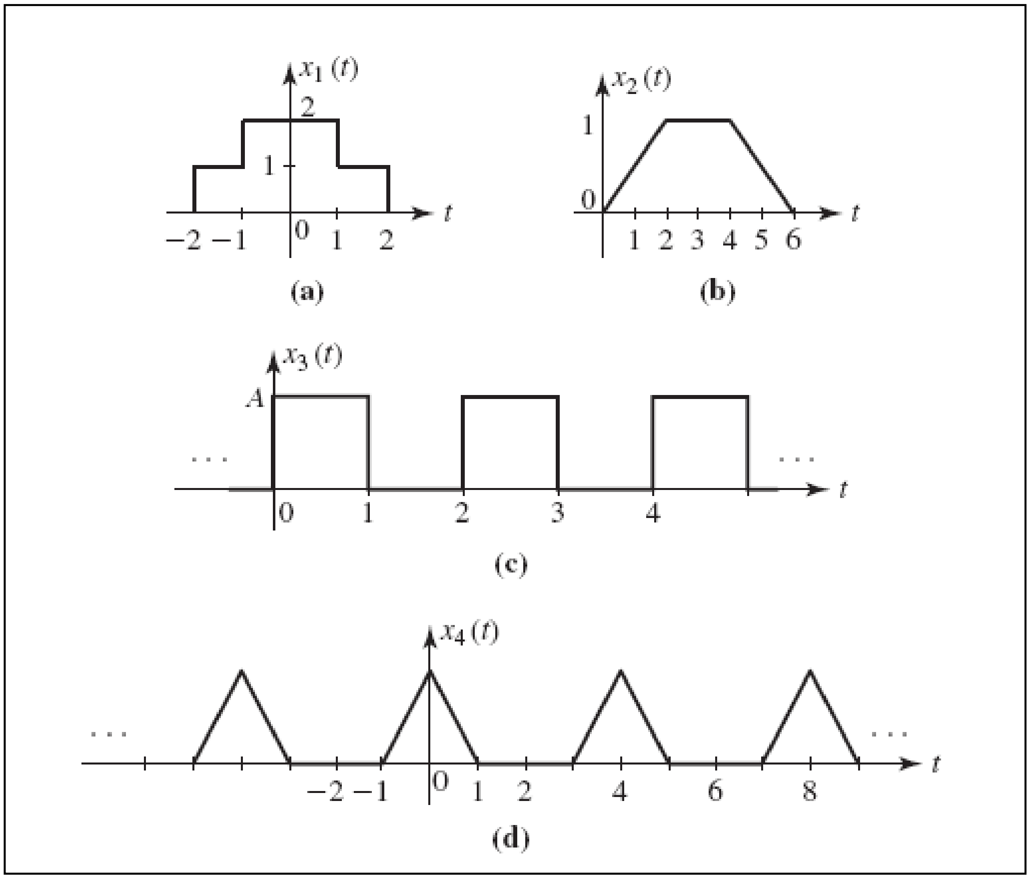
Chapter 2

2.1 Consider the signals displayed in Figure P2.1. Show that each of these signals can be expressed as the sum of rectangular $\Pi(t)$ and/or triangular $\Lambda(t)$ pulses.

Figure P2.1



Solution:

a. $x_1(t) = \Pi\left(\frac{t}{2}\right) + \Pi\left(\frac{t}{4}\right)$

b. $x_2(t) = 2\Lambda\left(\frac{t-3}{6}\right) - \Lambda\left(\frac{t-3}{2}\right)$

c. $x_3(t) = \ldots + \Pi\left(t+\frac{11}{2}\right) + \Pi\left(t+\frac{7}{2}\right) + \Pi\left(t+\frac{3}{2}\right) + \Pi\left(t-\frac{1}{2}\right) + \Pi\left(t-\frac{5}{2}\right) + \Pi\left(t-\frac{9}{2}\right) + \ldots$

$$= \sum_{n=-\infty}^{\infty} \Pi\left[t - (2n+0.5)\right]$$

d. $x_4(t) = \ldots + \Lambda\left(\frac{t+4}{2}\right) + \Lambda\left(\frac{t}{2}\right) + \Lambda\left(\frac{t-4}{2}\right) + \ldots$

$$= \sum_{n=-\infty}^{\infty} \Lambda\left(\frac{t-4n}{2}\right)$$

2.2 For the signal $x_2(t)$ in Figure P2.1 (b) plot the following signals

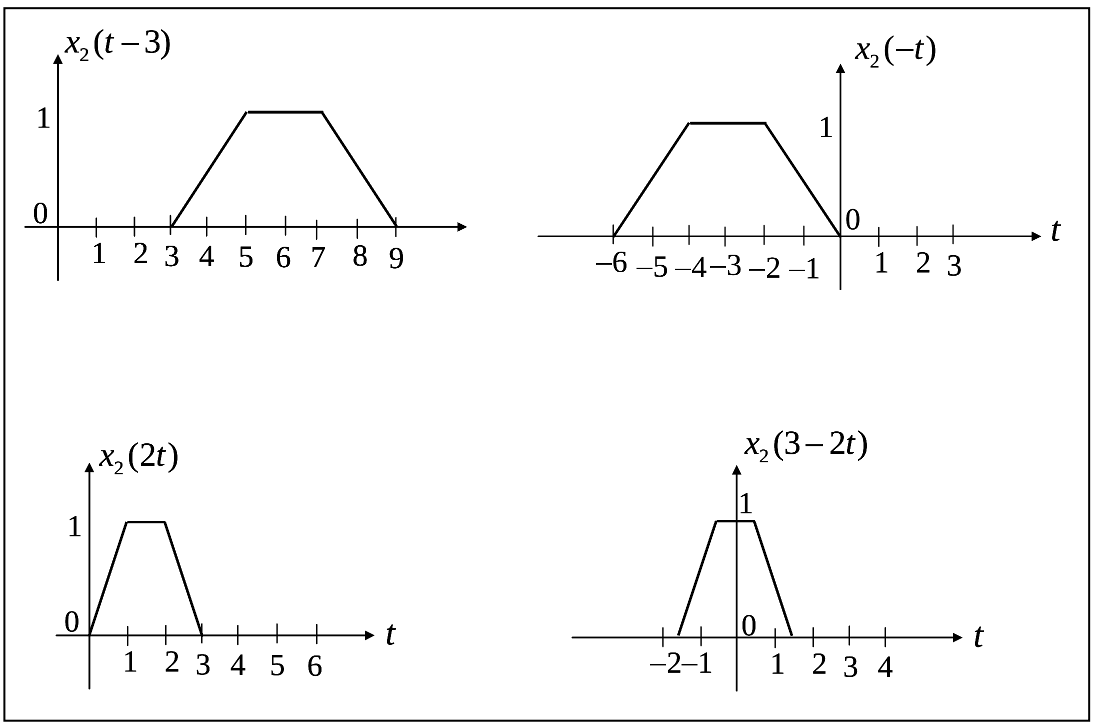
a. $x_2(t-3)$

b. $x_2(-t)$

c. $x_2(2t)$

d. $x_2(3-2t)$

Solution:



2.3 Plot the following signals

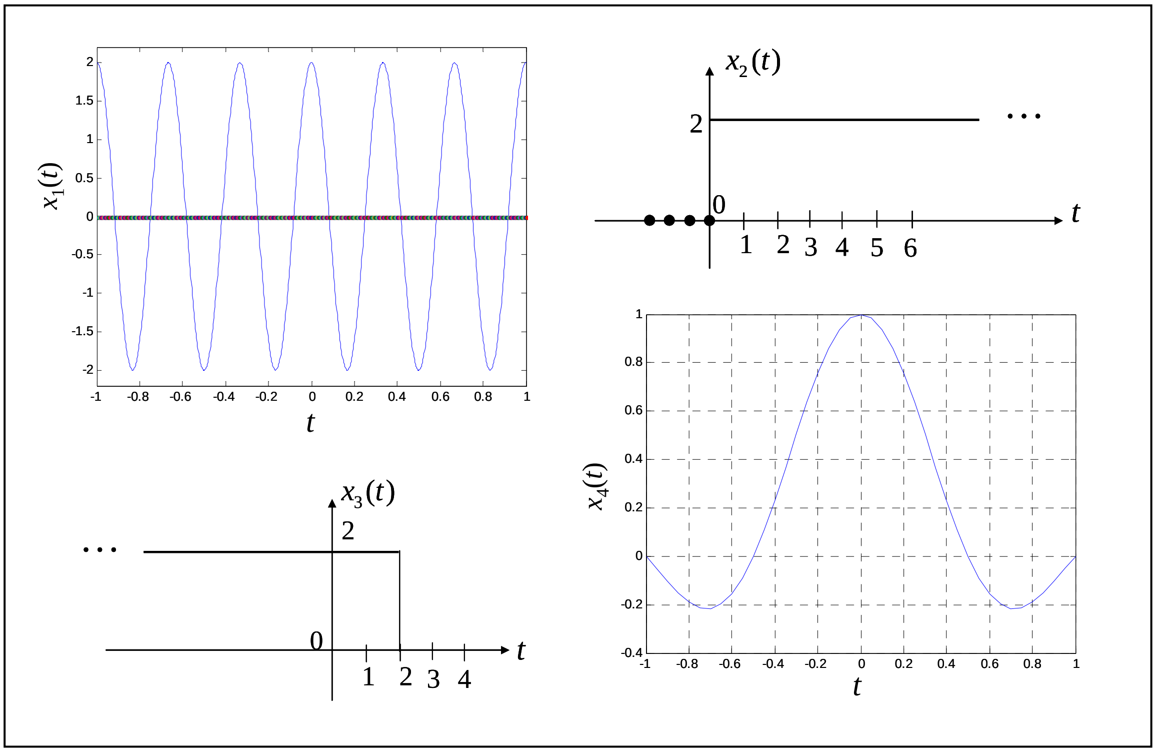
a. $x_1(t) = 2\Pi(t/2)\cos(6\pi t)$

b. $x_2(t) = 2\left[\frac{1}{2} + \frac{1}{2}\operatorname{sgn}(t)\right]$

c. $x_3(t) = x_2(-t+2)$

d. $x_4(t) = \text{sinc}(2t)\Pi(t/2)$

Solution:



2.4 Determine whether the following signals are periodic. For periodic signals, determine the fundamental period.

a. $x_1(t) = \sin(\pi t) + 5\cos(4\pi t/5)$

Solution:

$\sin(\pi t)$ is periodic with period $T_1 = \frac{2\pi}{\pi} = 2$. $\cos(4\pi t/5)$ is periodic with period $T_2 = \frac{2\pi}{4\pi/5} = \frac{5}{2}$. $x_1(t)$ is periodic if the ratio $\frac{T_1}{T_2}$ can be written as ratio of integers. In the present case,

$$\frac{T_1}{T_2} = \frac{2\times 2}{5} = \frac{4}{5}$$

Therefore, $x_1(t)$ is periodic with fundamental period T_o such that

$$T_o = 5T_1 = 4T_2 = 10$$

b. $x_2(t) = e^{j3t} + e^{j9t} + \cos(12t)$

Solution:

e^{j3t} is periodic with period $T_1 = \frac{2\pi}{3}$. Similarly, e^{j9t} is periodic with period $T_2 = \frac{2\pi}{9}$. $\cos(12t)$ is periodic with period $T_3 = \frac{2\pi}{12} = \frac{\pi}{6}$. $x_2(t)$ is periodic with fundamental period $T_o = LCM\left(T_1, T_2, T_3\right) = \frac{2\pi}{3}$.

