CHAPTER 3

LINEAR EQUATIONS OF HIGHER ORDER

SECTION 3.1

INTRODUCTION: SECOND-ORDER LINEAR EQUATIONS

In this section the central ideas of the theory of linear differential equations are introduced and illustrated concretely in the context of **second-order** equations. These key concepts include superposition of solutions (Theorem 1), existence and uniqueness of solutions (Theorem 2), linear independence, the Wronskian (Theorem 3), and general solutions (Theorem 4). This discussion of second-order equations serves as preparation for the treatment of *n*th order linear equations in Section 3.2. Although the concepts in this section may seem somewhat abstract to students, the problems set is quite tangible and largely computational.

In each of Problems 1–16 the verification that y_1 and y_2 satisfy the given differential equation is a routine matter. As in Example 2, we then impose the given initial conditions on the general solution $y = c_1y_1 + c_2y_2$. This yields two linear equations that determine the values of the constants c_1 and c_2.

1. Imposition of the initial conditions $y(0) = 0$, $y'(0) = 5$ on the general solution $y(x) = c_1e^x + c_2e^{-x}$ yields the two equations $c_1 + c_2 = 0$, $c_1 - c_2 = 5$ with solution $c_1 = \frac{5}{2}$, $c_2 = -\frac{5}{2}$. Hence the desired particular solution is $y(x) = \frac{5}{2}\left(e^x - e^{-x}\right)$.

2. Imposition of the initial conditions $y(0) = -1$, $y'(0) = 15$ on the general solution $y(x) = c_1e^{3x} + c_2e^{-3x}$ yields the two equations $c_1 + c_2 = -1$, $3c_1 - 3c_2 = 15$, with solution $c_1 = 2$, $c_2 = 3$. Hence the desired particular solution is $y(x) = 2e^{3x} - 3e^{-3x}$.

3. Imposition of the initial conditions $y(0) = 3$, $y'(0) = 8$ on the general solution $y(x) = c_1\cos 2x + c_2\sin 2x$ yields the two equations $c_1 = 3$, $2c_2 = 8$ with solution $c_1 = 3$, $c_2 = 4$. Hence the desired particular solution is $y(x) = 3\cos 2x + 4\sin 2x$.

4. Imposition of the initial conditions $y(0) = 10$, $y'(0) = -10$ on the general solution $y(x) = c_1\cos 5x + c_2\sin 5x$ yields the two equations $c_1 = 10$, $5c_2 = -10$ with solution $c_1 = 3$, $c_2 = 4$. Hence the desired particular solution is $y(x) = 10\cos 5x - 2\sin 5x$.

5. Imposition of the initial conditions $y(0)=1$, $y'(0)=0$ on the general solution $y(x)=c_1e^x+c_2e^{2x}$ yields the two equations $c_1+c_2=1$, $c_1+2c_2=0$ with solution $c_1=2$, $c_2=-1$. Hence the desired particular solution is $y(x)=2e^x-e^{2x}$.

6. Imposition of the initial conditions $y(0)=7$, $y'(0)=-1$ on the general solution $y(x)=c_1e^{2x}+c_2e^{-3x}$ yields the two equations $c_1+c_2=7$, $2c_1-3c_2=-1$ with solution $c_1=4$, $c_2=3$. Hence the desired particular solution is $y(x)=4e^{2x}+3e^{-3x}$.

7. Imposition of the initial conditions $y(0)=-2$, $y'(0)=8$ on the general solution $y(x)=c_1+c_2e^{-x}$ yields the two equations $c_1+c_2=-2$, $-c_2=8$ with solution $c_1=6$, $c_2=-8$. Hence the desired particular solution is $y(x)=6-8e^{-x}$.

8. Imposition of the initial conditions $y(0)=4$, $y'(0)=-2$ on the general solution $y(x)=c_1+c_2e^{3x}$ yields the two equations $c_1+c_2=4$, $3c_2=-2$ with solution $c_1=\frac{14}{3}$, $c_2=\frac{2}{3}$. Hence the desired particular solution is $y(x)=\frac{1}{3}\left(14-2e^{3x}\right)$.

9. Imposition of the initial conditions $y(0)=2$, $y'(0)=-1$ on the general solution $y(x)=c_1e^{-x}+c_2xe^{-x}$ yields the two equations $c_1=2$, $-c_1+c_2=-1$ with solution $c_1=2$ $c_2=1$. Hence the desired particular solution is $y(x)=2e^{-x}+xe^{-x}$.

10. Imposition of the initial conditions $y(0)=3$, $y'(0)=13$ on the general soution $y(x)=c_1e^{5x}+c_2xe^{5x}$ yields the two equations $c_1=3$, $5c_1+c_2=13$ with solution $c_1=3$, $c_2=-2$. Hence the desired particular solution is $y(x)=3e^{5x}-2xe^{5x}$.

11. Imposition of the initial conditions $y(0)=0$, $y'(0)=5$ on the general solution $y(x)=c_1e^x\cos x+c_2e^x\sin x$ yields the two equations $c_1=0$, $c_1+c_2=5$ with solution $c_1=0$, $c_2=5$. Hence the desired particular solution is $y(x)=5e^x\sin x$.

12. Imposition of the initial conditions $y(0)=2$, $y'(0)=0$ on the general solution $y(x)=c_1e^{-3x}\cos 2x+c_2e^{-3x}\sin 2x$ yields the two equations $c_1=2$, $-3c_1+2c_2=5$ with solution $c_1=2$, $c_2=3$. Hence the desired particular solution is $y(x)=e^{-3x}(2\cos 2x+3\sin 2x)$.

13. Imposition of the initial conditions $y(1)=3$, $y'(1)=1$ on the general solution $y(x)=c_1x+c_2x^2$ yields the two equations $c_1+c_2=3$, $c_1+2c_2=1$ with solution $c_1=5$, $c_2=-2$. Hence the desired particular solution is $y(x)=5x-2x^2$.

14. Imposition of the initial conditions $y(2)=10$, $y'(2)=15$ on the general solution $y(x)=c_1x^2+c_2x^{-3}$ yields the two equations $4c_1+\frac{c_2}{8}=10$, $4c_1-\frac{3c_2}{16}=15$ with solution $c_1=3$, $c_2=-16$. Hence the desired particular solution is $y(x)=3x^2-\frac{16}{x^3}$.

15. Imposition of the initial conditions $y(1)=7$, $y'(1)=2$ on the general solution $y(x)=c_1x+c_2x\ln x$ yields the two equations $c_1=7$, $c_1+c_2=2$ with solution $c_1=7$, $c_2=-5$. Hence the desired particular solution is $y(x)=7x-5x\ln x$.

16. Imposition of the initial conditions $y(1)=2$, $y'(1)=3$ on the general solution $y(x)=c_1\cos(\ln x)+c_2\sin(\ln x)$ yields the two equations $c_1=2$, $c_2=3$. Hence the desired particular solution is $y(x)=2\cos(\ln x)+3\sin(\ln x)$.

17. If $y=\frac{c}{x}$, then $y'+y^2=-\frac{c}{x^2}+\frac{c^2}{x^2}=\frac{c(c-1)}{x^2}\neq 0$ unless either $c=0$ or $c=1$.

18. If $y=cx^3$, then $yy''=cx^3\cdot 6cx=6c^2x^4\neq 6x^4$ unless $c^2=1$.

19. If $y=1+\sqrt{x}$, then $yy''+(y')^2=\left(1+\sqrt{x}\right)\left(-\frac{x^{-3/2}}{4}\right)+\left(\frac{x^{-1/2}}{2}\right)^2=-\frac{x^{-3/2}}{4}\neq 0$.

20. Linearly dependent, because $f(x)=\pi=\pi\left(\cos^2 x+\sin^2 x\right)=\pi g(x)$.

21. Linearly independent, because $x^3=+x^2|x|$ if $x>0$, whereas $x^3=-x^2|x|$ if $x<0$.

22. Linearly independent, because $1+x=c\left(1+|x|\right)$ would require that $c=1$ with $x=0$, but $c=0$ with $x=-1$. Thus there is no such constant *c*.

23. Linearly independent, because $f(x)=+g(x)$ if $x>0$, whereas $f(x)=-g(x)$ if $x<0$.

24. Linearly dependent, because $g(x)=2f(x)$.

25. $f(x) = e^x \sin x$ and $g(x) = e^x \cos x$ are linearly independent, because $f(x) = kg(x)$ would imply that $\sin x = k \cos x$, whereas $\sin x$ and $\cos x$ are linearly independent, as noted in Example 3.

26. To see that $f(x)$ and $g(x)$ are linearly independent, assume that $f(x) = cg(x)$, and then substitute both $x = 0$ and $x = \frac{\pi}{2}$.

27. The operator notation used elsewhere in this chapter is convenient here. Let $L[y]$ denote $y'' + py' + qy$. Then $L[y_c] = 0$ and $L\left[y_p\right] = f$, so $L\left[y_c + y_p\right] = 0 + f = f$.

28. If $y(x) = 1 + c_1 \cos x + c_2 \sin x$, then $y'(x) = -c_1 \sin x + c_2 \cos x$, so the initial conditions $y(0) = y'(0) = -1$ yield $c_1 = -2$, $c_2 = -1$. Hence $y(x) = 1 - 2\cos x - \sin x$.

29. There is no contradiction, because if the given differential equation is divided by x^2 to get the form in Equation (8) in the text, then the resulting functions $p(x) = -\frac{4}{x}$ and $q(x) = \frac{6}{x^2}$ are not continuous at $x = 0$.

30. **(a)** $y_1 = x^3$ and $y_2 = \left|x^3\right|$ are linearly independent because $x^3 = c\left|x^3\right|$ would require that $c = 1$ with $x = 1$, but $c = -1$ with $x = -1$.

(b) For $x > 0$, $W(y_1, y_2) = 0$ because $y_2 = y_1$. For $x < 0$, $W(y_1, y_2) = \begin{vmatrix} x^3 & -x^3 \\ 3x^2 & -3x^2 \end{vmatrix} = 0$ because $y_2 = -y_1$. At $x = 0$, y_2 has left- and right-hand derivatives both equal to zero, so that $W(y_1, y_2) = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$ once again. Thus $W(y_1, y_2)$ is identically zero.

The fact that $W(y_1, y_2) = 0$ everywhere does not contradict Theorem 3, because when the given equation is written in the required form, namely $y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 0$, the coefficient functions $p(x) = -\frac{3}{x}$ and $q(x) = \frac{3}{x^2}$ are not continuous at $x = 0$.

31. $W(y_1, y_2) = -2x$ vanishes at $x = 0$, whereas if y_1 and y_2 were (linearly independent) solutions of an equation $y'' + py' + qy = 0$ with *p* and *q* both continuous on an open interval *I* containing $x = 0$, then Theorem 3 would imply that $W \neq 0$ on *I*.

32. **(a)** Because $W = y_1 y_2' - y_1' y_2$, we have

$$A\frac{dW}{dx} = A\left[\cancel{y_1'y_2'} + y_1y_2'' - y_1''y_2 - \cancel{y_1'y_2'}\right]$$
$$= y_1\left(Ay_2''\right) - y_2\left(Ay_1''\right)$$
$$= y_1\left(-By_2' - Cy_2\right) - y_2\left(-By_1' - Cy_1\right)$$
$$= -B\left(y_1y_2' - y_1'y_2\right)$$
$$= -BW\left(y_1, y_2\right),$$

and thus $A(x)\frac{dW}{dx} = -B(x)W(x)$.

(b) The differential equation for $W(x)$ found in **a** can be rewritten as $\frac{dW}{dx} + \frac{B(x)}{A(x)}W(x) = 0$, since $A(x)$ is never zero. We can solve this equation as a linear first-order equation: Multiplying by the integrating factor $\rho(x) = \exp\left(\int \frac{B(x)}{A(x)}dx\right)$ gives

$$\exp\left(\int \frac{B(x)}{A(x)}dx\right)\frac{dW}{dx} + \exp\left(\int \frac{B(x)}{A(x)}dx\right)\frac{B(x)}{A(x)}W(x) = 0,$$

or

$$\frac{d}{dx}\left[\exp\left(\int \frac{B(x)}{A(x)}dx\right)W(x)\right] = 0,$$

or

$$\exp\left(\int \frac{B(x)}{A(x)}dx\right)W(x) = K,$$

where K is a constant. Finally we find $W = K\exp\left(-\int \frac{B(x)}{A(x)}dx\right)$, as desired.

(c) Because the exponential factor is never zero.

In Problems 33–42 we give the characteristic equation, its roots, and the corresponding general solution.

33. $r^2 - 3r + 2 = 0$; $r = 1, 2$; $y(x) = c_1e^{x} + c_2e^{2x}$

34. $r^2 + 2r - 15 = 0$; $r = 3, -5$; $y(x) = c_1e^{-5x} + c_2e^{3x}$

35. $r^2 + 5r = 0$; $r = 0, -5$; $y(x) = c_1 + c_2e^{-5x}$

36. $2r^2+3r=0$; $r=0,-\frac{3}{2}$; $y(x)=c_1+c_2e^{-3x/2}$

37. $2r^2-r-1=0$; $r=1,-\frac{1}{2}$; $y(x)=c_1e^{-x/2}+c_2e^{x}$

38. $4r^2+8r+3=0$; $r=-\frac{1}{2},-\frac{3}{2}$; $y(x)=c_1e^{-x/2}+c_2e^{-3x/2}$

39. $4r^2+4r+1=0$; $r=-\frac{1}{2}$ (repeated); $y(x)=(c_1+c_2x)e^{-x/2}$

40. $9r^2-12r+4=0$; $r=\frac{2}{3}$ (repeated); $y(x)=(c_1+c_2x)e^{2x/3}$

41. $6r^2-7r-20=0$; $r=-\frac{4}{3},\frac{5}{2}$; $y(x)=c_1e^{-4x/3}+c_2e^{5x/2}$

42. $35r^2-r-12=0$; $r=-\frac{4}{7},\frac{3}{5}$; $y(x)=c_1e^{-4x/7}+c_2e^{3x/5}$

In Problems 43–48 we first write and simplify the equation with the indicated characteristic roots, and then write the corresponding differential equation.

43. $(r-0)(r+10)=r^2+10r=0$; $y''+10y'=0$

44. $(r-10)(r+10)=r^2-100=0$; $y''-100y=0$

45. $(r+10)(r+10)=r^2+20r+100=0$; $y''+20y'+100y=0$

46. $(r-10)(r-100)=r^2-110r+1000=0$; $y''-110y'+1000y=0$

47. $(r-0)(r-0)=r^2=0$; $y''=0$

48. $\left[r-\left(1+\sqrt{2}\right)\right]\left[r-\left(1-\sqrt{2}\right)\right]=r^2-2r-1=0$; $y''-2y'-y=0$

49. The solution curve with $y(0)=1$, $y'(0)=6$ is $y(x)=8e^{-x}-7e^{-2x}$. We find that $y'(x)=0$ when $x=\ln\frac{7}{4}$, so that $e^{-x}=\frac{4}{7}$ and $e^{-2x}=\frac{16}{49}$. It follows that $y\left(\ln\frac{7}{4}\right)=\frac{16}{7}$, so the high point on the curve is $\left(\ln\frac{7}{4}, \frac{16}{7}\right) \approx (0.56, 2.29)$, which looks consistent with Fig. 3.1.6.

50. The two solution curves satisfying $y(0) = a$ and $y(0) = b$, as well as $y'(0) = 1$, are given by

$$\begin{aligned} y &= (2a+1)e^{-x} - (a+1)e^{-2x} \\ y &= (2b+1)e^{-x} - (b+1)e^{-2x}. \end{aligned}$$

Subtraction, followed by division by $a - b$, gives $2e^{-x} = e^{-2x}$, so it follows that $x = -\ln 2$. Now substitution in either formula gives $y = -2$, so the common point of intersection is $(-\ln 2, -2)$.

51. **(a)** The substitution $v = \ln x$ gives

$$y' = \frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = \frac{1}{x}\frac{dy}{dv}.$$

Then another differentiation using the chain and product rules gives

$$\begin{aligned} y'' &= \frac{d^2y}{dx^2} \\ &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\ &= \frac{d}{dx}\left(\frac{1}{x}\cdot\frac{dy}{dv}\right) \\ &= -\frac{1}{x^2}\cdot\frac{dy}{dv} + \frac{1}{x}\cdot\frac{d}{dx}\left(\frac{dy}{dv}\right) \\ &= -\frac{1}{x^2}\cdot\frac{dy}{dv} + \frac{1}{x}\cdot\frac{d}{dv}\left(\frac{dy}{dv}\right)\cdot\frac{dv}{dx} \\ &= -\frac{1}{x^2}\cdot\frac{dy}{dv} + \frac{1}{x^2}\cdot\frac{d^2y}{dv^2}. \end{aligned}$$

Substitution of these expressions for y' and y'' into Eq. (21) in the text then yields immediately the desired Eq. (23):

$$a\frac{d^2y}{dv^2} + (b-a)\frac{dy}{dv} + cy = 0.$$

(b) If the roots r_1 and r_2 of the characteristic equation of Eq. (23) are real and distinct, then a general solution of the original Euler equation is

$$y(x) = c_1e^{r_1v} + c_2e^{r_2v} = c_1\left(e^v\right)^{r_1} + c_2\left(e^v\right)^{r_2} = c_1x^{r_1} + c_2x^{r_2}.$$

52. The substitution $v = \ln x$ yields the converted equation $\frac{dy^2}{dv^2} - y = 0$, whose characteristic equation $r^2 - 1 = 0$ has roots $r_1 = 1$ and $r_2 = -1$. Because $e^v = x$, the corresponding general solution is $y = c_1 e^v + c_2 e^{-v} = c_1 x + \frac{c_2}{x}$.

53. The substitution $v = \ln x$ yields the converted equation $\frac{dy^2}{dv^2} + \frac{dy}{dv} - 12y = 0$, whose characteristic equation $r^2 + r - 12 = 0$ has roots $r_1 = -4$ and $r_2 = 3$. Because $e^v = x$, the corresponding general solution is $y = c_1 e^{-4v} + c_2 e^{3v} = c_1 x^{-4} + c_2 x^3$.

54. The substitution $v = \ln x$ yields the converted equation $4\frac{dy^2}{dv^2} + 4\frac{dy}{dv} - 3y = 0$, whose characteristic equation $4r^2 + 4r - 3 = 0$ has roots $r_1 = -\frac{3}{2}$ and $r_2 = \frac{1}{2}$. Because $e^v = x$, the corresponding general solution is $y = c_1 e^{-3v/2} + c_2 e^{v/2} = c_1 x^{-3/2} + c_2 x^{1/2}$.

55. The substitution $v = \ln x$ yields the converted equation $\frac{dy^2}{dv^2} = 0$, whose characteristic equation $r^2 = 0$ has repeated roots $r_1, r_2 = 0$. Because $v = \ln x$, the corresponding general solution is $y = c_1 + c_2 v = c_1 + c_2 \ln x$.

56. The substitution $v = \ln x$ yields the converted equation $\frac{dy^2}{dv^2} - 4\frac{dy}{dv} + 4y = 0$, whose characteristic equation $r^2 - 4r + 4 = 0$ has roots $r_1, r_2 = 2$. Because $e^v = x$, the corresponding general solution is $y = c_1 e^{2v} + c_2 v e^{2v} = x^2 \left(c_1 + c_2 \ln v\right)$.

SECTION 3.2

GENERAL SOLUTIONS OF LINEAR EQUATIONS

Students should check each of Theorems 1 through 4 in this section to see that, in the case $n = 2$, it reduces to the corresponding theorem in Section 3.1. Similarly, the computational problems for this section largely parallel those for the previous section. By the end of Section 3.2 students should understand that, although we do not prove the existence-uniqueness theorem now, it provides the basis for everything we do with linear differential equations.

The linear combinations listed in Problems 1–6 were discovered "by inspection"—that is, by trial and error.

1. $\frac{5}{2}\cdot 2x+\left(-\frac{8}{3}\right)\cdot 3x^2+(-1)\left(5x-8x^2\right)=0$ for all x.

2. $-4\cdot 5+5\cdot\left(2-3x^2\right)+1\cdot\left(10+15x^2\right)=0$ for all x.

3. $1\cdot 0+0\cdot\sin x+0\cdot e^x=0$ for all x.

4. $1\cdot 17+\left(-\frac{17}{2}\right)\cdot 2\sin^2 x+\left(-\frac{17}{3}\right)\cdot 3\cos^2 x=0$ for all x, because $\sin^2 x+\cos^2 x=1$.

5. $1\cdot 17+(-34)\cdot\cos^2 x+17\cdot\cos 2x=0$ for all x, because $2\cos^2 x=1+\cos 2x$.

6. $(-1)\cdot e^x+1\cdot\cosh x+1\cdot\sinh x=0$, because $\cosh x=\frac{1}{2}\left(e^x+e^{-x}\right)$ and $\sinh x=\frac{1}{2}\left(e^x-e^{-x}\right)$.

7. $W=\begin{vmatrix}1 & x & x^2\\ 0 & 1 & 2x\\ 0 & 0 & 2\end{vmatrix}=2$ is nonzero everywhere.

8. $W=\begin{vmatrix}e^x & e^{2x} & e^{3x}\\ e^x & 2e^{2x} & 3e^{3x}\\ e^x & 4e^{2x} & 9e^{3x}\end{vmatrix}=2e^{6x}$ is never zero.

9. $W=e^x\left(\cos^2 x+\sin^2 x\right)=e^x$ is never zero.

10. $W=x^{-7}e^x(x+1)(x+4)$ is nonzero for $x>0$.

11. $W=x^3e^{2x}$ is nonzero if $x\neq 0$.

12. $W=x^{-2}\left[2\cos^2(\ln x)+2\sin^2(\ln x)\right]=2x^{-2}$ is nonzero for $x>0$.

In each of Problems 13-20 we first form the general solution $y(x)=c_1y_1(x)+c_2y_2(x)+c_3y_3(x)$, then calculate $y'(x)$ and $y''(x)$, and finally impose the given initial conditions to determine the values of the coefficients c_1,c_2,c_3.

13. Imposition of the initial conditions $y(0)=1$, $y'(0)=2$, $y''(0)=0$ on the general solution $y(x)=c_1e^x+c_2e^{-x}+c_3e^{-2x}$ yields the three equations

$$c_1+c_2+c_3=1,\quad c_1-c_2-2c_3=2,\quad c_1+c_2+4c_3=0,$$

with solution $c_1=\frac{4}{3}$, $c_2=0$, $c_3=-\frac{1}{3}$. Hence the desired particular solution is given by $y(x)=\frac{1}{3}\left(4e^x-e^{-2x}\right)$.

14. Imposition of the initial conditions $y(0)=0$, $y'(0)=0$, $y''(0)=3$ on the general solution $y(x)=c_1e^x+c_2e^{2x}+c_3e^{3x}$ yields the three equations

$$c_1+c_2+c_3=1,\quad c_1+2c_2+3c_3=2,\quad c_1+4c_2+9c_3=0,$$

with solution $c_1=\frac{3}{2}$, $c_2=-3$, $c_3=\frac{3}{2}$. Hence the desired particular solution is given by $y(x)=\frac{3}{2}e^x-3e^{2x}+\frac{3}{2}e^{3x}$.

15. Imposition of the initial conditions $y(0)=2$, $y'(0)=0$, $y''(0)=0$ on the general solution $y(x)=c_1e^x+c_2xe^x+c_3x^2e^x$ yields the three equations

$$c_1=2,\quad c_1+c_2=0,\quad c_1+2c_2+2c_3=0,$$

with solution $c_1=2$, $c_2=-2$, $c_3=1$. Hence the desired particular solution is given by $y(x)=\left(2-2x+x^2\right)e^x$.

16. Imposition of the initial conditions $y(0)=1$, $y'(0)=4$, $y''(0)=0$ on the general solution $y(x)=c_1e^x+c_2e^{2x}+c_3xe^{2x}$ yields the three equations

$$c_1+c_2=1,\quad c_1+2c_2+c_3=4,\quad c_1+4c_2+4c_3=0$$

with solution $c_1=-12$, $c_2=13$, $c_3=-10$. Hence the desired particular solution is given by $y(x)=-12e^x+13e^{2x}-10xe^{2x}$.

17. Imposition of the initial conditions $y(0)=3$, $y'(0)=1$, $y''(0)=2$ on the general solution $y(x)=c_1+c_2\cos 3x+c_3\sin 3x$ yields the three equations

$$c_1+c_3=3,\quad 3c_3=-1,\quad -9c_2=2$$

with solution $c_1 = \frac{29}{9}$, $c_2 = -\frac{2}{9}$, $c_3 = -\frac{1}{3}$. Hence the desired particular solution is given by $y(x) = \frac{29}{9} - \frac{2}{9}\cos 3x - \frac{1}{3}\sin 3x$.

18. Imposition of the initial conditions $y(0) = 1$, $y'(0) = 0$, $y''(0) = 0$ on the general solution $y(x) = e^x(c_1 + c_2\cos x + c_3\sin x)$ yields the three equations

$$c_1 + c_2 = 1, \quad c_1 + c_2 + c_3 = 0, \quad c_1 + 2c_3 = 0$$

with solution $c_1 = 2$, $c_2 = -1$, $c_3 = -1$. Hence the desired particular solution is given by $y(x) = e^x(2 - \cos x - \sin x)$.

19. Imposition of the initial conditions $y(1) = 6$, $y'(1) = 14$, $y''(1) = 22$ on the general solution $y(x) = c_1x + c_2x^2 + c_3x^3$ yields the three equations

$$c_1 + c_2 + c_3 = 6, \quad c_1 + 2c_2 + 3c_3 = 14, \quad 2c_2 + 6c_3 = 22,$$

with solution $c_1 = 1$, $c_2 = 2$, $c_3 = 3$. Hence the desired particular solution is given by $y(x) = x + 2x^2 + 3x^3$.

