CHAPTER
3

Second-Order Linear Equations

3.1

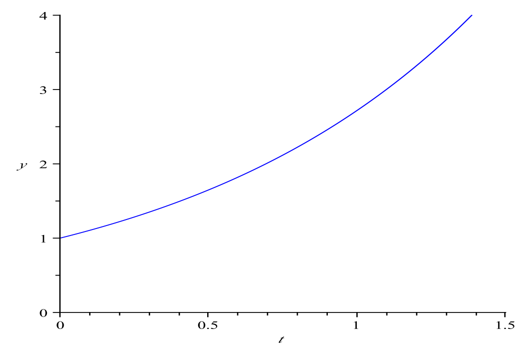
1. Let $y = e^{rt}$, so that $y' = r\,e^{rt}$ and $y'' = r^2\,e^{rt}$. Direct substitution into the differential equation yields $(r^2 + 2r - 3)e^{rt} = 0$. Canceling the exponential, the characteristic equation is $r^2 + 2r - 3 = 0$. The roots of the equation are $r = -3\,,1$. Hence the general solution is $y = c_1e^t + c_2e^{-3t}$.

2. Let $y = e^{rt}$. Substitution of the assumed solution results in the characteristic equation $r^2 + 3r + 2 = 0$. The roots of the equation are $r = -2\,,-1$. Hence the general solution is $y = c_1e^{-t} + c_2e^{-2t}$.

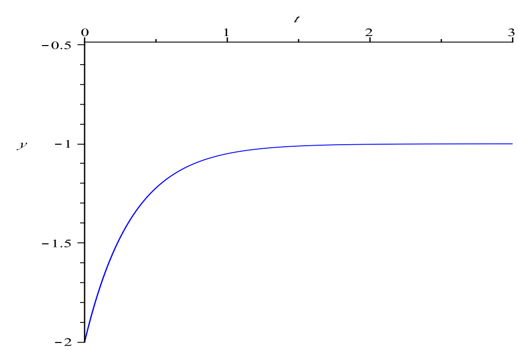
5. The characteristic equation is $4r^2 - 9 = 0$, with roots $r = \pm 3/2$. Therefore the general solution is $y = c_1e^{-3t/2} + c_2e^{3t/2}$.

6. The characteristic equation is $r^2 - 2r - 2 = 0$, with roots $r = 1 \pm \sqrt{3}$. Hence the general solution is $y = c_1e^{(1-\sqrt{3})t} + c_2e^{(1+\sqrt{3})t}$.

7. Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $r^2 + r - 2 = 0$. The roots of the equation are $r = -2\,,1$. Hence the general solution is $y = c_1e^{-2t} + c_2e^t$. Its derivative is $y' = -2c_1e^{-2t} + c_2e^t$. Based on the first condition, $y(0) = 1$, we require that $c_1 + c_2 = 1$. In order to satisfy $y'(0) = 1$, we find that $-2c_1 + c_2 = 1$. Solving for the constants, $c_1 = 0$ and $c_2 = 1$. Hence the specific solution is $y(t) = e^t$. It clearly increases without bound as $t \to \infty$.



9. The characteristic equation is $r^2 + 3r = 0$, with roots $r = -3\,,\ 0$. Therefore the general solution is $y = c_1 + c_2 e^{-3t}$, with derivative $y' = -3\,c_2 e^{-3t}$. In order to satisfy the initial conditions, we find that $c_1 + c_2 = -2$, and $-3\,c_2 = 3$. Hence the specific solution is $y(t) = -1 - e^{-3t}$. This converges to -1 as $t \to \infty$.



10. The characteristic equation is $2r^2 + r - 4 = 0$, with roots $r = (-1 \pm \sqrt{33}\,)/4$. The general solution is $y = c_1 e^{(-1-\sqrt{33})t/4} + c_2 e^{(-1+\sqrt{33})t/4}$, with derivative

$$y' = \frac{-1-\sqrt{33}}{4}\, c_1 e^{(-1-\sqrt{33})t/4} + \frac{-1+\sqrt{33}}{4}\, c_2 e^{(-1+\sqrt{33})t/4}\,.$$

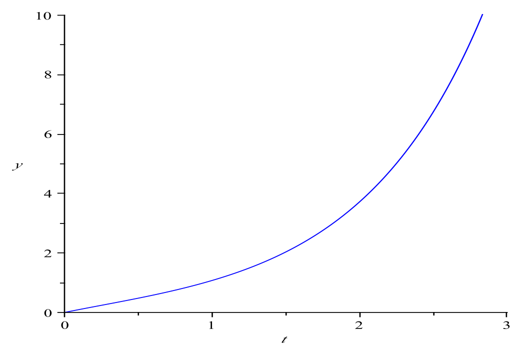
In order to satisfy the initial conditions, we require that

$$c_1 + c_2 = 0 \quad \text{and} \quad \frac{-1-\sqrt{33}}{4}\, c_1 + \frac{-1+\sqrt{33}}{4}\, c_2 = 1.$$

Solving for the coefficients, $c_1 = -2/\sqrt{33}$ and $c_2 = 2/\sqrt{33}$. The specific solution is

$$y(t) = -2\left[e^{(-1-\sqrt{33})t/4} - e^{(-1+\sqrt{33})t/4}\right]/\sqrt{33}\,.$$

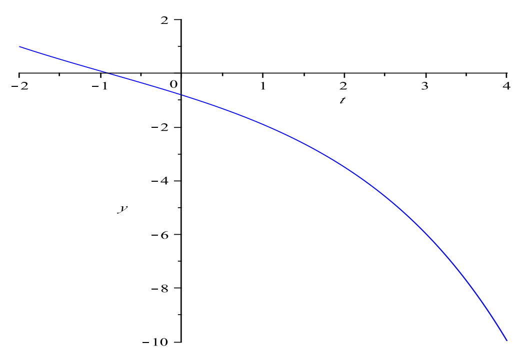
It clearly increases without bound as $t \to \infty$.



12. The characteristic equation is $4r^2 - 1 = 0$, with roots $r = \pm 1/2$. Therefore the general solution is $y = c_1 e^{-t/2} + c_2 e^{t/2}$. Since the initial conditions are specified at $t = -2$, is more convenient to write $y = d_1 e^{-(t+2)/2} + d_2 e^{(t+2)/2}$. The derivative is given by $y' = -\left[d_1 e^{-(t+2)/2}\right]/2 + \left[d_2 e^{(t+2)/2}\right]/2$. In order to satisfy the initial conditions, we find that $d_1 + d_2 = 1$, and $-d_1/2 + d_2/2 = -1$. Solving for the coefficients, $d_1 = 3/2$, and $d_2 = -1/2$. The specific solution is

$$y(t) = \frac{3}{2}e^{-(t+2)/2} - \frac{1}{2}e^{(t+2)/2} = \frac{3}{2e}e^{-t/2} - \frac{e}{2}e^{t/2}.$$

It clearly decreases without bound as $t \to \infty$.



15. The characteristic equation is $2r^2 - 3r + 1 = 0$, with roots $r = 1/2, 1$. Therefore the general solution is $y = c_1 e^{t/2} + c_2 e^{t}$, with derivative $y' = c_1 e^{t/2}/2 + c_2 e^{t}$. In order to satisfy the initial conditions, we require $c_1 + c_2 = 2$ and $c_1/2 + c_2 = 1/2$. Solving for the coefficients, $c_1 = 3$, and $c_2 = -1$. The specific solution is $y(t) = 3e^{t/2} - e^{t}$. To find the stationary point, set $y' = 3e^{t/2}/2 - e^{t} = 0$. There is a unique solution, with $t_1 = \ln(9/4)$. The maximum value is then $y(t_1) = 9/4$. To find the x-intercept, solve the equation $3e^{t/2} - e^{t} = 0$. The solution is readily found to be $t_2 = \ln 9 \approx 2.1972$.

17. The characteristic equation is $r^2 - (2\alpha - 1)r + \alpha(\alpha - 1) = 0$. Examining the coefficients, the roots are $r = \alpha$, $\alpha - 1$. Hence the general solution of the differential equation is $y(t) = c_1 e^{\alpha t} + c_2 e^{(\alpha - 1)t}$. Assuming $\alpha \in \mathbb{R}$, all solutions will tend to zero as long as $\alpha < 0$. On the other hand, all solutions will become unbounded as long as $\alpha - 1 > 0$, that is, $\alpha > 1$.

19.(a) The characteristic roots are $r = -3, -2$. The solution of the initial value problem is $y(t) = (6+\beta)e^{-2t} - (4+\beta)e^{-3t}$.

(b) The maximum point has coordinates $t_0 = \ln[(3(4+\beta))/(2(6+\beta))]$, $y_0 = 4(6+\beta)^3/(27(4+\beta)^2)$.

(c) $y_0 = 4(6+\beta)^3/(27(4+\beta)^2) \geq 4$, as long as $\beta \geq 6 + 6\sqrt{3}$.

(d) $\lim_{\beta\to\infty} t_0 = \ln(3/2)$, $\lim_{\beta\to\infty} y_0 = \infty$.

20.(a) Assuming that y is a constant, the differential equation reduces to $c\,y = d$. Hence the only equilibrium solution is $y = d/c$.

(b) Setting $y = Y + d/c$, substitution into the differential equation results in the equation $a\,Y'' + b\,Y' + c(Y + d/c) = d$. The equation satisfied by Y is $a\,Y'' + b\,Y' + cY = 0$.

3.2

1.
$$W(e^{2t}, e^{-3t/2}) = \begin{vmatrix} e^{2t} & e^{-3t/2} \\ 2e^{2t} & -\frac{3}{2}e^{-3t/2} \end{vmatrix} = -\frac{7}{2}e^{t/2}.$$

3.
$$W(e^{-2t}, t\,e^{-2t}) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} = e^{-4t}.$$

4.
$$W(e^t \sin t\,, e^t \cos t) = \begin{vmatrix} e^t \sin t & e^t \cos t \\ e^t(\sin t + \cos t) & e^t(\cos t - \sin t) \end{vmatrix} = -e^{2t}.$$

5.
$$W(\cos^2\theta\,, 1 + \cos 2\theta) = \begin{vmatrix} \cos^2\theta & 1 + \cos 2\theta \\ -2\sin\theta\cos\theta & -2\sin 2\theta \end{vmatrix} = 0.$$

6. Write the equation as $y'' + (3/t)y' = 1$. $p(t) = 3/t$ is continuous for all $t > 0$. Since $t_0 > 0$, the IVP has a unique solution for all $t > 0$.

7. Write the equation as $y'' + (3/(t-4))y' + (4/t(t-4))y = 2/t(t-4)$. The coefficients are not continuous at $t = 0$ and $t = 4$. Since $t_0 \in (0, 4)$, the largest interval is $0 < t < 4$.

8. The coefficient $3\ln|t|$ is discontinuous at $t = 0$. Since $t_0 > 0$, the largest interval of existence is $0 < t < \infty$.

10. $y_1'' = 2$. We see that $t^2(2) - 2(t^2) = 0$. $y_2'' = 2\,t^{-3}$, with $t^2(y_2'') - 2(y_2) = 0$. Let $y_3 = c_1 t^2 + c_2 t^{-1}$, then $y_3'' = 2c_1 + 2c_2 t^{-3}$. It is evident that y_3 is also a solution.

13. No. Substituting $y = \sin(t^2)$ into the differential equation,

$$-4t^2 \sin(t^2) + 2\cos(t^2) + 2t\,\cos(t^2)p(t) + \sin(t^2)q(t) = 0\,.$$

At $t = 0$, this equation becomes $2 = 0$ (if we suppose that $p(t)$ and $q(t)$ are continuous), which is impossible.

14. $W(e^{2t}, g(t)) = e^{2t}g'(t) - 2e^{2t}g(t) = 3e^{4t}$. Dividing both sides by e^{2t}, we find that g must satisfy the ODE $g' - 2g = 3e^{2t}$. Hence $g(t) = 3t\,e^{2t} + c\,e^{2t}$.

15. $W(f\,,g) = fg' - f'g = t\,\cos\,t - \sin\,t$, and $W(u\,,v) = -4fg' + 4f'g$. Hence $W(u\,,v) = -4t\,\cos\,t + 4\sin\,t$.