c. $x_3(t) = \sin(2\pi t) + \cos(10t)$

Solution:

$\sin(2\pi t)$ is periodic with period $T_1 = \frac{2\pi}{2\pi} = 1$. $\cos(10t)$ is periodic with period $T_2 = \frac{2\pi}{10} = \frac{\pi}{5}$. $x_3(t)$ is periodic if the ratio $\frac{T_1}{T_2}$ can be written as ratio of integers. In the present case,

$$\frac{T_1}{T_2} = \frac{1 \times 5}{\pi} = \frac{5}{\pi}$$

Since π is an irrational number, the ratio is not rational. Therefore, $x_3(t)$ is not periodic.

d. $x_4(t) = \cos\left(2\pi t - \frac{\pi}{4}\right) + \sin(5\pi t)$

Solution:

$\cos\left(2\pi t - \frac{\pi}{4}\right)$ is periodic with period $T_1 = \frac{2\pi}{2\pi} = 1$. $\sin(5\pi t)$ is periodic with period $T_2 = \frac{2\pi}{5\pi} = \frac{2}{5}$. $x_4(t)$ is periodic if the ratio $\frac{T_1}{T_2}$ can be written as ratio of integers. In the present case,

$$\frac{T_1}{T_2} = \frac{1 \times 5}{2} = \frac{5}{2}$$

Therefore, $x_4(t)$ is periodic with fundamental period T_o such that

$$T_o = 2T_1 = 5T_2 = 2$$

2.5 Classify the following signals as odd or even or neither.

a. $x(t) = -4t$

Solution:

$x(-t) = 4t = -(-4t) = -x(t)$. So $x(t)$ is odd.

b. $x(t) = e^{-|t|}$

Solution:

$e^{-|t|} = e^{-|-t|}$. So $x(t)$ is even.

c. $x(t) = 5\cos(3t)$

Solution:

Since $\cos(t)$ is even, $5\cos(3t)$ is also even.

d. $x(t) = \sin(3t - \frac{\pi}{2})$

Solution:

$x(t) = \sin(3t - \frac{\pi}{2}) = -\cos(3t)$ which is even.

e. $x(t) = u(t)$

Solution:

$u(t)$ is neither even nor odd; For example, $u(1) = 1$ but $u(-1) = 0 \neq -u(1)$.

f. $x(t) = \sin(2t) + \cos(2t)$

Solution:

$x(t)$ is neither even nor odd; For example,

$x(\pi/8) = \sqrt{2}$ but $x(-\pi/8) = 0 \neq -x(\pi/8)$.

2.6 Determine whether the following signals are energy or power, or neither and calculate the corresponding energy or power in the signal:

a. $x_1(t) = u(t)$

Solution:

The normalized average power of a signal $x_1(t)$ is defined as

$$P_{x_1} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |u(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T/2} 1\, dt = \lim_{T\to\infty} \frac{1}{T}\frac{T}{2} = \frac{1}{2}$$

Therefore, $x_1(t)$ is a power signal.

b. $x_2(t) = 4\cos(2\pi t) + 3\cos\left(4\pi t\right)$

Solution:

$4\cos(2\pi t)$ is a power signal. $3\cos\left(4\pi t\right)$ is also a power signal. Since $x_2(t)$ is sum of two power signals, it is a power signal.

$$P_{x_2} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left[4\cos(2\pi t) + 3\cos\left(4\pi t\right)\right]^2 dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left[4\cos(2\pi t) + 3\cos\left(4\pi t\right)\right]^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left[16\cos^2(2\pi t) + 9\cos^2\left(4\pi t\right) + 24\cos(2\pi t)\cos\left(4\pi t\right)\right] dt$$

Now

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} 16\cos^2(2\pi t) dt = \lim_{T\to\infty} \frac{8}{T} \int_{-T/2}^{T/2} \left[1 + \cos(4\pi t)\right] dt = \lim_{T\to\infty} \frac{8T}{T} = 8$$

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} 9\cos^2(4\pi t) dt = 4.5$$

$$\lim_{T\to\infty}\frac{24}{T}\int_{-T/2}^{T/2}\cos(2\pi t)\cos\left(4\pi t\right)dt=\lim_{T\to\infty}\frac{24}{T}\int_{-T/2}^{T/2}\left[\cos(6\pi t)+\cos\left(2\pi t\right)\right]dt=0$$

Therefore,

$$P_{x_2}=8+4.5=12.5$$

c. $x_3(t)=\dfrac{1}{t}$

Solution:

$$E_{x_3}=\lim_{T\to\infty}\int_{-T/2}^{T/2}\left|1/t\right|^2dt=\lim_{T\to\infty}\int_{-T/2}^{T/2}t^{-2}dt=\lim_{T\to\infty}\left(-\frac{1}{t}\Big|_{-T/2}^{T/2}\right)=\lim_{T\to\infty}\left(\frac{-2}{T/2}\right)=0$$

$$P_{x_3}=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\left|1/t\right|^2dt=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}t^{-2}dt=\lim_{T\to\infty}\frac{1}{T}\left(-\frac{1}{t}\Big|_{-T/2}^{T/2}\right)=\lim_{T\to\infty}\frac{1}{T}\left(\frac{-2}{T/2}\right)=0$$

$x_3(t)$ is neither an energy nor a power signal.

d. $x_4(t)=e^{-\alpha t}u(t)$

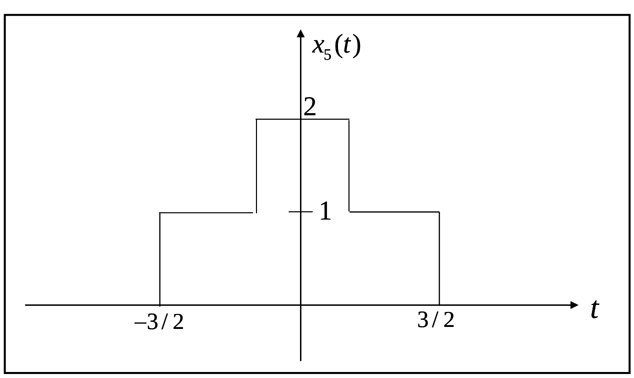
Solution:

$$E_{x_4}=\lim_{T\to\infty}\int_{-T/2}^{T/2}\left|e^{-\alpha t}u(t)\right|^2dt=\lim_{T\to\infty}\int_{0}^{T/2}e^{-2\alpha t}dt=\lim_{T\to\infty}\left(-\frac{e^{-2\alpha t}}{2\alpha}\Big|_{0}^{T/2}\right)=\frac{1}{2\alpha}\lim_{T\to\infty}\left(-e^{-\alpha T}+1\right)=\frac{1}{2\alpha}$$

Thus $x_4(t)$ is an energy signal.

e. $x_5(t)=\Pi(t/3)+\Pi(t)$

Solution:



$$E_{x_5} = \lim_{T\to\infty} \int_{-T/2}^{T/2} \left|\Pi(t/3)+\Pi(t)\right|^2 dt = \int_{-3/2}^{-1/2} 1dt + \int_{-1/2}^{1/2} 4dt + \int_{-3/2}^{-1/2} 1dt$$
$$= 1+4+1 = 6$$

Thus $x_5(t)$ is an energy signal.

f. $x_6(t) = 5e^{(-2t+j10\pi t)}u(t)$

Solution:

$$E_{x_6} = \lim_{T\to\infty} \int_{-T/2}^{T/2} \left|5e^{(-2t+j10\pi t)}u(t)\right|^2 dt = \lim_{T\to\infty} \int_{0}^{T/2} \left|5e^{-2t}e^{j10\pi t})\right|^2 dt = \lim_{T\to\infty} 25\int_{0}^{T/2} e^{-4t}dt$$
$$= 25\lim_{T\to\infty}\left(-\frac{e^{-4t}}{4}\bigg|_0^{T/2}\right) = \frac{25}{4}\lim_{T\to\infty}\left(-e^{-2T}+1\right) = \frac{25}{4}$$

Thus $x_6(t)$ is an energy signal.

g. $x_7(t) = \sum_{n=-\infty}^{\infty} \Lambda\left[\left(t-4n\right)/2\right]$

Solution:

$x_7(t)$ is a periodic signal with period $T_o = 4$.

$$P_{x_7} = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2}\left[\Lambda\left(t/2\right)\right]^2 dt = \frac{2}{T_o}\int_0^1 \left(1-t\right)^2 dt = \frac{1}{2}\int_0^1\left(1-2t+t^2\right)dt$$
$$= \frac{1}{2}\left(t-t^2+\frac{t^3}{3}\right)\bigg|_0^1 = \frac{1}{2}\left(1-1+\frac{1}{3}\right) = \frac{1}{6}$$

2.7 Evaluate the following expressions by using the properties of the delta function:

a. $x_1(t) = \delta(4t)\sin(2t)$

Solution:

$$\delta(4t) = \frac{1}{4}\delta(t)$$

$$x_1(t) = \frac{1}{4}\delta(t)\sin(2t) = \frac{1}{4}\delta(t)\sin(0) = 0$$

b. $x_2(t) = \delta(t)\cos\left(30\pi t + \frac{\pi}{4}\right)$

Solution:

$$x_2(t) = \delta(t)\cos\left(30\pi t + \frac{\pi}{4}\right) = \delta(t)\cos\left(0 + \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)\delta(t)$$

c. $x_3(t) = \delta(t)\text{sinc}(t+1)$

Solution:

$$x_3(t) = \delta(t)\text{sinc}(t+1) = \delta(t)\text{sinc}(0+1) = \delta(t)\text{sinc}(1) = \delta(t)\times 0 = 0$$

d. $x_4(t) = \delta(t-2)e^{-t}\sin(2.5\pi t)$

Solution:

$$\begin{aligned} x_4(t) &= \delta(t-2)e^{-t}\sin(2.5\pi t) \\ &= \delta(t-2)e^{-2}\sin(2.5\pi \times 2) = \delta(t-2)e^{-2}\sin(\pi) = 0 \end{aligned}$$

e. $x_5(t) = \int_{-\infty}^{\infty} \delta(2t)\text{sinc}(t)dt$

Solution:

$$\begin{aligned} x_5(t) &= \int_{-\infty}^{\infty} \delta(2t)\text{sinc}(t)dt = \frac{1}{2}\int_{-\infty}^{\infty} \delta(t)\text{sinc}(t)dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \delta(t)\text{sinc}(0)dt = \frac{1}{2}\int_{-\infty}^{\infty} \delta(t)dt = \frac{1}{2} \end{aligned}$$

f. $x_6(t) = \int_{-\infty}^{\infty} \delta(t-3)\cos(t)dt$

Solution:

$$x_6(t) = \int_{-\infty}^{\infty} \delta(t-3)\cos(t)dt = \cos(3)\int_{-\infty}^{\infty} \delta(t-3)dt = \cos(3)$$

g. $x_7(t)=\int_{-\infty}^{\infty}\delta(2-t)\frac{1}{1-t^3}dt$

Solution:

$$x_7(t)=\int_{-\infty}^{\infty}\delta(2-t)\frac{1}{1-t^3}dt=\int_{-\infty}^{\infty}\delta(t-2)\frac{1}{1-t^3}dt$$
$$=\int_{-\infty}^{\infty}\delta(t-2)\frac{1}{1-2^3}dt=-\frac{1}{7}\int_{-\infty}^{\infty}\delta(t-2)dt=-\frac{1}{7}$$

h. $x_8(t)=\int_{-\infty}^{\infty}\delta(3t-4)e^{-3t}dt$

Solution:

$$x_8(t)=\int_{-\infty}^{\infty}\delta(3t-4)e^{-3t}dt=\frac{1}{3}\int_{-\infty}^{\infty}\delta\left(t-\frac{4}{3}\right)e^{-3t}dt$$
$$=\frac{1}{3}\int_{-\infty}^{\infty}\delta\left(t-\frac{4}{3}\right)e^{-3\times\frac{4}{3}}dt=\frac{e^{-4}}{3}\int_{-\infty}^{\infty}\delta\left(t-\frac{4}{3}\right)dt=\frac{e^{-4}}{3}$$

i. $x_9(t)=\delta'(t)\otimes\Pi(t)$

Solution:

$$x_9(t)=\int_{-\infty}^{\infty}\Pi(t-\tau)\delta'(\tau)d\tau=(-1)\frac{d}{d\tau}\Pi(t-\tau)\Big|_{\tau=0}$$
$$=\Pi'(t)=\delta\left(t+0.5\right)-\delta\left(t-0.5\right)$$