20. Imposition of the initial conditions $y(1) = 1$, $y'(1) = 5$, $y''(1) = -11$ on the general solution $y(x) = c_1x + c_2x^{-2} + c_3x^{-2}\ln x$ yields the three equations

$$c_1 + c_2 = 1, \quad c_1 - 2c_2 + c_3 = 5, \quad 6c_2 - 5c_3 = -11,$$

with solution $c_1 = 2$, $c_2 = -1$, $c_3 = 1$. Hence the desired particular solution is given by $y(x) = 2x - x^{-2} + x^{-2}\ln x$.

In each of Problems 21 - 24 we first form the general solution

$$y(x) = y_c(x) + y_p(x) = c_1y_1(x) + c_2y_2(x) + y_p(x),$$

then calculate $y'(x)$, and finally impose the given initial conditions to determine the values of the coefficients c_1 and c_2.

21. Imposition of the initial conditions $y(0) = 2$, $y'(0) = -2$ on the general solution $y(x) = c_1\cos x + c_2\sin x + 3x$ yields the two equations $c_1 = 2$, $c_2 + 3 = -2$ with solution $c_1 = 2$, $c_2 = -5$. Hence the desired particular solution is given by $y(x) = 2\cos x - 5\sin x + 3x$.

22. Imposition of the initial conditions $y(0)=0$, $y'(0)=10$ on the general solution $y(x)=c_1e^{2x}+c_2e^{-2x}-3$ yields the two equations $c_1+c_2-3=0$, $2c_1-2c_2=10$ with solution $c_1=4$, $c_2=-1$. Hence the desired particular solution is given by $y(x)=4e^{2x}-e^{-2x}-3$.

23. Imposition of the initial conditions $y(0)=3$, $y'(0)=11$ on the general solution $y(x)=c_1e^{-x}+c_2e^{3x}-2$ yields the two equations $c_1+c_2-2=3$, $-c_1+3c_2=11$ with solution $c_1=1$, $c_2=4$. Hence the desired particular solution is given by $y(x)=e^{-x}+4e^{3x}-2$.

24. Imposition of the initial conditions $y(0)=4$, $y'(0)=8$ on the general solution $y(x)=c_1e^x\cos x+c_2e^x\cos x+x+1$ yields the two equations $c_1+1=4$, $c_1+c_2+1=8$ with solution $c_1=3$, $c_2=4$. Hence the desired particular solution is given by $y(x)=e^x(3\cos x+4\sin x)+x+1$.

25. $Ly=L[y_1+y_2]=Ly_1+Ly_2=f+g$

26. **(a)** $y_1=2$ and $y_2=3x$

(b) $y=y_1+y_2=2+3x$

27. The equations

$$c_1+c_2x+c_3x^2=10,\quad c_2+2c_3x=0,\quad 2c_3=0$$

(the latter two obtained by successive differentiation of the first one) evidently imply that $c_1=c_2=c_3=0$.

28. If we differentiate the equation $c_0+c_1x+c_2x^2+\cdots+c_nx^n=0$ repeatedly, n times in succession, the result is the system

$$\begin{aligned} c_0+c_1x+c_2x^2+\cdots+c_nx^n &= 0\\ c_1+2c_2x+\cdots+nc_nx^{n-1} &= 0\\ &\vdots\\ (n-1)!c_{n-1}+n!c_nx &= 0\\ n!c_n &= 0 \end{aligned}$$

of $n+1$ equations in the $n+1$ coefficients $c_0,c_1,c_2,\cdots,c_n$. The last equation implies that $c_n=0$, whereupon the preceding equation gives $c_{n-1}=0$, and so forth. Thus it follows that all of the coefficients must vanish.

29. If $c_0e^{rx}+c_1xe^{rx}+\cdots+c_nx^ne^{rx}=0$, then division by e^{rx} yields $c_0+c_1x+\cdots+c_nx^n=0$, so the result of Problem 28 applies.

30. When the equation $x^2y''-2xy'+2y=0$ is rewritten in standard form

$$y''+\left(-\frac{2}{x}\right)y'+\frac{2}{x^2}y=0,$$

we see that the coefficient functions $p_1=-\frac{2}{x}$ and $p_2(x)=\frac{2}{x^2}$ are not continuous at $x=0$. Thus the hypotheses of Theorem 3 are not satisfied.

31. **(a)** Substitution of $x=a$ in the differential equation gives $y''(a)=-py'(a)-q(a)$.

(b) If $y(0)=1$ and $y'(0)=0$, then the equation $y''-2y'-5y=0$ implies that $y''(0)=2y'(0)+5y(0)=5$.

32. Let the functions $y_1,y_2,\ldots,y_n$ be chosen as indicated. Then evaluation at $x=a$ of the $(k-1)$st derivative of the equation $c_1y_1+c_2y_2+\cdots c_ny_n=0$ yields $c_k=0$. Thus $c_1=c_2=\cdots c_n=0$, so the functions are linearly independent.

33. This follows from the fact that

$$\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}=(b-a)(c-b)(c-a)$$

when a, b, and c are distinct, which can be verified by expanding both sides of the equation.

34. $W(f_1,f_2,\ldots,f_n)=V\exp\left[\left(\sum_{i=1}^{n}r_i\right)x\right]$, and neither V nor $\exp\left[\left(\sum_{i=1}^{n}r_i\right)x\right]$ vanishes.

36. If $y=vy_1$, then substitution of the derivatives $y'=vy_1'+v'y_1$, $y''=vy_1''+2v'y_1'+v''y_1$ in the differential equation $y''+py'+qy=0$ gives

$$[vy_1''+2v'y_1'+v''y_1]+p[vy_1'+v'y_1]+q[vy_1]=0,$$

or

$$v[y_1''+py_1'+qy_1]+v''y_1+2v'y_1'+pv'y_1=0.$$

But the terms within brackets vanish because y_1 is a solution, and this leaves the equation

$$y_1v''+(2y_1'+py_1)v'=0.$$

We can solve this by separating variables and integrating: $\dfrac{v''}{v'} = -2\dfrac{y_1'}{y_1} - p$ leads to

$$\ln v' = -2\ln y_1' - \int p(x)\,dx + \ln C,$$

or

$$v'(x) = \frac{C}{y_1^2}e^{-\int p(x)\,dx},$$

or

$$v(x) = C\int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx + K.$$

With $C = 1$ and $K = 0$ this gives the second solution

$$y_2(x) = y_1(x)\int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx.$$

37. When we substitute $y = vx^3$ in the given differential equation and simplify, we get the separable equation $xv'' + v' = 0$, which we write as $\dfrac{v''}{v'} = -\dfrac{1}{x}$. Integrating gives $\ln v' = -\ln x + \ln A$, and then solving for v' leads to $v' = \dfrac{A}{x}$, or finally $v(x) = A\ln x + B$. With $A = 1$ and $B = 0$ we get $v(x) = \ln x$, and thus $y_2(x) = x^3\ln x$.

38. When we substitute $y = vx^3$ in the given differential equation and simplify, we get the separable equation $xv'' + 7v' = 0$, which we write as $\dfrac{v''}{v'} = -\dfrac{7}{x}$. Integrating gives $\ln v' = -7\ln x + \ln A$, and then solving for v' leads to $v' = \dfrac{A}{x^7}$, or finally $v(x) = -\dfrac{A}{6x^6} + B$. With $A = -6$ and $B = 0$ we get $v(x) = \dfrac{1}{x^6}$, and hence $y_2(x) = \dfrac{1}{x^3}$.

39. When we substitute $y = ve^{x/2}$ in the given differential equation and simplify, we eventually get the simple equation $v'' = 0$, with general solution $v(x) = Ax + B$. With $A = 1$ and $B = 0$ we get $v(x) = x$, and hence $y_2(x) = xe^{x/2}$.

40. When we substitute $y = vx$ in the given differential equation and simplify, we get the separable equation $v'' - v' = 0$, which we write as $\dfrac{v''}{v'} = 1$. Integrating gives

$\ln v' = x + \ln A$, and then solving for v' leads to $v' = Ae^x$, or finally $v(x) = Ae^x + B$. With $A = 1$ and $B = 0$ we get $v(x) = e^x$, and hence $y_2(x) = xe^x$.

41. When we substitute $y = ve^x$ in the given differential equation and simplify, we get the separable equation $(1+x)v'' + xv' = 0$, which we write as $\frac{v''}{v'} = -\frac{x}{1+x} = -1 + \frac{1}{1+x}$. Integrating gives $\ln v' = -x + \ln(1+x) + \ln A$, and then solving for v' leads to $v' = A(1+x)e^{-x}$, or finally $v(x) = A\int(1+x)e^{-x}\,dx = -A(2+x)e^{-x} + B$. With $A = -1$ and $B = 0$ we get $v(x) = (2+x)e^{-x}$, and hence $y_2(x) = 2 + x$.

42. When we substitute $y = vx$ in the given differential equation and simplify, we get the separable equation $x(x^2-1)v'' = 2v'$, which we write as

$$\frac{v''}{v'} = \frac{2}{x(x^2-1)} = -\frac{2}{x} + \frac{1}{1+x} - \frac{1}{1-x}.$$

Integrating gives

$$\ln v' = -2\ln x + \ln(1+x) + \ln(1-x) + \ln A,$$

and then solving for v' leads to $v' = \frac{A(1-x^2)}{x^2} = A\left(\frac{1}{x^2} - 1\right)$, or finally $v(x) = A\left(-\frac{1}{x} - x\right) + B$. With $A = -1$ and $B = 0$ we get $v(x) = x + \frac{1}{x}$, and hence $y_2(x) = x^2 + 1$.

43. When we substitute $y = vx$ in the given differential equation and simplify, we get the separable equation $x(x^2-1)v'' = (2-4x^2)v'$, which we write using the method of partial fractions as

$$\frac{v''}{v'} = \frac{2-4x^2}{x(x^2-1)} = -\frac{2}{x} - \frac{1}{1+x} + \frac{1}{1-x}.$$

Integrating gives

$$\ln v' = -2\ln x - \ln(1+x) - \ln(1-x) + \ln A,$$

and then solving for v' leads to

$$v' = \frac{A}{x^2(1-x^2)} = A\left[\frac{1}{x^2} + \frac{1}{2(1+x)} + \frac{1}{2(1-x)}\right],$$

or finally

$$v(x) = A\left[-\frac{1}{x} + \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x)\right] + B.$$

With $A = -1$ and $B = 0$ we get $v(x) = \frac{1}{x} - \frac{1}{2}\ln(1+x) + \frac{1}{2}\ln(1-x)$, and hence $y_2(x) = 1 - \frac{x}{2}\ln\frac{1+x}{1-x}$.

44. When we substitute $y = vx^{-1/2}\cos x$ in the given differential equation and simplify, we eventually get the separable equation $(\cos x)v'' = 2(\sin x)v'$, which we write as $\frac{v''}{v'} = \frac{2\sin x}{\cos x}$. Integrating gives

$$\ln v' = -2\ln|\cos x| + \ln A = \ln\sec^2 x + \ln A,$$

and then solving for v' leads to $v' = A\sec^2 x$, or finally $v(x) = A\tan x + B$. With $A = 1$ and $B = 0$ we get $v(x) = \tan x$, and hence $y_2(x) = (\tan x)(x^{-1/2}\cos x) = x^{-1/2}\sin x$.

SECTION 3.3

HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

This is a purely computational section devoted to the single most widely applicable type of higher order differential equations—linear ones with constant coefficients. In Problems 1–20, we first write the characteristic equation and list its roots, then give the corresponding general solution of the given differential equation. Explanatory comments are included only when the solution of the characteristic equation is not routine.

1. $r^2 - 4 = (r-2)(r+2) = 0$; $r = -2, 2$; $y(x) = c_1 e^{2x} + c_2 e^{-2x}$

2. $2r^2 - 3r = r(2r-3) = 0$; $r = 0, \frac{3}{2}$; $y(x) = c_1 + c_2 e^{3x/2}$

3. $r^2 + 3r - 10 = (r+5)(r-2) = 0$; $r = -5, 2$; $y(x) = c_1 e^{2x} + c_2 e^{-5x}$

4. $2r^2 - 7r + 3 = (2r-1)(r-3) = 0$; $r = \frac{1}{2}, 3$; $y(x) = c_1 e^{x/2} + c_2 e^{3x}$

5. $r^2 + 6r + 9 = (r+3)^2 = 0$; $r = -3$ (repeated); $y(x) = c_1 e^{-3x} + c_2 x e^{-3x}$

6. $r^2 + 5r + 5 = 0$; $r = \frac{-5 \pm \sqrt{5}}{2}$; $y(x) = c_1 e^{\frac{-5+\sqrt{5}}{2}x} + c_2 e^{\frac{-5-\sqrt{5}}{2}x}$

7. $4r^2 - 12r + 9 = (2r-3)^2 = 0$; $r = \dfrac{3}{2}$ (repeated); $y(x) = c_1 e^{3x/2} + c_2 x e^{3x/2}$

8. $r^2 - 6r + 13 = 0$; $r = \dfrac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i$; $y(x) = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$

9. $r^2 + 8r + 25 = 0$; $r = \dfrac{-8 \pm \sqrt{-36}}{2} = -4 \pm 3i$; $y(x) = e^{-4x}(c_1 \cos 3x + c_2 \sin 3x)$

10. $5r^4 + 3r^3 = r^3(5r+3) = 0$; $r = 0,0,0,-\dfrac{3}{5}$; $y(x) = c_1 + c_2 x + c_3 x^2 + c_4 e^{-3x/5}$

11. $r^4 - 8r^3 + 16r^2 = r^2(r-4)^2 = 0$; $r = 0,0,4,4$; $y(x) = c_1 + c_2 x + c_3 e^{4x} + c_4 x e^{4x}$

12. $r^4 - 3r^3 + 3r^2 - r = r(r-1)^3 = 0$; $r = 0,1,1,1$; $y(x) = c_1 + c_2 e^x + c_3 x e^x + c_4 x^2 e^x$

13. $9r^3 + 12r^2 + 4r = r(3r+2)^2 = 0$; $r = 0,-\dfrac{2}{3},-\dfrac{2}{3}$; $y(x) = c_1 + c_2 e^{-2x/3} + c_3 x e^{-2x/3}$

14. $r^4 + 3r^2 - 4 = (r^2-1)(r^2+4) = 0$; $r = -1,1,\pm 2i$; $y(x) = c_1 e^x + c_2 e^{-x} + c_3 \cos 2x + c_4 \sin 2x$

15. $4r^4 - 8r^2 + 16 = (r^2-4)^2 = (r-2)^2(r+2)^2 = 0$; $r = 2,2,-2,-2$;

$$y(x) = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-2x} + c_4 x e^{-2x}$$

16. $r^4 + 18r^2 + 81 = (r^2+9)^2 = 0$; $r = \pm 3i, \pm 3i$; $y(x) = (c_1 + c_2 x)\cos 3x + (c_3 + c_4 x)\sin 3x$

17. $6r^4 + 11r^2 + 4 = (2r^2+1)(3r^2+4) = 0$; $r = \pm\dfrac{i}{\sqrt{2}}, \pm\dfrac{2i}{\sqrt{3}}$;

$$y(x) = c_1 \cos\frac{x}{\sqrt{2}} + c_2 \sin\frac{x}{\sqrt{2}} + c_3 \cos\frac{2x}{\sqrt{3}} + c_4 \sin\frac{2x}{\sqrt{3}}$$

18. $r^4 - 16 = (r^2-4)(r^2+4) = 0$; $r = -2,2,\pm 2i$; $y(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$

19. Factoring by grouping gives $r^3 + r^2 - r - 1 = r(r^2-1) + (r^2-1) = (r-1)(r+1)^2 = 0$; $r = 1,-1,-1$; $y(x) = c_1 e^x + c_2 e^{-x} + c_3 x e^{-x}$.

20. $r^4 + 2r^3 + 3r^2 + 2r + 1 = (r^2 + r + 1)^2 = 0$; $\dfrac{-1 \pm \sqrt{3}i}{2}, \dfrac{-1 \pm \sqrt{3}i}{2}$;

$$y(x) = e^{-x/2}(c_1 + c_2 x)\cos\left(\frac{\sqrt{3}}{2}x\right) + e^{-x/2}(c_3 + c_4 x)\sin\left(\frac{\sqrt{3}}{2}x\right)$$

21. Imposition of the initial conditions $y(0) = 7$, $y'(0) = 11$ on the general solution $y(x) = c_1 e^x + c_2 e^{3x}$ yields the two equations $c_1 + c_2 = 7$, $c_1 + 3c_2 = 11$ with solution $c_1 = 5$, $c_2 = 2$. Hence the desired particular solution is $y(x) = 5e^x + 2e^{3x}$.

22. Imposition of the initial conditions $y(0) = 3$, $y'(0) = 4$ on the general solution $y(x) = e^{-x/3}\left(c_1 \cos\dfrac{x}{\sqrt{3}} + c_2 \sin\dfrac{x}{\sqrt{3}}\right)$ yields the two equations $c_1 = 3$, $-\dfrac{c_1}{3} + \dfrac{c_2}{\sqrt{3}} = 4$ with solution $c_1 = 3$, $c_2 = 5\sqrt{3}$. Hence the desired particular solution is

$$y(x) = e^{-x/3}\left(3\cos\frac{x}{\sqrt{3}} + 5\sqrt{3}\sin\frac{x}{\sqrt{3}}\right).$$

23. Imposition of the initial conditions $y(0) = 3$, $y'(0) = 1$ on the general solution $y(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$ yields the two equations $c_1 = 3$, $3c_1 + 4c_2 = 1$ with solution $c_1 = 3$, $c_2 = -2$. Hence the desired particular solution is $y(x) = e^{3x}(3\cos 4x - 2\sin 4x)$.

24. Imposition of the initial conditions $y(0) = 1$, $y'(0) = -1$, $y''(0) = 3$ on the general solution $y(x) = c_1 + c_2 e^{2x} + c_3 e^{-x/2}$ yields the three equations

$$c_1 + c_2 + c_3 = 1, \quad 2c_2 - \frac{c_3}{2} = -1, \quad 4c_2 + \frac{c_3}{4} = 3,$$

with solution $c_1 = -\dfrac{7}{2}$, $c_2 = \dfrac{1}{2}$, $c_3 = 4$. Hence the desired particular solution is $y(x) = -\dfrac{7}{2} + \dfrac{1}{2}e^{2x} + 4e^{-x/2}$.

25. Imposition of the initial conditions $y(0) = -1$, $y'(0) = 0$, $y''(0) = 1$ on the general solution $y(x) = c_1 + c_2 x + c_3 e^{-2x/3}$ yields the three equations

$$c_1 + c_3 = -1, \quad c_2 - \frac{2c_3}{3} = 0, \quad \frac{4c_3}{9} = 1,$$

with solution $c_1 = -\frac{13}{4}$, $c_2 = \frac{3}{2}$, $c_3 = \frac{9}{4}$. Hence the desired particular solution is $y(x) = -\frac{13}{4} + \frac{3}{2}x + \frac{9}{4}e^{-2x/3}$.

26. Imposition of the initial conditions $y(0) = 1$, $y'(0) = -1$, $y''(0) = 3$ on the general solution $y(x) = c_1 + c_2 e^{-5x} + c_3 x e^{-5x}$ yields the three equations

$$c_1 + c_2 = 3, \quad -5c_2 + c_3 = 4, \quad 25c_2 - 10c_3 = 5,$$

with solution $c_1 = \frac{24}{5}$, $c_2 = -\frac{9}{5}$, $c_3 = -5$. Hence the desired particular solution is $y(x) = \frac{24}{5} - \frac{9}{5}e^{-5x} - 5xe^{-5x}$.

27. First we spot the root $r = 1$. Then long division of the polynomial $r^3 + 3r^2 - 4$ by $r - 1$ yields the quadratic factor $r^2 + 4r + 4 = (r+2)^2$, with roots $r = -2, -2$. Hence the general solution is $y(x) = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x}$.

28. First we spot the root $r = 2$. Then long division of the polynomial $2r^3 - r^2 - 5r - 2$ by the factor $r - 2$ yields the quadratic factor $2r^2 + 3r + 1 = (2r+1)(r+1)$, with roots $r = -1, -\frac{1}{2}$. Hence the general solution is $y(x) = c_1 e^{2x} + c_2 e^{-x} + c_3 e^{-x/2}$.

29. First we spot the root $r = -3$. Then long division of the polynomial $r^3 + 27$ by $r + 3$ yields the quadratic factor $r^2 - 3r + 9$, with roots $r = \frac{3}{2} \pm i\frac{3\sqrt{3}}{2}$. Hence the general solution is $y(x) = c_1 e^{-3x} + e^{3x/2}\left(c_2 \cos\frac{3\sqrt{3}}{2}x + c_3 \sin\frac{3\sqrt{3}}{2}x\right)$.

30. First we spot the root $r = -1$. Then long division of the polynomial $r^4 - r^3 + r^2 - 3r - 6$ by $r + 1$ yields the cubic factor $r^3 - 2r^2 + 3r - 6$. Next we spot the root $r = 2$, and another long division yields the quadratic factor $r^2 + 3$, with roots $r = \pm i\sqrt{3}$. Hence the general solution is $y(x) = c_1 e^{-x} + c_2 e^{2x} + c_3 \cos\sqrt{3}x + c_4 \sin\sqrt{3}x$.

31. The characteristic equation $r^3 + 3r^2 + 4r - 8 = 0$ has the evident root $r = 1$, and long division then yields the quadratic factor $r^2 + 4r + 8 = (r+2)^2 + 4$, corresponding to the complex conjugate roots $-2 \pm 2i$. Hence the general solution is $y(x) = c_1 e^x + e^{-2x}(c_2 \cos 2x + c_3 \sin 2x)$.

32. The characteristic equation $r^4 + r^3 - 3r^2 - 5r - 2 = 0$ has the root $r = 2$, as is readily found by trial and error, and long division then yields the factorization $(r-2)(r+1)^3 = 0$. Thus we obtain the general solution $y(x) = c_1 e^{2x} + \left(c_2 + c_3 x + c_4 x^2\right)e^{-x}$.

33. Knowing that $y = e^{3x}$ is one solution, we divide the characteristic polynomial $r^3 + 3r^2 - 54$ by $r - 3$ and get the quadratic factor $r^2 + 6r + 18 = (r+3)^2 + 9$. Hence the general solution is $y(x) = c_1 e^{3x} + e^{-3x}\left(c_2 \cos 3x + c_3 \sin 3x\right)$.

34. Knowing that $y = e^{2x/3}$ is one solution, we divide the characteristic polynomial $3r^3 - 2r^2 + 12r - 8$ by $3r - 2$ and get the quadratic factor $r^2 + 4$. Hence the general solution is $y(x) = c_1 e^{2x/3} + c_2 \cos 2x + c_3 \sin 2x$.

35. The fact that $y = \cos 2x$ is one solution tells us that $r^2 + 4$ is a factor of the characteristic polynomial $6r^4 + 5r^3 + 25r^2 + 20r + 4$. Then long division yields the quadratic factor $6r^2 + 5r + 1 = (3r+1)(2r+1)$, with roots $r = -\frac{1}{2}, -\frac{1}{3}$. Hence the general solution is $y(x) = c_1 e^{-x/2} + c_2 e^{-x/3} + c_3 \cos 2x + c_4 \sin 2x$.

36. The fact that $y = e^{-x} \sin x$ is one solution tells us that $(r+1)^2 + 1 = r^2 + 2r + 2$ is a factor of the characteristic polynomial $9r^3 + 11r^2 + 4r - 14$. Then long division yields the linear factor $9r - 7$. Hence the general solution is $y(x) = c_1 e^{7x/9} + e^{-x}\left(c_2 \cos x + c_3 \sin x\right)$.

37. The characteristic equation is $r^4 - r^3 = r^3(r-1) = 0$, so the general solution is $y(x) = A + Bx + Cx^2 + De^x$. Imposition of the given initial conditions yields the equations

$$A + D = 18, \quad B + D = 12, \quad 2C + D = 13, \quad D = 7$$

with solution $A = 11$, $B = 5$, $C = 3$, $D = 7$. Hence the desired particular solution is $y(x) = 11 + 5x + 3x^2 + 7e^x$.

38. Given that $r = 5$ is one characteristic root, we divide $r - 5$ into the characteristic polynomial $r^3 - 5r^2 + 100r - 500$ and get the remaining factor $r^2 + 100$. Thus the general solution is $y(x) = Ae^{5x} + B\cos 10x + C\sin 10x$. Imposition of the given initial conditions yields the equations

$$A + B = 0, \quad 5A + 10C = 10, \quad 25A - 100B = 250,$$

with solution $A = 2$, $B = -2$, $C = 0$. Hence the desired particular solution is $y(x) = 2e^{5x} - 2\cos 10x$.

39. The characteristic polynomial is $(r-2)^3 = r^3 - 6r^2 + 12r - 8$, so the differential equation is $y''' - 6y'' + 12y' - 8y = 0$.

40. The characteristic polynomial is $(r-2)(r^2+4) = r^3 - 2r^2 + 4r - 8$, so the differential equation is $y''' - 2y'' + 4y' - 8y = 0$.

41. The characteristic polynomial is $(r^2+4)(r^2-4) = r^4 - 16$, so the differential equation is $y^{(4)} - 16y = 0$.

42. The characteristic polynomial is $(r^2+4)^3 = r^6 + 12r^4 + 48r^2 + 64$, so the differential equation is $y^{(6)} + 12y^{(4)} + 48y'' + 64y = 0$.

43. **(a)** Given a complex number $z = x + iy$ we define r to be $\sqrt{x^2 + y^2}$ and θ to bethe unique angle satisfying $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$, and $-\pi < \theta \le \pi$. Then Euler's formula gives

$$re^{i\theta} = r(\cos\theta + i\sin\theta) = r\left(\frac{x}{r} + i\frac{y}{r}\right) = x + iy = z.$$

(b) $4 = 4e^{i\cdot 0}$; $-2 = 2e^{i\pi}$; $3i = 3e^{i\pi/2}$; $1 + i = \sqrt{2}e^{i\pi/4}$; $-1 + i\sqrt{3} = 2e^{i\cdot 2\pi/3}$

(c) Because $2 - 2i\sqrt{3} = 4e^{-i\pi/3}$, the square roots of $2 - 2i\sqrt{3}$ are $\pm 2e^{-i\pi/6}$. Likewise, because $-2 + 2i\sqrt{3} = 4e^{i\cdot 2\pi/3}$, the square roots of $-2 + 2i\sqrt{3}$ are $\pm 2e^{i\cdot\pi/3}$.