16. We compute

$$W(a_1y_1 + a_2y_2, b_1y_1 + b_2y_2) = \begin{vmatrix} a_1y_1 + a_2y_2 & b_1y_1 + b_2y_2 \\ a_1y_1' + a_2y_2' & b_1y_1' + b_2y_2' \end{vmatrix} =$$

$$= (a_1y_1 + a_2y_2)(b_1y_1' + b_2y_2') - (b_1y_1 + b_2y_2)(a_1y_1' + a_2y_2') =$$

$$= a_1b_2(y_1y_2' - y_1'y_2) - a_2b_1(y_1y_2' - y_1'y_2) = (a_1b_2 - a_2b_1)W(y_1, y_2).$$

This now readily shows that y_3 and y_4 form a fundamental set of solutions if and only if $a_1b_2 - a_2b_1 \neq 0$.

18. The general solution is $y = c_1e^{-3t} + c_2e^{-t}$. $W(e^{-3t}, e^{-t}) = 2e^{-4t}$, and hence the exponentials form a fundamental set of solutions. On the other hand, the fundamental solutions must also satisfy the conditions $y_1(1) = 1$, $y_1'(1) = 0$; $y_2(1) = 0$, $y_2'(1) = 1$. For y_1, the initial conditions require $c_1 + c_2 = e$, $-3c_1 - c_2 = 0$. The coefficients are $c_1 = -e^3/2$, $c_2 = 3e/2$. For the solution y_2, the initial conditions require $c_1 + c_2 = 0$, $-3c_1 - c_2 = e$. The coefficients are $c_1 = -e^3/2$, $c_2 = e/2$. Hence the fundamental solutions are

$$y_1 = -\frac{1}{2}e^{-3(t-1)} + \frac{3}{2}e^{-(t-1)} \quad \text{and} \quad y_2 = -\frac{1}{2}e^{-3(t-1)} + \frac{1}{2}e^{-(t-1)}.$$

19. Yes. $y_1'' = -4\,\cos\,2t$; $y_2'' = -4\,\sin\,2t$. $W(\cos\,2t\,,\sin\,2t) = 2$.

20. Clearly, $y_1 = e^t$ is a solution. $y_2' = (1+t)e^t$, $y_2'' = (2+t)e^t$. Substitution into the ODE results in $(2+t)e^t - 2(1+t)e^t + t\,e^t = 0$. Furthermore, $W(e^t, te^t) = e^{2t}$. Hence the solutions form a fundamental set of solutions.

24. Writing the equation in standard form, we find that $P(t) = \sin\,t/\cos\,t$. Hence the Wronskian is $W(t) = c\,e^{-\int(\sin\,t/\cos\,t)\,dt} = c\,e^{\ln|\cos\,t|} = c\,\cos\,t$, in which c is some constant.

25. Writing the equation in standard form, we find that $P(x) = -2x/(1-x^2)$. The Wronskian is $W(x) = c\,e^{-\int -2x/(1-x^2)\,dx} = c\,e^{-\ln|1-x^2|} = c/(1-x^2)$, in which c is some constant.

26. Rewrite the equation as $p(t)y'' + p'(t)y' + q(t)y = 0$. After writing the equation in standard form, we have $P(t) = p'(t)/p(t)$. Hence the Wronskian is

$$W(t) = c\,e^{-\int p'(t)/p(t)\,dt} = c\,e^{-\ln p(t)} = c/p(t).$$

28. For the given differential equation, the Wronskian satisfies the first order differential equation $W' + p(t)W = 0$. Given that W is constant, it is necessary that $p(t) \equiv 0$.

32. $P = 1$, $Q = x$, $R = 1$. We have $P'' - Q' + R = 0$. The equation is exact. Note that $(y')' + (xy)' = 0$. Hence $y' + xy = c_1$. This equation is linear, with integrating factor $\mu = e^{x^2/2}$. Therefore the general solution is

$$y(x) = c_1 e^{-x^2/2}\int_{x_0}^{x} e^{u^2/2} du + c_2 e^{-x^2/2}.$$

34. $P = x^2$, $Q = x$, $R = -1$. We have $P'' - Q' + R = 0$. The equation is exact. Write the equation as $(x^2y')' - (xy)' = 0$. After integration, we conclude that $x^2y' - xy = c$. Divide both sides of the differential equation by x^2. The resulting equation is linear, with integrating factor $\mu = 1/x$. Hence $(y/x)' = c\,x^{-3}$. The solution is $y(t) = c_1x^{-1} + c_2x$.

36. $P = x^2$, $Q = x$, $R = x^2 - \nu^2$. Hence the coefficients are $2P' - Q = 3x$ and $P'' - Q' + R = x^2 + 1 - \nu^2$. The adjoint of the original differential equation is given by $x^2\mu'' + 3x\,\mu' + (x^2 + 1 - \nu^2)\mu = 0$.

37. $P = 1$, $Q = 0$, $R = -x$. Hence the coefficients are given by $2P' - Q = 0$ and $P'' - Q' + R = -x$. Therefore the adjoint of the original equation is $\mu'' - x\,\mu = 0$.

3.3

1. $e^{2-3i} = e^2e^{-3i} = e^2(\cos 3 - i\sin 3)$.

2. $e^{i\pi} = \cos\pi + i\sin\pi = -1$.

3. $e^{2-(\pi/2)\,i} = e^2(\cos(\pi/2) - i\sin(\pi/2)) = -e^2\,i$.

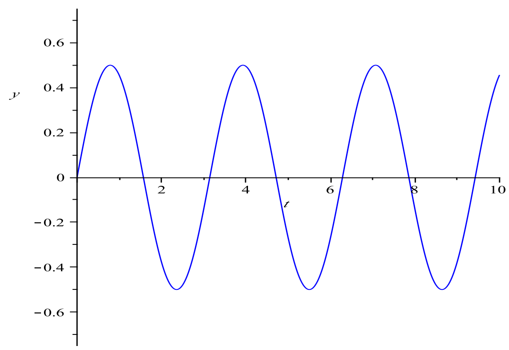
6. The characteristic equation is $r^2 - 2r + 6 = 0$, with roots $r = 1 \pm i\sqrt{5}$. Hence the general solution is $y = c_1e^t\cos\sqrt{5}\,t + c_2\,e^t\sin\sqrt{5}\,t$.

7. The characteristic equation is $r^2 + 2r + 2 = 0$, with roots $r = -1 \pm i$. Hence the general solution is $y = c_1e^{-t}\cos t + c_2\,e^{-t}\sin t$.

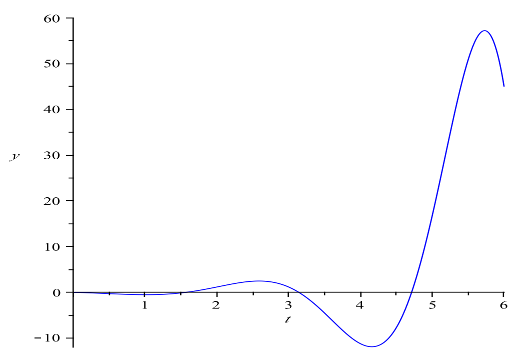
9. The characteristic equation is $r^2 + 2r + 1.25 = 0$, with roots $r = -1 \pm i/2$. Hence the general solution is $y = c_1 e^{-t}\cos(t/2) + c_2\, e^{-t}\sin(t/2)$.

11. The characteristic equation is $r^2 + 4r + 6.25 = 0$, with roots $r = -2 \pm (3/2)\, i$. Hence the general solution is $y = c_1 e^{-2t}\cos(3t/2) + c_2\, e^{-2t}\sin(3t/2)$.

12. The characteristic equation is $r^2 + 4 = 0$, with roots $r = \pm 2i$. Hence the general solution is $y = c_1 \cos 2t + c_2 \sin 2t$. Now $y' = -2c_1 \sin 2t + 2c_2 \cos 2t$. Based on the first condition, $y(0) = 0$, we require that $c_1 = 0$. In order to satisfy the condition $y'(0) = 1$, we find that $2c_2 = 1$. The constants are $c_1 = 0$ and $c_2 = 1/2$. Hence the specific solution is $y(t) = \sin 2t\,/2$. The solution is periodic.

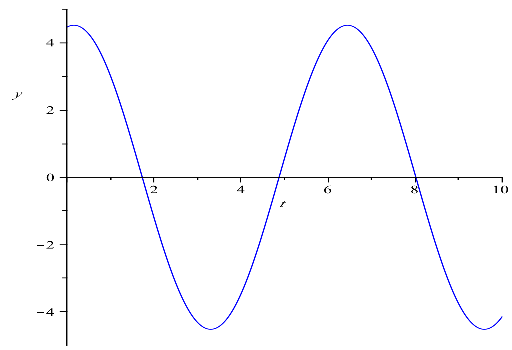


13. The characteristic equation is $r^2 - 2r + 5 = 0$, with roots $r = 1 \pm 2i$. Hence the general solution is $y = c_1 e^t \cos 2t + c_2\, e^t \sin 2t$. Based on the initial condition $y(\pi/2) = 0$, we require that $c_1 = 0$. It follows that $y = c_2\, e^t \sin 2t$, and so the first derivative is $y' = c_2\, e^t \sin 2t + 2c_2\, e^t \cos 2t$. In order to satisfy the condition $y'(\pi/2) = 2$, we find that $-2e^{\pi/2} c_2 = 2$. Hence we have $c_2 = -e^{-\pi/2}$. Therefore the specific solution is $y(t) = -e^{t-\pi/2} \sin 2t$. The solution oscillates with an exponentially growing amplitude.



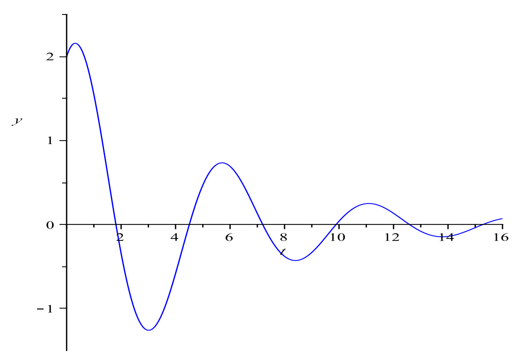
14. The characteristic equation is $r^2 + 1 = 0$, with roots $r = \pm i$. Hence the general solution is $y = c_1 \cos t + c_2 \sin t$. Its derivative is $y' = -c_1 \sin t + c_2 \cos t$. Based on the first condition, $y(\pi/3) = 2$, we require that $c_1 + \sqrt{3}\, c_2 = 4$. In order to satisfy the condition $y'(\pi/3) = -4$, we find that $-\sqrt{3}\, c_1 + c_2 = -8$. Solving

these for the constants, $c_1 = 1 + 2\sqrt{3}$ and $c_2 = \sqrt{3} - 2$. Hence the specific solution is a steady oscillation, given by $y(t) = (1 + 2\sqrt{3})\cos t + (\sqrt{3} - 2)\sin t$.



17.(a) The characteristic equation is $5r^2 + 2r + 7 = 0$, with roots $r = -(1 \pm i\sqrt{34})/5$. The solution is $u = c_1 e^{-t/5}\cos\sqrt{34}t/5 + c_2 e^{-t/5}\sin\sqrt{34}t/5$. Invoking the given initial conditions, we obtain the equations for the coefficients: $c_1 = 2$, $-2 + \sqrt{34}\,c_2 = 5$. That is, $c_1 = 2$, $c_2 = 7/\sqrt{34}$. Hence the specific solution is

$$u(t) = 2e^{-t/5}\cos\frac{\sqrt{34}}{5}t + \frac{7}{\sqrt{34}}e^{-t/5}\sin\frac{\sqrt{34}}{5}t.$$



(b) Based on the graph of $u(t)$, T is in the interval $14 < t < 16$. A numerical solution on that interval yields $T \approx 14.5115$.

19. Direct calculation gives the result. On the other hand, it can be shown that $W(fg, fh) = f^2 W(g, h)$. Hence $W(e^{\lambda t}\cos\mu t, e^{\lambda t}\sin\mu t) = e^{2\lambda t}\,W(\cos\mu t, \sin\mu t) = e^{2\lambda t}\left[\cos\mu t(\sin\mu t)' - (\cos\mu t)'\sin\mu t\right] = \mu\, e^{2\lambda t}$.

20.(a) Clearly, y_1 and y_2 are solutions. Also, $W(\cos t, \sin t) = \cos^2 t + \sin^2 t = 1$.

(b) $y' = i\,e^{it}$, $y'' = i^2 e^{it} = -e^{it}$. Evidently, y is a solution and so $y = c_1 y_1 + c_2 y_2$.