2.8 For each of the following continuous-time systems, determine whether or not the system is (1) linear, (2) time-invariant, (3) memoryless, and (4) casual.

a. $y(t)=x(t-1)$

Solution:

The system is linear, time-invariant, causal, and has memory. The system has memory because current value of the output depends on the previous value of the input. The system is causal because current value of the output does not depend on future inputs. To prove linearity, let $x(t)=\alpha x_1(t)+\beta x_2(t)$. The response of the system to $x(t)$ is

$$y(t) = x(t-1) = \alpha x_1(t-1) + \beta x_2(t-1)$$
$$= \alpha y_1(t) + \beta y_2(t)$$

where $x_1(t) \xrightarrow{\mathcal{T}} y_1(t)$ and $x_2(t) \xrightarrow{\mathcal{T}} y_2(t)$.

To show that the system is time-invariant, let $x_1(t) = x(t-t_o)$ be the system input. The corresponding output is

$$y_1(t) = x_1(t-1) = x(t-1-t_o) = y(t-t_o)$$

b. $y(t) = 3x(t) - 2$

Solution:

The system is nonlinear, time-invariant, causal, and memoryless. The system is memoryless because current value of the output depends only on the current value of the input. The system is causal because current value of the output does not depend on future inputs.

To prove nonlinearity, let $x(t) = \alpha x_1(t) + \beta x_2(t)$. The response of the system to $x(t)$ is

$$y(t) = 3x(t) - 2 = 3\left[\alpha x_1(t) + \beta x_2(t)\right] - 2$$
$$\neq \alpha y_1(t) + \beta y_2(t) = \alpha\left[3x_1(t) - 2\right] + \beta\left[3x_2(t) - 2\right]$$

To show that the system is time-invariant, let $x_1(t) = x(t-t_o)$ be the system input. The corresponding output is

$$y_1(t) = 3x_1(t) - 2 = 3x(t-t_o) - 2 = y(t-t_o)$$

c. $y(t) = \left|x(t)\right|$

Solution:

The system is nonlinear, time-invariant, causal, and memoryless. The system is memoryless because current value of the output depends only on the current value of the input. The system is causal because current value of the output does not depend on future inputs.

To prove nonlinearity, let $x(t) = \alpha x_1(t) + \beta x_2(t)$. The response of the system to $x(t)$ is

$$y(t) = |\alpha x_1(t) + \beta x_2(t)| \le |\alpha x_1(t)| + |\beta x_2(t)|$$

Because

$$|\alpha x_1(t)| + |\beta x_2(t)| = |\alpha||x_1(t)| + |\beta||x_2(t)| \ne \alpha y_1(t) + \beta y_2(t) \text{ for all } \alpha \text{ and } \beta,$$

the system is nonlinear.

To show that the system is time-invariant, let $x_1(t) = x(t - t_o)$ be the system input. The corresponding output is

$$y_1(t) = |x_1(t)| = |x(t - t_o)| = y(t - t_o)$$

d. $y(t) = [\cos(2t)]x(t)$

Solution:

The system is linear, time-variant, causal, and memoryless. The system is memoryless because current value of the output depends only on the current value of the input. The system is causal because current value of the output does not depend on future inputs.

To prove nonlinearity, let $x(t) = \alpha x_1(t) + \beta x_2(t)$. The response of the system to $x(t)$ is

$$\begin{aligned} y(t) &= [\cos(2t)]x(t) = [\cos(2t)][\alpha x_1(t) + \beta x_2(t)] \\ &= \alpha[\cos(2t)]x_1(t) + \beta[\cos(2t)]x_2(t) = \alpha y_1(t) + \beta y_2(t) \end{aligned}$$

To prove time-variance, let $x_1(t) = x(t - t_o)$ be the system input. The corresponding output is

$$y_1(t) = [\cos(2t)]x_1(t) = [\cos(2t)]x(t - t_o) \ne y(t - t_o) = \cos[2(t - t_o)]x(t - t_o)$$

e. $y(t) = e^{x(t)}$

Solution:

The system is nonlinear, time-invariant, causal, and memoryless. The system is memoryless because current value of the output depends only on the current value of the input. The system is causal because current value of the output does not depend on future inputs.

To prove nonlinearity, let $x(t) = \alpha x_1(t) + \beta x_2(t)$. The response of the system to $x(t)$ is

$$y(t) = e^{\alpha x_1(t) + \beta x_2(t)} = e^{\alpha x_1(t)} e^{\beta x_2(t)} \neq \alpha y_1(t) + \beta y_2(t) = \alpha e^{x_1(t)} + \beta e^{x_2(t)}$$

for all α and β

To show that the system is time-invariant, let $x_1(t) = x(t - t_o)$ be the system input. The corresponding output is

$$y_1(t) = e^{x_1(t)} = e^{x(t-t_o)} = y(t - t_o)$$

f. $y(t) = tx(t)$

Solution:

The system is linear, time-variant, causal, and memoryless. The system is memoryless because current value of the output depends only on the current value of the input. The system is causal because current value of the output does not depend on future inputs. To prove nonlinearity, let $x(t) = \alpha x_1(t) + \beta x_2(t)$. The response of the system to $x(t)$ is

$$y(t) = t\left[\alpha x_1(t) + \beta x_2(t)\right] = \alpha t x_1(t) + \beta t x_2(t) = \alpha y_1(t) + \beta y_2(t)$$

To prove time-variance, let $x_1(t) = x(t - t_o)$ be the system input. The corresponding output is

$$y_1(t) = t x_1(t) = t x\left(t - t_o\right) \neq y(t - t_o) = \left(t - t_o\right) x\left(t - t_o\right)$$

g. $y(t) = \int_{-\infty}^{t} x(2\tau) d\tau$

Solution:

The system is linear, time-invariant, causal, and has memory. The system has memory because current value of the output depends only on the past values of the input. The system is causal because current value of the output does not depend on future inputs. To prove linearity, let $x(t) = \alpha x_1(t) + \beta x_2(t)$. The response of the system to $x(t)$ is

$$y(t)=\int_{-\infty}^{t}\left[\alpha x_1(2\tau)+\beta x_2(2\tau)\right]d\tau=\alpha\int_{-\infty}^{t}x_1(2\tau)d\tau+\beta\int_{-\infty}^{t}x_2(2\tau)d\tau=\alpha y_1(t)+\beta y_2(t)$$

To check for time-variance, let $x_1(t)=x(t-t_o)$ be the system input. The corresponding output is

$$y_1(t)=\int_{-\infty}^{t}x_1(2\tau)d\tau=\int_{-\infty}^{t}x\left[2(\tau-t_o)\right]d\tau=\int_{-\infty}^{t-t_o}x\left(2\alpha\right)d\alpha=y\left(t-t_o\right)$$

2.9 Calculate the output $y(t)$ of the LTI system for the following cases:

a. $x(t)=e^{-2t}u(t)$ and $h(t)=u(t-2)-u(t-4)$

Solution:

$$h(t)=u(t-2)-u(t-4)=\Pi\left(\frac{t-3}{2}\right)$$

$$y(t)=\int_{-\infty}^{\infty}h(\tau)x(t-\tau)d\tau=\int_{-\infty}^{\infty}e^{-2(t-\tau)}u(t-\tau)\Pi\left(\frac{\tau-3}{2}\right)d\tau$$

For $t<2$, there is no overlap and $y(t)=0$.

For $2\le t<4$, $y(t)=\int_{2}^{t}e^{-2(t-\tau)}d\tau=\int_{0}^{t-2}e^{-2\tau}d\tau=\left.\frac{e^{-2\tau}}{-2}\right|_{0}^{t-2}=\frac{1-e^{-2(t-2)}}{2}$

For $t\ge 4$,

$$y(t)=\int_{2}^{4}e^{-2(t-\tau)}d\tau=e^{-2t}\int_{2}^{4}e^{2\tau}d\tau=e^{-2t}\left.\frac{e^{2\tau}}{2}\right|_{2}^{4}=e^{-2t}\frac{e^{8}-e^{4}}{2}=e^{-2(t-2)}\frac{\left(e^{4}-1\right)}{2}$$

$$y(t)=\begin{cases}\dfrac{1-e^{-2(t-2)}}{2}, & 2\le t\le 4\\ e^{-2(t-2)}\dfrac{\left(e^{4}-1\right)}{2}, & t>4\\ 0, & \text{otherwise}\end{cases}$$

b. $x(t)=e^{-t}u(t)$ and $h(t)=e^{-2t}u(t)$

Solution:

$$y(t)=\int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-2(t-\tau)}u(t-\tau)d\tau=\int_{0}^{t} e^{-\tau}e^{-2(t-\tau)}d\tau$$

For $t<0$, there is no overlap and $y(t)=0$.

$$y(t)=e^{-2t}\int_{0}^{t} e^{\tau}d\tau = e^{-2t}\, e^{\tau}\Big|_{0}^{t} = e^{-2t}\left(e^{t}-1\right)=e^{-t}-e^{-2t},\quad t\ge 0$$

c. $x(t)=u(-t)$ and $h(t)=\delta(t)-3e^{-2t}u(t)$

Solution:

$$y(t)=\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau=\int_{-\infty}^{\infty}\left[\delta(\tau)-3e^{-2\tau}u(\tau)\right]u(\tau-t)d\tau$$

Now

$$\int_{-\infty}^{\infty}\delta(\tau)u(\tau-t)d\tau=u(-t)$$

For $t<0$, $\int_{0}^{\infty}3e^{-2\tau}d\tau=\frac{3e^{-2\tau}}{-2}\Big|_{0}^{\infty}=\frac{3}{2}$