44. **(a)** $x = \dfrac{-i \pm \sqrt{i^2 - 4\cdot 2}}{2} = \left(\dfrac{-1 \pm 3}{2}\right)i = i, -2i$

(b) $x = \dfrac{2i \pm \sqrt{(-2i)^2 - 4\cdot 3}}{2} = \left(\dfrac{2 \pm 4}{2}\right)i = -i, 3i$

45. The characteristic polynomial is the quadratic polynomial of Problem 44(b). Hence the general solution is

$$y(x) = c_1 e^{-ix} + c_2 e^{3ix} = c_1(\cos x - i\sin x) + c_2(\cos 3x + i\sin 3x).$$

46. The characteristic polynomial is $r^2 - ir + 6 = (r + 2i)(r - 3i)$, so the general solution is

$$y(x) = c_1 e^{3ix} + c_2 e^{-2ix} = c_1(\cos 3x + i\sin 3x) + c_2(\cos 2x - i\sin 2x).$$

47. The characteristic roots are $r = \pm\sqrt{-2+2i\sqrt{3}} = \pm\left(1+i\sqrt{3}\right)$, so the general solution is

$$y(x) = c_1 e^{\left(1+i\sqrt{3}\right)x} + c_2 e^{-\left(1+i\sqrt{3}\right)x} = c_1 e^x\left(\cos\sqrt{3}x + i\sin\sqrt{3}x\right) + c_2 e^{-x}\left(\cos\sqrt{3}x - i\sin\sqrt{3}x\right).$$

48. The general solution is $y(x) = Ae^x + Be^{\alpha x} + Ce^{\beta x}$, where $\alpha = \dfrac{-1+i\sqrt{3}}{2}$ and $\beta = \dfrac{-1-i\sqrt{3}}{2}$. Imposition of the given initial conditions yields the equations

$$\begin{aligned} A + \quad B + \quad C &= 17 \\ A + \alpha B + \beta C &= 0 \\ A + \alpha^2 B + \beta^2 C &= 0 \end{aligned}$$

that we solve for $A = B = C = \dfrac{1}{3}$. Thus the desired particular solution is given by $y(x) = \dfrac{1}{3}\left[e^x + e^{\left(-1+i\sqrt{3}\right)x/2} + e^{\left(-1-i\sqrt{3}\right)x/2}\right]$, which (using Euler's relation) reduces to the given real-valued solution.

49. We adopt the same strategy as was used in Problem 48. The general solution is $y(x) = Ae^{2x} + Be^{-x} + C\cos x + D\sin x$. Imposition of the given initial conditions yields the equations

$$\begin{aligned} A + B + C \quad\quad &= 0 \\ 2A - B \quad\quad + D &= 0 \\ 4A + B - C \quad\quad &= 0 \\ 8A - B \quad\quad - D &= 30 \end{aligned}$$

that we solve for $A = 2$, $B = -5$, $C = 3$, and $D = -9$. Thus

$$y(x) = 2e^{2x} - 5e^{-x} + 3\cos x - 9\sin x.$$

50. If $x > 0$, then the differential equation is $y'' + y = 0$, with general solution $y = A\cos x + B\sin x$. But if $x < 0$, then it is $y'' - y = 0$, with general solution $y = C\cosh x + D\sinh x$. To satisfy the initial conditions $y_1(0) = 1$, $y_1'(0) = 0$ we choose $A = C = 1$ and $B = D = 0$. But to satisfy the initial conditions $y_2(0) = 0$, $y_2'(0) = 1$ we choose $A = C = 0$ and $B = D = 1$. The corresponding solutions are defined by

$$y_1(x) = \begin{cases} \cos x, & x \ge 0 \\ \cosh x, & x \le 0 \end{cases}; \text{ and } y_2(x) = \begin{cases} \sin x, & x \ge 0 \\ \sinh x, & x \le 0 \end{cases}.$$

Examination of left- and right-hand derivatives at $x=0$ shows not only that $y_1(x)$ and $y_2(x)$ are differentiable at $x=0$, but that y_1' and y_2' are in fact continuous there.

51. In the solution of Problem 51 in Section 3.1 we showed that the substitution $v=\ln x$ gives $y'=\frac{dy}{dx}=\frac{1}{x}\frac{dy}{dv}$ and $y''=\frac{d^2y}{dx^2}=-\frac{1}{x^2}\cdot\frac{dy}{dv}+\frac{1}{x^2}\cdot\frac{d^2y}{dv^2}$. A further differentiation using the chain rule gives

$$y'''=\frac{d^3y}{dx^3}=\frac{2}{x^3}\cdot\frac{dy}{dv}-\frac{3}{x^3}\cdot\frac{d^2y}{dv^2}+\frac{1}{x^3}\cdot\frac{d^3y}{dv^3}.$$

Substitution of these expressions for y', y'', and y''' into the third-order Euler equation $ax^3y'''+bx^2y''+cxy'+dy=0$, together with collection of coefficients, yields the desired constant-coefficient equation

$$a\frac{d^3y}{dv^3}+(b-3a)\frac{d^2y}{dv^2}+(c-b+2a)\frac{dy}{dv}+d\cdot y=0.$$

In Problems 52 through 58 we list first the transformed constant-coefficient equation, then its characteristic equation and roots, and finally the corresponding general solution with $v=\ln x$ and $e^v=x$.

52. $\frac{d^2y}{dv^2}+9y=0$; $r^2+9=0$; $r=\pm 3i$;

$$y(x)=c_1\cos 3v+c_2\sin 3v=c_1\cos(3\ln x)+c_2\sin(3\ln x)$$

53. $\frac{d^2y}{dv^2}+6\frac{dy}{dv}+25y=0$; $r^2+6r+25=0$; $r=-3\pm 4i$;

$$y(x)=e^{-3v}(c_1\cos 4v+c_2\sin 4v)=x^{-3}\left[c_1\cos(4\ln x)+c_2\sin(4\ln x)\right]$$

54. $\frac{d^3y}{dv^3}+3\frac{d^2y}{dv^2}=0$; $r^3+3r^2=0$; $r=0,0,-3$;

$$y(x)=c_1+c_2v+c_3e^{-3v}=c_1+c_2\ln x+c_3x^{-3}$$

55. $\frac{d^3y}{dv^3}-4\frac{d^2y}{dv^2}+4\frac{dy}{dv}=0$; $r^3-4r^2+4r=0$; $r=0,2,2$;

$$y(x)=c_1+c_2e^{2v}+c_3ve^{2v}=c_1+x^2(c_2+c_3\ln x)$$

56. $\dfrac{d^3y}{dv^3}=0$**;** $r^3=0$**;** $r=0,0,0$**;**

$$y(x)=c_1+c_2v+c_3v^2=c_1+c_2\ln x+c_3(\ln x)^2$$

57. $\dfrac{d^3y}{dv^3}-5\dfrac{d^2y}{dv^2}+5\dfrac{dy}{dv}=0$**;** $r^3-4r^2+4r=0$**;** $r=0,3\pm\sqrt{3}$**;**

$$y(x)=c_1+c_2e^{(3-\sqrt{3})v}+c_3ve^{(3+\sqrt{3})v}=c_1+x^3\left(c_2x^{-\sqrt{3}}+c_3x^{+\sqrt{3}}\right)$$

58. $\dfrac{d^3y}{dv^3}+3\dfrac{d^2y}{dv^2}+3\dfrac{dy}{dv}+y=0$**;** $r^3+3r^2+3r+1=0$**;** $r=-1,-1,-1$**;**

$$y(x)=c_1e^{-v}+c_2ve^{-v}+c_3v^2e^{-v}=x^{-1}\left[c_1+c_2\ln x+c_3(\ln x)^2\right]$$

SECTION 3.4

MECHANICAL VIBRATIONS

In this section we discuss four types of free motion of a mass on a spring—undamped, underdamped, critically damped, and overdamped. However, the undamped and underdamped cases—in which actual oscillations occur—are emphasized because they are both the most interesting and the most important cases for applications.

1. Frequency: $\omega_0=\sqrt{\dfrac{k}{m}}=\sqrt{\dfrac{16}{4}}=2\,\text{rad/sec}=\dfrac{1}{\pi}\,\text{Hz}$; period: $P=\dfrac{2\pi}{\omega_0}=\dfrac{2\pi}{2}=\pi\sec$

2. Frequency $\omega_0=\sqrt{\dfrac{k}{m}}=\sqrt{\dfrac{48}{0.75}}=8\,\text{rad/sec}=\dfrac{4}{\pi}\,\text{Hz}$; period: $P=\dfrac{2\pi}{\omega_0}=\dfrac{2\pi}{8}=\dfrac{\pi}{4}\sec$

3. The spring constant is $k=\dfrac{15\,N}{0.20\,\text{m}}=75\,\text{N/m}$. The solution of $3x''+75x=0$ with $x(0)=0$ and $x'(0)=-10$ is $x(t)=-2\sin 5t$. Thus the amplitude is 2 m, the frequency is $\omega_0=\sqrt{\dfrac{k}{m}}=\sqrt{\dfrac{75}{3}}=5\,\text{rad/sec}=\dfrac{2.5}{\pi}\,\text{Hz}$, and the period is $\dfrac{2\pi}{5}\sec$.

4. **(a)** With $m=\dfrac{1}{4}\,\text{kg}$ and $k=9\,\text{N}\cdot 0.25\,\text{m}{=}36\,\text{N/m}$, we find that $\omega_0=12\,\text{rad/sec}$. The solution of $x''+144x=0$ with $x(0)=1$ and $x'(0)=-5$ is

$$x(t) = \cos 12 - \frac{5}{12}\sin 12t = \frac{13}{12}\left(\frac{12}{13}\cos 12t - \frac{5}{13}\sin 2t\right) = \frac{13}{12}\cos(12t - \alpha),$$

where $\alpha = 2\pi - \tan^{-1}\frac{5}{12} \approx 5.8884\,\text{rad}$.

(b) $C = \frac{13}{12} \approx 1.0833\,\text{m}$ and $T = \frac{2\pi}{12} \approx 0.5236\,\text{sec}$.

5. The gravitational acceleration at distance R from the center of the earth is $g = \frac{GM}{R^2}$. According to Equation (6) in the text, the (circular) frequency ω of a (linearized) pendulum is given by $\omega^2 = \frac{g}{L} = \frac{GM}{R^2 L}$, so its period is $p = \frac{2\pi}{\omega} = 2\pi R\sqrt{\frac{L}{GM}}$.

6. If the pendulum in the clock executes n cycles per day (86400 sec) at Paris, then its period is $p_1 = \frac{86400}{n}\text{sec}$. At the equatorial location it takes $24\,\text{hr}\,2\,\text{min}\,40\,\text{sec} = 86560\,\text{sec}$ for the same number of cycles, so its period there is $p_1 = \frac{86560}{n}\text{sec}$. Now let $R_1 = 3956\,\text{mi}$ be the Earth's "radius" at Paris, and R_2 its "radius" at the equator. Then substitution in the equation $\frac{p_1}{p_2} = \frac{R_1}{R_2}$ of Problem 5 (with $L_1 = L_2$) yields $R_2 = 3963.33\,\text{mi}$. Thus this (rather simplistic) calculation gives $7.33\,\text{mi}$ as the thickness of the Earth's equatorial bulge.

7. The period equation $p = 3960\sqrt{100.10} = (3960 + x)\sqrt{100}$ yields $x \approx 1.9795\,\text{mi} \approx 10.450\,\text{ft}$ for the altitude of the mountain.

8. Let n be the number of cycles required for a correct clock with unknown pendulum length L_1 and period p_1 to register $24\,\text{hrs} = 86400\,\text{sec}$, so $np_1 = 86400$. The given clock with length $L_2 = 30\,\text{in}$ and period p_2 loses $10\,\text{min} = 600\,\text{sec}$ per day, so $np_2 = 87000$. Then the formula of Problem 5 yields $\sqrt{\frac{L_1}{L_2}} = \frac{p_1}{p_2} = \frac{np_1}{np_2} = \frac{86400}{87000}$, so

$$L_1 = 30 \cdot \left(\frac{86400}{87000}\right)^2 \approx 29.59\,\text{in}.$$

9. Designating $x(t)$ as in the suggestion, we see that the mass is subject to a restorative force $F_S = -kx$ together with the force of gravity $W = mg$. We also assume that the mass is subject to a damping force $F_R = -cx'$. Applying Newton's law then gives

$mx'' = -kx + mg - cx'$, or $mx'' + cx' + kx = mg$. Finally, substituting $y = x - s_0$, so that $x = y + s_0$ and thus $x' = y'$ and $x'' = y''$, yields $my'' + cy' + k(y + s_0) = mg$, or $my'' + cy' + ky = mg - ks_0$, which is Equation (5) with $F(t)$ assuming the constant value $mg - ks_0$.

10. The mass of the buoy is given by $m = \rho\pi r^2 h$, and the net downward force on the buoy is $F = \rho\pi r^2 hg - \pi r^2 g \cdot x$. (Note that the depth $x(t)$ is taken to be positive.) Therefore Newton's second law $ma = F$ gives

$$\rho\pi r^2 h \cdot x'' = \rho\pi r^2 hg - \pi r^2 g \cdot x,$$

which simplifies to

$$x'' + \frac{g}{\rho h}x = g.$$

The complementary function for this equation is $x_c(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$, where $\omega_0 = \sqrt{\dfrac{g}{\rho h}}$. A particular solution is given by $x_p(t) = A$, where A is a constant, and substituting into the differential equation shows that $A = \rho h$. Thus the general solution of the differential equation is

$$x(t) = x_c(t) + x_p(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t + \rho h.$$

Applying the initial conditions $x(0) = x'(0) = 0$ gives $c_1 = -\rho h$ and $c_2 = 0$. All told, the motion of the buoy is given by

$$x(t) = -\rho h \cos\omega_0 t + \rho h.$$

Thus the buoy undergoes simple harmonic motion about an equilibrium of $x_e = \rho h$. Further, with the given numerical values of ρ, h, and g, the amplitude of oscillation is $\rho h = 100\,\text{cm}$ and the period is $p = \dfrac{2\pi}{\omega_0} = \dfrac{2\pi}{\sqrt{\dfrac{g}{\rho h}}} = 2\pi\sqrt{\dfrac{\rho h}{g}} \approx 2.01\,\text{sec}$.

11. The differential equation from Problem 10 must be modified to reflect the fact that the weight density of water is $62.4\,\text{lb}/\text{ft}^3$ (as opposed to $1\,\text{g}/\text{cm}^3$ in the cgs system). Thus the weight of water displaced by the buoy is given by $62.4\pi r^2 \cdot x$. Moreover, the mass and weight of the buoy are given to be 3.125 slugs and 100 lb, respectively. Applying $ma = F$ then gives $3.125x'' = 100 - 62.4\pi r^2 \cdot x$, or $x'' + \dfrac{62.4\pi}{3.125}r^2 \cdot x = 32$. The frequency of the oscillations of the buoy is therefore $\dfrac{\omega_0}{2\pi}$, where $\omega_0 = \sqrt{\dfrac{62.4\pi}{3.125}} \cdot r$. Since the frequency of the buoy's motion is observed to be $\frac{4\,\text{cycles}}{10\,\text{sec}} = 0.4\,\text{cycles/sec}$, we can equate the two to conclude that $\frac{1}{2\pi}\sqrt{\frac{62.4\pi}{3.125}\cdot r} = 0.4$, which gives

$r = 0.8\sqrt{\frac{3.125\pi}{62.4}} \approx 0.3173\,\text{ft} \approx 3.8\,\text{in}$.

12. **(a)** Substitution of $M_r = \left(\frac{r}{R}\right)^3 M$ in $F_r = -\frac{GM_r m}{r^2}$ yields $F_r = -\frac{GMm}{R^3}r$.

(b) Because $\frac{GM}{R^3} = \frac{g}{R}$, the equation $mr'' = F_r$ yields the differential equation

$r'' + \frac{g}{R}r = 0$.

(c) The solution of this equation with $r(0) = R$ and $r'(0) = 0$ is $r(t) = R\cos\omega_0 t$, where $\omega_0 = \sqrt{\frac{g}{R}}$. Hence, with $g = 32.2\,\text{ft/sec}^2$ and $R = 3960\cdot 5280\,\text{ft}$, we find that the period of the particle's simple harmonic motion is

$$p = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{R}{g}} \approx 5063.10\,\text{sec} \approx 84.38\,\text{min}.$$

(d) The orbital velocity v of such a satellite must be such that the centrifugal force $\frac{mv^2}{R}$ on the satellite just offsets the weight mg of the satellite at the surface of the earth. Thus $\frac{mv^2}{R} = mg$, which implies that

$v = \sqrt{gR} \approx \sqrt{32.2\,\text{ft/sec}^2\cdot 3960\cdot 5280\,\text{ft}} \approx 2.5947\times 10^4\,\text{ft/sec} \approx 1.7691\times 10^4\,\text{mi/hr}$.

Because the circumference of the earth is $2\pi R$, the period of the satellite's orbit is $\frac{2\pi R}{\sqrt{gR}} = 2\pi\sqrt{\frac{R}{g}}$, which is equal to the period of the particle found in part **c**. This is not a coincidence. Imagine that at time $t = 0$ the satellite is directly over the hole in the earth at the top of Figure 3.4.13, and that its orbit proceeds in a clockwise direction. We found in part **c** that the distance r of the particle from the center of the earth is $r(t) = R\cos\omega_0 t$. The key observation is that $\omega_0 t$ is the angle drawn clockwise from the vertical to the radius vector of the satellite at time t; thus, the distance $r(t)$ is simply the vertical component of the satellite's position. It follows that $r(t)$ completes one cycle through the earth (and back) in the same length of time required for the satellite to complete one orbit around the earth.

(e) The particle passes through the center of the earth when $r(t) = R\cos\omega_0 t = 0$, that is, when $\omega_0 t = \dfrac{\pi}{2}$, or $t = \dfrac{\pi}{2\omega_0}$. At this time the speed of the particle is

$$|r'(t)| = |-R\omega_0 \sin\omega_0 t| = \left|-R\omega_0 \sin\left(\omega_0 \cdot \frac{\pi}{2\omega_0}\right)\right| = R\sqrt{\frac{g}{R}} = \sqrt{gR} \approx 1.7691\times 10^4 \text{ mi/hr}.$$

(f) In part **d** we found the orbital velocity to be $v = \sqrt{gR}$, in agreement with part **e**. Again this is not a concidence. The vertical component of the satellite's velocity vector $\mathbf{v}(t)$ at any given time t is equal to the speed $|r'(t)|$ of the particle at that time. At the moment when the particle passes through the center of earth, the satellite is travelling straight downward, and hence $\mathbf{v}(t)$ is vertical. Therefore the orbital velocity v of the satellite, which is the magnitude of $\mathbf{v}(t)$, is equal to the speed of the particle at this moment.

13. **(a)** The characteristic equation $10r^2 + 9r + 2 = (5r+2)(2r+1) = 0$ has roots $r = -\dfrac{2}{5}, -\dfrac{1}{2}$. When we impose the initial conditions $x(0) = 0$, $x'(0) = 5$ on the general solution $x(t) = c_1 e^{-2t/5} + c_2 e^{-t/2}$ we get the particular solution $x(t) = 50\left(e^{-2t/5} - e^{-t/2}\right)$.

(b) The derivative $x'(t) = 25e^{-t/2} - 20e^{-2t/5} = 5e^{-2t/5}\left(5e^{-t/10} - 4\right) = 0$ when $t = 10\ln\dfrac{5}{4} \approx 2.23144$. Hence the mass's farthest distance to the right is given by $x\left(10\ln\dfrac{5}{4}\right) = \dfrac{512}{125} = 4.096$.

14. **(a)** The characteristic equation $25r^2 + 10r + 226 = (5r+1)^2 + 15^2 = 0$ has roots $r = \dfrac{-1 \pm 15i}{5} = -\dfrac{1}{5} \pm 3i$. When we impose the initial conditions $x(0) = 20$, $x'(0) = 41$ on the general solution $x(t) = e^{-t/5}(A\cos 3t + B\sin 3t)$ we get $A = 20$, $B = 15$. The corresponding particular solution is given by

$$x(t) = e^{-t/5}(20\cos 3t + 15\sin 3t) = 25e^{-t/5}\cos(3t - \alpha),$$

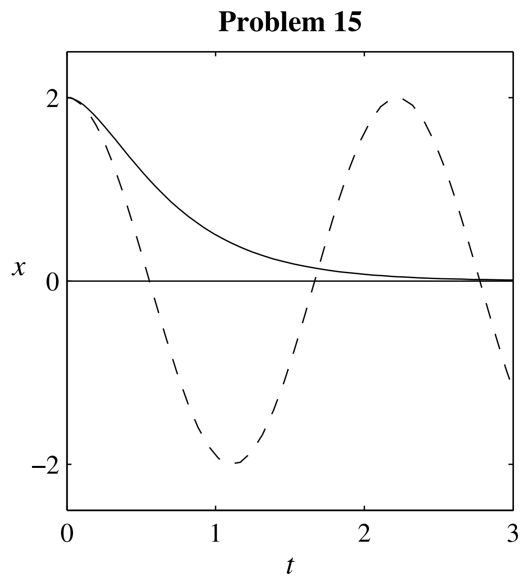
where $\alpha = \tan^{-1}\dfrac{3}{4} \approx 0.6435$.

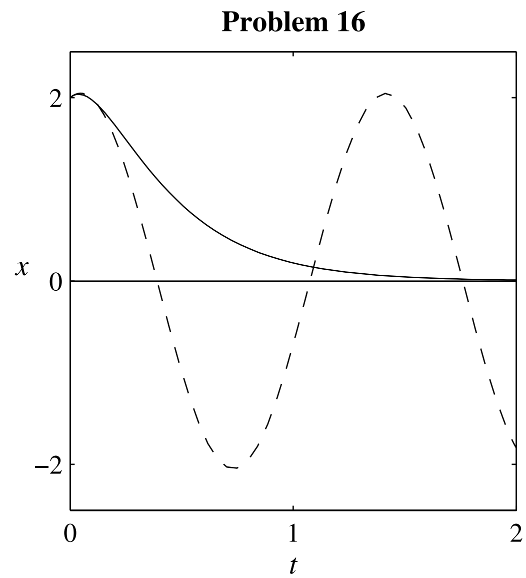
(b) Thus the oscillations are "bounded" by the curves $x = \pm 25e^{-t/5}$, and the pseudoperiod of oscillation is $T = \dfrac{2\pi}{3}$ (because $\omega = 3$).

In Problems 15-21 the graph of the damped motion $x(t)$, that is, with the dashpot attached, is shown as a solid line; the graph of the corresponding undamped motion $u(t)$ is dashed.

15. **With damping:** The characteristic equation $\frac{1}{2}r^2 + 3r + 4 = 0$ has roots $r = -2, -4$. When we impose the initial conditions $x(0) = 2$, $x'(0) = 0$ on the general solution $x(t) = c_1 e^{-2t} + c_2 e^{-4t}$ we get the particular solution $x(t) = 4e^{-2t} - 2e^{-4t}$ that describes overdamped motion.

Without damping: The characteristic equation $\frac{1}{2}r^2 + 4 = 0$ has roots $r = \pm 2i\sqrt{2}$. When we impose the initial conditions $x(0) = 2$, $x'(0) = 0$ on the general solution $u(t) = A\cos\left(2\sqrt{2}t\right) + B\sin\left(2\sqrt{2}t\right)$ we get the particular solution $u(t) = 2\cos\left(2\sqrt{2}t\right)$.





16. **With damping:** The characteristic equation $3r^2 + 30r + 63 = 0$ has roots $r = -3, -7$. When we impose the initial conditions $x(0) = 2$, $x'(0) = 2$ on the general solution $x(t) = c_1 e^{-3t} + c_2 e^{-7t}$ we get the particular solution $x(t) = 4e^{-3t} - 2e^{-7t}$ that describes overdamped motion.

Without damping: The characteristic equation $3r^2 + 63 = 0$ has roots $r = \pm i\sqrt{21}$. When we impose the initial conditions $x(0) = 2$, $x'(0) = 2$ on the general solution $u(t) = A\cos\left(\sqrt{21}t\right) + B\sin\left(\sqrt{21}t\right)$ we get the particular solution

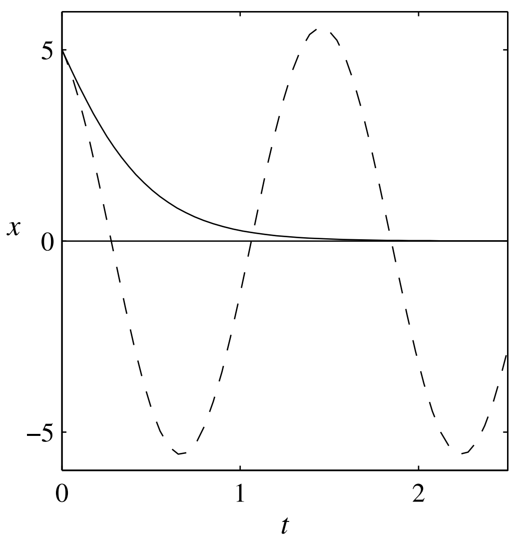
$$u(t) = 2\cos\left(\sqrt{21}t\right) + \frac{2}{\sqrt{21}}\sin\left(\sqrt{21}t\right) \approx 2\sqrt{\frac{22}{21}}\cos\left(\sqrt{21}t - 0.2149\right).$$

17. **With damping:** The characteristic equation $r^2 + 8r + 16 = 0$ has roots $r = -4, -4$. When we impose the initial conditions $x(0) = 5$, $x'(0) = -10$ on the general solution $x(t) = (c_1 + c_2 t)e^{-4t}$ we get the particular solution $x(t) = 5e^{-4t}(2t+1)$ that describes critically damped motion.

