf{Z} \to \mathbf{Z}$ is defined by $g(x) = \begin{cases} x & \text{if } x \text{ is even} \\ \dfrac{x+1}{2} & \text{if } x \text{ is odd.} \end{cases}$

h. A left inverse does not exist, since f is not one-to-one.

i. A left inverse does not exist, since f is not one-to-one.

j. A left inverse does not exist, since f is not one-to-one.

k. A left inverse does not exist, since f is not one-to-one.

l. A left inverse $g : \mathbf{Z} \to \mathbf{Z}$ is defined by: $g(x) = \begin{cases} x+1 & \text{if } x \text{ is odd} \\ \frac{x}{2} & \text{if } x \text{ is even.} \end{cases}$

m. A left inverse does not exist, since f is not one-to-one.

n. A left inverse does not exist, since f is not one-to-one.

3. $n!$

4. Let $f : A \to A$, where A is nonempty.

$$\begin{aligned} f \text{ has a left inverse } &\Leftrightarrow f \text{ is one-to-one, by Lemma 1.24} \\ &\Leftrightarrow f^{-1}(f(S)) = S \text{ for every subset } S \text{ of } A, \text{ by} \\ &\qquad\qquad \text{Exercise 27 of Section 1.2.} \end{aligned}$$

5. Let $f : A \to A$, where A is nonempty.

$$\begin{aligned} f \text{ has a right inverse } &\Leftrightarrow f \text{ is onto, by Lemma 1.25} \\ &\Leftrightarrow f\left(f^{-1}(T)\right) = T \text{ for every subset } T \text{ of } A, \text{ by} \\ &\qquad\qquad \text{Exercise 28 of Section 1.2.} \end{aligned}$$

Section 1.6

1. True **2**. False **3**. False **4**. False **5**. False **6**. False **7**. True

8. False **9**. False **10**. False **11**. True **12**. True

Exercises 1.6

1. **a**. $A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 5 & 4 \end{bmatrix}$ **b**. $A = \begin{bmatrix} -1 & -2 \\ 1 & 2 \\ -1 & -2 \\ 1 & 2 \end{bmatrix}$ **c**. $B = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$

d. $B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **e**. $C = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix}$ **f**. $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

2. **a**. $\begin{bmatrix} 3 & 0 & -4 \\ 8 & -8 & 6 \end{bmatrix}$ **b**. $\begin{bmatrix} 1 & 9 \\ -3 & 2 \end{bmatrix}$ **c**. Not possible **d**. Not possible

3. **a**. $\begin{bmatrix} -5 & 7 \\ 8 & -1 \end{bmatrix}$ **b**. $\begin{bmatrix} -10 & 2 & 1 \\ -14 & 6 & -21 \\ 6 & -1 & -2 \end{bmatrix}$ **c**. Not possible **d**. $\begin{bmatrix} 7 & -11 \\ 12 & 6 \\ -2 & 20 \end{bmatrix}$

e. $\begin{bmatrix} 4 & 2 \\ 3 & 7 \end{bmatrix}$ **f.** $\begin{bmatrix} 1 & 3 \\ -4 & 10 \end{bmatrix}$ **g.** Not possible **h.** Not possible

i. $[4]$ **j.** $\begin{bmatrix} -12 & 8 & -4 \\ -15 & 10 & -5 \\ 18 & -12 & 6 \end{bmatrix}$

4.
$$\begin{aligned} c_{ij} &= \sum_{k=1}^{3} (i+k)(2k-j) \\ &= (i+1)(2-j) + (i+2)(4-j) + (i+3)(6-j) \\ &= 12i - 6j - 3ij + 28 \end{aligned}$$

6. $\begin{bmatrix} 1 & 6 & -3 & 2 \\ 4 & -7 & 1 & 5 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$

7. **a.** n **b.** $n(n-1)$ **c.** 0

d. δ_{ik}, if $1 \le i \le n, 1 \le k \le n; 0$ if $i > n$ or $k > n$

8.

$\cdot$	I	A	B	C
I	I	A	B	C
A	A	B	C	I
B	B	C	I	A
C	C	I	A	B

9. (Answer not unique) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

10. A trivial example is with $A = I_2$ and B an arbitrary 2×2 matrix. Another example is provided by $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$.