For $t\ge 0$, $\int_{-\infty}^{\infty}3e^{-2\tau}u(\tau)u(\tau-t)d\tau=\int_{t}^{\infty}3e^{-2\tau}d\tau=\frac{3e^{-2\tau}}{-2}\Big|_{t}^{\infty}=\frac{3}{2}e^{-2t}$

$$y(t)=\begin{cases}\frac{3}{2}+1=\frac{5}{2}, & t<0\\ \frac{3}{2}e^{-2t}, & t\ge 0\end{cases}$$

d. $x(t)=\delta(t-2)+3e^{3t}u(-t)$ and $h(t)=u(t)-u(t-1)$

Solution:

$$y(t)=\int_{-\infty}^{\infty}h(\tau)x(t-\tau)d\tau=\int_{-\infty}^{\infty}\left[\delta(t-\tau-2)+3e^{3(t-\tau)}u(-t+\tau)\right]\left[u(\tau)-u(\tau-1)\right]d\tau$$

$$=\int_{-\infty}^{\infty}\delta(t-\tau-2)\left[u(\tau)-u(\tau-1)\right]d\tau+\int_{-\infty}^{\infty}3e^{3(t-\tau)}u(-t+\tau)\left[u(\tau)-u(\tau-1)\right]d\tau$$

Now

$$\int_{-\infty}^{\infty}\delta(t-\tau-2)\left[u(\tau)-u(\tau-1)\right]d\tau=u(t-2)-u(t-2-1)=\Pi\left(t-2.5\right)$$

For $t<0$,

$$\int_{-\infty}^{\infty}3e^{3(t-\tau)}u(-t+\tau)\left[u(\tau)-u(\tau-1)\right]d\tau=\int_{0}^{1}3e^{3(t-\tau)}d\tau=3e^{3t}\int_{0}^{1}e^{-3\tau}d\tau$$

$$=3e^{3t}\left.\frac{e^{-3\tau}}{-3}\right|_{0}^{1}=e^{3t}\left(1-\frac{1}{e^{3}}\right)$$

For $0<t\le 1$,

$$\int_{-\infty}^{\infty}3e^{3(t-\tau)}u(-t+\tau)\left[u(\tau)-u(\tau-1)\right]d\tau=\int_{t}^{1}3e^{3(t-\tau)}d\tau=3e^{3t}\int_{t}^{1}e^{-3\tau}d\tau$$

$$=3e^{3t}\left.\frac{e^{-3\tau}}{-3}\right|_{t}^{1}=e^{3t}\left(e^{-3t}-\frac{1}{e^{3}}\right)$$

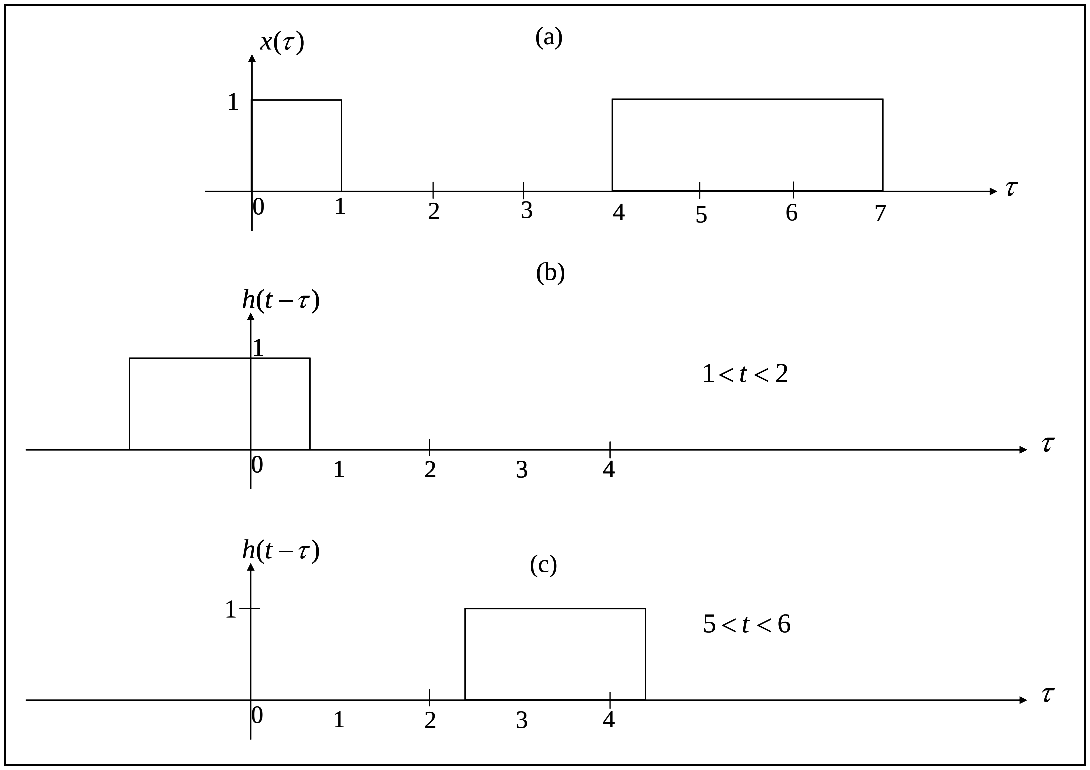
$$y(t)=\begin{cases}e^{3t}\left(1-\frac{1}{e^{3}}\right), & t\le 0\\ e^{3t}\left(e^{-3t}-\frac{1}{e^{3}}\right), & 0<t\le 1\\ 1, & 2\le t\le 3\\ 0, & \text{otherwise}\end{cases}$$

2.10 The impulse response function of a continuous-time LTI is displayed in Figure P2.2(b). Assuming the input $x(t)$ to the system is waveform illustrated in Figure P2.2(a), determine the system output waveform $y(t)$ and sketch it.

Solution:

For $t<1$, there is no overlap and $y(t)=0$.

Figure P2.2



As shown in Figure (b), $y(t) = \int_0^{t-1} d\tau = t - 1$ for $1 \le t < 2$

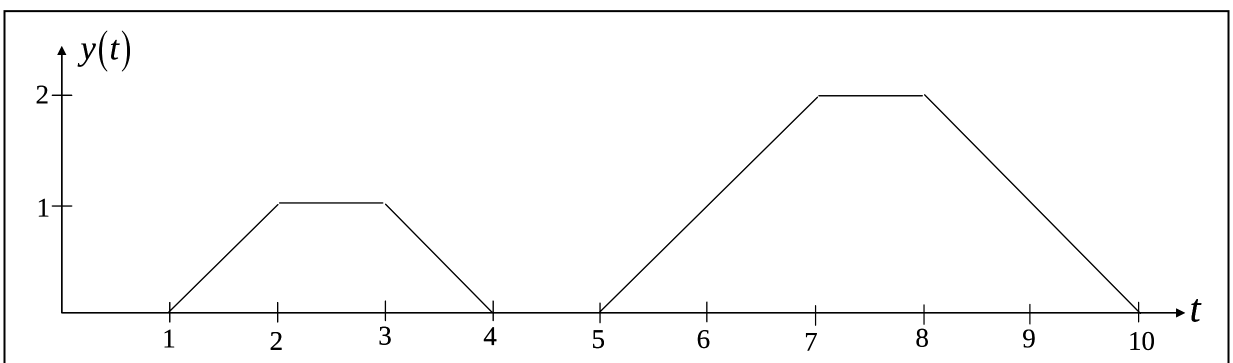
For $2 \le t < 3$, $y(t) = \int_0^1 d\tau = 1$

For $3 \le t < 4$, $y(t) = \int_{t-3}^{1} d\tau = 1 - t + 3 = 4 - t$

Referring to Figure (c), $y(t) = \int_5^t d\tau = t - 5$ for $5 \le t < 7$

For $7 \le t < 8$, $y(t) = \int_{t-2}^{t} d\tau = t - t + 2 = 2$

For $8 \le t < 10$, $y(t) = \int_{t-3}^{7} d\tau = 7 - t + 3 = 10 - t$



2.11 An LTI system has the impulse response $h(t)=e^{-0.5(t-2)}u(t-2)$.

a. Is the system casual?

Solution:

Yes. The system is casual because $h(t)=0$ for $t<0$.

b. Is the system stable?

Solution:

The system is stable because $\int_{-\infty}^{\infty}|h(t)|dt = \int_{0}^{\infty} e^{-0.5x}dx = \left.\frac{e^{-0.5x}}{-0.5}\right|_{0}^{\infty} = 2$

c. Repeat parts (a) and (b) for $h(t)=e^{-0.5(t+2)}u(t+2)$.

Solution:

The system is not causal but stable.

2.12 Write down the exponential Fourier series coefficients of the signal

$x(t)=5\sin(40\pi t)+7\cos\left(80\pi t-\pi/2\right)-\cos\left(160\pi t+\pi/4\right)$

Solution:

Applying the Euler's formula, we get the following terms:

$$x(t)=5\left[\frac{e^{j40\pi t}-e^{-j40\pi t}}{2j}\right]+7\left[\frac{e^{j80\pi t}e^{-j\pi/2}+e^{-j80\pi t}e^{j\pi/2}}{2}\right]-\left[\frac{e^{j160\pi t}e^{j\pi/4}+e^{-j160\pi t}e^{-j\pi/4}}{2}\right]$$
$$=-j2.5e^{j40\pi t}+j2.5e^{-j40\pi t}-j3.5e^{j80\pi t}+j3.5e^{-j80\pi t}-0.5e^{j\pi/4}e^{j160\pi t}-0.5e^{-j\pi/4}e^{-j160\pi t}$$

The Fourier coefficients are

$C_1=-j2.5, C_{-1}=j2.5$
$C_2=-j3.5, C_{-2}=j3.5$
$C_3=-0.5e^{j\pi/4}, C_{-3}=-0.5e^{-j\pi/4}$

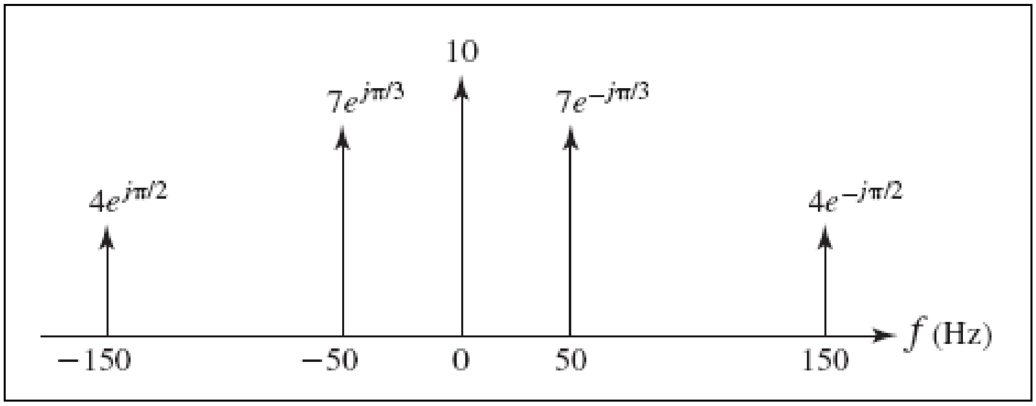
a. Is $x(t)$ periodic? If so, what is its period?

Solution:

Yes. $\omega_o = 40\pi \Rightarrow f_o = 20, T_o = \frac{1}{20}$. The period is $\frac{1}{20} = 0.05\,\text{sec}$.

2.13 A signal has the two-sided spectrum representation shown in Figure P2.3.

Figure P2.3



a. Write the equation for $x(t)$.

Solution:

$$x(t) = 14\cos\left(100\pi t - \frac{\pi}{3}\right) + 10 + 8\cos\left(300\pi t - \frac{\pi}{2}\right)$$

b. Is the signal periodic? If so, what is its period?

Solution:

Yes. It is periodic with fundamental period $T_o = \frac{1}{50} = 0.02$.

c. Does the signal have energy at DC?