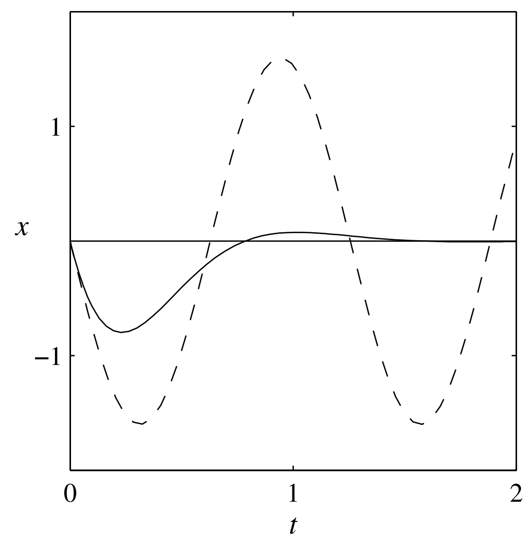
Without damping: The characteristic equation $r^2 + 16 = 0$ has roots $r = \pm 4i$. When we impose the initial conditions $x(0) = 5$, $x'(0) = -10$ on the general solution $u(t) = A\cos 4t + B\sin 4t$ we get the particular solution

$$u(t) = 5\cos 4t - \frac{5}{2}\sin 4t \approx \frac{5}{2}\sqrt{5}\cos(4t - 5.8195).$$

Problem 17



Problem 18



18. **With damping:** The characteristic equation $2r^2 + 12r + 50 = 0$ has roots $r = -3 \pm 4i$. When we impose the initial conditions $x(0) = 0$, $x'(0) = -8$ on the general solution $x(t) = e^{-3t}(A\cos 4t + B\sin 4t)$ we get the particular solution

$$x(t) = -2e^{-3t}\sin 4t = 2e^{-3t}\cos\left(4t - \frac{3\pi}{2}\right)$$

that describes underdamped motion.

Without damping: The characteristic equation $2r^2 + 50 = 0$ has roots $r = \pm 5i$. When we impose the initial conditions $x(0) = 0$, $x'(0) = -8$ on the general solution $u(t) = A\cos 5t + B\sin 5t$ we get the particular solution

$$u(t) = -\frac{8}{5}\sin 5t = \frac{8}{5}\cos\left(5t - \frac{3\pi}{2}\right).$$

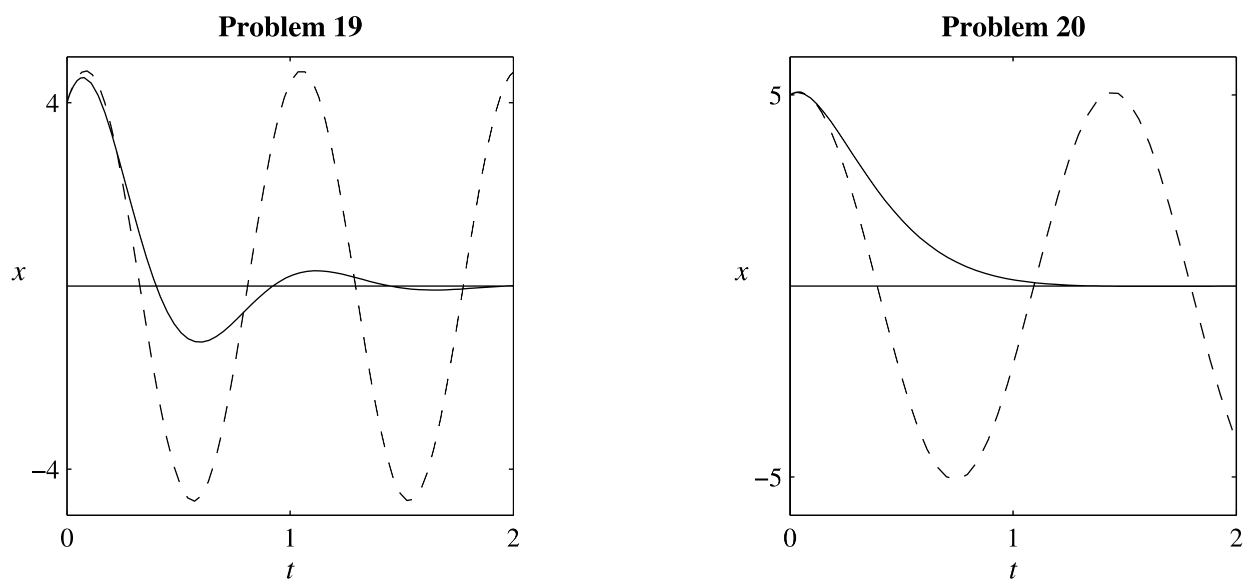
19. The characteristic equation $4r^2 + 20r + 169 = 0$ has roots $r = -\frac{5}{2} \pm 6i$. When we impose the initial conditions $x(0) = 4$, $x'(0) = 16$ on the general solution $x(t) = e^{-5t/2}(A\cos 6t + B\sin 6t)$ we get the particular solution

$$x(t) = e^{-5t/2}\left(4\cos 6t + \frac{13}{3}\sin 6t\right) \approx \frac{1}{3}\sqrt{313}\, e^{-5t/2}\cos(6t - 0.8254)$$

that describes underdamped motion.

Without damping: The characteristic equation $4r^2 + 169 = 0$ has roots $r = \pm\frac{13}{2}i$. When we impose the initial conditions $x(0) = 4$, $x'(0) = 16$ on the general solution $u(t) = A\cos\left(\frac{13}{2}t\right) + B\sin\left(\frac{13}{2}t\right)$ we get the particular solution

$$u(t) = 4\cos\left(\frac{13t}{2}\right) + \frac{32}{13}\sin\left(\frac{13t}{2}\right) \approx \frac{4}{13}\sqrt{233}\cos\left(\frac{13}{2}t - 0.5517\right).$$



20. **With damping:** The characteristic equation $2r^2 + 16r + 40 = 0$ has roots $r = -4 \pm 2i$. When we impose the initial conditions $x(0) = 5$, $x'(0) = 4$ on the general solution $x(t) = e^{-4t}(A\cos 2t + B\sin 2t)$ we get the particular solution

$$x(t) = e^{-4t}(5\cos 2t + 12\sin 2t) \approx 13e^{-4t}\cos(2t - 1.1760)$$

that describes underdamped motion.

Without damping: The characteristic equation $2r^2 + 40 = 0$ has roots $r = \pm 2\sqrt{5}i$. When we impose the initial conditions $x(0) = 5$, $x'(0) = 4$ on the general solution $u(t) = A\cos\left(2\sqrt{5}t\right) + B\sin\left(2\sqrt{5}t\right)$ we get the particular solution

$$u(t) = 5\cos\left(2\sqrt{5}t\right) + \frac{2}{\sqrt{5}}\sin\left(2\sqrt{5}t\right) \approx \sqrt{\frac{129}{5}}\cos\left(2\sqrt{5}t - 0.1770\right).$$

21. **With damping:** The characteristic equation $r^2 + 10r + 125 = 0$ has roots $r = -5 \pm 10i$. When we impose the initial conditions $x(0) = 6$, $x'(0) = 50$ on the general solution $x(t) = e^{-5t}\left(A\cos 10t + B\sin 10t\right)$ we get the particular solution

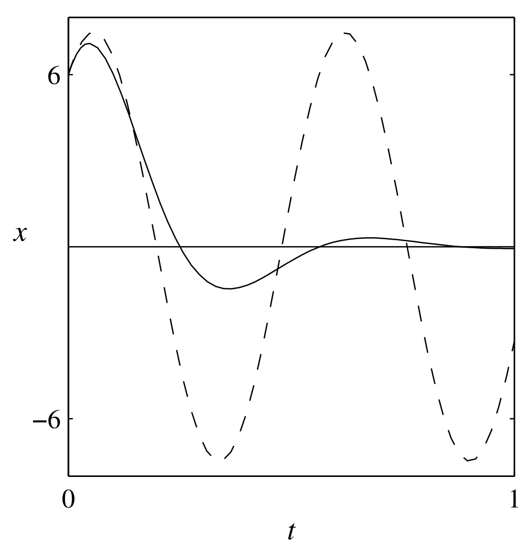
$$x(t) = e^{-5t}\left(6\cos 10t + 8\sin 10t\right) \approx 10e^{-5t}\cos\left(10t - 0.9273\right)$$

that describes underdamped motion.

Without damping: The characteristic equation $r^2 + 125 = 0$ has roots $r = \pm 5\sqrt{5}i$. When we impose the initial conditions $x(0) = 6$, $x'(0) = 50$ on the general solution $u(t) = A\cos\left(5\sqrt{5}\,t\right) + B\sin\left(5\sqrt{5}\,t\right)$ we get the particular solution

$$u(t) = 6\cos\left(5\sqrt{5}\,t\right) + 2\sqrt{5}\sin\left(5\sqrt{5}\,t\right) \approx 2\sqrt{14}\cos\left(5\sqrt{5}\,t - 0.6405\right).$$

Problem 21



22. **(a)** With $m = \frac{12}{32} = 0.375\,\text{slug}$, $c = 3\,\text{lb-sec/ft}$, and $k = 24\,\text{lb/ft}$, the differential equation is equivalent to $3x'' + 24x' + 192x = 0$. The characteristic equation $3r^2 + 24r + 192 = 0$ has roots $r = -4 \pm 4\sqrt{3}i$. When we impose the initial conditions $x(0) = 1$, $x'(0) = 0$ on the general solution $x(t)=e^{-4t}\left[A\cos\left(4\sqrt{3}\,t\right)+B\sin\left(4\sqrt{3}\,t\right)\right]$ we get the particular solution

$$\begin{aligned} x(t) &= e^{-4t}\left[\cos\left(4\sqrt{3}\,t\right)+\frac{1}{\sqrt{3}}\sin\left(4\sqrt{3}\,t\right)\right] \\ &= \frac{2}{\sqrt{3}}e^{-4t}\left[\frac{\sqrt{3}}{2}\cos\left(4\sqrt{3}\,t\right)+\frac{1}{2}\sin\left(4\sqrt{3}\,t\right)\right] \\ &= \frac{2}{\sqrt{3}}e^{-4t}\cos\left(4\sqrt{3}\,t-\frac{\pi}{6}\right). \end{aligned}$$

(b) The time-varying amplitude is $\frac{2}{\sqrt{3}}\approx 1.15\,\text{ft}$, the frequency is $4\sqrt{3}\approx 6.93\,\text{rad/sec}$, and the phase angle is $\frac{\pi}{6}$.

23. **(a)** With $m=100\,\text{slug}$ we get $\omega=\sqrt{\frac{k}{100}}$. But we are given that

$$\omega=(80\,\text{cycles/min})\cdot 2\pi\cdot(1\,\text{min}/60\,\text{sec})=\frac{8\pi}{3},$$

and equating the two values yields $k\approx 7018\,\text{lb/ft}$.

(b) With $\omega_1=2\pi\cdot\frac{78}{60}\,\text{cycles/sec}$, Equation (21) in the text yields $c\approx 372.31\,\text{lb}/(\text{ft/sec})$. Hence $p=\frac{c}{2m}\approx 1.8615$. Finally $e^{-pt}=0.01$ gives $t\approx 2.47\,\text{sec}$.

30. In the underdamped case we have

$$x(t)=e^{-pt}\left(A\cos\omega_1 t+B\sin\omega_1 t\right)$$

and

$$x'(t)=-pe^{-pt}\left(A\cos\omega_1 t+B\sin\omega_1 t\right)+e^{-pt}\left(-A\omega_1\sin\omega_1 t+B\omega_1\cos\omega_1 t\right).$$

The conditions $x(0)=x_0$, $x'(0)=v_0$ yield the equations $A=x_0$ and $-pA+B\omega_1=v_0$, whence $B=\frac{v_0+px_0}{\omega_1}$.

31. The binomial series

$$(1+x)^{\alpha}=1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3+\cdots$$

converges if $|x|<1$. (See, for instance, Section 10.8 of Edwards and Penney, Calculus: Early Transcendentals, 7th edition, Pearson, 2008.) With $\alpha=\frac{1}{2}$ and $x=-\frac{c^2}{4mk}$ in Eq. (22) of Section 3.4 in the differential equations text, the binomial series gives

$$\omega_1=\sqrt{\omega_0^2-p^2}=\sqrt{\frac{k}{m}-\frac{c^2}{4m^2}}=\sqrt{\frac{k}{m}}\sqrt{1-\frac{c^2}{4mk}}=\sqrt{\frac{k}{m}}\left(1-\frac{c^2}{8mk}-\frac{c^4}{128m^2k^2}-\cdots\right)\approx\omega_0\left(1-\frac{c^2}{8mk}\right).$$

32. If $x(t)=Ce^{-pt}\cos(\omega_1 t-\alpha)$, then

$$x'(t)=-pCe^{-pt}\cos(\omega_1 t-\alpha)+C\omega_1 e^{-pt}\sin(\omega_1 t-\alpha)=0$$

yields $\tan(\omega_1 t-\alpha)=-\frac{p}{\omega_1}$. Because the tangent function is periodic with period π and the local maxima and minima of $x(t)$ are interlaced, successive maxima are separated by a distance of $\frac{2\pi}{\omega_1}$.

33. If $x_1=x(t_1)$ and $x_2=x(t_2)$ are two successive local maxima, then $\omega_1 t_2=\omega_1 t_1+2\pi$, and so $x_1=Ce^{-pt_1}\cos(\omega_1 t_1-\alpha)$ and $x_2=Ce^{-pt_2}\cos(\omega_1 t_2-\alpha)=Ce^{-pt_2}\cos(\omega_1 t_1-\alpha)$. Hence $\frac{x_1}{x_2}=e^{-p(t_1-t_2)}$ and therefore $\ln\left(\frac{x_1}{x_2}\right)=-p(t_1-t_2)=\frac{2\pi p}{\omega_1}$.

34. With $t_1=0.34$ and $t_2=1.17$ we first use the equation $\omega_1 t_2=\omega_1 t_1+2\pi$ from Problem 33 to calculate $\omega_1=\frac{2\pi}{0.83}\approx 7.57\text{ rad/sec}$. Next, with $x_1=6.73$ and $x_2=1.46$, the result of Problem 33 yields $p=\frac{1}{0.83}\ln\left(\frac{6.73}{1.46}\right)\approx 1.84$. Then Equation (16) in this section gives $c=2mp=2\cdot\frac{100}{32}\cdot 1.84\approx 11.51\text{ lb-sec/ft}$, and finally Equation (22) yields $k=\frac{4m^2\omega_1^2+c^2}{4m}\approx 189.68\text{ lb/ft}$.

35. The characteristic equation $r^2+2r+1=0$ has roots $r=-1,-1$. When we impose the initial conditions $x(0)=0$, $x'(1)=0$ on the general solution $x(t)=(c_1+c_2t)e^{-t}$ we get the particular solution $x_1(t)=te^{-t}$.

36. The characteristic equation $r^2+2r+\left(1-10^{-2n}\right)=0$ has roots $r=-1\pm 10^{-n}$. When we impose the initial conditions $x(0)=0$, $x'(1)=0$ on the general solution

$$x(t) = c_1 \exp\left[\left(-1+10^{-n}\right)t\right] + c_2 \exp\left[\left(-1-10^{-n}\right)t\right]$$

we get the equations

$$c_1 + c_2 = 0, \quad \left(-1+10^{-n}\right)c_1 + \left(-1-10^{-n}\right)c_2 = 1$$

with solution $c_1 = 2^{n-1}5^n$, $c_2 = 2^{n-1}5^n$. This gives the particular solution

$$x_2(t) = 10^n e^{-t}\left[\frac{\exp\left(10^{-n}t\right) - \exp\left(-10^{-n}t\right)}{2}\right] = 10^n e^{-t}\sinh\left(10^{-n}t\right).$$

37. The characteristic equation $r^2 + 2r + \left(1+10^{-2n}\right) = 0$ has roots $r = -1 \pm 10^{-n}i$. When we impose the initial conditions $x(0) = 0$, $x'(1) = 0$ on the general solution

$$x(t) = e^{-t}\left[A\cos\left(10^{-n}t\right) + B\sin\left(10^{-n}t\right)\right]$$

we get the equations $c_1 = 0$, $-c_1 + 10^{-n}c_2 = 1$ with solution $c_1 = 0$, $c_2 = 10^n$. This gives the particular solution $x_3(t) = 10^n e^{-t}\sin\left(10^{-n}t\right)$.

38. This follows from

$$\lim_{n\to\infty} x_2(t) = \lim_{n\to\infty} 10^n e^{-t}\sinh\left(10^{-n}t\right) = te^{-t}\cdot\lim_{n\to\infty}\frac{\sinh\left(10^{-n}t\right)}{10^{-n}t} = te^{-t}$$

and

$$\lim_{n\to\infty} x_3(t) = \lim_{n\to\infty} 10^n e^{-t}\sin\left(10^{-n}t\right) = te^{-t}\cdot\lim_{n\to\infty}\frac{\sin\left(10^{-n}t\right)}{10^{-n}t} = te^{-t},$$

using the fact that $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \lim_{\theta\to 0}\frac{\sinh\theta}{\theta} = 0$ (by L'Hôpital's rule, for instance).

SECTION 3.5

NONHOMOGENEOUS EQUATIONS AND UNDETERMINED COEFFICIENTS

The method of undetermined coefficients is based on "educated guessing". If we can guess correctly the **form** of a particular solution of a nonhomogeneous linear equation with constant coefficients, then we can determine the particular solution explicitly by substitution in the given differential equation. It is pointed out at the end of Section 3.5 that this simple approach is not always successful—in which case the method of variation of parameters is available if a complementary function is known. However, undetermined coefficients *does* turn out to work well with a surprisingly large number of the nonhomogeneous linear differential equations that arise in elementary scientific applications.

In each of Problems 1-20 we give first the form of the trial solution y_{trial}, then the equations in the coefficients we get when we substitute y_{trial} into the differential equation and collect like terms, and finally the resulting particular solution y_p.

1. $y_{\text{trial}} = Ae^{3x}$; $25A = 1$; $y_p = \dfrac{1}{25}e^{3x}$.

2. $y_{\text{trial}} = A + Bx$; $-2A - B = 4$, $-2B = 3$; $y_p = -\dfrac{1}{4}(5 + 6x)$.

3. $y_{\text{trial}} = A\cos 3x + B\sin 3x$; $-15A - 3B = 0$, $3A - 15B = 2$; $y_p = \dfrac{1}{39}\cos 3x - \dfrac{5}{39}\sin 3x$.

4. $y_{\text{trial}} = Ae^x + Bxe^x$; $9A + 12B = 0$, $9B = 3$, $y_p = -\dfrac{4}{9}e^x + \dfrac{1}{3}xe^x$.

5. First we substitute $\dfrac{1 - \cos 2x}{2}$ for $\sin^2 x$ on the right-hand side of the differential equation, leading to $y_{\text{trial}} = A + B\cos 2x + C\sin 2x$, and then

$$A = \frac{1}{2}, \quad -3B + 2C = -\frac{1}{2}, \quad -2B - 3C = 0;$$

$$y_p = \frac{1}{2} + \frac{3}{26}\cos 2x - \frac{1}{13}\sin 2x.$$

6. $y_{\text{trial}} = A + Bx + Cx^2$; $7A + 4B + 4C = 0$, $7B + 8C = 0$, $7C = 1$; $y_p = \dfrac{4}{343} - \dfrac{8}{49}x + \dfrac{1}{7}x^2$.

7. First we substitute $\dfrac{e^x - e^{-x}}{2}$ for $\sinh x$ on the right-hand side of the differential equatio, leading to $y_{\text{trial}} = Ae^x + Be^{-x}$; $-3A = \dfrac{1}{2}$, $-3B = -\dfrac{1}{2}$; $y_p = \dfrac{1}{6}e^{-x} - \dfrac{1}{6}e^x = -\dfrac{1}{3}\sinh x$. (Note that according to Rule 1 in the text, we could also have started with $y_{\text{trial}} = A\cosh x + B\sinh x$.)

8. First we note that $\cosh 2x = \dfrac{e^{2x} + e^{-2x}}{2}$ is part of the complementary function $y_c = c_1e^{2x} + c_2e^{-2x}$, leading to $y_{\text{trial}} = x\left(Ae^{2x} + Be^{-2x}\right)$; $A = \dfrac{1}{8}$, $B = -\dfrac{1}{8}$;

$y_p = x\left(\frac{1}{8}e^{2x} - \frac{1}{8}e^{-2x}\right) = \frac{1}{4}x\sinh 2x$. (As an extension of Rule 2 in the text, we could also have started with $y_{\text{trial}} = x\left(A\cosh 2x + B\sinh 2x\right)$.)

9. First we note that e^x is part of the complementary function $y_c = c_1e^x + c_2e^{-3x}$. Then $y_{\text{trial}} = A + x\left(B + Cx\right)e^x$, and then

$$-3A = 1 \quad 4B + 2C = 0 \quad 8C = 1;$$

$$y_p = -\frac{1}{3} + x\left(-\frac{1}{16} + \frac{1}{8}x\right)e^x.$$

10. First we note the duplication with the complementary function $y_c = c_1\cos 3x + c_2\sin 3x$. Then $y_{\text{trial}} = x\left(A\cos 3x + B\sin 3x\right)$; $6B = 2$; $-6A = 3$;

$$y_p = x\left(\frac{1}{3}\cos 3x - \frac{1}{2}\sin 3x\right) = \frac{1}{6}\left(2x\sin 3x - 3x\cos 3x\right).$$

11. First we note the duplication with the complementary function $y_c = c_1 + c_2\cos 2x + c_3\sin 2x$. Then $y_{\text{trial}} = x\left(A + Bx\right)$; $4A = -1$, $8B = 3$; $y_p = x\left(-\frac{1}{4} + \frac{3}{8}x\right) = \frac{1}{8}\left(3x^2 - 2x\right)$.

12. First we note the duplication with the complementary function $y_c = c_1 + c_2\cos x + c_3\sin x$. Then $y_{\text{trial}} = Ax + x\left(B\cos x + C\sin x\right)$; $A = 2$, $-2B = 0$, $-2C = -1$; $y_p = 2x + \frac{1}{2}x\sin x$.

13. $y_{\text{trial}} = e^x\left(A\cos x + B\sin x\right)$; $7A + 4B = 0$, $-4A + 7B = 1$; $y_p = \frac{1}{65}e^x\left(7\sin x - 4\cos x\right)$.

14. First we note the duplication with the complementary function $y_c = \left(c_1 + c_2x\right)e^{-x} + \left(c_3 + c_4x\right)e^x$. Then $y_{\text{trial}} = x^2\left(A + Bx\right)e^x$; $8A + 24B = 0$, $24B = 1$; $y_p = x^2\left(-\frac{1}{8} + \frac{1}{24}x\right)e^x = \frac{1}{24}\left(-3x^2e^x + x^3e^x\right)$.

15. This is something of a trick problem. We cannot solve the characteristic equation $r^5 + 5r^4 - 1 = 0$ to find the complementary function, but we can see that the complementary function contains no constant term (why?). Hence we can take $y_{\text{trial}} = A$, leading immediately to the particular solution $y_p = -17$.

16. $y_{\text{trial}} = A + \left(B + Cx + Dx^2\right)e^{3x}$;

$$9A = 5, \quad 18B + 6C + 2D = 0, \quad 18C + 12D = 0, \quad 18D = 2;$$

$$y_p = \frac{5}{9} + \left(\frac{1}{81} - \frac{2}{27}x + \frac{1}{9}x^2\right)e^{3x} = \frac{1}{81}\left(45 + e^{3x} - 6xe^{3x} + 9x^2e^{3x}\right).$$

17. First we note the duplication with the complementary function $y_c = c_1 \cos x + c_2 \sin x$. Then $y_{\text{trial}} = x\left[(A + Bx)\cos x + (C + Dx)\sin x\right]$;

$$2B + 2C = 0 \quad 4D = 1 \quad -2A + 2D = 1 \quad -4B = 0;$$

$$y_p = x\left(-\frac{1}{4}\cos x + \frac{1}{4}x \sin x\right) = \frac{1}{4}\left(x^2 \sin x - x \cos x\right).$$

18. First we note the duplication with the complementary function $y_c = c_1e^{-x} + c_2e^{x} + c_3e^{-2x} + c_4e^{2x}$. Then $y_{\text{trial}} = x \cdot Ae^x + x(B + Cx)e^{2x}$;

$$-6A = 1, \quad 12B + 38C = 0, \quad 24C = -1;$$

$$y_p = x \cdot \left(-\frac{1}{6}\right)e^x + x\left(\frac{19}{144} - \frac{1}{24}x\right)e^{2x} = \frac{1}{144}\left(-24xe^x + 19xe^{2x} - 6x^2e^{2x}\right).$$

19. First we note the duplication with the part $c_1 + c_2x$ of the complementary function (which corresponds to the factor r^2 of the characteristic polynomial). Then $y_{\text{trial}} = x^2\left(A + Bx + Cx^2\right)$;

$$4A + 12B = -1, \quad 12B + 48C = 0, \quad 24C = 3;$$

$$y_p = x^2\left(\frac{5}{4} - \frac{1}{2}x + \frac{1}{8}x^2\right) = \frac{1}{8}\left(10x^2 - 4x^3 + x^4\right).$$

20. First we note that the characteristic polynomial $r^3 - r$ has the zero $r = 1$ corresponding to the duplicating part e^x of the complementary function. Then: $y_{\text{trial}} = A + x \cdot Be^x$;

$-A = 7$; $3B = 1$; $y_p = -7 + \frac{1}{3}xe^x$.

In Problems 21-30 we list first the complementary function y_c, then the initially proposed trial function y_i, and finally the actual trial function y_p, in which duplication with the complementary function has been eliminated.