11. (Answer not unique) $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} -6 & -6 \\ 3 & 3 \end{bmatrix}$

12. $(A-B)(A+B) = \begin{bmatrix} 10 & 1 \\ 2 & 1 \end{bmatrix}$ and $A^2 - B^2 = \begin{bmatrix} 2 & 6 \\ -4 & 9 \end{bmatrix}$, $(A-B)(A+B) \neq A^2 - B^2$.

13. $(A+B)^2 = \begin{bmatrix} 22 & 5 \\ 30 & 7 \end{bmatrix}$, $A^2+2AB+B^2 = \begin{bmatrix} 30 & 0 \\ 36 & -1 \end{bmatrix}$, $(A+B)^2 \neq A^2+2AB+B^2$.

14. $X = A^{-1}B$ **15**. $X = A^{-1}BC^{-1}$

22. **b**. For each x in G of the form $\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$, then $y = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. For each x in G of the form $\begin{bmatrix} 0 & 0 \\ a & a \end{bmatrix}$, then $y = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$.

25. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$. Then the product $AB = \begin{bmatrix} 2 & 7 \\ 2 & 7 \end{bmatrix}$ is not diagonal even though B is diagonal.

26. Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Then the product $AB = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ is diagonal but neither A nor B is diagonal .

27. **c**. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Then the product $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is upper triangular but neither A nor B is upper triangular.

30. (Answer not unique) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix}$.

Section 1.7

1. True **2**. False **3**. True **4**. False **5**. True **6**. False

Exercises 1.7

1. **a**. This is a mapping, since for every $a \in A$ there is a unique $b \in A$ such that (a, b) is an element of the relation.

b. This is a mapping, since for every $a \in A$ there is $1 \in A$ such that $(a, 1)$ is an element of the relation.

c. This is not a mapping, since the element 1 is related to three different values; $1R1$, $1R3$, and $1R5$.

d. This is a mapping, since for every $a \in A$ there is a unique $b \in A$ such that (a, b) is an element of the relation.

e. This is a mapping, since for every $a \in A$ there is a unique $b \in A$ such that (a, b) is an element of the relation.

f. This is not a mapping, since the element 5 is related to three different values: $5R1, 5R3$, and $5R5$.

2. **a.** The relation R is not reflexive, since $2 \not{R} 2$. It is not symmetric, since $4R2$ but $2 \not{R} 4$. It is not transitive, since $4R2$ and $2R1$ but $4 \not{R} 1$.

b. The relation R is not reflexive, since $2 \not{R} 2$. It is symmetric, since $x = -y \Rightarrow y = -x$. It is not transitive, since $2R(-2)$ and $(-2)R2$, but $2 \not{R} 2$.

c. The relation R is reflexive and transitive, but not symmetric, since for arbitrary x, y, and z in $\mathbf{Z}$ we have:

(1) $x = x \cdot 1$ with $1 \in \mathbf{Z}$

(2) $6 = 3\,(2)$ with $2 \in \mathbf{Z}$ but $3 \neq 6k$ where $k \in \mathbf{Z}$

(3) $y = xk_1$ for some $k_1 \in \mathbf{Z}$ and $z = yk_2$ for some $k_2 \in \mathbf{Z}$ imply $z = yk_2 = x\,(k_1k_2)$ with $k_1k_2 \in \mathbf{Z}$.

d. The relation R is not reflexive, since $1 \not{R} 1$. It is not symmetric, since $1R2$, but $2 \not{R} 1$. It is transitive, since $x < y$ and $y < z \Rightarrow x < z$ for all x, y, and $z \in \mathbf{Z}$.

e. The relation R is reflexive, since $x \geq x$ for all $x \in \mathbf{Z}$. It is not symmetric, since $5R3$ but $3 \not{R} 5$. It is transitive, since $x \geq y$ and $y \geq z$ imply $x \geq z$ for all x, y, z in $\mathbf{Z}$.

f. The relation R is not reflexive, since $(-1) \not{R} (-1)$. It is not symmetric, since $1R\,(-1)$ but $(-1) \not{R} 1$. It is transitive, since $x = |y|$ and $y = |z|$ implies $x = |y| = ||z|| = |z|$ for all x, y, and $z \in \mathbf{Z}$.