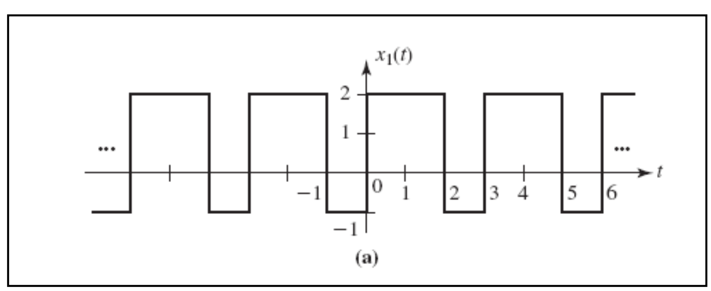
Solution:

Yes as indicated by the presence of DC term $C_0 = 10$.

2.14 Write down the complex exponential Fourier series for each of the periodic signals shown in Figure P2.4. Use odd or even symmetry whenever possible.

Solution:

a. $T_o = 3, f_o = 1/3$



$$C_0 = \frac{1}{3}\left[\int_0^2 2dt - \int_2^3 1dt\right] = \frac{1}{3}(4-1) = 1$$

$$C_n = \frac{1}{3}\left[\int_0^2 2e^{-j\frac{2\pi nt}{3}}dt - \int_2^3 e^{-j\frac{2\pi nt}{3}}dt\right]$$

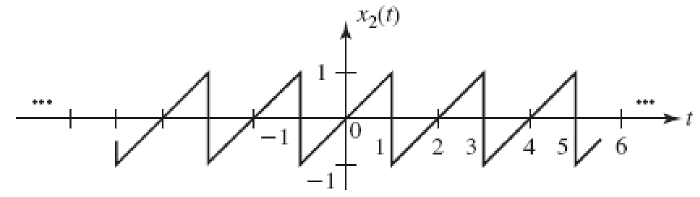
$$= \frac{j3}{3\times 2\pi n}\left[2e^{-j\frac{2\pi nt}{3}}\Big|_0^2 - e^{-j\frac{2\pi nt}{3}}\Big|_2^3\right]$$

$$= \frac{j}{2\pi n}\left[2e^{-j4\pi n/3} - 2 - e^{-j2\pi n} + e^{-j4\pi n/3}\right]$$

$$= \frac{j3}{2\pi n}\left(e^{-j4\pi n/3} - 1\right) = \frac{3e^{-j2\pi n/3}}{\pi n}\left(\frac{e^{+j2\pi n/3} - e^{-j2\pi n/3}}{2j}\right)$$

$$= \frac{3e^{-j2\pi n/3}}{\pi n}\sin\left(\frac{2\pi n}{3}\right), \quad n = \pm 1, \pm 2, ...$$

b. $T_o = 2,\ f_o = 1/2$



(b)

$$C_0 = \frac{1}{2}\left[\int_{-1}^1 tdt\right] = \frac{1}{2}\frac{t^2}{2}\Big|_{-1}^1 = \frac{1}{4}(1-1) = 0$$

$$C_n = \frac{1}{2}\left[\int_{-1}^1 te^{-j\pi nt}dt\right] = -\frac{1}{j2\pi n}\left[\int_{-1}^1 td\left(e^{-j\pi nt}\right)\right]$$

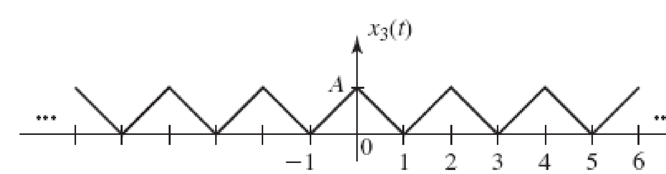
$$= \frac{j}{2\pi n}\left[te^{-j\pi nt}\Big|_{-1}^1 - \int_{-1}^1 e^{-j\pi nt}dt\right]$$

$$= \frac{j}{2\pi n}\left[e^{-j\pi n} + e^{+j\pi n} + \frac{1}{j\pi n}e^{-j\pi nt}\Big|_{-1}^1\right]$$

$$= \frac{j}{2}\left[\frac{e^{-j\pi n}}{\pi n} + \frac{e^{+j\pi n}}{\pi n} + \frac{e^{-j\pi n}}{j\pi^2 n^2} - \frac{e^{j\pi n}}{j\pi^2 n^2}\right]$$

$$= \frac{je^{-j\pi n}}{\pi n} = \frac{j(-1)^n}{\pi n}, \quad n = \pm 1, \pm 2, ...$$

c. $T_o = 2,\ f_o = 1/2$



(c)

$$C_0 = \frac{2A}{2}\left[\int_0^1 (1-t)dt\right] = A\left(t - \frac{t^2}{2}\right)\Bigg|_0^1 = A\left(1 - \frac{1}{2}\right) = \frac{A}{2}$$

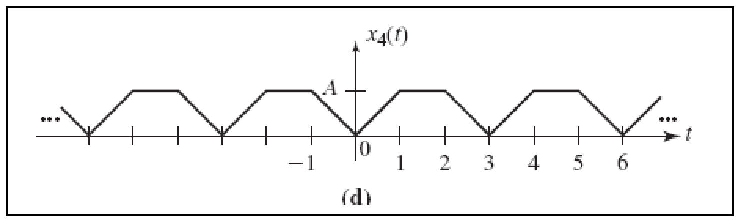
Since $x_3(t)$ is an even function of time,

$$C_n = \frac{A_n}{2} = \left[\frac{2A}{2}\int_0^1 (1-t)\cos(\pi nt)dt\right] = A\int_0^1 \cos(\pi nt)dt - A\int_0^1 t\cos(\pi nt)dt$$

$$= 0 - \frac{A}{\pi n}\int_0^1 td\left(\sin(\pi nt)\right)dt = \frac{A}{\pi n}\left[-t\sin(\pi nt)\Big|_0^1 + \int_0^1 \sin(\pi nt)dt\right]$$

$$= \frac{A}{\pi n}\left[0 - \frac{\cos(\pi nt)}{\pi n}\Bigg|_0^1\right] = -\frac{A}{\pi^2 n^2}\left[\cos(\pi n) - 1\right] = \frac{A}{\pi^2 n^2}\left[1 - \cos(\pi n)\right]$$

$$= \begin{cases} 0, & n \text{ even} \\ \dfrac{2A}{\pi^2 n^2}, & n \text{ odd} \end{cases}$$



d. $T_o = 3, f_o = 1/3$

$$C_0 = \frac{2A}{3}\left[\int_0^1 tdt + \int_1^{3/2} 1dt\right] = \frac{2A}{3}\left[\frac{t^2}{2}\Bigg|_0^1 + t\Big|_1^{3/2}\right] = \frac{1}{3}\left[\frac{1}{2} + \frac{1}{2}\right] = \frac{2A}{3}$$

Since $x_4(t)$ is an even function of time,

$$C_n = \frac{A_n}{2} = \left[\frac{2A}{3}\int_0^{1.5} x_4(t)\cos(2\pi nt/3)dt\right] = \frac{2A}{3}\left[\int_0^1 t\cos(2\pi nt/3)dt + \int_1^{3/2}\cos(2\pi nt/3)dt\right]$$

Now

$$\int_0^1 t\cos(2\pi nt/3)dt = \frac{3}{2\pi n}\left[\int_0^1 td\left(\sin(2\pi nt/3)\right)\right] = \frac{3}{2\pi n}\left[t\sin(2\pi nt/3)\Big|_0^1 - \int_0^1 \sin(2\pi nt/3)dt\right]$$

$$= \frac{3}{2\pi n}\left[\sin(2\pi n/3) + \frac{3}{2\pi n}\cos(2\pi nt/3)\Big|_0^1\right]$$

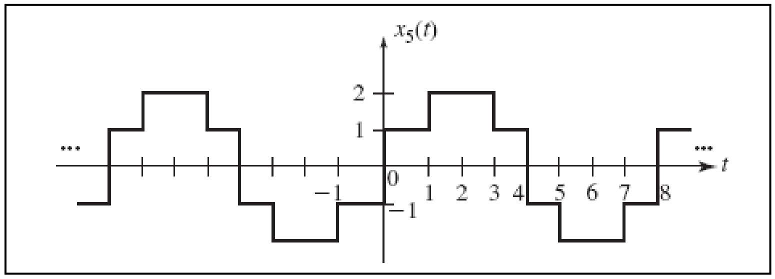
$$= \frac{3}{2\pi n}\left[\sin(2\pi n/3) + \frac{3}{2\pi n}\left[\cos(2\pi n/3) - 1\right]\right]$$

$$\int_1^{3/2}\cos(2\pi nt/3)dt = \frac{3}{2\pi n}\sin(2\pi nt/3)\Big|_1^{3/2} = -\frac{3}{2\pi n}\sin(2\pi n/3)$$

Substituting yields

$$C_n = \frac{2A}{3}\left(\frac{3}{2\pi n}\right)^2\left[\cos\left(\frac{2\pi n}{3}\right)-1\right] = \frac{3A}{2\pi^2 n^2}\left[\cos\left(\frac{2\pi n}{3}\right)-1\right]$$

e. $x_5(t) = \sum_{n=-\infty}^{\infty} p(t-8n)$



where

$$p(t) = \Pi\left[(t-2)/2\right] + \Pi\left[(t-2)/4\right] - \Pi\left[(t-6)/2\right] - \Pi\left[(t-6)/4\right]$$

The FT of the pulse shape $p(t)$ over $[0,\ T_o]$ is given by

$$P(f) = \int_0^{T_o} p(t)e^{-j2\pi ft}dt$$

The FS coefficients of a periodic signal with basic pulse shape $p(t)$ are given by

$$C_n = \frac{1}{T_o}\int_0^{T_o} p(t)e^{-j2\pi nf_o t}dt$$

Comparing yields

$$C_n = \frac{1}{T_o}P(f)\Big|_{f=nf_o}$$

Now

$$\Pi\left[(t-2)/2\right] \overset{\Im}{\longleftrightarrow} 2\mathrm{sinc}(2f)e^{-j4\pi f}$$
$$\Pi\left[(t-2)/4\right] \overset{\Im}{\longleftrightarrow} 4\mathrm{sinc}(4f)e^{-j4\pi f}$$
$$\Pi\left[(t-6)/2\right] \overset{\Im}{\longleftrightarrow} 2\mathrm{sinc}(2f)e^{-j12\pi f}$$
$$\Pi\left[(t-6)/4\right] \overset{\Im}{\longleftrightarrow} 4\mathrm{sinc}(4f)e^{-j12\pi f}$$

Therefore,

$$P(f) = 2\mathrm{sinc}(2f)e^{-j4\pi f}\left[1-e^{-j8\pi f}\right] + 4\mathrm{sinc}(4f)e^{-j4\pi f}\left[1-e^{-j8\pi f}\right]$$

The FS coefficients of a periodic signal $x_5(t)$ are now obtained as

$$C_n = \frac{1}{T_o}\left\{2\text{sinc}\left(2nf_o\right)e^{-j4\pi nf_o}\left[1-e^{-j8\pi nf_o}\right]+4\text{sinc}\left(4nf_o\right)e^{-j4\pi nf_o}\left[1-e^{-j8\pi nf_o}\right]\right\}$$
$$= \frac{1}{4}\left\{\text{sinc}\left(n/4\right)e^{-j\pi n/2}\left[1-e^{-j\pi n}\right]+2\text{sinc}\left(n/2\right)e^{-j\pi n/2}\left[1-e^{-j\pi n}\right]\right\}$$