21. $y_c = e^x\left(c_1 \cos x + c_2 \sin x\right)$; $y_i = e^x\left(A\cos x + B\sin x\right)$; $y_p = x \cdot e^x\left(A\cos x + B\sin x\right)$

22. $y_c = \left(c_1 + c_2 x + c_3 x^2\right) + c_4 e^x + c_5 e^{-x}$; $y_i = \left(A + Bx + Cx^2\right) + De^x$;
$y_p = x^3 \cdot \left(A + Bx + Cx^2\right) + x \cdot De^x$

23. $y_c = c_1 \cos 2x + c_2 \sin 2x$; $y_i = (A + Bx)\cos 2x + (C + Dx)\sin 2x$;
$y_p = x \cdot \left[(A + Bx)\cos 2x + (C + Dx)\sin 2x\right]$

24. $y_c = c_1 + c_2 e^{-3x} + c_3 e^{4x}$; $y_i = (A + Bx) + (C + Dx)e^{-3x}$; $y_p = x \cdot (A + Bx) + x \cdot (C + Dx)e^{-3x}$

25. $y_c = c_1 e^{-x} + c_2 e^{-2x}$; $y_i = (A + Bx)e^{-x} + (C + Dx)e^{-2x}$;
$y_p = x \cdot (A + Bx)e^{-x} + x \cdot (C + Dx)e^{-2x}$

26. $y_c = e^{3x}\left(c_1 \cos 2x + c_2 \sin 2x\right)$; $y_i = (A + Bx)e^{3x}\cos 2x + (C + Dx)e^{3x}\sin 2x$;
$y_p = x \cdot \left[(A + Bx)e^{3x}\cos 2x + (C + Dx)e^{3x}\sin 2x\right]$

27. $y_c = \left(c_1 \cos x + c_2 \sin x\right) + \left(c_3 \cos 2x + c_4 \sin 2x\right)$;
$y_i = (A\cos x + B\sin x) + (C\cos 2x + D\sin 2x)$;
$y_p = x \cdot \left[(A\cos x + B\sin x) + (C\cos 2x + D\sin 2x)\right]$

28. $y_c = \left(c_1 + c_2 x\right) + \left(c_3 \cos 3x + c_4 \sin 3x\right)$;
$y_i = \left(A + Bx + Cx^2\right)\cos 3x + \left(D + Ex + Fx^2\right)\sin 3x$;
$y_p = x \cdot \left[\left(A + Bx + Cx^2\right)\cos 3x + \left(D + Ex + Fx^2\right)\sin 3x\right]$

29. $y_c = \left(c_1 + c_2 x + c_3 x^2\right)e^x + c_4 e^{2x} + c_5 e^{-2x}$; $y_i = (A + Bx)e^x + Ce^{2x} + De^{-2x}$;
$y_p = x^3 \cdot (A + Bx)e^x + x \cdot \left(Ce^{2x}\right) + x \cdot \left(De^{-2x}\right)$

30. $y_c = \left(c_1 + c_2 x\right)e^{-x} + \left(c_3 + c_4 x\right)e^x$; $y_i = y_p = \left(A + Bx + Cx^2\right)\cos x + \left(D + Ex + Fx^2\right)\sin x$

In Problems 31-40 we list first the complementary function y_c, the trial solution y_{tr} for the method of undetermined coefficients, and the corresponding general solution $y_g = y_c + y_p$, where y_p results from determining the coefficients in y_{tr} so as to satisfy the given nonhomogeneous differential equation. Then we list the linear equations obtained by imposing the given initial conditions, and finally the resulting particular solution $y(x)$.

31. $y_c = c_1\cos 2x + c_2\sin 2x$; $y_{\text{tr}} = A + Bx$; $y_{\text{g}} = c_1\cos 2x + c_2\sin 2x + \dfrac{x}{2}$; $c_1 = 1$, $2c_2 + \dfrac{1}{2} = 2$;
$y(x) = \cos 2x + (3/4)\sin 2x + x/2$

32. $y_c = c_1e^{-x} + c_2e^{-2x}$; $y_{\text{tr}} = Ae^x$; $y_{\text{g}} = c_1e^{-x} + c_2e^{-2x} + \dfrac{1}{6}e^x$; $c_1 + c_2 + \dfrac{1}{6} = 0$,
$-c_1 - 2c_2 + \dfrac{1}{6} = 3$; $y(x) = \dfrac{5}{2}e^{-x} - \dfrac{8}{3}e^{-2x} + \dfrac{1}{6}e^x$

33. $y_c = c_1\cos 3x + c_2\sin 3x$; $y_{\text{tr}} = A\cos 2x + B\sin 2x$; $y_{\text{g}} = c_1\cos 3x + c_2\sin 3x + \dfrac{1}{5}\sin 2x$;
$c_1 = 1$, $3c_2 + \dfrac{2}{5} = 0$, $y(x) = \cos 3x - \dfrac{2}{15}\sin 3x + \dfrac{1}{5}\sin 2x$

34. $y_c = c_1\cos x + c_2\sin x$; $y_{\text{tr}} = x\cdot(A\cos x + B\sin x)$; $y_{\text{g}} = c_1\cos x + c_2\sin x + \dfrac{1}{2}x\sin x$;
$c_1 = 1$, $c_2 = -1$, $y(x) = \cos x - \sin x + \dfrac{1}{2}x\sin x$

35. $y_c = e^x(c_1\cos x + c_2\sin x)$; $y_{\text{tr}} = A + Bx$; $y_{\text{g}} = e^x(c_1\cos x + c_2\sin x) + 1 + \dfrac{x}{2}$; $c_1 + 1 = 3$,
$c_1 + c_2 + \dfrac{1}{2} = 0$; $y(x) = e^x\left(2\cos x - \dfrac{5}{2}\sin x\right) + 1 + \dfrac{x}{2}$

36. $y_c = c_1 + c_2x + c_3e^{-2x} + c_4e^{2x}$; $y_{\text{tr}} = x^2\cdot(A + Bx + Cx^2)$

$$y_{\text{g}} = c_1 + c_2x + c_3e^{-2x} + c_4e^{2x} - \frac{x^2}{16} - \frac{x^4}{48}$$

$$c_1 + c_3 + c_4 = 1, \quad c_2 - 2c_3 + 2c_4 = 1, \quad 4c_3 + 4c_4 - \frac{1}{8} = -1, \quad -8c_3 + 8c_4 = -1$$

$$y(x) = \frac{39}{32} + \frac{5}{4}x - \frac{3}{64}e^{-2x} - \frac{11}{64}e^{2x} - \frac{1}{16}x^2 - \frac{1}{48}x^4$$

37. $y_c = c_1 + c_2e^x + c_3xe^x$; $y_{\text{tr}} = x\cdot(A) + x^2\cdot(B + Cx)e^x$

$$y_{\text{g}} = c_1 + c_2e^x + c_3xe^x + x - \frac{1}{2}x^2e^x + \frac{1}{6}x^3e^x$$

$$c_1 + c_2 = 0, \quad c_2 + c_3 + 1 = 0, \quad c_2 + 2c_3 - 1 = 1$$

$$y(x) = 4 + x + e^x\left(-4 + 3x - \frac{1}{2}x^2 + \frac{1}{6}x^3\right)$$

38. $y_c = e^{-x}\left(c_1 \cos x + c_2 \sin x\right)$; $y_{\text{tr}} = A\cos 3x + B\sin 3x$

$$y_{\text{g}} = e^{-x}\left(c_1 \cos x + c_2 \sin x\right) - \frac{6}{85}\cos 3x - \frac{7}{85}\sin 3x$$

$$c_1 - \frac{6}{185} = 2, \quad -c_1 + c_2 - \frac{21}{85} = 0$$

$$y(x) = e^{-x}\left(\frac{176}{85}\cos x + \frac{197}{85}\sin x\right) - \frac{6}{85}\cos 3x - \frac{7}{85}\sin 3x$$

39. $y_c = c_1 + c_2 x + c_3 e^{-x}$; $y_{\text{tr}} = x^2 \cdot (A + Bx) + x \cdot \left(Ce^{-x}\right)$

$$y_{\text{g}} = c_1 + c_2 x + c_3 e^{-x} - \frac{x^2}{2} + \frac{x^3}{6} + xe^{-x}$$

$$c_1 + c_3 = 1, \quad c_2 - c_3 + 1 = 0, \quad c_3 - 3 = 1$$

$$y(x) = \frac{1}{6}\left(-18 + 18x - 3x^2 + x^3\right) + (4 + x)e^{-x}$$

40. $y_c = c_1 e^{-x} + c_2 e^{x} + c_3 \cos x + c_4 \sin x$; $y_{\text{tr}} = A$

$$y_{\text{g}} = c_1 e^{-x} + c_2 e^{x} + c_3 \cos x + c_4 \sin x - 5$$

$$c_1 + c_2 + c_3 - 5 = 0, \quad -c_1 + c_2 + c_4 = 0, \quad c_1 + c_2 - c_3 = 0, \quad -c_1 + c_2 - c_4 = 0$$

$$y(x) = \frac{1}{4}\left(5e^{-x} + 5e^{x} + 10\cos x - 20\right)$$

41. The trial solution $y_{\text{tr}} = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5$ leads to the equations

$$\begin{aligned}
2A - B - 2C - 6D + 24E \phantom{{}+120F} &= 0\\
-2B - 2C - 6D - 24E + 120F &= 0\\
-2C - 3D - 12E - 60F &= 0\\
-2D - 4E - 20F &= 0\\
-2E - 5F &= 0\\
-2F &= 8
\end{aligned}$$

that are readily solved by back-substitution. The resulting particular solution is

$$y(x) = -255 - 450x + 30x^2 + 20x^3 + 10x^4 - 4x^5.$$

42. The characteristic equation $r^4 - r^3 - r^2 - r - 2 = 0$ has roots $r = -1, 2$, and $\pm i$, so the complementary function is $y_c = c_1 e^{-x} + c_2 e^{2x} + c_3 \cos x + c_4 \sin x$. We find that the coefficients satisfy the equations

$$\begin{aligned} c_1 + c_2 + c_3 \quad\quad - 255 &= 0 \\ -c_1 + c_2 \quad\quad + c_4 - 450 &= 0 \\ c_1 + 4c_2 - c_3 \quad\quad + 60 &= 0 \\ -c_1 + 8c_2 \quad\quad - c_4 + 120 &= 0 \end{aligned}.$$

Solution of this system gives finally the particular solution $y = y_c + y_p$, where y_p is the particular solution of Problem 41 and

$$y_c = 10e^{-x} + 35e^{2x} + 210\cos x + 390\sin x .$$

43. **(a)** Applying Euler's formula gives

$$\cos 3x + i\sin 3x = \left(\cos x + i\sin x\right)^3 = \cos^3 x + 3i\cos^2 x\sin x - 3\cos x\sin^2 x - i\sin^3 x .$$

When we equate real parts we get the equation

$$\cos^3 x - 3\left(\cos x\right)\left(1 - \cos^2 x\right) = 4\cos^3 x - 3\cos x$$

and readily solve for $\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$. The formula for $\sin^3 x$ is derived similarly by equating imaginary parts in the first equation above.

(b) Upon substituting the trial solution $y_p = A\cos x + B\sin x + C\cos 3x + D\sin 3x$ in the differential equation $y'' + 4y = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$, we find that

$$A = \frac{1}{4}, \quad B = 0, \quad C = -\frac{1}{20}, \quad D = 0 .$$

The resulting general solution is

$$y(x) = c_1\cos 2x + c_2\sin 2x + \frac{1}{4}\cos x - \frac{1}{20}\cos 3x .$$

44. We use the identity $\sin x\sin 3x = \frac{1}{2}\cos 2x - \frac{1}{2}\cos 4x$, and hence substitute the trial solution $y_p = A\cos 2x + B\sin 2x + C\cos 4x + D\sin 4x$ in the differential equation $y'' + y' + y = \frac{1}{2}\cos 2x - \frac{1}{2}\cos 4x$. We find that

$$A = -\frac{3}{26}, \quad B = \frac{1}{13}, \quad C = \frac{15}{482}, \quad D = \frac{2}{241} .$$

The resulting general solution is

$$y(x) = e^{-x/2}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right) + \frac{1}{26}\left(-3\cos 2x + 2\sin 2x\right) + \frac{1}{482}\left(15\cos 4x + 4\sin 4x\right).$$

45. We substitute

$$\sin^4 x = \frac{1}{4}\left(1 - \cos 2x\right)^2 = \frac{1}{4}\left(1 - 2\cos 2x + \cos^2 2x\right) = \frac{1}{8}\left(3 - 4\cos 2x + \cos 4x\right)$$

on the right-hand side of the differential equation, and then substitute the trial solution

$$y_p = A\cos 2x + B\sin 2x + C\cos 4x + D\sin 4x + E\,.$$

We find that

$$A = -\frac{1}{10}, \quad B = 0, \quad C = -\frac{1}{56}, \quad D = 0, \quad E = \frac{1}{24}.$$

The resulting general solution is

$$y = c_1\cos 3x + c_2\sin 3x + \frac{1}{24} - \frac{1}{10}\cos 2x - \frac{1}{56}\cos 4x\,.$$

46. By the formula for $\cos^3 x$ in Problem 43, the differential equation can be written as

$$y'' + y = \frac{3}{4}x\cos x + \frac{1}{4}x\cos 3x\,.$$

The complementary solution is $y_c = c_1\cos x + c_2\sin x$, so we substitute the trial solution

$$y_p = x\cdot\left[(A+Bx)\cos x + (C+Dx)\sin x\right] + \left[(E+Fx)\cos 3x + (G+Hx)\sin 3x\right].$$

We find that

$$A = \frac{3}{16}, \quad B = C = 0, \quad D = \frac{3}{16}, \quad E = 0, \quad F = -\frac{1}{32}, \quad G = \frac{3}{128}, \quad H = 0\,.$$

Hence the general solution is given by $y = y_c + y_1 + y_2$, where

$$y_1 = \frac{1}{16}\left(3x\cos x + 3x^2\sin x\right) \quad \text{and} \quad y_2 = \frac{1}{128}\left(3\sin 3x - 4x\cos 3x\right).$$

In Problems 47–49 we list the independent solutions y_1 and y_2 of the associated homogeneous equation, their Wronskian $W = W(y_1, y_2)$, the coefficient functions

$$u_1(x) = -\int\frac{y_2(x)f(x)}{W(x)}dx \quad \text{and} \quad u_2(x) = \int\frac{y_1(x)f(x)}{W(x)}dx$$

in the particular solution $y_p = u_1y_1 + u_2y_2$ of Eq. (32) in the text, and finally y_p itself.

47. $y_1 = e^{-2x}$, $y_2 = e^{-x}$, $W = e^{-3x}$, $u_1 = -\frac{4}{3}e^{3x}$, $u_2 = 2e^{2x}$, $y_p = \frac{2}{3}e^{x}$

48. $y_1 = e^{-2x}$, $y_2 = e^{4x}$, $W = 6e^{2x}$, $u_1 = -\frac{x}{2}$, $u_2 = -\frac{1}{12}e^{-6x}$, $y_p = -\frac{1}{12}(6x+1)e^{-2x}$

49. $y_1 = e^{2x}$, $y_2 = xe^{2x}$, $W = e^{4x}$, $u_1 = -x^2$, $u_2 = 2x$, $y_p = x^2e^{2x}$.

50. The complementary function is $y_1 = c_1 \cosh 2x + c_2 \sinh 2x$, so the Wronskian is $W = 2\cosh^2 2x - 2\sinh^2 2x = 2$, so when we solve Equations (31) simultaneously for u_1' u_2', integrate each, and substitute in $y_p = y_1 u_1 + y_2 u_2$, the result is

$$y_p = -(\cosh 2x)\int \frac{1}{2}(\sinh 2x)(\sinh 2x)\,dx + (\sinh 2x)\int \frac{1}{2}(\cosh 2x)(\sinh 2x)\,dx.$$

Using the identities $2\sinh^2 x = \cosh 2x - 1$ and $2\sinh x \cosh x = \sinh 2x$, we evaluate the integrals and find that

$$y_p = \frac{1}{16}(4x\cosh 2x - \sinh 4x \cosh 2x + \cosh 4x \sinh 2x).$$

Using the identity

$$\cosh 4x \sinh 2x - \sinh 4x \cosh 2x = \sinh(2x - 4x) = -\sinh 2x$$

we can reduce y_p to simply

$$y_p = \frac{1}{16}(4x\cosh 2x - \sinh 2x).$$

51. $y_1 = \cos 2x$, $y_2 = \sin 2x$, $W = 2$. Liberal use of trigonometric sum and product identities yields

$$u_1 = \frac{1}{20}(\cos 5x - 5\cos x) \quad \text{and} \quad u_2 = \frac{1}{20}(\sin 5x + 5\sin x).$$

Thus

$$\begin{aligned} y_p &= \frac{1}{20}(\cos 5x - 5\cos x)\cos 2x + \frac{1}{20}(\sin 5x + 5\sin x)\sin 2x \\ &= \frac{1}{20}\left[(\cos 5x \cos 2x + \sin 5x \sin 2x) - 5(\cos x \cos 2x - \sin x \sin 2x)\right] \\ &= \frac{1}{20}(\cos 3x - 5\cos 3x) \\ &= -\frac{1}{5}\cos 3x \text{ (!)} \end{aligned}$$

52. $y_1 = \cos 3x$, $y_2 = \sin 3x$, $W = 3$; $u_1 = -\frac{1}{36}(6x - \sin 6x)$, $u_2 = -\frac{1}{36}(1 + \cos 6x)$;

$$\begin{aligned}
y_p &= -\frac{1}{36}(6x-\sin 6x)\cos 3x + -\frac{1}{36}(1+\cos 6x)\sin 3x \\
&= -\frac{1}{36}\left[6x\cos 3x + \sin 3x + (\cos 6x \sin 3x - \sin 6x \cos 3x)\right] \\
&= -\frac{1}{36}\left[6x\cos 3x + \cancel{\sin 3x} - \cancel{\sin 3x}\right] \\
&= -\frac{1}{6}x\cos 3x.
\end{aligned}$$

53. $y_1 = \cos 3x$, $y_2 = \sin 3x$, $W = 3$; $u_1' = -\frac{2}{3}\tan 3x$, so $u_1 = \frac{2}{9}\ln|\cos 3x|$; $u_2' = \frac{2}{3}$, so $u_2 = \frac{2}{3}x$. Thus

$$\begin{aligned}
y_p &= (\cos 3x)\cdot\frac{2}{9}\ln|\cos 3x| + (\sin 3x)\cdot\frac{2}{3}x \\
&= \frac{2}{9}\left[3x\sin 3x + (\cos 3x)\ln|\cos 3x|\right].
\end{aligned}$$

54. $y_1 = \cos x$, $y_2 = \sin x$, $W = 1$; $u_1' = -\csc x$, so $u_1 = -\ln|\csc x - \cot x|$; $u_2' = \cos x \csc^2 x$, so $u_2 = -\csc x$. Thus

$$y_p = (-\ln|\csc x - \cot x|)\cos x - \csc x(\sin x) = -1 - (\cos x)\cdot\ln|\csc x - \cot x|.$$

55. $y_1 = \cos 2x$, $y_2 = \sin 2x$, $W = 2$;

$$u_1' = -\frac{1}{2}\sin^2 x \sin 2x = -\frac{1}{2}\cdot\frac{1-\cos 2x}{2}\cdot\sin 2x = -\frac{1}{4}(\sin 2x - \cos 2x \sin 2x),$$

so $u_1 = \frac{1}{16}(2\cos 2x - \cos^2 2x)$;

$$u_2' = \frac{1}{2}\sin^2 x\cos 2x = \frac{1}{2}\cdot\frac{1-\cos 2x}{2}\cdot\cos 2x = \frac{1}{4}(\cos 2x - \cos^2 2x) = \frac{1}{8}\left[2\cos 2x - (1+\cos 4x)\right],$$

so $u_2 = \frac{1}{8}\left(\sin 2x - x - \frac{\sin 4x}{4}\right)$. Thus

$$y_p = \frac{1}{16}\left(2\cos 2x - \cos^2 2x\right)\cos 2x + \frac{1}{8}\left(\sin 2x - x - \frac{\sin 4x}{4}\right)\sin 2x$$
$$= \frac{1}{16}\left(\underline{2\cos^2 2x} - \cos^3 2x + \underline{2\sin^2 2x} - 2x\sin 2x - \frac{1}{2}\sin 4x \sin 2x\right)$$
$$= \frac{1}{16}\left(2 - 2x\sin 2x \underline{\underline{- \cos^3 2x - \frac{1}{2}\sin 4x\sin 2x}}\right),$$

because the single-underlined terms sum to 2. The double-underlined terms reduce to

$$-\cos^3 2x - \frac{1}{2}\sin 4x \sin 2x = -\cos^3 2x - \frac{1}{2}\left(2\sin 2x\cos 2x\right)\sin 2x$$
$$= -\cos^3 2x - \sin^2 2x\cos 2x$$
$$= -\cos^3 2x - \left(1 - \cos^2 2x\right)\cos 2x$$
$$= -\cos 2x.$$

Therefore we can take $y_p = \frac{1}{16}\left(2 - 2x\sin 2x - \cos 2x\right)$. However, because $\cos 2x$ is a solution of the associated homogeneous equation $y'' + y = 0$—that is, because $\cos 2x$ is part of the complimentary function y_c—we can in fact omit the $\cos 2x$ term from y_p, leading to the simpler version $y_p = \frac{1}{8}\left(1 - x\sin 2x\right)$.

56. $y_1 = e^{-2x}$, $y_2 = e^{2x}$, $W = 4$; $u_1 = -\frac{1}{36}\left(3x-1\right)e^{3x}$; $u_2 = -\frac{1}{4}\left(x+1\right)e^{-x}$;
$y_p = -\frac{1}{9}\left(3x+2\right)e^{x}$.

57. With $y_1 = x$, $y_2 = x^{-1}$, and $f(x) = 72x^3$, Equations (31) in the text take the form

$$xu_1' + x^{-1}u_2' = 0,$$
$$u_1' - x^{-2}u_2' = 72x^3.$$

Upon multiplying the second equation by x and then adding, we readily solve first for $u_1' = 36x^3$, so $u_1 = 9x^4$; then $u_2' = -x^2 u_1'$, so $u_2 = -6x^6$. It follows that

$$y_p = y_1u_1 + y_2u_2 = x\left(9x^4\right) + \left(x^{-1}\right)\left(-6x^6\right) = 3x^5.$$

58. Here it is important to remember that in order to apply the method of variation of parameters, the differential equation must be written in standard form with leading coefficient 1. We therefore rewrite the given equation with complementary function $y_c = c_1x^2 + c_2x^3$ as

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = x.$$

Thus $f(x) = x$ and $W = x^4$, so simultaneous solution of Equations (31) as in Problem 50 (followed by integration of u_1' and u_2') yields

$$y_p = -x^2 \int x^3 \cdot x \cdot x^{-4} dx + x^3 \int x^2 \cdot x \cdot x^{-4} dx = -x^2 \int dx + x^3 \int \frac{1}{x} dx = x^3 (\ln x - 1).$$

59. $y_1 = x^2$, $y_2 = x^2 \ln x$, $W = x^3$, $f(x) = x^2$; $u_1' = -x \ln x$, $u_2' = x$; $y_p = \frac{1}{4} x^4$.

60. $y_1 = x^{1/2}$, $y_2 = x^{3/2}$, $f(x) = 2x^{-2/3}$, $W = x$; $u_1 = -\frac{12}{5} x^{5/6}$, $u_2 = -12 x^{-1/6}$; $y_p = -\frac{72}{5} x^{4/3}$.

61. $y_1 = \cos(\ln x)$, $y_2 = \sin(\ln x)$, $W = \frac{1}{x}$, $f(x) = \frac{\ln x}{x^2}$; from $u_1' = -\frac{(\ln x)\sin(\ln x)}{x}$ and $u_2' = \frac{(\ln x)\cos(\ln x)}{x}$ integration by parts yields

$$\begin{aligned} u_1 &= -\int \frac{(\ln x)\sin(\ln x)}{x} dx = -\int \frac{\sin(\ln x)}{x} \cdot \ln x \, dx \\ &= \cos(\ln x) \cdot \ln x - \int \frac{\cos(\ln x)}{x} dx = \cos(\ln x) \cdot \ln x - \sin(\ln x) \end{aligned}$$

and

$$\begin{aligned} u_2 &= \int \frac{(\ln x)\cos(\ln x)}{x} dx = \int \frac{\cos(\ln x)}{x} \cdot \ln x \, dx \\ &= \sin(\ln x) \cdot \ln x - \int \frac{\sin(\ln x)}{x} dx = \sin(\ln x) \cdot \ln x + \cos(\ln x). \end{aligned}$$

Thus

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= [\cos(\ln x) \cdot \ln x - \sin(\ln x)]\cos(\ln x) + [\sin(\ln x) \cdot \ln x + \cos(\ln x)]\sin(\ln x) \\ &= [\cos^2(\ln x) + \sin^2(\ln x)]\ln x - \cancel{\sin(\ln x)\cos(\ln x)} + \cancel{\cos(\ln x)\sin(\ln x)} \\ &= \ln x \ (!) \end{aligned}$$

62. $y_1 = x$, $y_2 = 1 + x^2$, $W = x^2 - 1$, $f(x) = 1$. From $u_1' = \frac{1+x^2}{1-x^2}$, long division and the method of partial fractions yield

$$u_1 = \int \frac{1+x^2}{1-x^2} dx = \int -1 + \frac{1}{1-x} + \frac{1}{1+x} dx = -x + \ln\left|\frac{1+x}{1-x}\right|;$$

likewise $u_2' = \dfrac{x}{x^2-1}$ gives $u_2 = \dfrac{1}{2}\ln\left|x^2-1\right|$. Altogether

$$y_p = u_1y_1 + u_2y_2 = -x^2 + x\ln\left|\frac{1+x}{1-x}\right| + \frac{1}{2}\left(1+x^2\right)\ln\left|x^2-1\right|.$$

63. This is simply a matter of solving the equations in (31) for the derivatives

$$u_1' = -\frac{y_2(x)f(x)}{W(x)} \quad \text{and} \quad u_2' = \frac{y_1(x)f(x)}{W(x)},$$

integrating each, and then substituting the results in (32).