(c) Setting $t = 0$, $1 = c_1\cos 0 + c_2\sin 0$, and $c_1 = 1$.

(d) Differentiating, $i\,e^{it} = c_2 \cos t$. Setting $t = 0$, $i = c_2 \cos 0$ and hence $c_2 = i$. Therefore $e^{it} = \cos t + i \sin t$.

21. Euler's formula is $e^{it} = \cos t + i \sin t$. It follows that $e^{-it} = \cos t - i \sin t$. Adding these equation, $e^{it} + e^{-it} = 2 \cos t$. Subtracting the two equations results in $e^{it} - e^{-it} = 2i \sin t$.

22. Let $r_1 = \lambda_1 + i\mu_1$, and $r_2 = \lambda_2 + i\mu_2$. Then

$$\begin{aligned} e^{(r_1+r_2)t} &= e^{(\lambda_1+\lambda_2)t+i(\mu_1+\mu_2)t} = e^{(\lambda_1+\lambda_2)t}\left[\cos(\mu_1+\mu_2)t + i\,\sin(\mu_1+\mu_2)t\right] = \\ &= e^{(\lambda_1+\lambda_2)t}\left[(\cos\,\mu_1 t + \,i \sin\,\mu_1 t)(\cos\,\mu_2 t + \,i \sin\,\mu_2 t)\right] = \\ &= e^{\lambda_1 t}(\cos\,\mu_1 t + \,i \sin\,\mu_1 t)\cdot e^{\lambda_2 t}(\cos\,\mu_1 t + \,i \sin\,\mu_1 t) = e^{r_1 t}\,e^{r_2 t}. \end{aligned}$$

Hence $e^{(r_1+r_2)t} = e^{r_1 t}\,e^{r_2 t}$.

23. Clearly, $u' = \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t = e^{\lambda t}(\lambda\cos\mu t - \mu\sin\mu t)$ and then $u'' = \lambda e^{\lambda t}(\lambda\cos\mu t - \mu\sin\mu t) + e^{\lambda t}(-\lambda\mu\sin\mu t - \mu^2\cos\mu t)$. Plugging these into the differential equation, dividing by $e^{\lambda t} \neq 0$ and arranging the sine and cosine terms we obtain that the identity to prove is

$$(a(\lambda^2-\mu^2) + b\lambda + c)\cos\mu t + (-2\lambda\mu a - b\mu)\sin\mu t = 0.$$

We know that $\lambda \pm i\mu$ solves the characteristic equation $ar^2 + br + c = 0$, so $a(\lambda - i\mu)^2 + b(\lambda - i\mu) + c = a(\lambda^2 - \mu^2) + b\lambda + c + i(-2\lambda\mu a - \mu b) = 0$. If this complex number is zero, then both the real and imaginary parts of it are zero, but those are the coefficients of $\cos\mu t$ and $\sin\mu t$ in the above identity, which proves that $au'' + bu' + cu = 0$. The solution for v is analogous.

26. The equation transforms into $y'' + y = 0$. The characteristic roots are $r = \pm i$. The solution is $y = c_1\cos(x) + c_2\sin(x) = c_1\cos(\ln t) + c_2\sin(\ln t)$.

28. The equation transforms into $y'' - 5y' - 6y = 0$. The characteristic roots are $r = -1,\, 6$. The solution is $y = c_1 e^{-x} + c_2 e^{6x} = c_1 e^{-\ln t} + c_2 e^{6\ln t} = c_1/t + c_2 t^6$.

29. The equation transforms into $y'' - 5y' + 6y = 0$. The characteristic roots are $r = 2,\, 3$. The solution is $y = c_1 e^{2x} + c_2 e^{3x} = c_1 e^{2\ln t} + c_2 e^{3\ln t} = c_1 t^2 + c_2 t^3$.

30. The equation transforms into $y'' + 2y' - 3y = 0$. The characteristic roots are $r = 1,\, -3$. The solution is $y = c_1 e^{x} + c_2 e^{-3x} = c_1 e^{\ln t} + c_2 e^{-3\ln t} = c_1 t + c_2/t^3$.

31. The equation transforms into $y'' + 6y' + 10y = 0$. The characteristic roots are $r = -3 \pm i$. The solution is

$$y = c_1 e^{-3x}\cos(x) + c_2 e^{-3x}\sin(x) = c_1\frac{1}{t^3}\cos(\ln t) + c_2\frac{1}{t^3}\sin(\ln t).$$

32.(a) By the chain rule, $y'(x) = (dy/dx)x'$. In general, $dz/dt = (dz/dx)(dx/dt)$. Setting $z = (dy/dt)$, we have

$$\frac{d^2y}{dt^2} = \frac{dz}{dx}\frac{dx}{dt} = \frac{d}{dx}\left[\frac{dy}{dx}\frac{dx}{dt}\right]\frac{dx}{dt} = \left[\frac{d^2y}{dx^2}\frac{dx}{dt}\right]\frac{dx}{dt} + \frac{dy}{dx}\frac{d}{dx}\left[\frac{dx}{dt}\right]\frac{dx}{dt}.$$

However,

$$\frac{d}{dx}\left[\frac{dx}{dt}\right]\frac{dx}{dt} = \left[\frac{d^2x}{dt^2}\right]\frac{dt}{dx}\cdot\frac{dx}{dt} = \frac{d^2x}{dt^2}.$$

Hence

$$\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}\left[\frac{dx}{dt}\right]^2 + \frac{dy}{dx}\frac{d^2x}{dt^2}.$$

(b) Substituting the results in part (a) into the general differential equation, $y'' + p(t)y' + q(t)y = 0$, we find that

$$\frac{d^2y}{dx^2}\left[\frac{dx}{dt}\right]^2 + \frac{dy}{dx}\frac{d^2x}{dt^2} + p(t)\frac{dy}{dx}\frac{dx}{dt} + q(t)y = 0\,.$$

Collecting the terms,

$$\left[\frac{dx}{dt}\right]^2\frac{d^2y}{dx^2} + \left[\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}\right]\frac{dy}{dx} + q(t)y = 0\,.$$

(c) Assuming $(dx/dt)^2 = k\,q(t)$, and $q(t) > 0$, we find that $dx/dt = \sqrt{k\,q(t)}$, which can be integrated. That is, $x = u(t) = \int \sqrt{k\,q(t)}\;dt = \int \sqrt{q(t)}\;dt$, since $k = 1$.

(d) Let $k = 1$. It follows that $d^2x/dt^2 + p(t)dx/dt = du/dt + p(t)u(t) = q'/2\sqrt{q} + p\sqrt{q}$. Hence

$$\left[\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}\right] / \left[\frac{dx}{dt}\right]^2 = \frac{q'(t) + 2p(t)q(t)}{2\left[q(t)\right]^{3/2}}\,.$$

As long as $dx/dt \neq 0$, the differential equation can be expressed as

$$\frac{d^2y}{dx^2} + \left[\frac{q'(t) + 2p(t)q(t)}{2\left[q(t)\right]^{3/2}}\right]\frac{dy}{dx} + y = 0.$$

(e) To find the analogue to the condition found in part d) for the case when $q(t) < 0$ we return to the conditions that make the coefficients on y, dy/dt and d^2y/dt^2 proportional to each other. Since the coefficients on y and d^2y/dt^2 are proportional, $(dx/dt)^2 = \alpha q(t)$, and we may take $\alpha = -1$. Thus $dx/dt = (-q(t))^{1/2}$ and $d^2y/dt^2 = (-q'/2)(-q)^{-1/2}$. Since the coefficients on y and dy/dt are proportional, there is a constant β with

$$\beta\; q = \frac{d^2y}{dt^2} + p(t)\frac{dx}{dt} = \frac{-q'}{2}(-q)^{-1/2} + p(-q)^{1/2} = \frac{-q' - 2pq}{2(-q)^{1/2}}$$

and dividing each side of the equation by $-q$ gives

$$-\beta = \frac{-q' - 2pq}{2(-q)^{3/2}},\ \text{or}\ 2\beta = \frac{q' + 2pq}{(-q)^{3/2}}$$

Thus the desired condition is that $(q' + 2pq)/(-q)^{3/2}$ must be a constant.

34. Note that $p(t) = 3t$ and $q(t) = t^2$. We have $x = \int t\,dt = t^2/2$. Furthermore,

$$\frac{q'(t) + 2p(t)q(t)}{2\,[q(t)]^{3/2}} = \frac{1+3t^2}{t^2}\,.$$

The ratio is not constant, and therefore the equation cannot be transformed.

35. Note that $p(t) = t - 1/t$ and $q(t) = t^2$. We have $x = \int t\,dt = t^2/2$. Furthermore,

$$\frac{q'(t) + 2p(t)q(t)}{2\,[q(t)]^{3/2}} = 1\,.$$

The ratio is constant, and therefore the equation can be transformed. From Problem 32, the transformed equation is

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0\,.$$

Based on the methods in this section, the characteristic equation is $r^2 + r + 1 = 0$, with roots $r = (-1 \pm i\sqrt{3})/2$. The general solution is $y(x) = c_1 e^{-x/2} \cos \sqrt{3}\,x/2 + c_2\, e^{-x/2} \sin \sqrt{3}\,x/2$. Since $x = t^2/2$, the solution in the original variable t is

$$y(t) = e^{-t^2/4}\left[c_1 \cos(\sqrt{3}\;t^2/4) + c_2\,\sin(\sqrt{3}\;t^2/4)\right].$$

36. Note that $p(t) = t$ and $q(t) = -e^{-t^2} < 0$ for $-\infty < t < \infty$. To proceed we must confirm that $(q' + 2pq)/(-q)^{3/2}$ is a constant:

$$\frac{q' + 2pq}{(-q)^{3/2}} = \frac{2te^{-t^2} + 2t(-e^{t^2})}{(e^{-t^2})^{3/2}} = 0.$$

Thus the differential equation can be transformed into an equation with constant coefficients by letting $x = u(t) = \int e^{-t^2/2}\,dt$. Substituting $x = u(t)$ in the differential equation found in part (b) of Problem 32 we obtain, after dividing by the coefficient of d^2y/dx^2, the differential equation $(d^2y/dx^2) - y = 0$. Hence the general solution of the original differential equation is $y(t) = c_1 e^{x(t)} + c_2 e^{-x(t)}$, where $x(t) = \int e^{-t^2/2}\,dt$.

3.4

2. The characteristic equation is $9r^2 + 6r + 1 = 0$, with the double root $r = -1/3$. The general solution is $y(t) = c_1 e^{-t/3} + c_2 t\, e^{-t/3}$.

3. The characteristic equation is $4r^2 - 4r - 3 = 0$, with roots $r = -1/2\,,\,3/2$. The general solution is $y(t) = c_1 e^{-t/2} + c_2 e^{3t/2}$.

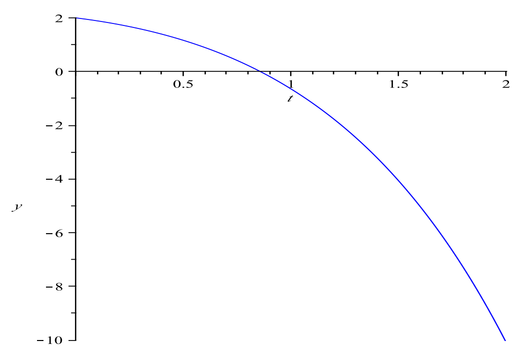
5. The characteristic equation is $r^2 - 6r + 9 = 0$, with the double root $r = 3$. The general solution is $y(t) = c_1 e^{3t} + c_2 t\, e^{3t}$.

6. The characteristic equation is $4r^2 + 17r + 4 = 0$, with roots $r = -1/4$, -4. The general solution is $y(t) = c_1e^{-t/4} + c_2e^{-4t}$.

7. The characteristic equation is $16r^2 + 24r + 9 = 0$, with double root $r = -3/4$. The general solution is $y(t) = c_1e^{-3t/4} + c_2t\,e^{-3t/4}$.

8. The characteristic equation is $2r^2 + 2r + 1 = 0$. We obtain the complex roots $r = (-1 \pm i)/2$. The general solution is $y(t) = c_1e^{-t/2}\cos(t/2) + c_2e^{-t/2}\sin(t/2)$.