2.15 For the rectangular pulse train in Figure 2.23, compute the Fourier coefficients of the new periodic signal $y(t)$ given by

a. $y(t) = x(t-0.5T_o)$

Solution:

The FS expansion of a periodic pulse train $x(t) = \sum_{n=-\infty}^{\infty} \Pi\left[\frac{(t-nT_o)}{\tau}\right]$ of rectangular pulses is given by

$$x(t) = \sum_{n=-\infty}^{\infty} C_n^x e^{j2\pi nf_o t}$$

where the exponential FS coefficients are

$$C_n^x = \frac{\tau}{T_o}\text{sinc}\left(nf_o\tau\right)$$

Let the FS expansion of a periodic pulse train $y(t) = x(t-0.5T_o)$ be expressed as

$$y(t) = \sum_{n=-\infty}^{\infty} C_n^y e^{j2\pi nf_o t}$$

where

$$C_n^y = \frac{1}{T_o}\int_{T_o} y(t)e^{-j2\pi nf_o t}dt = \frac{1}{T_o}\int_{T_o} x(t-0.5T_o)e^{-j2\pi nf_o t}dt$$
$$= \frac{1}{T_o}\int_{T_o} x(\nu)e^{-j2\pi nf_o(\nu+0.5T_o)}d\nu = e^{-j2\pi nf_o(0.5T_o)}\underbrace{\frac{1}{T_o}\int_{T_o} x(\nu)e^{-j2\pi nf_o\nu}d\nu}_{C_n^x}$$
$$= e^{-j\pi n}C_n^x$$

Time Shifting introduces a linear phase shift in the FS coefficients; their magnitudes are not changed.

b. $y(t) = x(t)e^{j2\pi t/T_o}$

Solution:

$$C_n^y = \frac{1}{T_o}\int_{T_o} y(t)e^{-j2\pi n f_o t}dt = \frac{1}{T_o}\int_{T_o} x(t)e^{j2\pi t f_o}e^{-j2\pi n f_o t}dt$$
$$= \frac{1}{T_o}\int_{T_o} x(t)e^{-j2\pi t f_o(n-1)}dt$$
$$= C_{n-1}^x$$

c. $y(t) = x(\alpha t)$

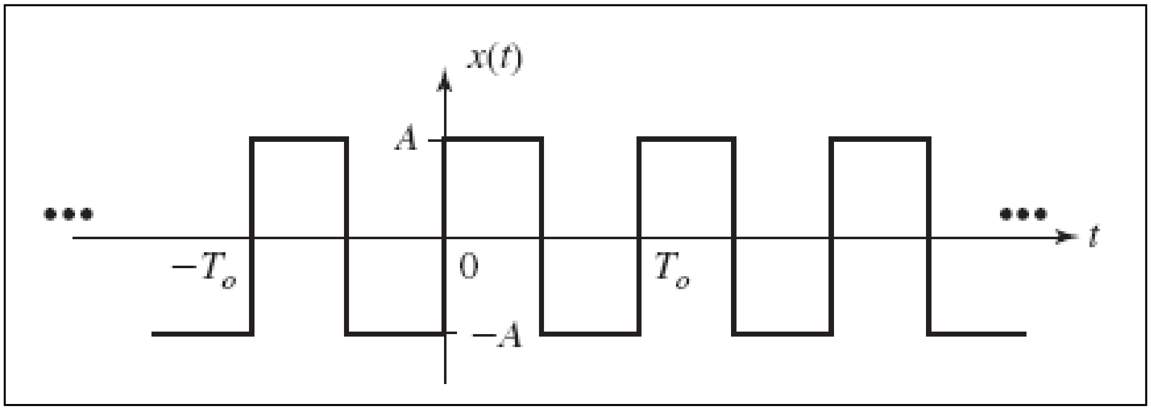
Solution:

$$C_n^y = \frac{1}{T_o}\int_{T_o} y(t)e^{-j2\pi n f_o t}dt = \frac{1}{T_o}\int_{T_o} x(\alpha t)e^{-j2\pi n f_o t}dt$$
$$= \frac{1}{\alpha T_o}\int_{\alpha T_o} x(\nu)e^{-j2\pi n\left(\frac{f_o}{\alpha}\right)\nu}d\nu$$
$$= C_n^x$$

Time Scaling does not change FS coefficients but the FS itself has changed as the harmonic components are now at the frequencies $\pm\frac{f_o}{\alpha}, \pm\frac{2f_o}{\alpha}, \pm\frac{3f_o}{\alpha}, \ldots$

2.16 Draw the one-sided power spectrum for the square wave in Figure P2.5 with duty cycle 50%.

Figure P2.5



Solution:

The FS expansion for the square wave is given by

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_o t}$$

where

$$C_n = \frac{1}{T_o} \int_{T_o} x(t) e^{-j2\pi n f_o t} dt$$

$$= \frac{1}{T_o} \left\{ A \int_0^{T_o/2} e^{-j2\pi n f_o t} dt - A \int_{T_o/2}^{T_o} e^{-j2\pi n f_o t} dt \right\}$$

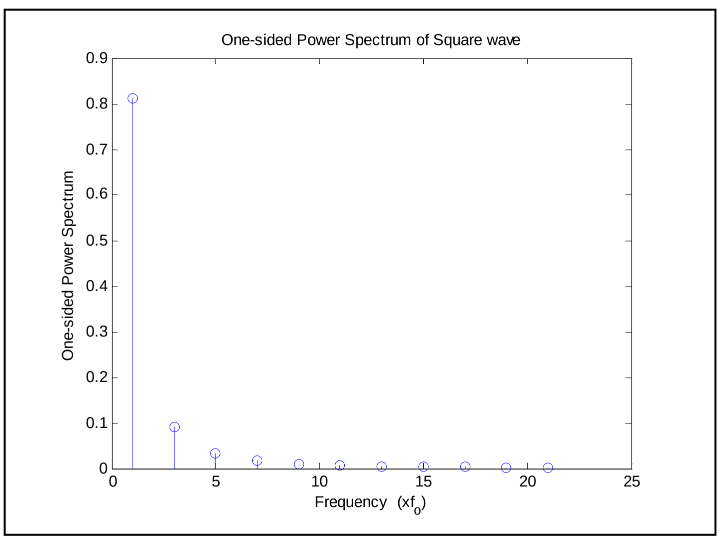
$$= \frac{A}{T_o(-j2\pi n f_o)} \left\{ e^{-j2\pi n f_o t} \Big|_0^{T_o/2} - e^{-j2\pi n f_o t} \Big|_{T_o/2}^{T_o} \right\}$$

$$= \frac{jA}{2\pi n} \left(e^{-j\pi n} - 1 - e^{-j2\pi n} + e^{-j\pi n} \right)$$

Therefore,

$$C_n = \begin{cases} -\frac{j2A}{\pi n}, & n \text{ odd} \\ 0, & \text{otherwise} \end{cases}$$

Average power in the frequency component at $f = nf_o$ equals $|C_n|^2$. Figure displays the one-sided power spectrum for the square wave.



a. Calculate the normalized average power.

Solution:

The normalized average power for a periodic signal $x(t)$ is given by

$$P_x = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} |x(t)|^2 dt = \frac{A^2 T_o}{T_o} = A^2$$

b. Determine the 98% power bandwidth of the pulse train.

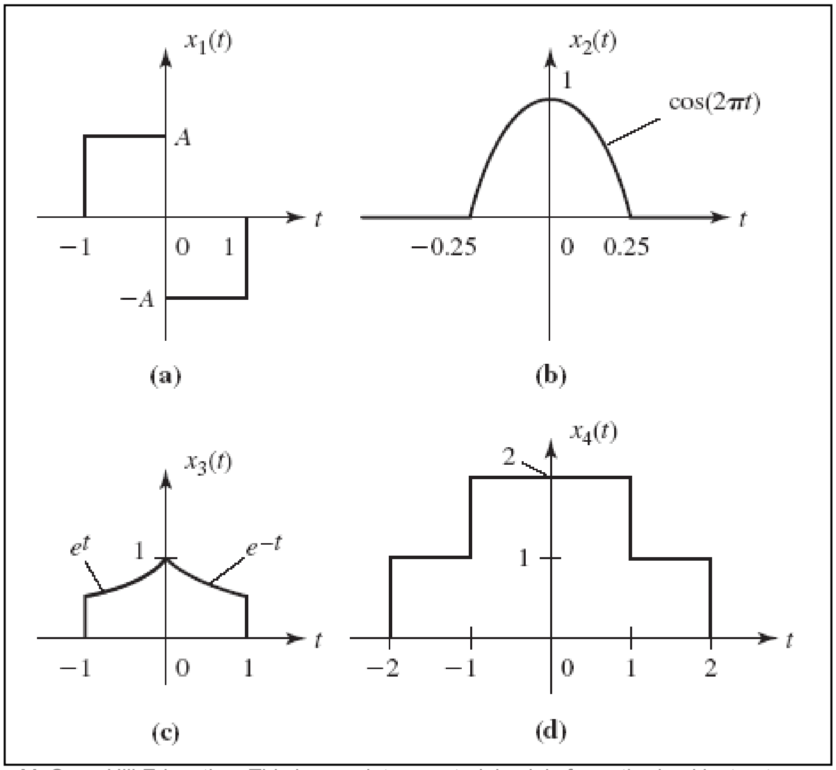
Solution:

n	Total Power including current Fourier coefficient
1	0.8106
3	0.9006
5	0.9331
7	0.9496
9	0.9596
11	0.9663
13	0.9711
15	0.9747
17	0.9775
19	0.9798
21	0.9816

98% Power bandwidth = $21\times f_o$

2.17 Determine the Fourier transforms of the signals shown in Figure P2.6.

Figure P2.6



Solution (a)

The pulse $x_1(t)$ can be expressed as

$$x_1(t) = A\left[\Pi\left(t+0.5\right) - \Pi\left(t-0.5\right)\right]$$

Now

$$\Pi\left(t\right) \xleftrightarrow{\Im} \text{sinc}\left(f\right)$$

Applying the time-shifting property of the FT, we obtain

$$\Pi\left(t+0.5\right) \xleftrightarrow{\Im} \text{sinc}\left(f\right)e^{j\pi f}$$
$$\Pi\left(t-0.5\right) \xleftrightarrow{\Im} \text{sinc}\left(f\right)e^{-j\pi f}$$

Adding

$$X_1(f) = A\left[\text{sinc}\left(f\right)e^{j\pi f} - \text{sinc}\left(f\right)e^{-j\pi f}\right] = A\text{sinc}\left(f\right)\left[e^{j\pi f} - e^{-j\pi f}\right]$$
$$= Aj2\text{sinc}\left(f\right)\sin\left(\pi f\right) = j2\pi fA\left[\text{sinc}^2\left(f\right)\right]$$

Solution (b)