64. Here we have $y_1(x) = \cos x$, $y_2(x) = \sin x$, $W(x) = 1$, and $f(x) = 2\sin x$, so (33) gives

$$\begin{aligned} y_p(x) &= -(\cos x)\int \sin x \cdot 2\sin x\,dx + (\sin x)\int \cos x \cdot 2\sin x\,dx \\ &= -(\cos x)\int (1-\cos 2x)\,dx + (\sin x)\int 2(\sin x)\cdot \cos x\,dx \\ &= -(\cos x)(x - \sin x\cos x) + (\sin x)\left(\sin^2 x\right) \\ &= -x\cos x + (\sin x)\left(\cos^2 x + \sin^2 x\right) \\ &= -x\cos x + \sin x. \end{aligned}$$

We can drop the term $\sin x$ because it satisfies the associated homogeneous equation $y'' + y = 0$, leaving simply $y_p(x) = -x\cos x$, as desired.

SECTION 3.6

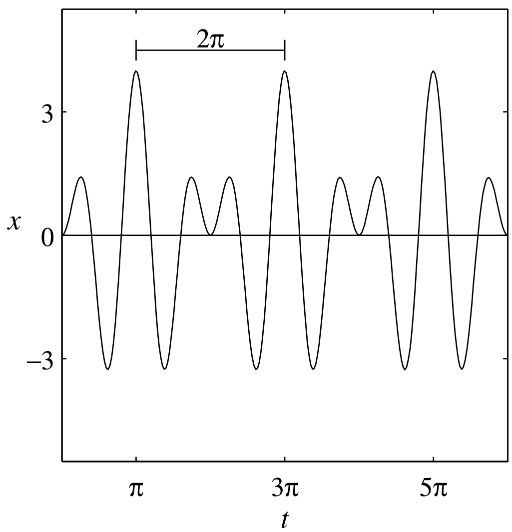
FORCED OSCILLATIONS AND RESONANCE

1. Trial of $x = A\cos 2t$ yields the particular solution $x_p = 2\cos 2t$. (Can you see that because the differential equation contains no first-derivative term, there is no need to include a $\sin 2t$ term in the trial solution?) Hence the general solution is

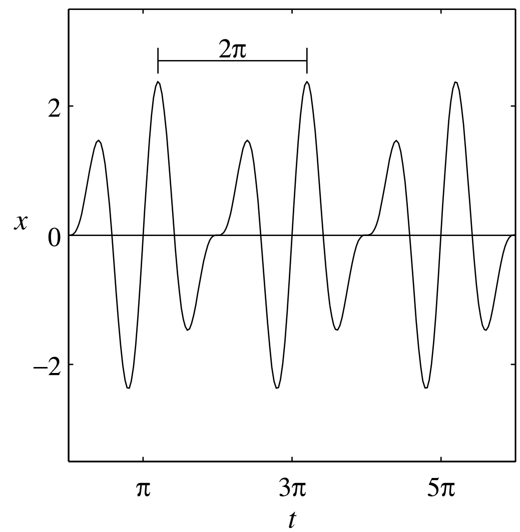
$$x(t) = c_1\cos 3t + c_2\sin 3t + 2\cos 2t.$$

The initial conditions imply that $c_1 = -2$ and $c_2 = 0$, so $x(t) = 2\cos 2t - 2\cos 3t$. The figure shows the graph of $x(t)$.

Problem 1



Problem 2



2. Trial of $x = A\sin 3t$ yields the particular solution $x_p = -\sin 3t$. Then we impose the initial conditions $x(0) = x'(0) = 0$ on the general solution

$$x(t) = c_1 \cos 2t + c_2 \sin 2t - \sin 3t$$

and find that $x(t) = \frac{3}{2}\sin 2t - \sin 3t$. The figure shows the graph of $x(t)$.

3. First we apply the method of undetermined coefficients with trial solution $x = A\cos 5t + B\sin 5t$ to find the particular solution

$$x_p = 3\cos 5t + 4\sin 5t = 5\left(\frac{3}{5}\cos 5t + \frac{4}{5}\sin 5t\right) = 5\cos(5t - \beta),$$

where $\beta = \tan^{-1}\frac{4}{3} \approx 0.9273$. Hence the general solution is

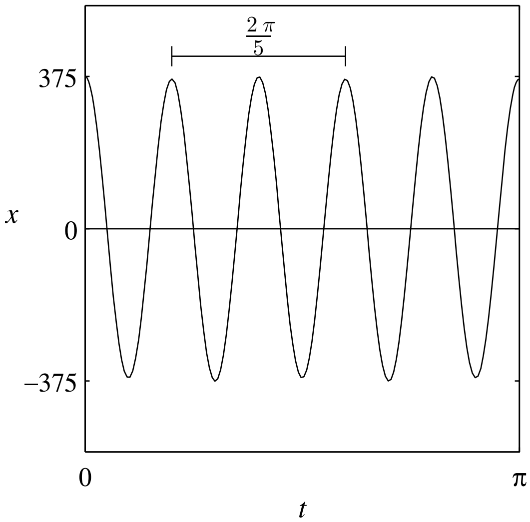
$$x(t) = c_1 \cos 10t + c_2 \sin 10t + 3\cos 5t + 4\sin 5t\,.$$

The initial conditions $x(0) = 375$, $x'(0) = 0$ now yield $c_1 = 372$ and $c_2 = -2$, so the part of the solution with frequency $\omega = 10$ is

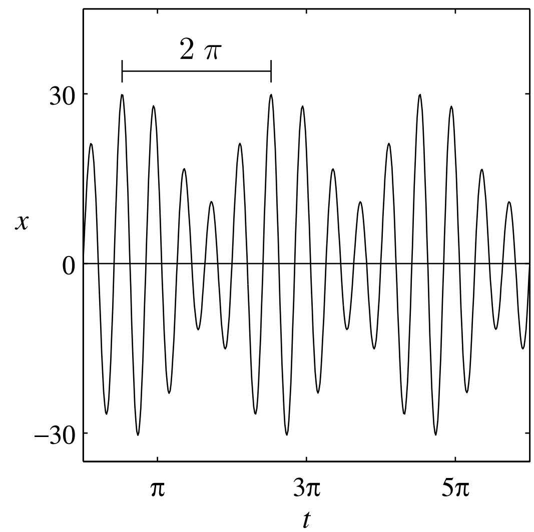
$$\begin{aligned} x_c &= 372\cos 10t - 2\sin 10t \\ &= \sqrt{138388}\left(\frac{372}{\sqrt{138388}}\cos 10t - \frac{2}{\sqrt{138388}}\sin 10t\right) \\ &= \sqrt{138388}\cos(10t - \alpha), \end{aligned}$$

where $\alpha = 2\pi - \tan^{-1}\frac{1}{186} \approx 6.2778$ is a fourth-quadrant angle. The figure shows the graph of $x(t)$.

Problem 3



Problem 4



4. Noting that there is no first-derivative term, we try $x = A\cos 4t$ and find the particular solution $x_p = 10\cos 4t$. Then imposition of the initial conditions on the general solution $x(t) = c_1\cos 5t + c_2\sin 5t + 10\cos 4t$ yields

$$\begin{aligned} x(t) &= (-10\cos 5t + 18\sin 5t) + 10\cos 4t \\ &= 2(-5\cos 5t + 9\sin 5t) + 10\cos 4t \\ &= 2\sqrt{106}\left(-\frac{5}{\sqrt{106}}\cos 5t + \frac{9}{\sqrt{106}}\sin 5t\right) + 10\cos 4t \\ &= 2\sqrt{106}\cos(5t-\alpha) + 10\cos 4t, \end{aligned}$$

where $\alpha = \pi - \tan^{-1}\frac{9}{5} \approx 2.0779$ is a second-quadrant angle. The figure shows the graph of $x(t)$.

5. Substitution of the trial solution $x = C\cos\omega t$ gives $C = \dfrac{F_0}{k - m\omega^2}$. Then imposition of the initial conditions $x(0) = x_0$, $x'(0) = 0$ on the general solution

$$x(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t + C\cos\omega t$$

(where $\omega_0 = \sqrt{k/m}$) gives the particular solution $x(t) = (x_0 - C)\cos\omega_0 t + C\cos\omega t$.

6. First, let's write the differential equation in the form $x'' + \omega_0^2 x = \dfrac{F_0}{m}\cos\omega_0 t$, which is the same as Eq. (13) in the text, and therefore has the particular solution $x_p = \dfrac{F_0}{2m\omega_0} t\sin\omega_0 t$

given in Eq. (14). When we impose the initial conditions $x(0)=0$, $x'(0)=v_0$ on the general solution

$$x(t)=c_1\cos\omega_0 t+c_2\sin\omega_0 t+\frac{F_0}{2m\omega_0}t\sin\omega_0 t,$$

we find that $c_1=0$, $c_2=\dfrac{v_0}{\omega_0}$. The resulting resonance solution of our initial value problem is $x(t)=\dfrac{2mv_0+F_0t}{2m\omega_0}\sin\omega_0 t$.

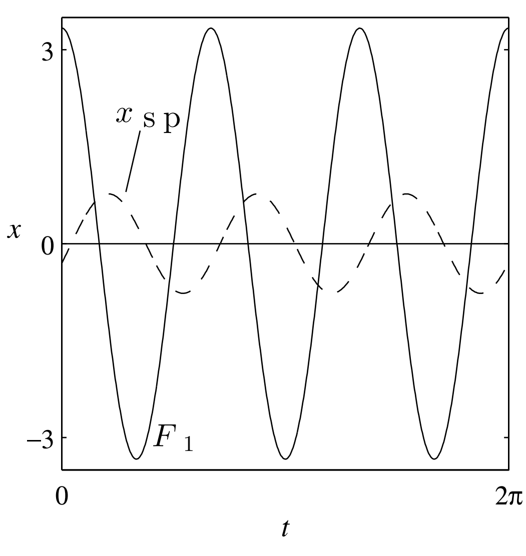
In Problems 7–10 we give first the trial solution x_p involving undetermined coefficients *A* and *B*, then the equations that determine these coefficients, and finally the resulting steady periodic solution x_{sp}. In each case the figure shows the graphs of $x_{sp}(t)$ and the adjusted forcing function $F_1(t)=F(t)/m\omega$.

7. $x_p=A\cos 3t+B\sin 3t$; $-5A+12B=10$, $12A+5B=0$.

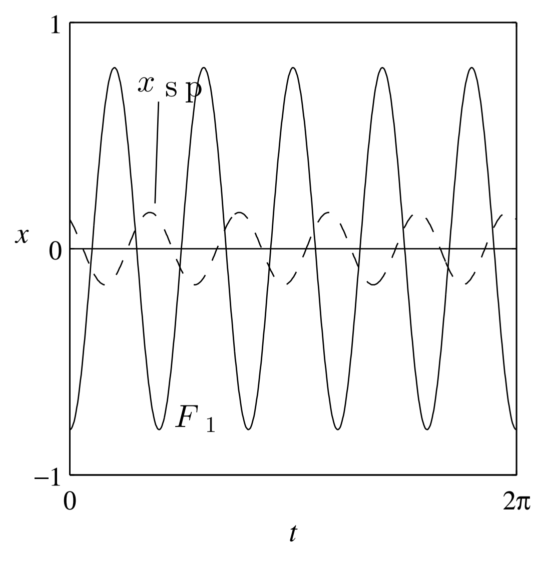
$$x_{sp}(t)=-\frac{50}{169}\cos 3t+\frac{120}{169}\sin 3t=\frac{10}{13}\left(-\frac{5}{13}\cos 3t+\frac{12}{13}\sin 3t\right)=\frac{10}{13}\cos(3t-\alpha),$$

where $\alpha=\pi-\tan^{-1}\dfrac{12}{5}\approx 1.9656$, a 2nd-quadrant angle.

Problem 7



Problem 8



8. $x_p=A\cos 5t+B\sin 5t$; $-20A+15B=-4$, $15A+20B=0$.

$$x_{sp}(t)=\frac{16}{125}\cos 5t-\frac{12}{125}\sin 5t=\frac{4}{25}\left(\frac{4}{5}\cos 5t-\frac{3}{5}\sin 5t\right)=\frac{4}{25}\cos(5t-\alpha),$$

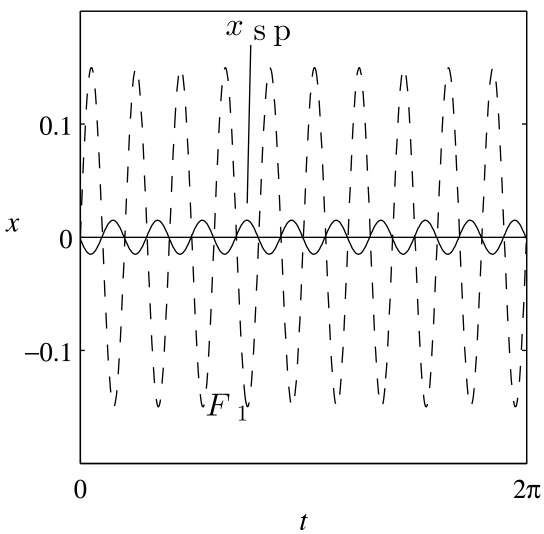
where $\alpha = 2\pi - \tan^{-1}\frac{3}{4} \approx 5.6397$, a 4th-quadrant angle.

9. $x_p = A\cos 10t + 10\sin 10t$; $-199A + 20B = 0$, $20A + 199B = -3$.

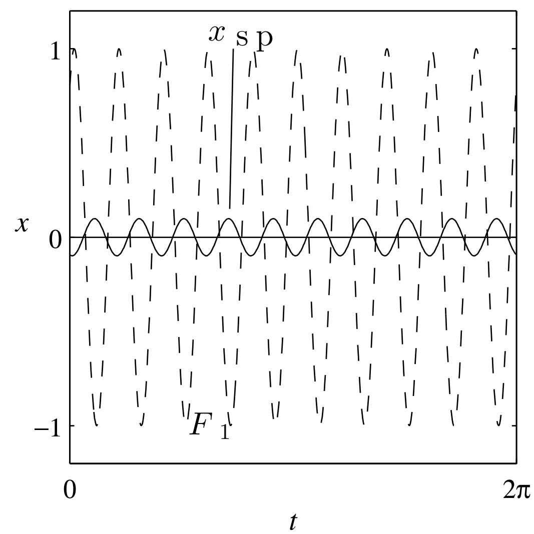
$$x_{sp}(t) = -\frac{60}{40001}\cos 10t - \frac{597}{40001}\sin 10t$$
$$= \frac{3}{\sqrt{40001}}\left(-\frac{20}{\sqrt{40001}}\cos 10t - \frac{199}{\sqrt{40001}}\sin 10t\right)$$
$$= \frac{3}{\sqrt{40001}}\cos(10t - \alpha),$$

where $\alpha = \pi + \tan^{-1}\frac{199}{20} \approx 4.6122$, a 3rd-quadrant angle.

Problem 9



Problem 10



10. $x_p = A\cos 10t + 10\sin 5t$; $-97A + 30B = 8$, $30A + 97B = -6$.

$$x_{sp}(t) = -\frac{956}{10309}\cos 10t - \frac{342}{10309}\sin 10t$$
$$= \frac{2\sqrt{257725}}{10309}\left(-\frac{478}{\sqrt{257725}}\cos 10t - \frac{171}{\sqrt{257725}}\sin 10t\right) = \frac{10}{793}\sqrt{61}\cos(10t - \alpha),$$

where $\alpha = \pi + \tan^{-1}\frac{171}{478} \approx 3.4851$, a 3rd-quadrant angle.

Each solution in Problems 11–14 has two parts. For the first part, we give first the trial solution x_p involving undetermined coefficients *A* and *B*, then the equations that determine these coefficients, and finally the resulting steady periodic solution x_{sp}. For the second part, we give first the general solution $x(t)$ involving the coefficients c_1 and c_2 in the transient solution, then the equations that determine these coefficients, and finally the resulting transient solution x_{tr}, so that $x(t) = x_{tr}(t) + x_{sp}(t)$ satisfies the given initial conditions. For each problem, the graph shows the graphs of both $x(t)$ and $x_{sp}(t)$.

11. $x_p = A\cos 3t + B\sin 3t$; $-4A + 12B = 10$, $12A + 4B = 0$.

$$x_{sp}(t) = -\frac{1}{4}\cos 3t + \frac{3}{4}\sin 3t = \frac{\sqrt{10}}{4}\left(-\frac{1}{\sqrt{10}}\cos 3t + \frac{3}{\sqrt{10}}\sin 3t\right) = \frac{\sqrt{10}}{4}\cos(3t - \alpha),$$

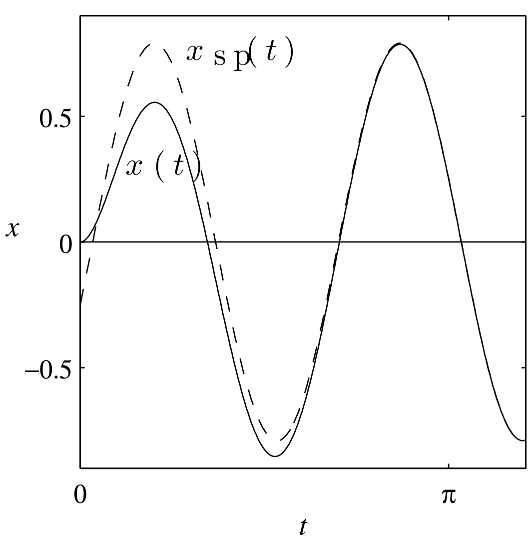
where $\alpha = \pi - \tan^{-1} 3 \approx 1.8925$, a 2nd-quadrant angle.

$$x(t) = e^{-2t}(c_1\cos t + c_2\sin t) + x_{sp}(t);\ c_1 - \frac{1}{4} = 0,\ -2c_1 + c_2 + \frac{9}{4} = 0.$$

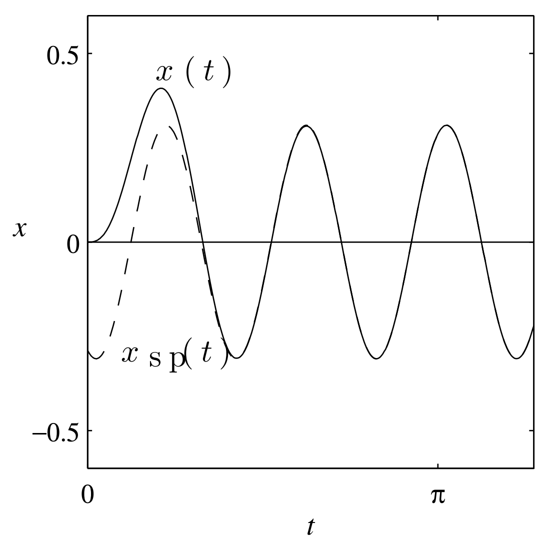
$$x_{tr}(t) = e^{-2t}\left(\frac{1}{4}\cos t - \frac{7}{4}\sin t\right) = \frac{\sqrt{50}}{4}e^{-2t}\left(\frac{1}{\sqrt{50}}\cos t - \frac{7}{\sqrt{50}}\sin t\right) = \frac{5}{4}\sqrt{2}e^{-2t}\cos(t - \beta),$$

where $\beta = 2\pi - \tan^{-1} 7 \approx 4.8543$, a 4th-quadrant angle.

Problem 11



Problem 12



12. $x_p = A\cos 5t + B\sin 5t$; $12A - 30B = 0$, $30A + 12B = -10$.

$$x_{sp}(t) = -\frac{25}{87}\cos 5t - \frac{10}{87}\sin 5t = \frac{5\sqrt{29}}{87}\left(-\frac{5}{\sqrt{29}}\cos 5t - \frac{2}{\sqrt{29}}\sin 5t\right) = \frac{5}{3\sqrt{29}}\cos(5t - \alpha),$$

where $\alpha = \pi + \tan^{-1}\frac{2}{5} \approx 3.5221$, a 3rd-quadrant angle.

$$x(t) = e^{-3t}(c_1\cos 2t + c_2\sin 2t) + x_{sp}(t);\ c_1 - \frac{25}{87} = 0,\ -3c_1 + 2c_2 - \frac{50}{87} = 0.$$

$$x_{\text{tr}}(t) = e^{-3t}\left(\frac{50}{174}\cos 2t + \frac{125}{174}\sin 2t\right)$$
$$= \frac{25\sqrt{29}}{174}e^{-3t}\left(\frac{2}{\sqrt{29}}\cos 2t + \frac{5}{\sqrt{29}}\sin 2t\right),$$
$$= \frac{25}{6\sqrt{29}}e^{-3t}\cos(2t-\beta),$$

where $\beta = \tan^{-1}\frac{5}{2} \approx 1.1903$, a 1st-quadrant angle.

13. $x_p = A\cos 10t + B\sin 10t$; $74A + 20B = 600$, $20A + 74B = 0$.

$$x_{\text{sp}}(t) = -\frac{11100}{1469}\cos 10t + \frac{3000}{1469}\sin 10t$$
$$= \frac{300}{\sqrt{1469}}\left(-\frac{37}{\sqrt{1469}}\cos 10t + \frac{10}{\sqrt{1469}}\sin 10t\right)$$
$$= \frac{300}{\sqrt{1469}}\cos(10t-\alpha),$$

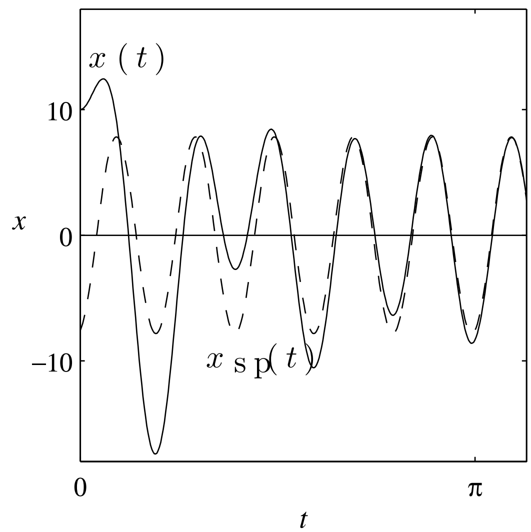
where $\alpha = \pi - \tan^{-1}\frac{10}{37} \approx 2.8776$, a 2nd-quadrant angle.

$x(t) = e^{-t}(c_1\cos 5t + c_2\sin 5t) + x_{\text{sp}}(t)$; $c_1 - \frac{11100}{1469} = 10$, $-c_1 + 5c_2 = -\frac{30000}{1469}$.

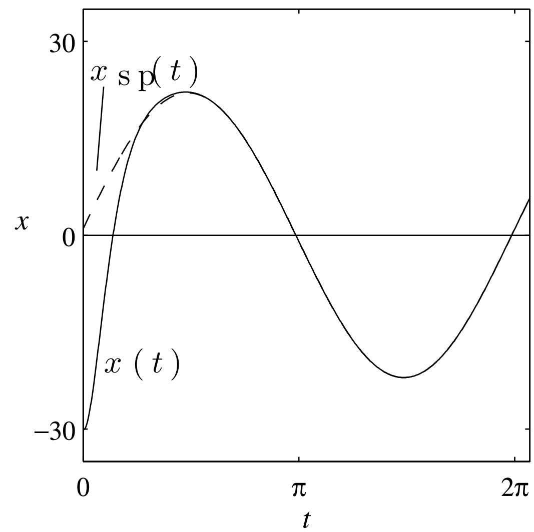
$$x_{\text{tr}}(t) = \frac{e^{-t}}{1469}(25790\cos 5t - 842\sin 5t)$$
$$= \frac{2\sqrt{166458266}}{1469}e^{-t}\left(\frac{12895}{\sqrt{166458266}}\cos 5t - \frac{421}{\sqrt{166458266}}\sin 5t\right)$$
$$= 2\sqrt{\frac{113314}{1469}}e^{-t}\cos(5t-\beta),$$

where $\beta = 2\pi - \tan^{-1}\frac{421}{12895} \approx 6.2505$, a 4th-quadrant angle.

Problem 13



Problem 14



14. $x_p = A\cos t + B\sin t$; $24A + 8B = 200$, $-8A + 24B = 520$.

$$x_{sp}(t) = \cos t + 22\sin t = \sqrt{485}\left(\frac{1}{\sqrt{485}}\cos t + \frac{22}{\sqrt{485}}\sin t\right) = \sqrt{485}\cos(t-\alpha),$$

where $\alpha = \tan^{-1} 22 \approx 1.5254$, a 1st-quadrant angle.

$x(t) = e^{-4t}(c_1\cos 3t + c_2\sin 3t) + x_{sp}(t)$; $c_1 + 1 = -30$, $-4c_1 + 3c_2 + 22 = -10$.