g. The relation R is not reflexive, since $(-6) \not{R} (-6)$. It is not symmetric, since $3R5$ but $5 \not{R} 3$. It is not transitive, since $4R3$ and $3R2$, but $4 \not{R} 2$.

h. The relation R is reflexive, since $x^2 \geq 0$ for all x in $\mathbf{Z}$. It is also symmetric, since $xy \geq 0$ implies that $yx \geq 0$. It is not transitive, since $(-2)\,R0$ and $0R4$ but $(-2) \not{R} 4$.

i. The relation R is not reflexive, since $2 \not{R} 2$. It is symmetric, since $xy \leq 0$ implies $yx \leq 0$ for all $x, y \in \mathbf{Z}$. It is not transitive, since $-1R2$ and $2R\,(-3)$ but $(-1) \not{R} (-3)$.

j. The relation R is not reflexive, since $|x - x| = 0 \neq 1$. It is symmetric, since $|x - y| = 1 \Rightarrow |y - x| = 1$. It is not transitive, since $|2 - 1| = 1$ and $|1 - 2| = 1$ but $|2 - 2| = 0 \neq 1$.

k. The relation R is reflexive, symmetric and transitive, since for arbitrary x, y, and z in $\mathbf{Z}$, we have:

(1) $|x-x| = |0| < 1$
(2) $|x-y| < 1 \Rightarrow |y-x| < 1$
(3) $|x-y| < 1$ and $|y-z| < 1 \Rightarrow x = y$ and $y = z \Rightarrow |x-z| < 1$.

3. **a.** $\{-3, 3\}$ **b.** $\{-5, -1, 3, 7, 11\} \subseteq [3]$

4. **b.** $[0] = \{\ldots, -10, -5, 0, 5, 10, \ldots\}$, $[1] = \{\ldots, -9, -4, 1, 6, 11, \ldots\}$,
$[2] = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$, $[8] = [3] = \{\ldots, -7, -2, 3, 8, 13, \ldots\}$
$[-4] = [1] = \{\ldots, -9, -4, 1, 6, 11, \ldots\}$

5. **b.** $[0] = \{\ldots, -14, -7, 0, 7, 14, \ldots\}$, $[1] = \{\ldots, -13, -6, 1, 8, 15, \ldots\}$
$[3] = \{\ldots, -11, -4, 3, 10, 17, \ldots\}$, $[9] = [2] = \{\ldots, -12, -5, 2, 9, 16, \ldots\}$
$[-2] = [5] = \{\ldots, -9, -2, 5, 12, 19, \ldots\}$

6. $[0] = \{\ldots, -2, 0, 2, 4, \ldots\}$, $[1] = \{\ldots, -3, -1, 1, 3, \ldots\}$

7. $[0] = \{0, \pm 5, \pm 10, \ldots\}$, $\{\pm 1, \pm 4, \pm 6, \pm 9\} \subseteq [1]$, $\{\pm 2, \pm 3, \pm 7, \pm 8\} \subseteq [2]$

8. $[0] = \{\ldots, -4, 0, 4, 8, \ldots\}$, $[1] = \{\ldots, -7, -3, 1, 5, \ldots\}$,
$[2] = \{\ldots, -6, -2, 2, 6, \ldots\}$, $[3] = \{\ldots, -5, -1, 3, 7, \ldots\}$,

9. $[0] = \{\ldots, -7, 0, 7, 14, \ldots\}$, $[1] = \{\ldots, -13, -6, 1, 8, \ldots\}$,
$[2] = \{\ldots, -12, -5, 2, 9, \ldots\}$, $[3] = \{\ldots, -11, -4, 3, 10, \ldots\}$,
$[4] = \{\ldots, -10, -3, 4, 11, \ldots\}$, $[5] = \{\ldots, -9, -2, 5, 12, \ldots\}$,
$[6] = \{\ldots, -8, -1, 6, 13, \ldots\}$

10. $[-1] = \{\ldots, -3, -1, 1, 3, \ldots\}$, $[0] = \{\ldots, -2, 0, 2, 4, \ldots\}$

11. The relation R is symmetric but not reflexive or transitive, since for arbitrary integers x, y, and z, we have the following:

(1) $x + x = 2x$ is not odd;
(2) $x + y$ is odd implies $y + x$ is odd;
(3) $x + y$ is odd and $y + z$ is odd does not imply that $x + z$ is odd. For example, take $x = 1, y = 2$ and $z = 3$.