9. The characteristic equation is $9r^2 - 12r + 4 = 0$, with the double root $r = 2/3$. The general solution is $y(t) = c_1e^{2t/3} + c_2t\,e^{2t/3}$. Invoking the first initial condition, it follows that $c_1 = 2$. Now $y'(t) = (4/3 + c_2)e^{2t/3} + 2c_2t\,e^{2t/3}/3$. Invoking the second initial condition, $4/3 + c_2 = -1$, or $c_2 = -7/3$. Hence we obtain the solution $y(t) = 2e^{2t/3} - (7/3)te^{2t/3}$. Since the second term dominates for large t, $y(t) \to -\infty$.



12. The characteristic roots are $r_1 = r_2 = 1/2$. Hence the general solution is given by $y(t) = c_1e^{t/2} + c_2t\,e^{t/2}$. Invoking the initial conditions, we require that $c_1 = 2$, and that $1 + c_2 = b$. The specific solution is $y(t) = 2e^{t/2} + (b-1)t\,e^{t/2}$. Since the second term dominates, the long-term solution depends on the sign of the coefficient $b - 1$. The critical value is $b = 1$.

15.(a) The characteristic equation is $r^2 + 2a\,r + a^2 = (r+a)^2 = 0$.

(b) With $p(t) = 2a$, Abel's Formula becomes $W(y_1\,, y_2) = c\,e^{-\int 2a\,dt} = c\,e^{-2at}$.

(c) $y_1(t) = e^{-at}$ is a solution. From part (b), with $c = 1$, $e^{-at}\,y_2'(t) + a\,e^{-at}y_2(t) = e^{-2at}$, which can be written as $(e^{at}\,y_2(t))' = 1$, resulting in $e^{at}\,y_2(t) = t$.

17.(a) If the characteristic equation $ar^2 + br + c$ has equal roots r_1, then $ar_1^2 + br_1 + c = a(r - r_1)^2 = 0$. Then clearly $L[e^{rt}] = (ar^2 + br + c)e^{rt} = a(r - r_1)^2e^{rt}$. This gives immediately that $L[e^{r_1t}] = 0$.

(b) Differentiating the identity in part (a) with respect to r we get $(2ar + b)e^{rt} + (ar^2 + br + c)te^{rt} = 2a(r - r_1)e^{rt} + a(r - r_1)^2te^{rt}$. Again, this gives $L[te^{r_1t}] = 0$.

18. Set $y_2(t) = t^2\, v(t)$. Substitution into the differential equation results in

$$t^2(t^2v'' + 4tv' + 2v) - 4t(t^2v' + 2tv) + 6t^2v = 0\,.$$

After collecting terms, we end up with $t^4v'' = 0$. Hence $v(t) = c_1 + c_2t$, and thus $y_2(t) = c_1t^2 + c_2t^3$. Setting $c_1 = 0$ and $c_2 = 1$, we obtain $y_2(t) = t^3$.

19. Set $y_2(t) = t\, v(t)$. Substitution into the differential equation results in

$$t^2(tv'' + 2v') + 2t(tv' + v) - 2tv = 0\,.$$

After collecting terms, we end up with $t^3v'' + 4t^2v' = 0$. This equation is linear in the variable $w = v'$. It follows that $v'(t) = c\,t^{-4}$, and $v(t) = c_1t^{-3} + c_2$. Thus $y_2(t) = c_1t^{-2} + c_2t$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(t) = t^{-2}$.

23. Direct substitution verifies that $y_1(t) = e^{-\delta x^2/2}$ is a solution of the differential equation. Now set $y_2(x) = y_1(x)\,v(x)$. Substitution of y_2 into the equation results in $v'' - \delta xv' = 0$. This equation is linear in the variable $w = v'$. An integrating factor is $\mu = e^{-\delta x^2/2}$. Rewrite the equation as $[\, e^{-\delta x^2/2}v']' = 0$, from which it follows that $v'(x) = c_1\, e^{\delta x^2/2}$. Integrating, we obtain

$$v(x) = c_1 \int_0^x e^{\delta u^2/2} du + v(0).$$

Hence

$$y_2(x) = c_1 e^{-\delta x^2/2} \int_0^x e^{\delta u^2/2} du + c_2 e^{-\delta x^2/2}.$$

Setting $c_2 = 0$, we obtain a second independent solution.

25. After writing the differential equation in standard form, we have $p(t) = 3/t$. Based on Abel's identity, $W(y_1, y_2) = c_1 e^{-\int 3/t\, dt} = c_1t^{-3}$. As shown in Problem 24, two solutions of a second order linear equation satisfy $(y_2/y_1)' = W(y_1, y_2)/y_1^2$. In the given problem, $y_1(t) = t^{-1}$. Hence $(t\,y_2)' = c_1t^{-1}$. Integrating both sides of the equation, $y_2(t) = c_1t^{-1}\ln\, t + c_2t^{-1}$. Setting $c_1 = 1$ and $c_2 = 0$ we obtain $y_2(t) = t^{-1}\ln t$.

27. Write the differential equation in standard form to find $p(x) = 1/x$. Based on Abel's identity, $W(y_1, y_2) = c\, e^{-\int 1/x\, dx} = c\, x^{-1}$. Two solutions of a second order linear differential equation satisfy $(y_2/y_1)' = W(y_1, y_2)/y_1^2$. In the given problem, $y_1(x) = x^{-1/2}\sin\, x$. Hence

$$(\frac{\sqrt{x}}{\sin\, x}\, y_2)' = c\,\frac{1}{\sin^2 x}\,.$$

Integrating both sides of the equation, $y_2(x) = c_1x^{-1/2}\cos\, x + c_2x^{-1/2}\sin\, x$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(x) = x^{-1/2}\cos\, x$.

29.(a) The characteristic equation is $ar^2 + c = 0$. If $a, c > 0$, then the roots are $r = \pm\, i\sqrt{c/a}$. The general solution is

$$y(t) = c_1 \cos \sqrt{\frac{c}{a}}\, t + c_2 \sin \sqrt{\frac{c}{a}}\, t\,,$$

which is bounded.

(b) The characteristic equation is $ar^2 + br = 0$. The roots are $r = 0, -b/a$, and hence the general solution is $y(t) = c_1 + c_2 e^{-bt/a}$. Clearly, $y(t) \to c_1$. With the given initial conditions, $c_1 = y_0 + (a/b)y_0'$.

30. Note that $2\cos t \sin t = \sin 2t$. Then $1 - k\cos t \sin t = 1 - (k/2)\sin 2t$. Now if $0 < k < 2$, then $(k/2)\sin 2t < |\sin 2t|$ and $-(k/2)\sin 2t > -|\sin 2t|$. Hence

$$1 - k\cos t \sin t = 1 - \frac{k}{2}\sin 2t > 1 - |\sin 2t| \geq 0\,.$$

31. The equation transforms into $y'' - 4y' + 4y = 0$. We obtain a double root $r = 2$. The solution is $y = c_1 e^{2x} + c_2 x e^{2x} = c_1 e^{2\ln t} + c_2 \ln t e^{2\ln t} = c_1 t^2 + c_2 t^2 \ln t$.

33. The equation transforms into $y'' + 2y' + y = 0$. We get a double root $r = -1$. The solution is $y = c_1 e^{-x} + c_2 x e^{-x} = c_1 e^{-\ln t} + c_2 \ln t e^{-\ln t} = c_1 t^{-1} + c_2 t^{-1} \ln t$.

34. The equation transforms into $y'' - 3y' + 9y/4 = 0$. We obtain the double root $r = 3/2$. The solution is $y = c_1 e^{3x/2} + c_2 x e^{3x/2} = c_1 e^{3\ln t/2} + c_2 \ln t e^{3\ln t/2} = c_1 t^{3/2} + c_2 t^{3/2} \ln t$.

3.5

2. The characteristic equation for the homogeneous problem is $r^2 - r - 2 = 0$, with roots $r = -1, 2$. Hence $y_c(t) = c_1 e^{-t} + c_2 e^{2t}$. Set $Y = At^2 + Bt + C$. Substitution into the given differential equation, and comparing the coefficients, results in the system of equations $-2A = 4$, $-2A - 2B = -2$ and $2A - B - 2C = 0$. Hence $Y = -2t^2 + 3t - 7/2$. The general solution is $y(t) = y_c(t) + Y$.

3. The characteristic equation for the homogeneous problem is $r^2 + r - 6 = 0$, with roots $r = -3, 2$. Hence $y_c(t) = c_1 e^{-3t} + c_2 e^{2t}$. Set $Y = Ae^{3t} + Be^{-2t}$. Substitution into the given differential equation, and comparing the coefficients, results in the system of equations $6A = 12$ and $-4B = 12$. Hence $Y = 2e^{3t} - 3e^{-2t}$. The general solution is $y(t) = y_c(t) + Y$.

4. The characteristic equation for the homogeneous problem is $r^2 - 2r - 3 = 0$, with roots $r = -1, 3$. Hence $y_c(t) = c_1 e^{-t} + c_2 e^{3t}$. Note that the assignment $Y = Ate^{-t}$ is not sufficient to match the coefficients. Try $Y = Ate^{-t} + Bt^2 e^{-t}$. Substitution into the differential equation, and comparing the coefficients, results in the system of equations $-4A + 2B = 0$ and $-8B = -3$. This implies that $Y = (3/16)te^{-t} + (3/8)t^2 e^{-t}$. The general solution is $y(t) = y_c(t) + Y$.

8. The characteristic equation for the homogeneous problem is $r^2 + \omega_0^2 = 0$, with complex roots $r = \pm\,\omega_0\, i$. Hence $y_c(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$. Since $\omega \neq \omega_0$, set $Y = A\cos \omega t + B \sin \omega t$. Substitution into the ODE and comparing the coefficients results in the system of equations $(\omega_0^2 - \omega^2)A = 1$ and $(\omega_0^2 - \omega^2)B = 0$.

Hence

$$Y = \frac{1}{\omega_0^2 - \omega^2} \cos \omega t \,.$$

The general solution is $y(t) = y_c(t) + Y$.

9. From Problem 8, $y_c(t)$ is known. Since $\cos \omega_0 t$ is a solution of the homogeneous problem, set $Y = At \cos \omega_0 t + Bt \sin \omega_0 t$. Substitution into the given ODE and comparing the coefficients results in $A = 0$ and $B = 1/2\omega_0$. Hence the general solution is $y(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + t \sin \omega_0 t/(2\omega_0)$.

12. The characteristic equation for the homogeneous problem is $r^2 + 4 = 0$, with roots $r = \pm 2i$. Hence $y_c(t) = c_1 \cos 2t + c_2 \sin 2t$. Set $Y_1 = A + Bt + Ct^2$. Comparing the coefficients of the respective terms, we find that $A = -1/8$, $B = 0$, $C = 1/4$. Now set $Y_2 = D\,e^t$, and obtain $D = 3/5$. Hence the general solution is $y(t) = c_1 \cos 2t + c_2 \sin 2t - 1/8 + t^2/4 + 3\,e^t/5$. Invoking the initial conditions, we require that $19/40 + c_1 = 0$ and $3/5 + 2c_2 = 2$. Hence $c_1 = -19/40$ and $c_2 = 7/10$.

13. The characteristic equation for the homogeneous problem is $r^2 - 2r + 1 = 0$, with a double root $r = 1$. Hence $y_c(t) = c_1 e^t + c_2 t\, e^t$. Consider $g_1(t) = t\, e^t$. Note that g_1 is a solution of the homogeneous problem. Set $Y_1 = At^2 e^t + Bt^3 e^t$ (the first term is not sufficient for a match). Upon substitution, we obtain $Y_1 = t^3 e^t/6$. By inspection, $Y_2 = 4$. Hence the general solution is $y(t) = c_1 e^t + c_2 t\, e^t + t^3 e^t/6 + 4$. Invoking the initial conditions, we require that $c_1 + 4 = 1$ and $c_1 + c_2 = 1$. Hence $c_1 = -3$ and $c_2 = 4$.

14. The characteristic equation for the homogeneous problem is $r^2 + 4 = 0$, with roots $r = \pm 2i$. Hence $y_c(t) = c_1 \cos 2t + c_2 \sin 2t$. Since the function $\sin 2t$ is a solution of the homogeneous problem, set $Y = At \cos 2t + Bt \sin 2t$. Upon substitution, we obtain $Y = -3t \cos 2t/4$. Hence the general solution is $y(t) = c_1 \cos 2t + c_2 \sin 2t - 3t \cos 2t/4$. Invoking the initial conditions, we require that $c_1 = 2$ and $2c_2 - (3/4) = -1$. Hence $c_1 = 2$ and $c_2 = -1/8$.