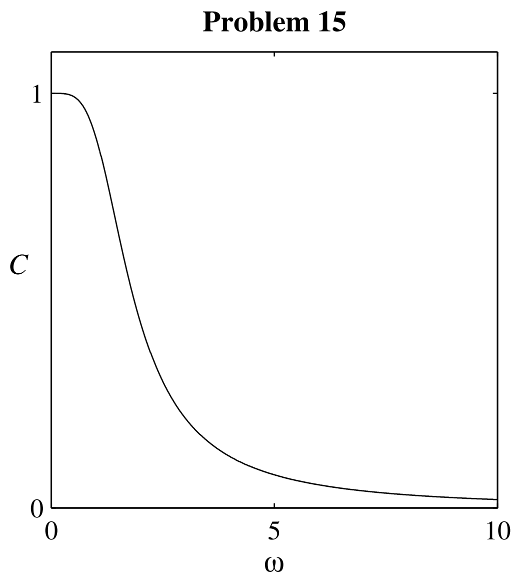
$$\begin{aligned} x_{tr}(t) &= e^{-4t}(-31\cos 3t - 52\sin 3t) \\ &= \sqrt{3665}e^{-4t}\left(-\frac{31}{\sqrt{3665}}\cos 3t - \frac{52}{\sqrt{3665}}\sin 3t\right) \\ &= \sqrt{3665}e^{-4t}\cos(3t-\beta), \end{aligned}$$

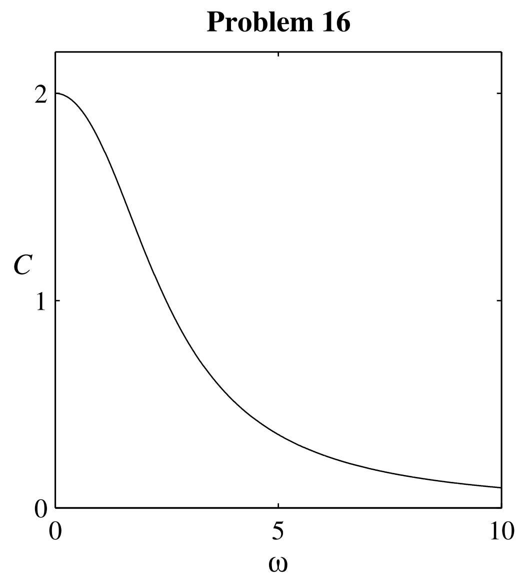
where $\beta = \pi + \tan^{-1}\frac{52}{31} \approx 4.1748$, a 3rd-quadrant angle.

In Problems 15-18 we substitute $x(t) = A(\omega)\cos\omega t + B(\omega)\sin\omega t$ into the differential equation $mx'' + cx' + kx = F_0\cos\omega t$ with the given numerical values of m, c, k, and F_0. We give first the equations in A and B that result upon collection of coefficients of $\cos\omega t$ and $\sin\omega t$, followed by the values of $A(\omega)$ and $B(\omega)$ that we get by solving these equations. Finally, $C = \sqrt{A^2 + B^2}$ gives the amplitude of the resulting forced oscillations as a function of the forcing frequency ω, and we show the graph of the function $C(\omega)$.

15. $\left(2-\omega^2\right)A+2\omega B=2$, $-2\omega A+\left(2-\omega^2\right)B=0$; $A=\dfrac{2\left(2-\omega^2\right)}{4+\omega^4}$, $B=\dfrac{4\omega}{4+\omega^4}$;

$C(\omega)=\dfrac{2}{\sqrt{4+\omega^4}}$ begins with $C(0)=1$ and steadily decreases as ω increases. Hence there is no practical resonance frequency.





16. $\left(5-\omega^2\right)A+4\omega B=10$, $-4\omega A+\left(5-\omega^2\right)B=0$; $A=\dfrac{10\left(5-\omega^2\right)}{25+6\omega^2+\omega^4}$, $B=\dfrac{40\omega}{25+6\omega^2+\omega^4}$;

$C(\omega)=\dfrac{10}{\sqrt{25+6\omega^2+\omega^4}}$ begins with $C(0)=2$ and steadily decreases as ω increases. Hence there is no practical resonance frequency.

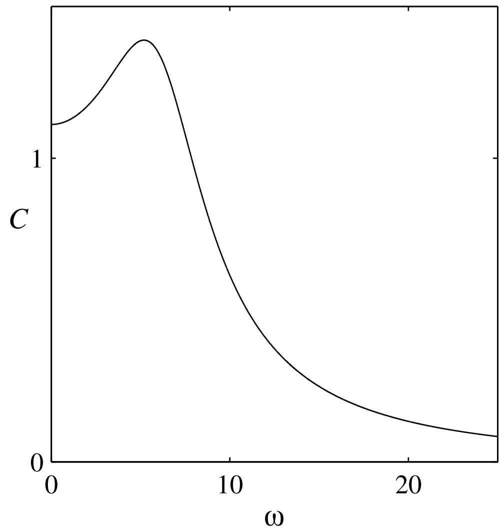
17. $\left(45-\omega^2\right)A+6\omega B=50$, $-6\omega A+\left(45-\omega^2\right)B=0$; $A=\dfrac{50\left(45-\omega^2\right)}{2025-54\omega^2+\omega^4}$,

$B=\dfrac{300\omega}{2025-54\omega^2+\omega^4}$; $C(\omega)=\dfrac{50}{\sqrt{2025-54\omega^2+\omega^4}}$. So, to find the maximum value of $C(\omega)$, we calculate its derivative

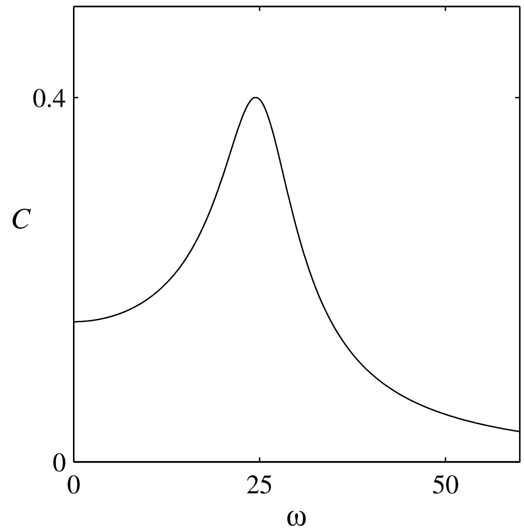
$$C'(\omega)=\frac{-100\omega\left(-27+\omega^2\right)}{\left(2025-54\omega^2+\omega^4\right)^{3/2}}.$$

Hence the practical resonance frequency (where the derivative vanishes) is $\omega=\sqrt{27}=3\sqrt{3}$.

Problem 17



Problem 18



18. $\left(650-\omega^2\right)A+10\omega B=100$, $-10\omega A+\left(650-\omega^2\right)B=0$; $A=\dfrac{100\left(650-\omega^2\right)}{422500-1200\omega^2+\omega^4}$,

$B=\dfrac{1000\omega}{422500-1200\omega^2+\omega^4}$; $C(\omega)=\dfrac{100}{\sqrt{422500-1200\omega^2+\omega^4}}$. So, to find the maximum value of $C(\omega)$, we calculate its derivative

$$C'(\omega)=\frac{-200\omega\left(-600+\omega^2\right)}{\left(422500-1200\omega^2+\omega^4\right)^{3/2}}.$$

Hence the practical resonance frequency (where the derivative vanishes) is $\omega=\sqrt{600}=10\sqrt{6}$.

19. $m=100/32$ slugs and $k=1200$ lb/ft, so the critical frequency is

$$\omega_0=\sqrt{\frac{k}{m}}=\sqrt{384}\ \text{rad/sec}=\frac{\sqrt{384}}{2\pi}\ \text{Hz}\approx 3.12\ \text{Hz}.$$

20. Let the machine have mass *m*. Then the force $F=mg=9.8\text{ m}$ (the machine's weight) causes a displacement of $x=0.5\text{ cm}=\dfrac{1}{200}\text{ m}$, so Hooke's law $F=kx$ gives $mg=k\cdot\dfrac{1}{200}$; that is, the spring constant is $k=200mg$ N/m. Hence the resonance frequency is

$$\omega=\sqrt{\frac{k}{m}}=\sqrt{200g}\approx\sqrt{200\times 9.8}\approx 44.27\ \text{rad/sec}\approx 7.05\ \text{Hz},$$

which is about 423 rpm (revolutions per minute).

21. If θ is the angular displacement from the vertical, then the (essentially horizontal) displacement of the mass is $x = L\theta$, so twice its total energy (KE + PE) is

$$m(x')^2 + kx^2 + 2mgh = mL^2(\theta')^2 + kL^2\theta^2 + 2mgL(1-\cos\theta) = C .$$

Differentiation, substitution of $\theta \approx \sin\theta$, and simplification yield $\theta'' + \left(\frac{k}{m} + \frac{g}{L}\right)\theta = 0$, so $\omega_0 = \sqrt{\frac{k}{m} + \frac{g}{L}}$.

22. Let x denote the displacement of the mass from its equilibrium position, $v = x'$ its velocity, and $\omega = v/a$ the angular velocity of the pulley. Then conservation of energy yields

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}kx^2 - mgx = C .$$

When we differentiate both sides with respect to t and simplify the result, we get the differential equation

$$\left(m + \frac{I}{a^2}\right)x'' + kx = mg .$$

Hence $\omega = \sqrt{\dfrac{k}{m + \frac{I}{a^2}}}$.

23. **(a)** In units of ft-lb-sec we have $m = 1000$ and $k = 10000$, so $\omega_0 = \sqrt{10}$ rad/sec ≈ 0.50 Hz.

(b) We are given that $\omega = 2\pi/2.25 \approx 2.79$ rad/sec, and the equation $mx'' + kx = F(t)$ simplifies to $x'' + 10x = \frac{1}{4}\omega^2 \sin\omega t$. When we substitute $x(t) = A\sin\omega t$ we find that the amplitude is $A = \dfrac{\omega^2}{4(10-\omega^2)} \approx 0.8854$ ft ≈ 10.63 in.

24. By the identity of Problem 43 in Section 3.5, the differential equation is

$$mx'' + kx = F_0\left(\frac{3}{4}\cos\omega t + \frac{1}{4}\cos 3\omega t\right).$$

Hence resonance occurs when either ω or 3ω equals $\omega_0 = \sqrt{k/m}$, that is, when either $\omega = \omega_0$ or $\omega = \omega_0/3$.

25. Substitution of the trial solution $x = A\cos\omega t + B\sin\omega t$ in the differential equation followed by collection of coefficients as usual yields the equations

$$\left(k - m\omega^2\right)A + (c\omega)B = 0, \quad -(c\omega)A + \left(k - m\omega^2\right)B = F_0$$

with coefficient determinant $\Delta = \left(k - m\omega^2\right)^2 + (c\omega)^2$ and solution

$$A = \frac{1}{\Delta}(-c\omega)F_0, \quad B = \frac{1}{\Delta}\left(k - m\omega^2\right)F_0 .$$

Hence

$$x(t) = \frac{F_0}{\sqrt{\Delta}}\left(\frac{k - m\omega^2}{\sqrt{\Delta}}\sin\omega t - \frac{c\omega}{\sqrt{\Delta}}\cos\omega t\right) = C\sin(\omega t - \alpha),$$

where $C = \frac{1}{\sqrt{\Delta}}F_0$, $\sin\alpha = \frac{c\omega}{\sqrt{\Delta}}$, and $\cos\alpha = \frac{1}{\sqrt{\Delta}}\left(k - m\omega^2\right)$.

26. Let $G_0 = \sqrt{E_0^2 + F_0^2}$ and $\rho = \dfrac{1}{\sqrt{\left(k - m\omega^2\right) + (c\omega)^2}}$. Then

$$\begin{aligned} x_{\text{sp}}(t) &= \rho E_0\cos(\omega t - \alpha) + \rho F_0\sin(\omega t - \alpha) \\ &= \rho G_0\left[\frac{E_0}{G_0}\cos(\omega t - \alpha) + \frac{F_0}{G_0}\sin(\omega t - \alpha)\right] \\ &= \rho G_0\left[\cos\beta\cos(\omega t - \alpha) + \sin\beta\sin(\omega t - \alpha)\right], \end{aligned}$$

or $x_{\text{sp}}(t) = \rho G_0\cos(\omega t - \alpha - \beta)$, where $\tan\beta = F_0/E_0$. The desired formula now results when we substitute the value of ρ defined above.

27. The derivative of $C(\omega) = \dfrac{F_0}{\sqrt{\left(k - m\omega^2\right)^2 + (c\omega)^2}}$ is given by

$$C'(\omega) = -\frac{\omega F_0}{2}\cdot\frac{\left(c^2 - 2km\right) + 2(m\omega)^2}{\left[\left(k - m\omega^2\right)^2 + (c\omega)^2\right]^{3/2}} .$$

(a) Therefore, if $c^2 \geq \left(c_{\text{cr}}/\sqrt{2}\right)^2 = 2km$, it is clear from the numerator that $C'(\omega) < 0$ for all ω, so that $C(\omega)$ steadily decreases as ω increases.

(b) If instead $c^2 < 2km$, however, then the numerator (and hence $C'(\omega)$) vanishes when

$$\omega = \omega_m = \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}} < \sqrt{\frac{k}{m}} = \omega_0 .$$

Calculation then shows that

$$C''(\omega_m) = \frac{16F_0 m^3\left(c^2 - 2km\right)}{c^3\left(4km - c^2\right)^{3/2}} < 0,$$

so it follows from the second-derivative test that $C(\omega_m)$ is a local maximum value.

28. **(a)** The given differential equation corresponds to Equation (17) with $F_0 = mA\omega^2$. It therefore follows from Equation (21) that the amplitude of the steady periodic vibrations at frequency ω is

$$C(\omega) = \frac{F_0}{\sqrt{\left(k - m\omega^2\right)^2 + (c\omega)^2}} = \frac{mA\omega^2}{\sqrt{\left(k - m\omega^2\right)^2 + (c\omega)^2}}.$$

(b) Now we calculate

$$C'(\omega) = \frac{mA\omega\left[2k^2 - \left(2mk - c^2\right)\omega^2\right]}{\left[\left(k - m\omega^2\right)^2 + (c\omega)^2\right]^{3/2}},$$

and we see that the numerator vanishes when

$$\omega = \sqrt{\frac{2k^2}{2mk - c^2}} = \sqrt{\frac{k}{m}\left(\frac{2mk}{2mk - c^2}\right)} > \sqrt{\frac{k}{m}} = \omega_0.$$

29. We need only substitute $E_0 = ac\omega$ and $F_0 = ak$ in the result of Problem 26.

30. When we substitute the values $\omega = 2\pi v/L$, $m = 800$, $k = 7\times10^4$, $c = 3000$, $L = 10$, and $a = 0.05$ in the formula of Problem 29, simplify, and square, we get the function

$$Csq(v) = \frac{25\left(9\pi^2 v^2 + 122500\right)}{16\left(16\pi^4 v^4 - 64375\pi^2 v^2 + 76562500\right)^2}$$

giving the square of the amplitude C (in meters) as a function of the velocity v (in meters per second). Differentiation gives

$$Csq'(v) = -\frac{50\pi^2 v\left(9\pi^4 v^4 + 245000\pi^2 v^2 - 535937500\right)}{\left(16\pi^4 v^4 - 64375\pi^2 v^2 - 76562500\right)^2}.$$

Because the principal factor in the numerator is quadratic in v^2 it is easy to solve the equation $Csq'(v) = 0$ to find where the maximum amplitude occurs; we find that the only positive solution is $v \approx 14.36$ m/sec ≈ 32.12 mi/hr. The corresponding amplitude of the car's vibrations is $\sqrt{Csq(14.36)} \approx 0.1364 \text{ m} = 13.64 \text{ cm}$.

SECTION 3.7

ELECTRICAL CIRCUITS

1. With $E(t) \equiv 0$ we have the simple exponential equation $5I' + 25I = 0$, whose solution with $I(0) = 4$ is $I(t) = 4e^{-5t}$.

2. With $E(t) \equiv 100$ we have the simple linear equation $5I' + 25I = 100$, whose solution with $I(0) = 0$ is $I(t) = 4\left(1 - e^{-5t}\right)$.

3. Now the differential equation is $5I' + 25I = 100\cos 60t$. Substitution of the trial solution $I_p = A\cos 60t + B\sin 60t$ yields

$$I_p = \frac{4}{145}\left(\cos 60t + 12\sin 60t\right).$$

The complementary function is $I_c = Ce^{-5t}$; the solution with $I(0) = 0$ is

$$I(t) = \frac{4}{145}\left(\cos 60t + 12\sin 60t - e^{-5t}\right).$$

4. The solution of the initial value problem $2I' + 40I = 100e^{-10t}$, $I(0) = 0$ is $I(t) = 5\left(e^{-10t} - e^{-20t}\right)$. To find the maximum current we solve the equation

$$I'(t) = -50e^{-10t} + 100e^{-20t} = -50e^{-20t}\left(e^{20t} - 2\right) = 0$$

for $t = \dfrac{\ln 2}{10}$. Then $I_{\max} = I\left(\dfrac{\ln 2}{10}\right) = \dfrac{5}{4}$.

5. The linear equation $I' + 10I = 50e^{-10t}\cos 60t$ has integrating factor $\rho = e^{10t}$. The resulting general solution is $I(t) = e^{-10t}\left(\dfrac{5}{6}\sin 60t + C\right)$. To satisfy the initial condition $I(0) = 0$, we take $C = 0$ and get $I(t) = \dfrac{5}{6}e^{-10t}\sin 60t$.

6. Substitution of the trial solution $I = A\cos 60t + B\sin 60t$ in the differential equation $I' + 10I = 30\cos 60t + 40\sin 60t$ gives the equations $10A + 60B = 30$, $-60A + 10B = 40$ with solution $A = -\dfrac{21}{37}$, $B = \dfrac{22}{37}$. The resulting steady periodic solution is

$$I_{\text{sp}} = \frac{1}{37}\left(-21\cos 60t + 22\sin 60t\right) = \frac{5}{\sqrt{37}}\cos\left(60t - \alpha\right),$$

where $\alpha = \pi - \tan^{-1}\frac{22}{21} \approx 2.3329$, a 2nd-quadrant angle.

7. **(a)** The linear differential equation $RQ' + \frac{1}{C}Q = E_0$ has integrating factor $\rho = e^{\frac{t}{RC}}$. The resulting solution with $Q(0) = 0$ is $Q(t) = E_0 C\left(1 - e^{-\frac{t}{RC}}\right)$. Then $I(t) = Q'(t) = \frac{E_0}{R}e^{-\frac{t}{RC}}$.

(b) These solutions make it obvious that $\lim_{t\to\infty} Q(t) = E_0 C$ and $\lim_{t\to\infty} I(t) = 0$.

8. **(a)** The linear equation $Q' + 5Q = 10e^{-5t}$ has integrating factor $\rho = e^{5t}$. The resulting solution with $Q(0) = 0$ is $Q(t) = 10te^{-5t}$, so $I(t) = Q'(t) = 10(1-5t)e^{-5t}$.

(b) $I(t) = 0$ when $t = \frac{1}{5}$, so $Q_{\max} = Q\left(\frac{1}{5}\right) = 2e^{-1}$.

9. Substitution of the trial solution $Q = A\cos 120t + B\sin 120t$ into the differential equation $200Q' + 4000Q = 100\cos 120t$ yields the equations

$$4000A + 24000B = 100, \quad -24000A + 4000B = 0$$

with solution $A = \frac{1}{1480}$, $B = \frac{3}{740}$. The complementary function is $Q_c = ce^{-20t}$, and imposition of the initial condition $Q(0) = 0$ yields the solution

$$Q(t) = \frac{1}{1480}\left(\cos 120t + 6\sin 120t - e^{-20t}\right).$$

The current function is then

$$I(t) = Q'(t) = \frac{1}{74}\left(36\cos 120t - 6\sin 120t + e^{-20t}\right).$$

Thus the steady-periodic current is

$$I_{\text{sp}} = \frac{6}{74}\left(6\cos 120t - \sin 120t\right) = \frac{6\sqrt{37}}{74}\left(\frac{6}{\sqrt{37}}\cos 120t - \frac{1}{\sqrt{37}}\sin 120t\right) = \frac{3}{\sqrt{37}}\cos\left(120t - \alpha\right)$$

(with $\alpha = 2\pi - \tan^{-1}\frac{1}{6}$), so the steady-state amplitude is $\frac{3}{\sqrt{37}}$.

10. Substitution of the trial solution $Q = A\cos\omega t + B\sin\omega t$ into the differential equation $RQ' + \frac{1}{C}Q = E_0\cos\omega t$ yields the equations

$$\frac{1}{C}A + \omega r B = E_0, \quad -r\omega A + \frac{1}{C}B = 0,$$

with solution

$$A = \frac{E_0 C}{1+\omega^2 R^2 C^2}, \quad B = \frac{E_0 \omega R C^2}{1+\omega^2 R^2 C^2},$$

so

$$\begin{aligned} Q_{sp}(t) &= \frac{E_0 C}{1+\omega^2 R^2 C^2}(\cos\omega t + \omega RC \sin\omega t) \\ &= \frac{E_0 C}{\sqrt{1+\omega^2 R^2 C^2}}\left(\frac{1}{\sqrt{1+\omega^2 R^2 C^2}}\cos\omega t + \frac{\omega RC}{\sqrt{1+\omega^2 R^2 C^2}}\sin\omega t\right) \\ &= \frac{E_0 C}{\sqrt{1+\omega^2 R^2 C^2}}\cos(\omega t - \beta), \end{aligned}$$

where $\beta = \tan^{-1}\omega RC$, a 1st-quadrant angle.

In Problems 11-16, we give first the trial solution $I_p = A\cos\omega t + B\sin\omega t$, then the equations in A and B that we get upon substituting this trial solution into the RLC equation $LI'' + RI' + \frac{1}{C}I = E'(t)$, and finally the resulting steady periodic solution.

11. $I_p = A\cos 2t + B\sin 2t$; $A + 6B = 10$, $-6A + B = 0$;

$$I_{sp}(t) = \frac{10}{37}\cos 2t + \frac{60}{37}\sin 2t = \frac{10}{\sqrt{37}}\left(\frac{1}{\sqrt{37}}\cos 2t + \frac{6}{\sqrt{37}}\sin 2t\right) = \frac{10}{\sqrt{37}}\sin(2t - \delta),$$

where $\delta = 2\pi - \tan^{-1}\frac{1}{6} \approx 6.1180$, a 4th-quadrant angle.

12. $I_p = A\cos 10t + B\sin 10t$; $A + 4B = 2$, $-4A + B = 0$;

$$I_{sp}(t) = \frac{2}{17}\cos 10t + \frac{8}{17}\sin 10t = \frac{2}{\sqrt{17}}\left(\frac{1}{\sqrt{17}}\cos 10t + \frac{4}{\sqrt{17}}\sin 10t\right) = \frac{2}{\sqrt{17}}\sin(10t - \delta),$$

where $\delta = 2\pi - \tan^{-1}\frac{1}{4} \approx 6.0382$, a 4th-quadrant angle.

13. $I_p = A\cos 5t + B\sin 5t$; $3A - 2B = 0$, $2A + 3B = 20$;

$$I_{sp}(t) = \frac{40}{13}\cos 5t + \frac{60}{13}\sin 5t = \frac{20}{\sqrt{13}}\left(\frac{2}{\sqrt{13}}\cos 5t + \frac{3}{\sqrt{13}}\sin 5t\right) = \frac{20}{\sqrt{13}}\sin(5t - \delta),$$

where $\delta = 2\pi - \tan^{-1}\frac{2}{3} \approx 5.6952$, a 4th-quadrant angle.

14. $I_p = A\cos 100t + B\sin 100t$; $-249A + 25B = 200$, $-25A - 249B = -150$;

$$\begin{aligned} I_{\text{sp}}(t) &= \frac{25}{31313}(-921\cos 100t + 847\sin 100t) \\ &= \frac{25\sqrt{1565650}}{31313}\left(-\frac{921}{\sqrt{1565650}}\cos 100t + \frac{847}{\sqrt{1565650}}\sin 100t\right) \\ &\approx 0.9990\sin(100t - \delta), \end{aligned}$$

where $\delta = \tan^{-1}\dfrac{921}{847} \approx 0.8272$, a 1st-quadrant angle.

15. $I_p = A\cos 60\pi t + B\sin 60\pi t$;

$$\left(1000 - 36\pi^2\right)A + 30\pi B = 33\pi, \quad 15\pi A - \left(500 - 18\pi^2\right)B = 0;$$

$$A = \frac{33\pi\left(250 - 9\pi^2\right)}{250000 - 17775\pi^2 + 324\pi^4}, \quad B = \frac{495\pi^2}{2\left(250000 - 17775\pi^2 + 324\pi^4\right)};$$

$I_{\text{sp}}(t) \approx I_0\sin(60\pi t - \delta)$, where $I_0 = \dfrac{33\pi}{2\sqrt{250000 - 17775\pi^2 + 324\pi^4}} \approx 0.1591$ and

$\delta = 2\pi - \tan^{-1}\dfrac{500 - 18\pi^2}{15\pi} \approx 4.8576$.

16. $I_p = A\cos 377t + B\sin 377t$;

$$-132129A + 47125B = 226200, \quad 47125A + 132129B = 0;$$

$$A = -\frac{14943789900}{9839419133}, \quad B = \frac{5329837500}{9839419133};$$

$I_{\text{sp}}(t) \approx I_0\sin(377t - \delta)$, where $I_0 = \sqrt{\dfrac{25583220000}{9839419133}} \approx 1.6125$ and

$\delta = \tan^{-1}\dfrac{132129}{47215} \approx 1.2282$.

In each of Problems 17–22, the first step is to substitute the given *RLC* parameters, the initial values $I(0)$ and $Q(0)$, and the voltage $E(t)$ into Eq. (16) and solve for the remaining initial value

$$I'(0) = \frac{1}{L}\left[E(0) - RI(0) - \frac{1}{C}Q(0)\right]. \qquad (*)$$

17. With $I(0)=0$ and $Q(0)=5$, Equation (*) gives $I'(0)=-75$. The solution of the *RLC* equation $2I''+16I'+50I=0$ with these initial conditions is $I(t)=-25e^{-4t}\sin 3t$.