The pulse $x_2(t)$ can be expressed as

$$x_2(t) = \Pi\left(2t\right)\cos\left(2\pi t\right)$$

Now

$$\Pi(2t) \xleftrightarrow{\Im} \frac{1}{2}\text{sinc}(f/2)$$
$$\cos(2\pi t) \xleftrightarrow{\Im} \frac{1}{2}\left[\delta\left(f-1\right) + \delta\left(f+1\right)\right]$$
$$X_2(f) = \Im\left\{\Pi(2t)\right\} \otimes \Im\left\{\cos(2\pi t)\right\} = \frac{1}{2}\text{sinc}(f/2) \otimes \frac{1}{2}\left[\delta\left(f-1\right) + \delta\left(f+1\right)\right]$$
$$= \frac{1}{4}\left[\text{sinc}\left[0.5\left(f-1\right)\right] + \text{sinc}\left[0.5\left(f+1\right)\right]\right]$$

Solution (c)

The pulse $x_3(t)$ can be expressed as

$$x_3(t) = p(t) + p(-t)$$

where

$$p(t) = e^{-t}\left[u(t) - u(t-1)\right]$$

From Table 2.2, we have

$$e^{-t}u(t) \overset{\Im}{\longleftrightarrow} \frac{1}{1 + j2\pi f}$$

Using the time-shifting property of the FT, we obtain

$$e^{-t}u(t-1) = e^{-1}e^{-(t-1)}u(t-1) \overset{\Im}{\longleftrightarrow} e^{-1}\frac{e^{-j2\pi f}}{1 + j2\pi f} = \frac{e^{-(1+j2\pi f)}}{1 + j2\pi f}$$

Combining

$$p(t) \overset{\Im}{\longleftrightarrow} P(f) = \frac{1}{1 + j2\pi f} - \frac{e^{-(1+j2\pi f)}}{1 + j2\pi f} = \frac{1}{1 + j2\pi f}\left[1 - e^{-(1+j2\pi f)}\right]$$

By time-reversal property,

$$p(-t) \overset{\Im}{\longleftrightarrow} P(-f) = \frac{1}{1 - j2\pi f}\left[1 - e^{-(1-j2\pi f)}\right]$$

$$\begin{aligned} X_3(f) = P(f) + P(-f) &= \frac{1}{1 + j2\pi f}\left[1 - e^{-(1+j2\pi f)}\right] + \frac{1}{1 - j2\pi f}\left[1 - e^{-(1-j2\pi f)}\right] \\ &= \frac{1}{1 + (2\pi f)^2}\left\{(1 - j2\pi f)\left[1 - e^{-(1+j2\pi f)}\right] + (1 + j2\pi f)\left[1 - e^{-(1-j2\pi f)}\right]\right\} \\ &= \frac{2}{1 + (2\pi f)^2}\left[1 - e^{-1}\cos(2\pi f) + 2\pi f e^{-1}\sin(2\pi f)\right] \end{aligned}$$

Solution (d)

The pulse $x_4(t)$ can be expressed as

$$x_4(t) = \Pi(t/2) + \Pi(t/4)$$

Now

$$\Pi\left(t/2\right) \overset{\Im}{\longleftrightarrow} 2\text{sinc}\left(2f\right)$$
$$\Pi\left(t/4\right) \overset{\Im}{\longleftrightarrow} 4\text{sinc}\left(4f\right)$$

Adding

$$X_4\left(f\right) = 2\text{sinc}\left(2f\right) + 4\text{sinc}\left(4f\right)$$

2.18 Use properties of the Fourier transform to compute the Fourier transform of following signals.

a. $\text{sinc}^2(Wt)$

Solution:

$$\Lambda(t/\tau) \overset{\Im}{\longleftrightarrow} \frac{\tau}{2}\text{sinc}^2\left(\frac{f\tau}{2}\right)$$

Using the duality property, we obtain

$$W\text{sinc}^2(Wt) \overset{\Im}{\longleftrightarrow} \Lambda\left(f/2W\right)$$

$$\text{sinc}^2(Wt) \overset{\Im}{\longleftrightarrow} \frac{1}{W}\Lambda\left(f/2W\right)$$

Thus the Fourier transform of a sinc^2 pulse is a triangular function in frequency.

b. $\Pi(t/T)\cos(2\pi f_c t)$

Solution:

$$\Pi\left(t/T\right) \overset{\Im}{\longleftrightarrow} T\text{sinc}\left(fT\right)$$
$$\cos(2\pi f_c t) \overset{\Im}{\longleftrightarrow} \frac{1}{2}\left[\delta\left(f-f_c\right)+\delta\left(f+f_c\right)\right]$$
$$X(f) = \Im\left\{\Pi\left(t/T\right)\right\} \otimes \Im\left\{\cos(2\pi f_c t)\right\} = T\text{sinc}\left(fT\right) \otimes \frac{1}{2}\left[\delta\left(f-f_c\right)+\delta\left(f+f_c\right)\right]$$
$$= \frac{T}{2}\left\{\text{sin c}\left[T\left(f-f_c\right)\right]+\text{sinc}\left[T\left(f+f_c\right)\right]\right\}$$

c. $\left(e^{-t}\cos 10\pi t\right)u(t)$

Solution:

From Table 2.2,

$$e^{-t}u(t) \xleftrightarrow{\Im} \frac{1}{1+j2\pi f}$$

Now

$$\begin{aligned}
\Im\left\{\left(e^{-t}u(t)\right)\cos 10\pi t\right\} &= \Im\left\{\left(e^{-t}u(t)\right)\right\} \otimes \Im\left\{\cos 10\pi t\right\} \\
&= \frac{1}{1+j2\pi f} \otimes \frac{1}{2}\left[\delta\left(f-5\right)+\delta\left(f+5\right)\right] \\
&= \frac{1}{2}\left[\frac{1}{1+j2\pi\left(f-5\right)}+\frac{1}{1+j2\pi\left(f+5\right)}\right] \\
&= \frac{1}{2}\left[\frac{1+j2\pi\left(f+5\right)+1+j2\pi\left(f-5\right)}{\left(1+j2\pi f\right)^2-\left(j2\pi 5\right)^2}\right] \\
&= \left[\frac{1+j2\pi f}{\left(1+j2\pi f\right)^2+100\pi^2}\right] \\
&= \left[\frac{1+j2\pi f}{\left(1+100\pi^2\right)+j4\pi f-4\pi^2 f^2}\right]
\end{aligned}$$

d. $te^{-t}u(t)$

Solution:

Applying the differentiation in frequency domain property in (2.85), we obtain

$$te^{-t}u(t) \xleftrightarrow{\Im} \frac{j}{2\pi}\frac{d}{df}\Im\left\{e^{-t}u(t)\right\}$$

That is,

$$\begin{aligned}
\Im\left\{te^{-t}u(t)\right\} &= \frac{j}{2\pi}\frac{d\left[\left(1+j2\pi f\right)^{-1}\right]}{df} = -\frac{j}{2\pi}j2\pi\frac{1}{\left(1+j2\pi f\right)^2} \\
&= \frac{1}{\left(1+j2\pi f\right)^2}
\end{aligned}$$

e. $e^{-\pi t^2}$

Solution:

$$X(f)=\int_{-\infty}^{\infty} e^{-\pi t^2} e^{-j2\pi ft}dt=\int_{-\infty}^{\infty} e^{-\pi\left(t^2+j2ft\right)}dt$$

Multiplying the right hand side by $e^{-\pi f^2}e^{\pi f^2}$ yields

$$X(f)=e^{-\pi f^2}\int_{-\infty}^{\infty} e^{-\pi\left(t^2+j2ft-f^2\right)}dt=e^{-\pi f^2}\int_{-\infty}^{\infty} e^{-\pi(t+jf)^2}dt$$

Substituting $t+jf=\nu$, we obtain

$$X(f)=e^{-\pi f^2}\underbrace{\int_{-\infty}^{\infty} e^{-\pi\nu^2}d\nu}_{1}$$

f. $4\text{sinc}^2(t)\cos\left(100\pi t\right)$

Solution:

$$4\text{sinc}^2(t)\overset{\Im}{\longleftrightarrow}4\Lambda\left(\frac{f}{2}\right)$$

$$\cos\left(100\pi t\right)\overset{\Im}{\longleftrightarrow}\frac{1}{2}\left[\delta(f-50)+\delta(f+50)\right]$$

$$X(f)=\Im\left\{4\text{sinc}^2(t)\right\}\otimes\Im\left\{\cos\left(100\pi t\right)\right\}=4\Lambda\left(\frac{f}{2}\right)\otimes\frac{1}{2}\left[\delta(f-50)+\delta(f+50)\right]$$

$$=2\left\{\Lambda\left[0.5(f-50)\right]+\Lambda\left[0.5(f-50)\right]\right\}$$

2.19 Find the following convolutions:

a. $\text{sinc}(Wt)\otimes\text{sinc}(2Wt)$

Solution:

We use the convolution property of Fourier transform in (2.79).

$$x(t)\otimes y(t)\overset{\Im}{\longleftrightarrow}X(f)Y(f)$$

Now

$$\text{sinc}(2Wt)\overset{\Im}{\longleftrightarrow}\frac{1}{2W}\Pi(f/2W)$$

Therefore,

$$\text{sinc}(Wt)\otimes\text{sinc}(2Wt)\overset{\Im}{\longleftrightarrow}\frac{1}{W}\Pi(f/W)\times\frac{1}{2W}\Pi(f/2W)$$
$$=\frac{1}{2W^2}\Pi(f/W)$$

b. $\text{sinc}^2(Wt)\otimes\text{sinc}(2Wt)$

Solution:

Again using the convolution property of Fourier transform

$$\text{sinc}^2(Wt)\otimes\text{sinc}(2Wt)\overset{\Im}{\longleftrightarrow}\frac{1}{W}\Lambda\left(f/2W\right)\times\frac{1}{2W}\Pi\left(f/2W\right)$$
$$=\frac{1}{2W^2}\Lambda\left(f/2W\right)$$

2.20 The FT of a signal $x(t)$ is described by

$$X(f)=\frac{1}{5+j2\pi f}$$

Determine the FT $V(f)$ of the following signals:

a. $v(t)=x(5t-1)$

Solution:

$$v(t)=x(5t-1)=x\left[5\left(t-\frac{1}{5}\right)\right]$$

Let

$$y(t)=x(5t)\overset{\Im}{\longleftrightarrow}Y(f)=\frac{1}{5}X\left(\frac{f}{5}\right)=\frac{1}{5}\frac{1}{\left(j2\pi f/5\right)+5}$$
$$=\frac{1}{j2\pi f+25}$$

$$v(t)=y\left(t-\frac{1}{5}\right)\overset{\Im}{\longleftrightarrow}V(f)=Y(f)e^{\frac{-j2\pi f}{5}}=\frac{1}{j2\pi f+25}e^{\frac{-j2\pi f}{5}}$$

b. $v(t) = x(t)\cos(100\pi t)$

Solution:

$$v(t) = x(t)\cos(100\pi t) = \frac{\left[x(t)e^{j100\pi t} + x(t)e^{-j100\pi t}\right]}{2}$$

Now

$$\frac{\left[x(t)e^{j100\pi t} + x(t)e^{-j100\pi t}\right]}{2} \overset{\Im}{\longleftrightarrow} V(f) = \frac{1}{2}\left[X(f-50) + X(f+50)\right]$$

$$= \frac{1}{j2\pi(f-50)+5} + \frac{1}{j2\pi(f+50)+5}$$

After simplification, we get

$$V(f) = \frac{j2\pi f + 5}{j20\pi f + \left(\pi^2 10^4 + 25 - 4\pi^2 f^2\right)}$$

c. $v(t) = x(t)e^{j10t}$

Solution:

$$v(t) = x(t)e^{j10t} \overset{\Im}{\longleftrightarrow} V(f) = X\left(f - \frac{5}{\pi}\right) = \frac{1}{j2\pi\left(f - \frac{5}{\pi}\right) + 5}$$

d. $v(t) = \frac{dx(t)}{dt}$

Solution:

Using the differentiation property of FT $\frac{d}{dt}x(t) \overset{\Im}{\longleftrightarrow} j2\pi f X(f)$, we obtain

$$V(f) = j2\pi f X(f) = \frac{j2\pi f}{5 + j2\pi f}$$

e. $v(t) = x(t) \otimes u(t)$

Solution:

Using the convolution property of FT $x(t) \otimes y(t) \overset{\Im}{\longleftrightarrow} X(f)Y(f)$, we can write

$$V(f)=X(f)U(f)=\frac{1}{5+j2\pi f}\left(\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}\right)$$
$$=\frac{1}{2}\delta(f)\frac{1}{5+j2\pi f}+\frac{1}{j2\pi f\left(5+j2\pi f\right)}$$
$$=\frac{1}{10}\delta(f)+\frac{1}{j2\pi f\left(5+j2\pi f\right)}$$

2.21 Consider the delay element $y(t)=x(t-3)$.

a. What is the impulse response $h(t)$?