15. The characteristic equation for the homogeneous problem is $r^2 + 2r + 5 = 0$, with complex roots $r = -1 \pm 2i$. Hence $y_c(t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t$. Based on the form of $g(t)$, set $Y = At\, e^{-t} \cos 2t + Bt\, e^{-t} \sin 2t$. After comparing coefficients, we obtain $Y = t\, e^{-t} \sin 2t$. Hence the general solution is $y(t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t + t\, e^{-t} \sin 2t$. Invoking the initial conditions, we require that $c_1 = 1$ and $-c_1 + 2c_2 = 0$. Hence $c_1 = 1$ and $c_2 = 1/2$.

17.(a) The characteristic equation for the homogeneous problem is $r^2 - 5r + 6 = 0$, with roots $r = 2, 3$. Hence $y_c(t) = c_1 e^{2t} + c_2 e^{3t}$. Consider $g_1(t) = e^{2t}(3t + 4) \sin t$, and $g_2(t) = e^t \cos 2t$. Based on the form of these functions on the right hand side of the ODE, set $Y_2(t) = e^t(A_1 \cos 2t + A_2 \sin 2t)$ and $Y_1(t) = (B_1 + B_2 t)e^{2t} \sin t + (C_1 + C_2 t)e^{2t} \cos t$.

(b) Substitution into the equation and comparing the coefficients results in

$$Y(t) = -\frac{1}{20}(e^t \cos 2t + 3e^t \sin 2t) + \frac{3}{2} t e^{2t}(\cos t - \sin t) + e^{2t}(\frac{1}{2} \cos t - 5 \sin t).$$

19.(a) The homogeneous solution is $y_c(t) = c_1 \cos 2t + c_2 \sin 2t$. Since $\cos 2t$ and $\sin 2t$ are both solutions of the homogeneous equation, set

$$Y(t) = t(A_0 + A_1 t + A_2 t^2) \cos 2t + t(B_0 + B_1 t + B_2 t^2) \sin 2t\,.$$

(b) Substitution into the equation and comparing the coefficients results in

$$Y(t) = (\frac{13}{32}t - \frac{1}{12}t^3\,) \cos 2t + \frac{1}{16}(28t + 13t^2) \sin 2t\,.$$

20.(a) The homogeneous solution is $y_c(t) = c_1 e^{-t} + c_2 t e^{-2t}$. None of the functions on the right hand side are solutions of the homogenous equation. In order to include all possible combinations of the derivatives, consider

$$\begin{aligned} Y(t) = {} & e^t(A_0 + A_1 t + A_2 t^2) \cos 2t + e^t(B_0 + B_1 t + B_2 t^2) \sin 2t + \\ & + e^{-t}(C_1 \cos t + C_2 \sin t) + De^t. \end{aligned}$$

(b) Substitution into the differential equation and comparing the coefficients results in

$$\begin{aligned} Y(t) = {} & e^t(A_0 + A_1 t + A_2 t^2) \cos 2t + e^t(B_0 + B_1 t + B_2 t^2) \sin 2t \\ & + e^{-t}(-\frac{3}{2} \cos t + \frac{3}{2} \sin t) + 2e^t/3, \end{aligned}$$

in which $A_0 = -4105/35152$, $A_1 = 73/676$, $A_2 = -5/52$, $B_0 = -1233/35152$, $B_1 = 10/169$, $B_2 = 1/52$.

21.(a) The homogeneous solution is $y_c(t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t$. None of the terms on the right hand side are solutions of the homogenous equation. In order to include the appropriate combinations of derivatives, consider

$$\begin{aligned} Y(t) = {} & e^{-t}(A_1 t + A_2 t^2) \cos 2t + e^{-t}(B_1 t + B_2 t^2) \sin 2t + \\ & + e^{-2t}(C_0 + C_1 t) \cos 2t + e^{-2t}(D_0 + D_1 t) \sin 2t. \end{aligned}$$

(b) Substitution into the differential equation and comparing the coefficients results in

$$\begin{aligned} Y(t) = {} & \frac{3}{16} t e^{-t} \cos 2t + \frac{3}{8} t^2 e^{-t} \sin 2t \\ & - \frac{1}{25} e^{-2t}(7 + 10t) \cos 2t + \frac{1}{25} e^{-2t}(1 + 5t) \sin 2t\,. \end{aligned}$$

23. The homogeneous solution is $y_c(t) = c_1 \cos \lambda t + c_2 \sin \lambda t$. Since the differential operator does not contain a first derivative (and $\lambda \neq m\pi$), we can set

$$Y(t) = \sum_{m=1}^{N} C_m \sin m\pi t\,.$$

Substitution into the differential equation yields

$$-\sum_{m=1}^{N} m^2\pi^2 C_m \sin m\pi t + \lambda^2 \sum_{m=1}^{N} C_m \sin m\pi t = \sum_{m=1}^{N} a_m \sin m\pi t\,.$$

Equating coefficients of the individual terms, we obtain

$$C_m = \frac{a_m}{\lambda^2 - m^2\pi^2}\,, \quad m = 1, 2 \ldots N.$$

25. Since $a\,, b, c > 0$, the roots of the characteristic equation have negative real parts. That is, $r = \alpha \pm \beta\, i$, where $\alpha < 0$. Hence the homogeneous solution is

$$y_c(t) = c_1 e^{\alpha t} \cos\, \beta t + c_2 e^{\alpha t} \sin\, \beta t\,.$$

If $g(t) = d$, then the general solution is

$$y(t) = d/c + c_1 e^{\alpha t} \cos\, \beta t + c_2 e^{\alpha t} \sin\, \beta t\,.$$

Since $\alpha < 0$, $y(t) \to d/c$ as $t \to \infty$. If $c = 0$, then the characteristic roots are $r = 0$ and $r = -b/a$. The ODE becomes $ay'' + by' = d$. Integrating both sides, we find that $ay' + by = d\,t + c_1$. The general solution can be expressed as

$$y(t) = d\,t/b + c_1 + c_2 e^{-bt/a}.$$

In this case, the solution grows without bound. If $b = 0$, also, then the differential equation can be written as $y'' = d/a$, which has general solution $y(t) = d\,t^2/2a + c_1 + c_2$. Hence the assertion is true only if the coefficients are positive.

27.(a) Since D is a linear operator, $D^2y + bDy + cy = D^2y - (r_1 + r_2)Dy + r_1 r_2 y = D^2y - r_2 Dy - r_1 Dy + r_1 r_2 y = D(Dy - r_2 y) - r_1(Dy - r_2 y) = (D - r_1)(D - r_2)y.$

(b) Let $u = (D - r_2)y$. Then the ODE (i) can be written as $(D - r_1)u = g(t)$, that is, $u' - r_1 u = g(t)$. The latter is a linear first order equation in u. Its general solution is

$$u(t) = e^{r_1 t} \int_{t_0}^{t} e^{-r_1 \tau} g(\tau) d\tau + c_1 e^{r_1 t}\,.$$

From above, we have $y' - r_2 y = u(t)$. This equation is also a first order ODE. Hence the general solution of the original second order equation is

$$y(t) = e^{r_2 t} \int_{t_0}^{t} e^{-r_2 \tau} u(\tau) d\tau + c_2 e^{r_2 t}\,.$$

Note that the solution $y(t)$ contains two arbitrary constants.

29. We have $(D^2 + 2D + 1)y = (D + 1)(D + 1)y$. Let $u = (D + 1)y$, and consider the ODE $u' + u = 2e^{-t}$. The general solution is $u(t) = 2t\,e^{-t} + c\,e^{-t}$. We therefore have the first order equation $u' + u = 2t\,e^{-t} + c_1 e^{-t}$. The general solution of the latter differential equation is

$$y(t) = e^{-t} \int_{t_0}^{t} [2\tau + c_1\,]\, d\tau + c_2 e^{-t} = e^{-t}(t^2 + c_1 t + c_2).$$

30. We have $(D^2+2D)y = D(D+2)y$. Let $u=(D+2)y$, and consider the equation $u' = 3+4\sin 2t$. Direct integration results in $u(t) = 3t - 2\cos 2t + c$. The problem is reduced to solving the ODE $y' + 2y = 3t - 2\cos 2t + c$. The general solution of this first order differential equation is

$$y(t) = e^{-2t}\int_{t_0}^{t} e^{2\tau}\left[3\tau - 2\cos 2\tau + c\right]d\tau + c_2e^{-2t} =$$

$$= \frac{3}{2}t - \frac{1}{2}(\cos 2t + \sin 2t) + c_1 + c_2e^{-2t}.$$

3.6

1. The solution of the homogeneous equation is $y_c(t) = c_1e^{2t} + c_2e^{3t}$. The functions $y_1(t) = e^{2t}$ and $y_2(t) = e^{3t}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^{5t}$. Using the method of variation of parameters, the particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$u_1(t) = -\int \frac{e^{3t}(2e^t)}{W(t)}dt = 2e^{-t} \quad \text{and} \quad u_2(t) = \int \frac{e^{2t}(2e^t)}{W(t)}dt = -e^{-2t}.$$

Hence the particular solution is $Y(t) = 2e^t - e^t = e^t$.

3. The functions $y_1(t) = e^{t/2}$ and $y_2(t) = te^{t/2}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^t$. First write the equation in standard form, so that $g(t) = 4e^{t/2}$. Using the method of variation of parameters, the particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$u_1(t) = -\int \frac{te^{t/2}(4e^{t/2})}{W(t)}dt = -2t^2 \quad \text{and} \quad u_2(t) = \int \frac{e^{t/2}(4e^{t/2})}{W(t)}dt = 4t.$$

Hence the particular solution is $Y(t) = -2t^2e^{t/2} + 4t^2e^{t/2} = 2t^2e^{t/2}$.

5. The solution of the homogeneous equation is $y_c(t) = c_1\cos 3t + c_2\sin 3t$. The two functions $y_1(t) = \cos 3t$ and $y_2(t) = \sin 3t$ form a fundamental set of solutions, with $W(y_1, y_2) = 3$. The particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$u_1(t) = -\int \frac{\sin 3t(9\sec^2 3t)}{W(t)}dt = -\csc 3t$$

$$u_2(t) = \int \frac{\cos 3t(9\sec^2 3t)}{W(t)}dt = \ln(\sec 3t + \tan 3t),$$

since $0 < t < \pi/6$. Hence $Y(t) = -1 + (\sin 3t)\ln(\sec 3t + \tan 3t)$. The general solution is given by

$$y(t) = c_1\cos 3t + c_2\sin 3t + (\sin 3t)\ln(\sec 3t + \tan 3t) - 1\,.$$

6. The functions $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^{-4t}$. The particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\int \frac{te^{-2t}(t^{-2}e^{-2t})}{W(t)}dt = -\ln t \quad \text{and} \quad u_2(t) = \int \frac{e^{-2t}(t^{-2}e^{-2t})}{W(t)}dt = -1/t.$$

Hence the particular solution is $Y(t) = -e^{-2t}\ln t - e^{-2t}$. Since the second term is a solution of the homogeneous equation, the general solution is given by

$$y(t) = c_1 e^{-2t} + c_2 te^{-2t} - e^{-2t}\ln t.$$

7. The functions $y_1(t) = \cos(t/2)$ and $y_2(t) = \sin(t/2)$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = 1/2$. First write the ODE in standard form, so that $g(t) = \sec(t/2)/2$. The particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\int \frac{\cos(t/2)\,[\sec(t/2)]}{2W(t)}dt = 2\,\ln(\cos(t/2))$$

$$u_2(t) = \int \frac{\sin(t/2)\,[\sec(t/2)]}{2W(t)}dt = t.$$

The particular solution is $Y(t) = 2\cos(t/2)\ln(\cos(t/2)) + t\,\sin(t/2)$. The general solution is given by

$$y(t) = c_1\cos(t/2) + c_2\sin(t/2) + 2\,\cos(t/2)\,\ln(\cos(t/2)) + t\,\sin(t/2).$$