Thus R is not an equivalence relation on $\mathbf{Z}$.

12. **a.** The relation R is symmetric but not reflexive or transitive, since for arbitrary lines l_1, l_2, and l_3 in a plane, we have the following:

(1) l_1 is not parallel to l_1, since parallel lines have no points in common;
(2) l_1 is parallel to l_2 implies that l_2 is parallel to l_1;
(3) l_1 is parallel to l_2 and l_2 is parallel to l_3 does not imply that l_1 is parallel to l_3. For example, take $l_3 = l_1$ with l_1 parallel to l_2.

Thus R is not an equivalence relation on $\mathbf{Z}$.

b. The relation R is symmetric but not reflexive or transitive, since for arbitrary lines l_1, l_2 and l_3 in a plane, we have the following:

(1) l_1 is not perpendicular to l_1;

(2) l_1 is perpendicular to l_2 implies that l_2 is perpendicular to l_1;

(3) l_1 is perpendicular to l_2 and l_2 is perpendicular to l_3 does not imply that l_1 is perpendicular to l_3.

Thus R is not an equivalence relation.

13. **a.** The relation R is reflexive and transitive but not symmetric, since for arbitrary nonempty subsets x, y, and z of A we have:

(1) x is a subset of x;

(2) x is a subset of y does not imply that y is a subset of x;

(3) x is a subset of y and y is a subset of z imply that x is a subset of z.

b. The relation R is not reflexive and not symmetric, but it is transitive, since for arbitrary nonempty subsets x, y, and z of A we have:

(1) x is not a proper subset of x;

(2) x is a proper subset of y implies that y is not a proper subset of x;

(3) x is a proper subset of y and y is a proper subset of z imply that x is a proper subset of z.

c. The relation R is reflexive, symmetric and transitive, since for arbitrary nonempty subsets x, y, and z of A we have:

(1) x and x have the same number of elements;

(2) If x and y have the same number of elements, then y and x have the same number of elements;

(3) If x and y have the same number of elements and y and z have the same number of elements, then x and z have the same number of elements.

14. **a.** The relation is reflexive and symmetric but not transitive, since if x, y, and z are human beings, we have:

(1) x lives within 400 miles of x;

(2) x lives within 400 miles of y implies that y lives within 400 miles of x;

(3) x lives within 400 miles of y and y lives within 400 miles of z do not imply that x lives within 400 miles of z.

b. The relation R is not reflexive, not symmetric, and not transitive, since if x, y, and z are human beings we have:

(1) x is not the father of x;

(2) x is the father of y implies that y is not the father of x;

(3) x is the father of y and y is the father of z imply that x is not the father of z.

c. The relation is symmetric but not reflexive and not transitive. Let x, y, and z be human beings, and we have:

(1) x is a first cousin of x is not a true statement;
(2) x is a first cousin of y implies that y is a first cousin of x;
(3) x is a first cousin of y and y is a first cousin of z do not imply that x is a first cousin of z.

d. The relation R is reflexive, symmetric, and transitive, since if x, y, and z are human beings we have:

(1) x and x were born in the same year;
(2) if x and y were born in the same year, then y and x were born in the same year;
(3) if x and y were born in the same year and if y and z were born in the same year, then x and z were born in the same year.

e. The relation is reflexive, symmetric, and transitive, since if x, y, and z are human beings, we have:

(1) x and x have the same mother;
(2) x and y have the same mother implies y and x have the same mother;
(3) x and y have the same mother and y and z have the same mother imply that x and z have the same mother.

f. The relation is reflexive, symmetric and transitive, since if x, y, and z are human beings we have:

(1) x and x have the same hair color;
(2) x and y have the same hair color implies that y and x have the same hair color;
(3) x and y have the same hair color and y and z have the same hair color imply that x and z have the same hair color.

15. **a**. The relation R is an equivalence relation on $A \times A$. Let a, b, c, d, p, and q be arbitrary elements of A.