18. Our differential equation to solve is

$$2I''+60I'+400I=-100e^{-t}.$$

We find the particular solution $I_p=-\frac{50}{171}e^{-t}$ by substituting the trial solution Ae^{-t}; the general solution is

$$I(t)=c_1e^{-10t}+c_2e^{-20t}-\frac{50}{171}e^{-t}.$$

The initial conditions are $I(0)=0$ and $I'(0)=50$, the latter found by substituting $L=2$, $R=60,$, $\frac{1}{C}=400$, $I(0)=Q(0)=0$, and $E(0)=100$ into Equation (*). Imposition of these initial values on the general solution above yields the equations

$$c_1+c_2-\frac{50}{171}=0,\quad -10c_1-20c_2+\frac{50}{171}=50,$$

with solution $c_1=\frac{50}{9}$, $c_2=-\frac{100}{19}$. This gives the solution

$$I(t)=\frac{50}{171}\left(19e^{-10t}-18e^{-20t}-e^{-t}\right).$$

19. Now our differential equation to solve is

$$2I''+60I'+400I=-1000e^{-10t}.$$

We find the particular solution $I_p=-50te^{-10t}$ by substituting the trial solution Ate^{-10t}; the general solution is

$$I(t)=c_1e^{-10t}+c_2e^{-20t}-50te^{-10t}.$$

The initial conditions are $I(0)=0$ and $I'(0)=-150$, the latter found by substituting $L=2$, $R=60$, $\frac{1}{C}=400$, $I(0)=0$, $Q(0)=-1$, and $E(0)=100$ into Equation (*). Imposition of these initial values on the general solution above yields the equations

$$c_1+c_2=0,\quad -10c_1-20c_2-50=-150,$$

with solution $c_1=-10$, $c_2=10$. Thus we get the solution

$$I(t)=10e^{-20t}-10e^{-10t}-50te^{-10t}.$$

20. The differential equation $10I''+30I'+50I=100\cos 2t$ has transient solution

$$I_{\text{tr}}(t) = e^{-3t/2}\left[c_1 \cos\left(\frac{\sqrt{11}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{11}}{2}t\right)\right],$$

and in Problem 11 we found the steady periodic solution

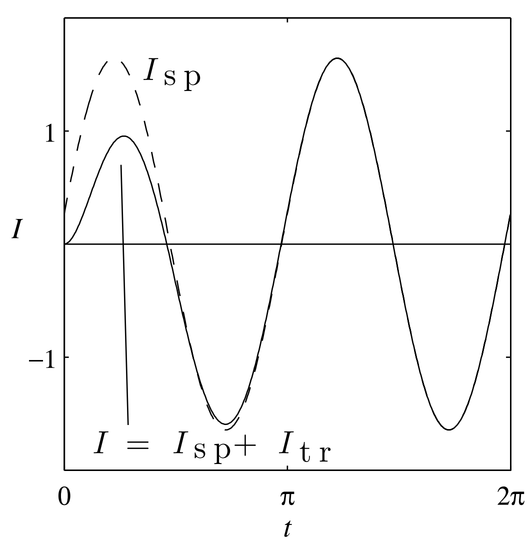
$$I_{\text{sp}}(t) = \frac{10}{37}(\cos 2t + 6\sin 2t).$$

When we impose the initial conditions $I(0) = I'(0) = 0$ on the general solution $I(t) = I_{\text{tr}}(t) + I_{\text{sp}}(t)$, we get the equations

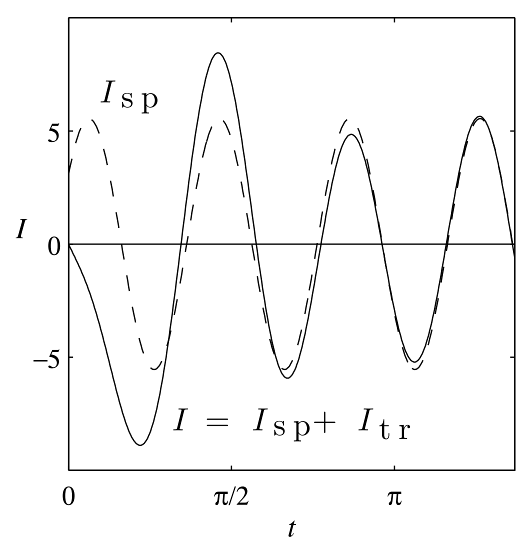
$$c_1 + \frac{10}{37} = 0, \quad -\frac{3}{2}c_1 + \frac{\sqrt{11}}{2}c_2 + \frac{120}{37} = 0,$$

with solution $c_1 = -\frac{10}{37}$, $c_2 = -\frac{270}{37\sqrt{11}}$. The figure shows the graphs of $I(t)$ and $I_{\text{sp}}(t)$.

Problem 20



Problem 21



21. The differential equation $10I'' + 20I' + 100I = -1000\sin 5t$ has transient solution $I_{\text{tr}}(t) = e^{-t}(c_1 \cos 3t + c_2 \sin 3t)$, and in Problem 13 we found the steady periodic solution $I_{\text{sp}}(t) = \frac{20}{13}(2\cos 5t + 3\sin 5t)$. When we impose the initial conditions $I(0) = 0$, $I'(0) = -10$ on the general solution $I(t) = I_{\text{tr}}(t) + I_{\text{sp}}(t)$, we get the equations

$$c_1 + \frac{40}{13} = 0, \quad -c_1 + 3c_2 + \frac{300}{13} = -10,$$

with solution $c_1 = -\frac{40}{13}$, $c_2 = -\frac{470}{39}$. The figure shows the graphs of $I(t)$ and $I_{\text{sp}}(t)$.

22. The differential equation $2I'' + 100I' + 200000I = 6600\pi \cos 60\pi t$ has transient solution

$$I_{tr}(t) = e^{-25t}\left[c_1\cos\left(25\sqrt{159}t\right) + c_2\sin\left(25\sqrt{159}t\right)\right],$$

and in Problem 15 we found the steady periodic solution

$$I_{sp}(t) = A\cos 60\pi t + B\sin 60\pi t \approx 0.157444\cos 60\pi t + 0.023017\sin 60\pi t$$

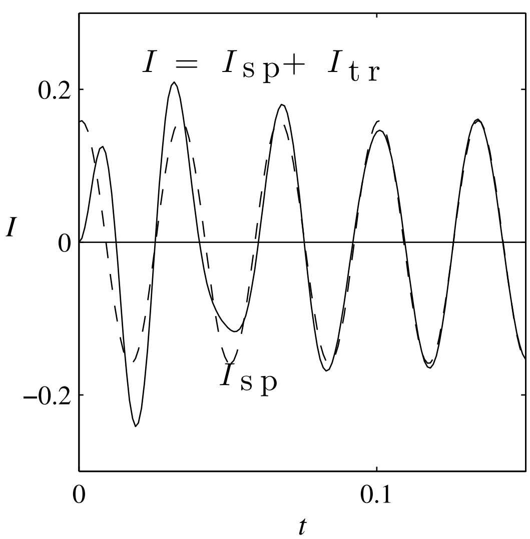
(with the exact values of *A* and *B* given there). When we impose the initial conditions $I(0) = I'(0) = 0$ on the general solution $I(t) = I_{tr}(t) + I_{sp}(t)$, we find (with the aid of a computer algebra system) that

$$c_1 = -\frac{33\pi\left(250 - 9\pi^2\right)}{250000 - 17775\pi^2 + 324\pi^4} \approx -0.157444,$$

$$c_2 = -\frac{11\pi\sqrt{159}\left(250 + 9\pi^2\right)}{53\left(250000 - 17775\pi^2 + 324\pi^4\right)} \approx -0.026249.$$

The figure shows the graphs of $I(t)$ and $I_{sp}(t)$.

Problem 22



23. The *LC* equation $LI'' + \frac{1}{C}I = 0$ has general solution $I(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t$ with critical frequency $\omega_0 = \frac{1}{\sqrt{LC}}$.

24. We need only observe that the roots $r = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}$ both necessarily have *negative* real parts.

25. According to Eq. (8) in the text, the amplitude of the steady periodic current is

$$\frac{E_0}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}}.$$

Because the radicand in the denominator is a sum of squares, it is obvious that the denominator is least when $\omega L - \dfrac{1}{\omega C} = 0$, that is, when $\omega = \dfrac{1}{\sqrt{LC}}$.

SECTION 3.8

ENDPOINT PROBLEMS AND EIGENVALUES

The material on eigenvalues and endpoint problems in Section 3.8 can be considered optional at this point in a first course. It will not be needed until we discuss boundary value problems in the last three sections of Chapter 9 and in Chapter 10. However, after the concentration thus far on initial value problems, the inclusion of this section can give students a view of a new class of problems that have diverse and important applications (as illustrated by the subsection on the whirling string). If Section 3.8 is not covered at this point in the course, then it can be inserted just prior to Section 9.5.

1. If $\lambda = 0$, then $y'' = 0$ implies that $y(x) = A + Bx$. The endpoint conditions $y'(0) = 0$ and $y(1) = 0$ yield $B = 0$ and $A = 0$, respectively. Hence $\lambda = 0$ is *not* an eigenvalue.

If $\lambda = \alpha^2 > 0$, then the general solution of $y'' + \alpha^2 y = 0$ is

$$y(x) = A\cos\alpha x + B\sin\alpha x,$$

so

$$y'(x) = -A\alpha\sin\alpha x + B\alpha\cos\alpha x.$$

Then $y'(0) = 0$ yields $B = 0$, so $y(x) = A\cos\alpha x$. Next $y(1) = 0$ implies that $\cos\alpha = 0$, so α is an odd multiple of $\dfrac{\pi}{2}$. Hence the positive eigenvalues are $\dfrac{(2n-1)^2\pi^2}{4}$, with associated eigenfunctions $\cos\dfrac{(2n-1)\pi x}{2}$, for $n = 1, 2, 3, \ldots$.

2. If $\lambda = 0$, then $y'' = 0$ implies that $y(x) = A + Bx$. The endpoint conditions $y'(0) = y'(\pi) = 0$ imply only that $B = 0$, so $\lambda_0 = 0$ is an eigenvalue with associated eigenfunction $y_0(x) = 1$.

If $\lambda = \alpha^2 > 0$, then the general solution of $y'' + \alpha^2 y = 0$ is

$$y(x) = A\cos\alpha x + B\sin\alpha x.$$

Then

$$y'(x) = -A\alpha \sin \alpha x + B\alpha \cos \alpha x,$$

so $y'(0) = 0$ implies that $B = 0$. Next, $y'(\pi) = 0$ implies that $\alpha\pi$ is an integral multiple of π. Hence the positive eigenvalues are n^2, with associated eigenfunctions $\cos nx$, for $n = 1, 2, 3, \ldots$.

3. Much as in Problem 1 we see that $\lambda = 0$ is not an eigenvalue. Suppose that $\lambda = \alpha^2 > 0$, so

$$y(x) = A\cos \alpha x + B\sin \alpha x.$$

Then the conditions $y(-\pi) = y(\pi) = 0$ yield

$$A\cos \alpha\pi + B\sin \alpha\pi = 0, \quad A\cos \alpha\pi - B\sin \alpha\pi = 0.$$

It follows that $A\cos \alpha\pi = 0 = B\sin \alpha\pi$. Hence either $A = 0$ and $B \neq 0$ with $\alpha\pi$ an even multiple of $\frac{\pi}{2}$, or $A \neq 0$ and $B = 0$ with $\alpha\pi$ an odd multiple of $\frac{\pi}{2}$. Thus the eigenvalues are $\frac{n^2}{4}$, $n = 1, 2, 3, \ldots$, and the n$^{\text{th}}$ eigenfunction is $y_n(x) = \cos\frac{nx}{2}$ if n is even, and $y_n(x) = \sin\frac{nx}{2}$ if n is odd.

4. Just as in Problem 2, $\lambda_0 = 0$ is an eigenvalue with associated eigenfunction $y_0(x) = 1$. If $\lambda = \alpha^2 > 0$ and

$$y(x) = A\cos \alpha x + B\sin \alpha x,$$

then the equations

$$y'(-\pi) = \alpha(A\sin \alpha\pi + B\cos \alpha\pi) = 0, \quad y'(\pi) = \alpha(-A\sin \alpha\pi + B\cos \alpha\pi) = 0$$

yield $A\sin \alpha\pi = B\cos \alpha\pi = 0$. If $A = 0$ and $B \neq 0$, then $\cos \alpha\pi = 0$, so $\alpha\pi$ must be an odd multiple of $\frac{\pi}{2}$. If $A \neq 0$ and $B = 0$, then $\sin \alpha\pi = 0$, so $\alpha\pi$ must be an even multiple of $\frac{\pi}{2}$. Therefore the positive eigenvalues are $\frac{n^2}{4}$, $n = 1, 2, 3, \ldots$, with associated eigenfunctions $y_n(x) = \cos\frac{nx}{2}$ if n is even and $y_n(x) = \sin\frac{nx}{2}$ if n is odd.

5. If $\lambda = \alpha^2 > 0$ and

$$y(x) = A\cos \alpha x + B\sin \alpha x,$$

so that

$$y'(x) = -A\alpha \sin \alpha x + B\alpha \cos \alpha x,$$

then the conditions $y(-2) = y'(2) = 0$ yield

$$A\cos 2\alpha - B\sin 2\alpha = 0, \quad A\sin 2\alpha + B\cos 2\alpha = 0.$$

It follows either that $A = B$ and $\cos 2\alpha = \sin 2\alpha$, or that $A = -B$ and $\cos 2\alpha = -\sin 2\alpha$. The former occurs if

$$2\alpha = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \ldots$$

and the latter if

$$2\alpha = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \ldots$$

Hence the n$^{\text{th}}$ eigenvalue is

$$\lambda_n = \alpha_n^2 = \frac{(2n-1)^2 \pi^2}{64}, n = 1, 2, 3, \ldots,$$

and the associated eigenfunction is

$$y_n(x) = \begin{cases} \cos\alpha_n x + \sin\alpha_n x, & n \text{ odd} \\ \cos\alpha_n x - \sin\alpha_n x, & n \text{ even} \end{cases}.$$

6. **(a)** If $\lambda = 0$ and $y(x) = A + Bx$, then $y'(0) = B = 0$, so $y(x) = A$. But then $y(1) + y'(1) = A = 0$ also, so $\lambda = 0$ is not an eigenvalue.

(b) If $\lambda = \alpha^2 > 0$ and

$$y(x) = A\cos\alpha x + B\sin\alpha x,$$

then

$$y'(x) = \alpha(-A\sin\alpha x + B\cos\alpha x),$$

so that $y'(0) = B\alpha = 0$. Hence $B = 0$, so $y(x) = A\cos\alpha x$. Then

$$y(1) + y'(1) = A(\cos\alpha - \alpha\sin\alpha) = 0,$$

so α must be a positive root of the equation $\tan\alpha = \dfrac{1}{\alpha}$.

7. **(a)** If $\lambda = 0$ and $y(x) = A + Bx$, then $y(0) = A = 0$, so $y(x) = Bx$. But then $y(1) + y'(1) = 2B = 0$, so $A = B = 0$ and $\lambda = 0$ is not an eigenvalue.

(b) If $\lambda = \alpha^2 > 0$ and

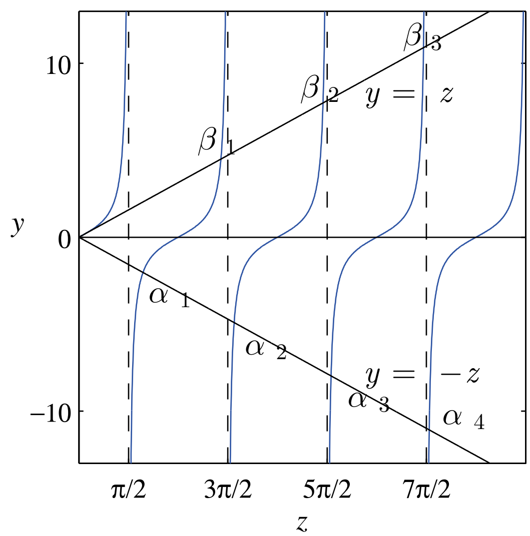
$$y(x) = A\cos\alpha x + B\sin\alpha x,$$

then $y(0) = A = 0$, so $y(x) = B\sin\alpha x$. Hence

$$y(1)+y'(1)=B(\sin\alpha+\alpha\cos\alpha)=0,$$

so α must be a positive root of the equation $\tan\alpha=-\alpha$, and hence the abscissa of a point of intersection of the lines $y=\tan z$ and $y=-z$. We see from the figure below that α_n lies just to the right of the vertical line $z=\dfrac{(2n-1)\pi}{2}$ and lies closer and closer to this line as n gets larger and larger.

Problem 7



8. **(a)** If $\lambda=0$ and $y(x)=A+Bx$, then $y(0)=A=0$, so $y(x)=Bx$. But then $y(1)=y'(1)$ says only that $B=B$. Hence $\lambda_0=0$ is an eigenvalue with associated eigenfunction $y_0(x)=x$.

(b) If $\lambda=\beta^2>0$ and $y(x)=A\cos\beta x+B\sin\beta x$, then $y(0)=A=0$, so $y(x)=B\sin\beta x$. Then $y(1)=y'(1)$ says that $B\sin\beta=B\beta\cos\beta$, so β must be a positive root of the equation $\tan\beta=\beta$, and hence the abscissa of a point of intersection of the lines $y=\tan z$ and $y=z$. We see from the figure above that β_n lies just to the left of the vertical line $z=\dfrac{(2n+1)\pi}{2}$ and lies closer and closer to this line as n gets larger and larger.

9. If $y''+\lambda y=0$ and $\lambda=-\alpha^2<0$, then $y(x)=Ae^{\alpha x}+Be^{-\alpha x}$. Then $y(0)=A+B=0$, so $B=-A$ and therefore $y(x)=A\left(e^{\alpha x}-e^{-\alpha x}\right)$. Hence $y'(L)=A\alpha\left(e^{\alpha L}+e^{-\alpha L}\right)=0$. But $\alpha\neq 0$ and $e^{\alpha L}+e^{-\alpha L}>0$, so $A=0$. Thus $\lambda=-\alpha^2$ is not an eigenvalue.

10. If $\lambda = -\alpha^2 < 0$, then the general solution of $y'' + \lambda y = 0$ is $y(x) = A\cosh\alpha x + B\sinh\alpha x$. Then $y(0) = 0$ implies that $A = 0$, so $y(x) = \sinh\alpha x$ (or a nonzero multiple thereof). Next,

$$y(1) + y'(1) = \sinh\alpha + \alpha\cosh\alpha = 0$$

implies that $\tanh\alpha = -\alpha$. But the graph of $y = \tanh\alpha$ lies in the first and third quadrants, while the graph of $y = -\alpha$ lies in the second and fourth quadrants. It follows that the only solution of $\tanh\alpha = -\alpha$ is $\alpha = 0$, and hence that our eigenvalue problem has no negative eigenvalues.

11. If $\lambda = -\alpha^2 < 0$, then the general solution of $y'' + \lambda y = 0$ is $y(x) = A\cosh\alpha x + B\sinh\alpha x$. Then $y'(0) = 0$ implies that $B = 0$, $y(x) = \cosh\alpha x$ so (or a nonzero multiple thereof). Next,

$$y(1) + y'(1) = \cosh\alpha + \alpha\sinh\alpha = 0$$

implies that $\tanh\alpha = -\dfrac{1}{\alpha}$. But the graph of $y = \tanh\alpha$ lies in the first and third quadrants, while the graph of $y = -\dfrac{1}{\alpha}$ lies in the second and fourth quadrants. It follows that the only solution of $\tanh\alpha = -\dfrac{1}{\alpha}$ is $\alpha = 0$, and hence that our eigenvalue problem has no negative eigenvalues.

12. **(a)** If $\lambda = 0$ and $y(x) = A + Bx$, then $y(-\pi) = y(\pi)$ means that $A + B\pi = A - B\pi$, so $B = 0$ and $y(x) = A$. But then $y'(-\pi) = y'(\pi)$ implies nothing about *A*. Hence $\lambda_0 = 0$ is an eigenvalue with $y_0(x) = 1$.

(b) If $\lambda = -\alpha^2 < 0$ and $y(x) = Ae^{\alpha x} + Be^{-\alpha x}$, then the conditions $y(-\pi) = y(\pi)$ and $y'(-\pi) = y'(\pi)$ yield the equations

$$\begin{aligned} Ae^{\alpha\pi} + Be^{-\alpha\pi} &= Ae^{-\alpha\pi} + Be^{\alpha\pi}, \\ Ae^{\alpha\pi} - Be^{-\alpha\pi} &= Ae^{-\alpha\pi} - Be^{\alpha\pi}. \end{aligned}$$

Addition of these equations yields $2Ae^{\alpha\pi} = 2Ae^{-\alpha\pi}$. Since $e^{\alpha\pi} \neq e^{-\alpha\pi}$ because $\alpha \neq 0$, it follows that $A = 0$. Similarly $B = 0$. Thus there are no negative eigenvalues.

(c) If $\lambda = \alpha^2 > 0$ and

$$y(x) = A\cos\alpha x + B\sin\alpha x,$$

then the endpoint conditions yield the equations

$$\begin{aligned} A\cos\alpha\pi + B\sin\alpha\pi &= A\cos\alpha\pi - B\sin\alpha\pi, \\ A\sin\alpha\pi + B\cos\alpha\pi &= A\sin\alpha\pi + B\cos\alpha\pi. \end{aligned}$$

The first equation implies that $B\sin\alpha\pi = 0$, the second that $A\sin\alpha\pi = 0$. If A and B are not both zero, then it follows that $\sin\alpha\pi = 0$, so $\alpha = n$, an integer. In this case A and B are both arbitrary. Thus $\cos nx$ and $\sin nx$ are two different eigenfunctions associated with the single eigenvalue n^2.

13. **(a)** With $\lambda = 1$, the general solution of $y'' + 2y' + y = 0$ is $y(x) = Ae^{-x} + Bxe^{-x}$. But then $y(0) = A = 0$ and $y(1) = e^{-1}(A+B) = 0$. Hence $\lambda = 1$ is not an eigenvalue.

(b) If $\lambda < 1$, then the equation $y'' + 2y' + \lambda y = 0$ has characteristic equation $r^2 + 2r + \lambda = 0$. This equation has the two distinct real roots $\dfrac{-2 \pm \sqrt{4-4\lambda}}{2}$; call them r and s. Then the general solution is

$$y(x) = Ae^{rx} + Be^{sx},$$

and the conditions $y(0) = y(1) = 0$ yield the equations

$$A + B = 0, \quad Ae^{r} + Be^{s} = 0.$$

If $A, B \neq 0$, then it follows that $e^r = e^s$. But $r \neq s$, so there is no eigenvalue $\lambda < 1$.

(c) If $\lambda > 1$, then let $\lambda - 1 = \alpha^2$, so $\lambda = 1 + \alpha^2$. Then the characteristic equation

$$r^2 + 2r + \lambda = (r+1)^2 + \alpha^2 = 0$$

has roots $-1 \pm \alpha i$, so

$$y(x) = e^{-x}(A\cos\alpha x + B\sin\alpha x).$$

Now $y(0) = A = 0$, so $y(x) = Ae^{-x}\sin\alpha x$. Next, $y(1) = Ae^{-1}\sin\alpha = 0$, so $\alpha = n\pi$ with n an integer. Thus the n^{th} positive eigenvalue is $\lambda_n = n^2\pi^2 + 1$. Because $\lambda = 1 + \alpha^2$, the eigenfunction associated with λ_n is

$$y_n(x) = e^{-x}\sin n\pi x.$$

14. If $\lambda = 1 + \alpha^2$, then we first impose the condition $y(0) = 0$ on the solution

$$y(x) = e^{-x}(A\cos\alpha x + B\sin\alpha x)$$

found in Problem 13, and find that $A = 0$. Hence $y(x) = Be^{-x}\sin\alpha x$, so that

$$y'(x) = B\left(-e^{-x}\sin\alpha x + e^{-x}\alpha\cos\alpha x\right),$$

so the condition $y'(1) = 0$ yields $-\sin\alpha + \alpha\cos\alpha = 0$, that is, $\tan\alpha = \alpha$.

15. **(a)** The endpoint conditions are

$$y(0) = y'(0) = y''(L) = y^{(3)}(L) = 0.$$

With these conditions, four successive integrations as in Example 5 yield the indicated shape function $y(x)$.

(b) The maximum value $y_{\max}$ of $y(x)$ on the closed interval $[0.L]$ must occur either at an interior point where $y'(x)=0$ or at one of the endpoints $x=0$ and $x=L$. Now

$$y'(x)=k\left(4x^3-12Lx^2+12L^2x\right)=4kx\left(x^2-3Lx+3L^2\right),$$

where $k=\dfrac{w}{24EI}$ and the quadratic factor has no real zero. Hence $x=0$ is the only zero of $y'(x)$. But $y(0)=0$, so it follows that $y_{\max}=y(L)$.

16. **(a)** The endpoint conditions are $y(0)=y'(0)=0$ and $y(L)=y'(L)=0$.

(b) The derivative

$$y'(x)=k\left(4x^3-6Lx^2+2L^2x\right)=2kx(2x-L)(x-L)$$

vanishes at $x=0,\dfrac{L}{2},L$. Because $y(0)=y(L)=0$, the argument of Problem 15(b) implies that $y_{\max}=y\left(\dfrac{L}{2}\right)$.

17. If $y(x)=k\left(x^4-2Lx^3+L^3x\right)$ with $k=\dfrac{w}{24EI}$, then

$$y'(x)=k\left(4x^3-6Lx^2+L^3\right)=0$$

has the solution $x=\dfrac{L}{2}$ that we can verify by inspection. Now long division of the cubic $4x^3-6Lx^2+L^3$ by $2x-L$ yields the quadratic factor $2x^2-2Lx-L^2$ whose zeros $\dfrac{2L\pm\sqrt{12L^2}}{4}=\dfrac{\left(1\pm\sqrt{3}\right)L}{2}$ both lie outside the interval $[0,L]$. Thus $x=\dfrac{L}{2}$ is, indeed, the only zero of $y'(x)=0$ in this interval.

18. **(a)** The endpoint conditions are $y(0)=y'(0)=0$ and $y(L)=y''(L)=0$.

(b) The only zero of the derivative

$$y'(x)=2kx\left(8x^2-15Lx+6L^2\right)$$

interior to the interval $[0,L]$ is $x_m=\dfrac{\left(15-\sqrt{33}\right)L}{16}$, and $y(0)=y(L)=0$, so it follows by the argument of Problem 15(b) that $y_{\max}=y(x_m)$.