8. The solution of the homogeneous equation is $y_c(t) = c_1 e^t + c_2 te^t$. The functions $y_1(t) = e^t$ and $y_2(t) = te^t$ form a fundamental set of solutions, with $W(y_1, y_2) = e^{2t}$. The particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\int \frac{te^t(e^t)}{W(t)(1+t^2)}dt = -\frac{1}{2}\ln(1+t^2)$$

$$u_2(t) = \int \frac{e^t(e^t)}{W(t)(1+t^2)}dt = \arctan t.$$

The particular solution is $Y(t) = -(1/2)e^t\,\ln(1+t^2) + te^t\,\arctan(t)$. Hence the general solution is given by

$$y(t) = c_1 e^t + c_2 te^t - \frac{1}{2}e^t\,\ln(1+t^2) + te^t\,\arctan(t).$$

10. Note first that $p(t) = 0\,, q(t) = -2/t^2$ and $g(t) = (3t^2 - 1)/t^2$. The functions $y_1(t)$ and $y_2(t)$ are solutions of the homogeneous equation, verified by substitution. The Wronskian of these two functions is $W(y_1, y_2) = -3$. Using the method of variation of parameters, the particular solution is $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\int \frac{t^{-1}(3t^2-1)}{t^2\,W(t)}dt = t^{-2}/6 + \ln t$$

$$u_2(t) = \int \frac{t^2(3t^2-1)}{t^2\,W(t)}dt = -t^3/3 + t/3.$$

Therefore $Y(t) = 1/6\, + t^2 \ln\, t - t^2/3 + 1/3\,.$

12. Observe that $g(t) = t\, e^{2t}$. The functions $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions. The Wronskian of these two functions is $W(y_1{,}y_2) = t\, e^t$. Using the method of variation of parameters, the particular solution is $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\int \frac{e^t(t\, e^{2t})}{W(t)}dt = -e^{2t}/2 \quad \text{and} \quad u_2(t) = \int \frac{(1+t)(t\, e^{2t})}{W(t)}dt = t\, e^t.$$

Therefore $Y(t) = -(1+t)e^{2t}/2 + t\, e^{2t} = -e^{2t}/2 + t\, e^{2t}/2\,.$

13. Note that $g(x) = \ln\, x\,$. The functions $y_1(x) = x^2$ and $y_2(x) = x^2 \ln\, x$ are solutions of the homogeneous equation, as verified by substitution. The Wronskian of the solutions is $W(y_1{,}y_2) = x^3$. Using the method of variation of parameters, the particular solution is $Y(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$, in which

$$u_1(x) = -\int \frac{x^2 \ln\, x(\ln\, x)}{W(x)}dx = -(\ln\, x)^3/3$$

$$u_2(x) = \int \frac{x^2(\ln\, x)}{W(x)}dx = (\ln\, x)^2/2.$$

Therefore $Y(x) = -x^2(\ln\, x)^3/3 + x^2(\ln\, x)^3/2 = x^2(\ln\, x)^3/6\,.$

15. First write the equation in standard form. The forcing function becomes $g(x)/x^2\,$. The functions $y_1(x) = x^{-1/2} \sin\, x$ and $y_2(x) = x^{-1/2} \cos\, x$ are a fundamental set of solutions. The Wronskian of the solutions is $W(y_1{,}y_2) = -1/x\,$. Using the method of variation of parameters, the particular solution is $Y(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$, in which

$$u_1(x) = \int_{x_0}^{x} \frac{\cos\, \tau\, (g(\tau))}{\tau\sqrt{\tau}}d\tau \quad \text{and} \quad u_2(x) = -\int_{x_0}^{x} \frac{\sin\, \tau\, (g(\tau))}{\tau\sqrt{\tau}}d\tau.$$

Therefore

$$Y(x) = \frac{\sin\, x}{\sqrt{x}} \int_{x_0}^{x} \frac{\cos\, \tau\, (g(\tau))}{\tau\sqrt{\tau}}dt - \frac{\cos\, x}{\sqrt{x}} \int_{x_0}^{x} \frac{\sin\, \tau\, (g(\tau))}{\tau\sqrt{\tau}}d\tau =$$

$$= \frac{1}{\sqrt{x}} \int_{x_0}^{x} \frac{\sin(x-\tau)\, g(\tau)}{\tau\sqrt{\tau}}d\tau\,.$$

16. Eq.(28) is

$$Y(t) = -y_1(t) \int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1,y_2)(s)}\, ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1,y_2)(s)}\, ds,$$

where t_0 is now considered the initial point. Bringing the terms $y_1(t)$ and $y_2(t)$ inside the integrals and using the fact that $W(y_1,y_2)(s) = y_1(s)y_2'(s) - y_1'(s)y_2(s)$, the desired result holds. To show that $Y(t)$ satisfies $L[y] = g(t)$ we must take the derivative using Leibniz's rule, which says that if $y(t) = \int_{t_0}^{t} G(t,s)\, ds$, then

$Y'(t) = G(t,t) + \int_{t_0}^{t} G_t(t,s)\,ds$. Letting $G(t,s)$ be the above integrand, we have that $G(t,t) = 0$ and

$$\frac{\partial G}{\partial t} = \frac{y_1(s)y_2'(t) - y_1'(t)y_2(s)}{W(y_1, y_2)(s)} g(s).$$

Likewise,

$$Y'' = \frac{\partial G(t,t)}{\partial t} + \int_{t_0}^{t} \frac{\partial^2 G}{\partial t^2}(t,s)\,ds = g(t) + \int_{t_0}^{t} \frac{y_1(s)y_2''(t) - y_1''(t)y_2(s)}{W(y_1,y_2)(s)} g(s)\,ds.$$

Since y_1 and y_2 are solutions of $L[y] = 0$, we have $L[Y] = g(t)$ since all the terms involving the integral will add to zero. Clearly $y(t_0) = 0$ and $y'(t_0) = 0$.

17. Let $y_1(t)$ and $y_2(t)$ be a fundamental set of solutions, and $W(t) = W(y_1, y_2)$ be the corresponding Wronskian. Any solution, $u(t)$, of the homogeneous equation is a linear combination $u(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$. Invoking the initial conditions, we require that

$$\begin{aligned} y_0 &= \alpha_1\, y_1(t_0) + \alpha_2\, y_2(t_0) \\ y_0' &= \alpha_1\, y_1'(t_0) + \alpha_2\, y_2'(t_0) \end{aligned}$$

Note that this system of equations has a unique solution, since $W(t_0) \neq 0$. Now consider the nonhomogeneous problem, $L\,[v] = g(t)$, with homogeneous initial conditions. Using the method of variation of parameters, the particular solution is given by

$$Y(t) = -y_1(t) \int_{t_0}^{t} \frac{y_2(s)\, g(s)}{W(s)} ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s)\, g(s)}{W(s)} ds\,.$$

The general solution of the IVP (iii) is

$$v(t) = \beta_1 y_1(t) + \beta_2 y_2(t) + Y(t) = \beta_1 y_1(t) + \beta_2 y_2(t) + y_1(t)u_1(t) + y_2(t)u_2(t)$$

in which u_1 and u_2 are defined above. Invoking the initial conditions, we require that

$$\begin{aligned} 0 &= \beta_1 y_1(t_0) + \beta_2 y_2(t_0) + Y(t_0) \\ 0 &= \beta_1 y_1'(t_0) + \beta_2 y_2'(t_0) + Y'(t_0) \end{aligned}$$

Based on the definition of u_1 and u_2, $Y(t_0) = 0$. Furthermore, since $y_1 u_1' + y_2 u_2' = 0$, it follows that $Y'(t_0) = 0$. Hence the only solution of the above system of equations is the trivial solution. Therefore $v(t) = Y(t)$. Now consider the function $y = u + v$. Then $L\,[y] = L\,[u + v] = L\,[u] + L\,[v] = g(t)$. That is, $y(t)$ is a solution of the nonhomogeneous problem. Further, $y(t_0) = u(t_0) + v(t_0) = y_0$, and similarly, $y'(t_0) = y_0'$. By the uniqueness theorems, $y(t)$ is the unique solution of the initial value problem.

18.(a) A fundamental set of solutions is $y_1(t) = \cos t$ and $y_2(t) = \sin t$. The Wronskian $W(t) = y_1 y_2' - y_1' y_2 = 1$. By the result in Problem 17,

$$\begin{aligned} Y(t) &= \int_{t_0}^{t} \frac{\cos(s)\, \sin(t) - \cos(t)\, \sin(s)}{W(s)} g(s) ds \\ &= \int_{t_0}^{t} \left[\cos(s)\, \sin(t) - \cos(t)\, \sin(s)\right] g(s) ds\,. \end{aligned}$$

Finally, we have $\cos(s)\,\sin(t) - \cos(t)\,\sin(s) = \sin(t-s)$.

(b) Using Problem 16 and part (a), the solution is

$$y(t) = y_0 \cos t + y_0' \sin t + \int_0^t \sin(t-s)g(s)ds\,.$$

19. A fundamental set of solutions is $y_1(t) = e^{at}$ and $y_2(t) = e^{bt}$. The Wronskian $W(t) = y_1 y_2' - y_1' y_2 = (b-a)e^{(a+b)t}$. By the result in Problem 17,

$$Y(t) = \int_{t_0}^t \frac{e^{as}e^{bt} - e^{at}e^{bs}}{W(s)} g(s)ds = \frac{1}{b-a}\int_{t_0}^t \frac{e^{as}e^{bt} - e^{at}e^{bs}}{e^{(a+b)s}} g(s)ds\,.$$

Hence the particular solution is

$$Y(t) = \frac{1}{b-a}\int_{t_0}^t \left[e^{b(t-s)} - e^{a(t-s)}\right] g(s)ds\,.$$

21. A fundamental set of solutions is $y_1(t) = e^{at}$ and $y_2(t) = te^{at}$. The Wronskian $W(t) = y_1 y_2' - y_1' y_2 = e^{2at}$. By the result in Problem 17,

$$Y(t) = \int_{t_0}^t \frac{te^{as+at} - s\,e^{at+as}}{W(s)} g(s)ds = \int_{t_0}^t \frac{(t-s)e^{as+at}}{e^{2as}} g(s)ds\,.$$

Hence the particular solution is

$$Y(t) = \int_{t_0}^t (t-s)e^{a(t-s)} g(s)ds\,.$$

22. The form of the kernel depends on the characteristic roots. If the roots are real and distinct,

$$K(t-s) = \frac{e^{b(t-s)} - e^{a(t-s)}}{b-a}\,.$$

If the roots are real and identical,

$$K(t-s) = (t-s)e^{a(t-s)}\,.$$

If the roots are complex conjugates,

$$K(t-s) = \frac{e^{\lambda(t-s)}\ \sin\mu(t-s)}{\mu}\,.$$

23. Let $y(t) = v(t)y_1(t)$, in which $y_1(t)$ is a solution of the homogeneous equation. Substitution into the given ODE results in

$$v''y_1 + 2v'y_1' + vy_1'' + p(t)\left[v'y_1 + vy_1'\right] + q(t)vy_1 = g(t)\,.$$

By assumption, $y_1'' + p(t)y_1 + q(t)y_1 = 0$, hence $v(t)$ must be a solution of the ODE

$$v''y_1 + \left[2y_1' + p(t)y_1\right]v' = g(t)\,.$$

Setting $w = v'$, we also have $w'y_1 + \left[2y_1' + p(t)y_1\right]w = g(t)$.

25. First write the equation as $y'' + 7t^{-1}y + 5t^{-2}\,y = t^{-1}$. As shown in Problem 23, the function $y(t) = t^{-1}v(t)$ is a solution of the given ODE as long as v is a solution of

$$t^{-1}\,v'' + \left[-2t^{-2} + 7t^{-2}\right]v' = t^{-1}\,,$$

that is, $v'' + 5t^{-1}\,v' = 1$. This ODE is linear and first order in v'. The integrating factor is $\mu = t^5$. The solution is $v' = t/6 + c\,t^{-5}$. Direct integration now results in $v(t) = t^2/12 + c_1t^{-4} + c_2$. Hence $y(t) = t/12 + c_1t^{-5} + c_2t^{-1}$.

26. Write the equation as $y'' - t^{-1}(1+t)y + t^{-1}\,y = t\,e^{2t}$. As shown in Problem 23, the function $y(t) = (1+t)v(t)$ is a solution of the given ODE as long as v is a solution of

$$(1+t)\,v'' + \left[2 - t^{-1}(1+t)^2\right]v' = t\,e^{2t},$$

that is,

$$v'' - \frac{1+t^2}{t(t+1)}\,v' = \frac{t}{t+1}e^{2t}.$$

This equation is first order linear in v', with integrating factor $\mu = t^{-1}(1+t)^2e^{-t}$. The solution is $v' = (t^2e^{2t} + c_1te^t)/(1+t)^2$. Integrating, we obtain $v(t) = e^{2t}/2 - e^{2t}/(t+1) + c_1e^t/(t+1) + c_2$. Hence the solution of the original ODE is $y(t) = (t-1)e^{2t}/2 + c_1e^t + c_2(t+1)$.