(1) $(a, b)\, R\, (a, b)$ since $ab = ba$.
(2) $(a, b)\, R\, (c, d) \Rightarrow ad = bc \Rightarrow (c, d)\, R\, (a, b)$.
(3)
$$\begin{aligned}(a, b)\, R\, (c, d) \text{ and } (c, d)\, R\, (p, q) &\Rightarrow ad = bc \text{ and } cq = dp \\ &\Rightarrow adcq = bcdp \\ &\Rightarrow aq = bp \text{ since } c \neq 0 \text{ and } d \neq 0 \\ &\Rightarrow (a, b)\, R\, (p, q).\end{aligned}$$

b. The relation R is an equivalence relation on $A \times A$. Let $(a, b), (c, d), (e, f)$ be arbitrary elements of $A \times A$.

(1) $(a, b)\, R\, (a, b)$, since $ab = ab$.

(2) $(a,b)\,R\,(c,d) \Rightarrow ab = cd \Rightarrow cd = ab \Rightarrow (c,d)\,R\,(a,b)$.

(3) $(a,b)\,R\,(c,d)$ and $(c,d)\,R\,(e,f) \Rightarrow ab = cd$ and $cd = ef \Rightarrow ab = ef \Rightarrow (a,b)\,R\,(e,f)$.

c. The relation R is an equivalence relation on $A \times A$. Let a, b, c, d, p, and q be arbitrary elements of A.

(1) $(a,b)\,R\,(a,b)$ since $a^2 + b^2 = a^2 + b^2$.

(2) $(a,b)\,R\,(c,d) \Rightarrow a^2 + b^2 = c^2 + d^2 \Rightarrow c^2 + d^2 = a^2 + b^2 \Rightarrow (c,d)\,R\,(a,b)$.

(3)
$$\begin{aligned} (a,b)\,R\,(c,d) \text{ and } (c,d)\,R\,(p,q) &\Rightarrow a^2 + b^2 = c^2 + d^2 \text{ and} \\ &\quad\; c^2 + d^2 = p^2 + q^2 \\ &\Rightarrow a^2 + b^2 = p^2 + q^2 \\ &\Rightarrow (a,b)\,R\,(p,q). \end{aligned}$$

d. The relation R is an equivalence relation on $A \times A$. Let $(a,b), (c,d)$, and (e,f) be arbitrary elements of $A \times A$.

(1) $(a,b)\,R\,(a,b)$, since $a - b = a - b$.

(2) $(a,b)\,R\,(c,d) \Rightarrow a - b = c - d \Rightarrow c - d = a - b \Rightarrow (c,d)\,R\,(a,b)$.

(3) $(a,b)\,R\,(c,d)$ and $(c,d)\,R\,(e,f) \Rightarrow a - b = c - d$ and $c - d = e - f \Rightarrow a - b = e - f \Rightarrow (a,b)\,R\,(e,f)$.

16. The relation R is reflexive and symmetric but not transitive.

17. **a.** The relation is symmetric but not reflexive and not transitive. Let x, y, and z be arbitrary elements of the power set $\mathcal{P}(A)$ of the nonempty set A.

(1) $x \cap x \neq \varnothing$ is not true if $x = \varnothing$.

(2) $x \cap y \neq \varnothing$ implies that $y \cap x \neq \varnothing$.

(3) $x \cap y \neq \varnothing$ and $y \cap z \neq \varnothing$ do not imply that $x \cap z \neq \varnothing$. For example, let $A = \{a,b,c,d\}$, $x = \{b,c\}$, $y = \{c,d\}$, and $z = \{d,a\}$. Then $x \cap y = \{c\} \neq \varnothing, y \cap z = \{d\} \neq \varnothing$ but $x \cap z = \varnothing$.

b. The relation R is reflexive and transitive but not symmetric, since for arbitrary subsets x, y, z of A we have:

(1) $x \subseteq x$;

(2) $\varnothing \subseteq A$ but $A \nsubseteq \varnothing$;

(3) $x \subseteq y$ and $y \subseteq z$ imply $x \subseteq z$.

18. The relation is reflexive, symmetric, and transitive. Let x, y, and z be arbitrary elements of the power set $\mathcal{P}(A)$ and C a fixed subset of A.

(1) xRx, since $x \cap C = x \cap C$.

(2) $xRy \Rightarrow x \cap C = y \cap C \Rightarrow y \cap C = x \cap C \Rightarrow yRx$.