3.7

1. $R\cos\delta = 3$ and $R\sin\delta = 4$, so $R = \sqrt{25} = 5$ and $\delta = \arctan(4/3)$. We obtain that $u = 5\,\cos(2t - \arctan(4/3))$.

2. $R\cos\delta = -2$ and $R\sin\delta = -3$, so $R = \sqrt{13}$ and $\delta = \pi + \arctan(3/2)$. We obtain that $u = \sqrt{13}\,\cos(\pi t - \pi - \arctan(3/2))$.

4. The spring constant is $k = 3/(1/4) = 12$ lb/ft. Mass $m = 3/32$ lb-s^2/ft. Since there is no damping, the equation of motion is $3u''/32 + 12u = 0$, that is, $u'' + 128u = 0$. The initial conditions are $u(0) = -1/12$ ft, $u'(0) = 2$ ft/s. The general solution is $u(t) = A\,\cos 8\sqrt{2}\,t + B\,\sin 8\sqrt{2}\,t$. Invoking the initial conditions, we have

$$u(t) = -\frac{1}{12}\,\cos 8\sqrt{2}\,t + \frac{1}{4\sqrt{2}}\,\sin 8\sqrt{2}\,t\,.$$

$R = \sqrt{11/288}$ ft, $\delta = \pi - \arctan(3/\sqrt{2})$ rad, $\omega_0 = 8\sqrt{2}$ rad/s, $T = \pi/(4\sqrt{2})$ s.

6. The spring constant is $k = 3/(.1) = 30$ N/m. The damping coefficient is given as $\gamma = 3/5$ N-s/m. Hence the equation of motion is $2u'' + 3u'/5 + 30u = 0$, that is, $u'' + 0.3u' + 15u = 0$. The initial conditions are $u(0) = 0.05$ m and $u'(0) = 0.01$ m/s. The general solution is $u(t) = A\,\cos\mu t + B\,\sin\mu t$, in which $\mu = 3.87008$ rad/s. Invoking the initial conditions, we have $u(t) = e^{-0.15t}(0.05\cos\mu t + 0.00452\sin\mu t)$. Also, $\mu/\omega_0 = 3.87008/\sqrt{15} \approx 0.99925$.

8. The frequency of the undamped motion is $\omega_0 = 1$. The quasi frequency of the damped motion is $\mu = \sqrt{4-\gamma^2}\,/2$. Setting $\mu = 2\omega_0\,/3$, we obtain $\gamma = 2\sqrt{5}\,/3$.

9. The spring constant is $k = mg/L$. The equation of motion for an undamped system is $mu'' + mgu/L = 0$. Hence the natural frequency of the system is $\omega_0 = \sqrt{g/L}$. The period is $T = 2\pi/\omega_0$.

10. The general solution of the system is $u(t) = A\,\cos\gamma(t-t_0) + B\,\sin\gamma(t-t_0)$. Invoking the initial conditions, we have $u(t) = u_0\cos\gamma(t-t_0) + (u_0'/\gamma)\sin\gamma(t-t_0)$. Clearly, the functions $v = u_0\cos\gamma(t-t_0)$ and $w = (u_0'/\gamma)\sin\gamma(t-t_0)$ satisfy the given criteria.

11. Note that $r\,\sin(\omega_0 t - \theta) = r\,\sin\omega_0 t\,\cos\theta - r\,\cos\omega_0 t\,\sin\theta$. Comparing the given expressions, we have $A = -r\,\sin\theta$ and $B = r\,\cos\theta$. That is, $r = R = \sqrt{A^2+B^2}$, and $\tan\theta = -A/B = -1/\tan\delta$. The latter relation is also $\tan\theta + \cot\delta = 1$.

12. The system is critically damped, when $R = 2\sqrt{L/C}$. Here $R = 1000$ ohms.

15.(a) Let $u = Re^{-\gamma t/2m}\cos(\mu t - \delta)$. Then attains a maximum when $\mu t_k - \delta = 2k\pi$. Hence $T_d = t_{k+1} - t_k = 2\pi/\mu$.

(b) $u(t_k)/u(t_{k+1}) = e^{-\gamma t_k/2m}/e^{-\gamma t_{k+1}/2m} = e^{(\gamma t_{k+1}-\gamma t_k)/2m}$. Hence $u(t_k)/u(t_{k+1}) = e^{\gamma(2\pi/\mu)/2m} = e^{\gamma\,T_d/2m}$.

(c) $\Delta = \ln\left[u(t_k)/u(t_{k+1})\right] = \gamma(2\pi/\mu)/2m = \pi\gamma/\mu m$.

16. The spring constant is $k = 16/(1/4) = 64$ lb/ft. Mass $m = 1/2$ lb-s^2/ft. The damping coefficient is $\gamma = 2$ lb-s/ft. The quasi frequency is $\mu = 2\sqrt{31}$ rad/s. Hence $\Delta = 2\pi/\sqrt{31} \approx 1.1285$.

18.(a) The characteristic equation is $mr^2 + \gamma r + k = 0$. Since $\gamma^2 < 4km$, the roots are $r_{1,2} = (-\gamma \pm i\sqrt{4mk-\gamma^2})/2m$. The general solution is

$$u(t) = e^{-\gamma t/2m}\left[A\,\cos\frac{\sqrt{4mk-\gamma^2}}{2m}t + B\,\sin\frac{\sqrt{4mk-\gamma^2}}{2m}t\right].$$

Invoking the initial conditions, $A = u_0$ and $B = (2mv_0 - \gamma u_0)/\sqrt{4mk-\gamma^2}$.

(b) We can write $u(t) = R\,e^{-\gamma t/2m}\cos(\mu t - \delta)$, in which

$$R = \sqrt{u_0^2 + \frac{(2mv_0-\gamma u_0)^2}{4mk-\gamma^2}} \quad \text{and} \quad \delta = \arctan\left[\frac{(2mv_0-\gamma u_0)}{u_0\sqrt{4mk-\gamma^2}}\right].$$

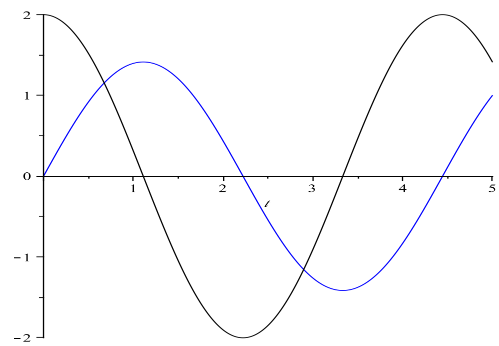
(c)

$$R = \sqrt{u_0^2 + \frac{(2mv_0-\gamma u_0)^2}{4mk-\gamma^2}} = 2\sqrt{\frac{m(ku_0^2+\gamma u_0 v_0 + mv_0^2)}{4mk-\gamma^2}} = \sqrt{\frac{a+b\gamma}{4mk-\gamma^2}}.$$

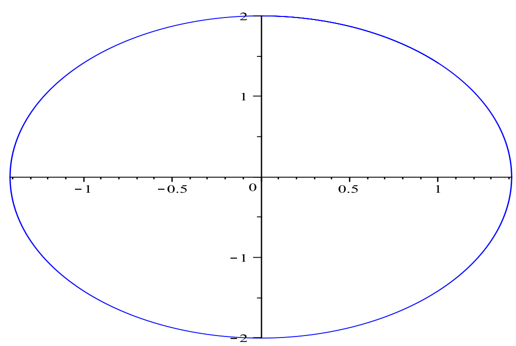
It is evident that R increases (monotonically) without bound as $\gamma \to (2\sqrt{mk})^-$.

20.(a) The general solution is $u(t) = A\cos\sqrt{2}\,t + B\sin\sqrt{2}\,t$. Invoking the initial conditions, we have $u(t) = \sqrt{2}\,\sin\sqrt{2}\,t$.

(b)



(c)



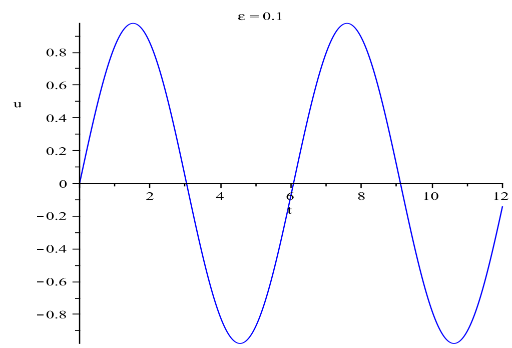
The condition $u'(0) = 2$ implies that $u(t)$ initially increases. Hence the phase point travels clockwise.

23. Based on Newton's second law, with the positive direction to the right, $\sum F = mu''$, where $\sum F = -ku - \gamma u'$. Hence the equation of motion is $mu'' + \gamma u' + ku = 0$. The only difference in this problem is that the equilibrium position is located at the unstretched configuration of the spring.

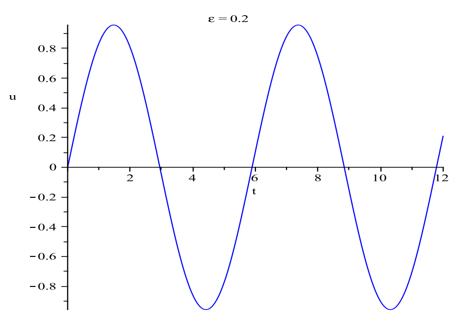
24.(a) The restoring force exerted by the spring is $F_s = -(ku + \epsilon u^3)$. The opposing viscous force is $F_d = -\gamma u'$. Based on Newton's second law, with the positive direction to the right, $F_s + F_d = mu''$. Hence the equation of motion is $mu'' + \gamma u' + ku + \epsilon u^3 = 0$.

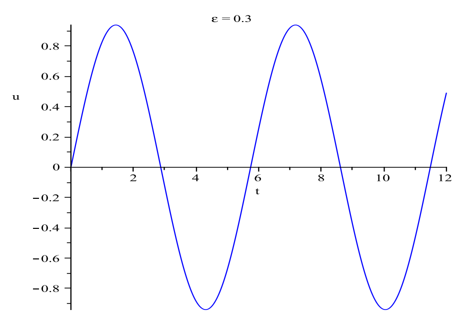
(b) With the specified parameter values, the equation of motion is $u'' + u = 0$. The general solution of this ODE is $u(t) = A\cos t + B\sin t$. Invoking the initial conditions, the specific solution is $u(t) = \sin t$. Clearly, the amplitude is $R = 1$, and the period of the motion is $T = 2\pi$.

(c) Given $\epsilon = 0.1$, the equation of motion is $u'' + u + 0.1\,u^3 = 0$. A solution of the IVP can be generated numerically. We estimate $A = 0.98$ and $T = 6.07$.



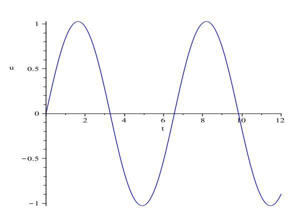
(d) For $\epsilon = 0.2$, $A = 0.96$ and $T = 5.90$. For $\epsilon = 0.3$, $A = 0.94$ and $T = 5.74$.



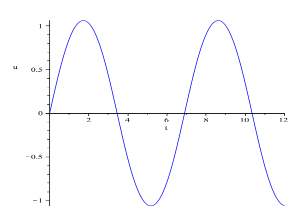


(e) The amplitude and period both seem to decrease.

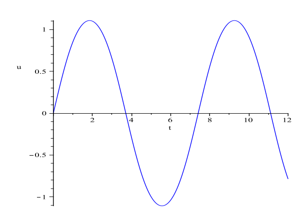
(f) For $\epsilon = -0.1$, $A = 1.03$ and $T = 6.55$. For $\epsilon = -0.2$, $A = 1.06$ and $T = 6.90$. For $\epsilon = -0.3$, $A = 1.11$ and $T = 7.41$. The amplitude and period both seem to increase.



(a) $\varepsilon = -0.1$



(b) $\varepsilon = -0.2$



(c) $\varepsilon = -0.3$

3.8

1. We have $\sin(\alpha \pm \beta) = \sin\,\alpha\,\cos\,\beta \pm \cos\,\alpha\,\sin\,\beta$. Subtracting the two identities, we obtain $\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\,\cos\,\alpha\,\sin\,\beta$. Setting $\alpha+\beta = 7t$ and $\alpha - \beta = 6t$, we get that $\alpha = 6.5t$ and $\beta = 0.5t$. This implies that $\sin\,7t - \sin\,6t = 2\,\sin\,(t/2)\cos\,(13t/2)$.

2. Consider the trigonometric identities $\cos(\alpha \pm \beta) = \cos\,\alpha\,\cos\,\beta \mp \sin\,\alpha\,\sin\,\beta$. Adding the two identities, we get $\cos(\alpha-\beta) + \cos(\alpha+\beta) = 2\,\cos\,\alpha\,\cos\,\beta$. Comparing the expressions, set $\alpha+\beta = 2\pi t$ and $\alpha - \beta = \pi t$. This means $\alpha = 3\pi t/2$ and $\beta = \pi t/2$. Upon substitution, we have $\cos(\pi t) + \cos(2\pi t) = 2\,\cos(3\pi t/2)\,\cos(\pi t/2)$.

3. Adding the two identities $\sin(\alpha \pm \beta) = \sin\,\alpha\,\cos\,\beta \pm \cos\,\alpha\,\sin\,\beta$, it follows that $\sin(\alpha-\beta) + \sin(\alpha+\beta) = 2\sin\,\alpha\,\cos\,\beta$. Setting $\alpha+\beta = 4t$ and $\alpha-\beta = 3t$, we have $\alpha = 7t/2$ and $\beta = t/2$. Hence $\sin\,3t + \sin\,4t = 2\,\sin(7t/2)\,\cos(t/2)$.

4. Using MKS units, the spring constant is $k = 5(9.8)/0.1 = 490$ N/m, and the damping coefficient is $\gamma = 2/0.04 = 50$ N-s/m. The equation of motion is

$$5u'' + 50u' + 490u = 10\,\sin(t/2)\,.$$

The initial conditions are $u(0) = 0$ m and $u'(0) = 0.03$ m/s.

5.(a) The homogeneous solution is $u_c(t) = Ae^{-5t}\cos\sqrt{73}\,t + Be^{-5t}\sin\sqrt{73}\,t$. Based on the method of undetermined coefficients, the particular solution is

$$U(t) = \frac{1}{153281}\left[-160\,\cos(t/2) + 3128\,\sin(t/2)\right].$$

Hence the general solution of the ODE is $u(t) = u_c(t) + U(t)$. Invoking the initial conditions, we find that

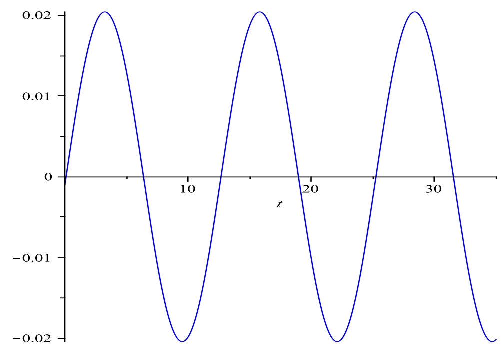
$$A = 160/153281 \text{ and } B = 383443\sqrt{73}\,/1118951300\,.$$

Hence the response is

$$u(t) = \frac{1}{153281}\left[160\,e^{-5t}\cos\,\sqrt{73}\,t + \frac{383443\sqrt{73}}{7300}e^{-5t}\sin\,\sqrt{73}\,t\right] + U(t).$$

(b) $u_c(t)$ is the transient part and $U(t)$ is the steady state part of the response.

(c)



(d) The amplitude of the forced response is given by $R = 2/\Delta$, in which

$$\Delta = \sqrt{25(98-\omega^2)^2 + 2500\,\omega^2}\,.$$

The maximum amplitude is attained when Δ is a minimum. Hence the amplitude is maximum at $\omega = 4\sqrt{3}$ rad/s.

8. The equation of motion is $2u'' + u' + 3u = 3\cos 3t - 2\sin 3t$. Since the system is damped, the steady state response is equal to the particular solution. Using the method of undetermined coefficients, we obtain $u_{ss}(t) = (\sin 3t - \cos 3t)/6$. Further, we find that $R = \sqrt{2}/6$ and $\delta = \arctan(-1) = 3\pi/4$. Hence we can write $u_{ss}(t) = (\sqrt{2}/6)\cos(3t - 3\pi/4)$.

9.(a) Plug in $u(t) = R\cos(\omega t - \delta)$ into the equation $mu'' + \gamma u' + ku = F_0\cos\omega t$, then use trigonometric identities and compare the coefficients of $\cos\omega t$ and $\sin\omega t$. The result follows.

(b) First note that since $R = F_0/\Delta$, $Rk/F_0 = k/\Delta$ and that since $\Gamma = \gamma^2/(mk)$, $(\gamma^2\omega^2)/m^2 = \Gamma\omega_0^2\omega^2$. Then using Eq.12,

$$\begin{aligned}\frac{Rk}{F_0} = \frac{k}{\Delta} &= \frac{m\omega_0^2}{\sqrt{m^2(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}} = \frac{m\omega_0^2}{\sqrt{m^2(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}} \\ &= \frac{\omega_0^2}{\sqrt{(\omega_0^2-\omega^2)^2+\frac{\gamma^2\omega^2}{m^2}}} = \frac{\omega_0^2}{\sqrt{(\omega_0^2-\omega^2)^2+\Gamma\omega_0^2\omega^2}} \\ &= \frac{1}{\sqrt{\left(\frac{\omega_0^2-\omega^2}{\omega_0^2}\right)^2+\Gamma\frac{\omega_0^2\omega^2}{\omega_0^4}}} = \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+\Gamma\frac{\omega^2}{\omega_0^2}}}\end{aligned}$$

(c) The amplitude of the steady-state response is given by

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2+\gamma^2\,\omega^2}}\,.$$

Since F_0 is constant, the amplitude is maximum when the denominator of R is minimum. Let $z = \omega^2$, and consider the function $f(z) = m^2(\omega_0^2 - z)^2 + \gamma^2 z$. Note that $f(z)$ is a quadratic, with minimum at $z = \omega_0^2 - \gamma^2/2m^2$. Hence the amplitude R attains a maximum at $\omega_{max}^2 = \omega_0^2 - \gamma^2/2m^2$. Furthermore, since $\omega_0^2 = k/m$,

$$\omega_{max}^2 = \omega_0^2 \left[1 - \frac{\gamma^2}{2km}\right].$$

(d) Substituting $\omega^2 = \omega_{max}^2$ into the expression for the amplitude R gives the maximum value for R:

$$R_{max} = \frac{F_0}{\sqrt{\gamma^4/4m^2 + \gamma^2\left(\omega_0^2 - \gamma^2/2m^2\right)}} = \frac{F_0}{\sqrt{\omega_0^2\gamma^2 - \gamma^4/4m^2}} = \frac{F_0}{\gamma\omega_0\sqrt{1 - \gamma^2/4mk}}.$$

To understand the approximation, note that

$$R_{max} = \frac{F_0}{\gamma\omega_0}\left(1 - \frac{\gamma^2}{4mk}\right)^{-1/2}$$

Recall that binomial theorem states that $(1+a)^p \approx 1 + pa$ when a is small. Applying this result with $a = -\gamma^2/(4mk)$ and $p = -1/2$ gives that

$$R_{max} = \frac{F_0}{\gamma\omega_0}\left(1 - \frac{\gamma^2}{4mk}\right)^{-1/2} \approx \frac{F_0}{\gamma\omega_0}\left(1 + \left(-\frac{1}{2}\right)\left(-\frac{\gamma^2}{4mk}\right)\right) = \frac{F_0}{\gamma\omega_0}\left(1 + \frac{\gamma^2}{8mk}\right)$$

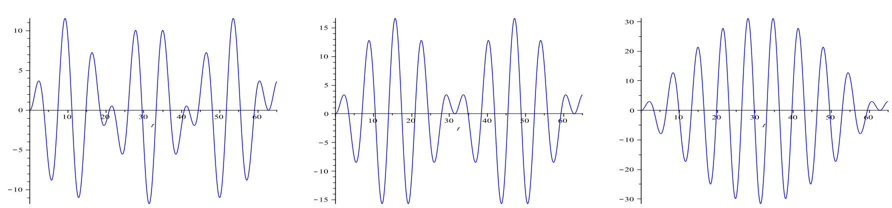
13.(a) The homogeneous solution is $u_c(t) = A\cos t + B\sin t$. Based on the method of undetermined coefficients, the particular solution is

$$U(t) = \frac{3}{1-\omega^2}\cos\omega t.$$

Hence the general solution of the ODE is $u(t) = u_c(t) + U(t)$. Invoking the initial conditions, we find that $A = 3/(\omega^2 - 1)$ and $B = 0$. Hence the response is

$$u(t) = \frac{3}{1-\omega^2}\left[\cos\omega t - \cos t\right].$$

(b)



(a) $\omega = 0.7$ (b) $\omega = 0.8$ (c) $\omega = 0.9$

Note that

$$u(t) = \frac{6}{1-\omega^2}\sin\left[\frac{(1-\omega)t}{2}\right]\sin\left[\frac{(\omega+1)t}{2}\right].$$

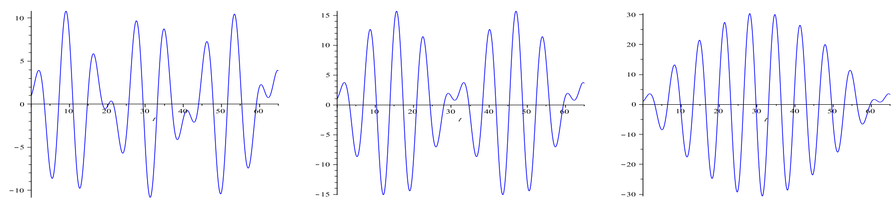
14.(a) The homogeneous solution is $u_c(t) = A\cos t + B\sin t$. Based on the method of undetermined coefficients, the particular solution is

$$U(t) = \frac{3}{1-\omega^2}\cos\omega t\,.$$

Hence the general solution is $u(t) = u_c(t) + U(t)$. Invoking the initial conditions, we find that $A = (\omega^2+2)/(\omega^2-1)$ and $B = 1$. Hence the response is

$$u(t) = \frac{1}{1-\omega^2}\left[3\cos\omega t - (\omega^2+2)\cos t\right] + \sin t\,.$$

(b)

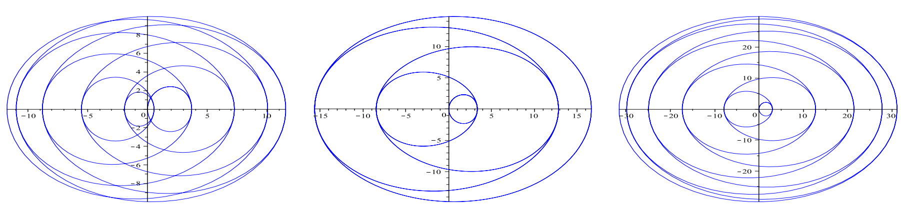


(a) $\omega = 0.7$ (b) $\omega = 0.8$ (c) $\omega = 0.9$

Note that

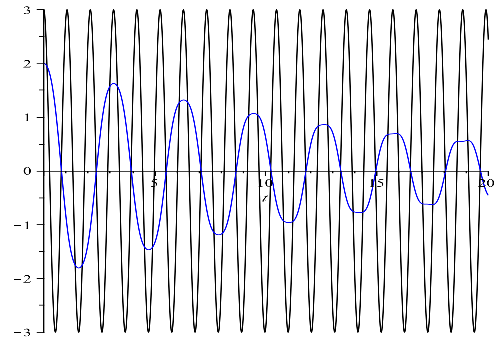
$$u(t) = \frac{6}{1-\omega^2}\sin\left[\frac{(1-\omega)t}{2}\right]\sin\left[\frac{(\omega+1)t}{2}\right] + \cos t + \sin t\,.$$

15.



(a) $\omega = 0.7$ (b) $\omega = 0.8$ (c) $\omega = 0.9$

18.(a)



(b) Phase plot - u' vs u :