Solution:

Since $y(t)=h(t)\otimes x(t)=x(t-3)$, the impulse response of the delay element is given from (2.16) as

$$h(t)=\delta(t-3)$$

b. What is the magnitude and phase response function of the system?

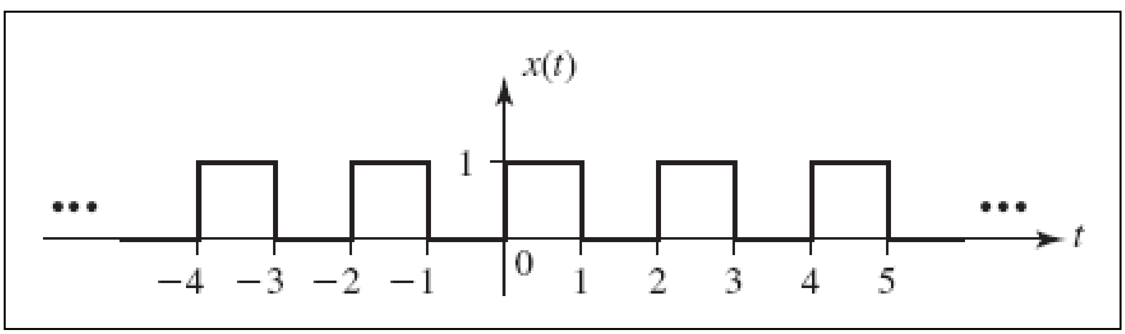
Solution:

$$H(f)=\Im\{\delta(t-3)\}=e^{-j6\pi f}$$
$$|H(f)|=1$$
$$\measuredangle H(f)=-6\pi f$$

2.22 The periodic input $x(t)$ to an LTI system is displayed in Figure P2.7. The frequency response function of the system is given by

$$H(f)=\frac{2}{2+j2\pi f}$$

Figure P2.7



a. Write the complex exponential FS of input $x(t)$.

Solution:

The input $x(t)$ is rectangular pulse train in Example 2.24 shifted by $T_o / 4$ and duty cycle $\frac{\tau}{T_o} = 0.5$. That is,

$$x(t) = g_{T_o}\left(t - T_o / 4\right) = \sum_{n=-\infty}^{\infty} \Pi\left[\frac{\left(t - nT_o - T_o / 4\right)}{0.5T_o}\right]$$

The complex exponential FS of $g_{T_o}(t)$ from Example 2.24 with $T_o = 2\left(f_o = 0.5\right)$ and $\frac{\tau}{T_o} = 0.5$ is given by

$$g_{T_o}(t) = 0.5 \sum_{n=-\infty}^{\infty} \operatorname{sinc}\left(0.5n\right) e^{j\pi nt}$$

In Exercise 2.15(a), we showed that time shifting introduces a linear phase shift in the FS coefficients; their magnitudes are not changed. The phase shift is equal to $e^{-j2\pi n f_o u}$ for a time shift of u. The exponential FS coefficients $x(t)$ are

$$C_n = 0.5\operatorname{sinc}\left(0.5n\right) e^{-j2\pi n f_o\left(0.25T_o\right)} = e^{-j\pi n/2} 0.5\operatorname{sinc}\left(0.5n\right)$$

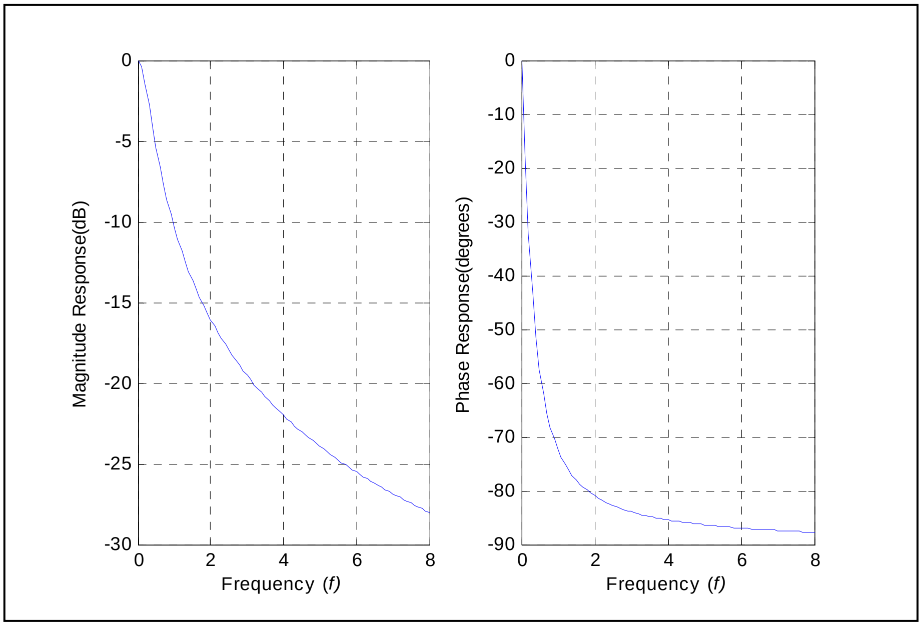
$$x(t) = 0.5 \sum_{n=-\infty}^{\infty} \left[e^{-j\pi n/2} \operatorname{sinc}\left(0.5n\right)\right] e^{j\pi nt}$$

b. Plot the magnitude and phase response functions for $H(f)$.

Solution:

$$H(f) = \frac{2}{2 + j2\pi f} = \frac{1}{1 + j\pi f}$$

$$H\left(nf_o\right) = \frac{1}{1 + j\pi f} = \frac{1}{1 + j\pi n f_o}$$



c. Compute the complex exponential FS of the output $y(t)$.

Solution:

Using (2.114), the output of an LTI system to the input

$$x(t) = \sum_{n=-\infty}^{\infty} \underbrace{\left[0.5e^{-j\pi n/2}\text{sinc}\left(0.5n\right)\right]}_{C_n} e^{j\pi nt}$$

is given by

$$y(t) = \sum_{n=-\infty}^{\infty} C_n H(0.5n) e^{j\pi nt} = \sum_{n=-\infty}^{\infty} \left[\underbrace{\frac{0.5e^{-j\pi n/2}}{1+j05\pi n}\text{sinc}\left(0.5n\right)}_{\text{FS coefficient of } y(t)}\right] e^{j\pi nt}$$

2.23 The frequency response of an ideal LP filter is given by

$$H(f) = \begin{cases} 5e^{-j0.0025\pi f}, & |f| < 1000 \text{ Hz} \\ 0, & |f| > 1000 \text{ Hz} \end{cases}$$

Determine the output signal in each of the following cases:

a. $x(t) = 5\sin\left(400\pi t\right) + 2\cos\left(1200\pi t - \pi/2\right) - \cos\left(2200\pi t + \pi/4\right)$

Solution:

$$|H(f)| = \begin{cases} 5, & |f| < 1000 \text{ Hz} \\ 0, & |f| > 1000 \text{ Hz} \end{cases}$$

$$\measuredangle H(f) = \begin{cases} -0.0025\pi f, & |f| < 1000 \text{ Hz} \\ 0, & |f| > 1000 \text{ Hz} \end{cases}$$

Since $|H(200)| = 5$ and $\measuredangle H(200) = -\pi/2$, the output of the system for an input $5\sin(400\pi t)$ can now be expressed using (2.117) as $25\sin(400\pi t - \pi/2)$.

Next $|H(600)| = 5$ and $\measuredangle H(600) = -3\pi/2 = \pi/2$, the output of the system for an input $2\cos(1200\pi t - \pi/2)$ can now be expressed using (2.117) as $10\cos(1200\pi t - \pi/2 + \pi/2) = 10\cos(1200\pi t)$.

Now $|H(1100)| = 0$. So the LP filter doesn't pass $\cos(2200\pi t + \pi/4)$. The output of the LP filter is therefore given by

$$y(t) = 25\sin(400\pi t - \pi/2) + 10\cos(1200\pi t)$$

b. $x(t) = 2\sin(400\pi t) + \dfrac{\sin(2200\pi t)}{\pi t}$

Solution:

Since $|H(200)| = 5$ and $\measuredangle H(200) = -\pi/2$, the output of the system for an input $2\sin(400\pi t)$ is $10\sin(400\pi t - \pi/2)$.

To calculate the response to $\dfrac{\sin(2200\pi t)}{\pi t}$, we note that

$$\frac{\sin(2200\pi t)}{\pi t} = 2200 \times \text{sinc}(2200t) \overset{\Im}{\longleftrightarrow} \Pi\left(\frac{f}{2200}\right)$$

In frequency domain, the output of LP filter to $\dfrac{\sin(2200\pi t)}{\pi t}$ is

$$\Pi\left(\frac{f}{2200}\right) 5e^{-j0.0025\pi f}\, \Pi\left(\frac{f}{2000}\right) = 5\Pi\left(\frac{f}{2000}\right) e^{-j0.0025\pi f}$$

Now

$$\Pi\left(\frac{f}{2000}\right)e^{-j0.0025\pi f} \overset{\Im}{\longleftrightarrow} 2000\text{sinc}\left[2000\left(t-0.00125\right)\right]$$

The output of the LP filter is therefore given by

$$y(t) = 10\sin\left(400\pi t - \pi/2\right) + 10000\text{sinc}\left[2000\left(t-0.00125\right)\right]$$

c. $x(t) = \cos(400\pi t) + \dfrac{\sin\left(1000\pi t\right)}{\pi t}$

Solution:

Since $\left|H(200)\right| = 5$ and $\measuredangle H(200) = -\pi/2$, the output of the system for an input $\cos(400\pi t)$ is $5\cos\left(400\pi t - \pi/2\right)$.

To calculate the response to $\dfrac{\sin\left(1000\pi t\right)}{\pi t}$, we note that

$$\frac{\sin\left(1000\pi t\right)}{\pi t} = 1000\times\text{sinc}\left(1000t\right) \overset{\Im}{\longleftrightarrow} \Pi\left(\frac{f}{1000}\right)$$

In frequency domain, the output of LP filter to $\dfrac{\sin\left(1000\pi t\right)}{\pi t}$ is

$$5e^{-j0.0025\pi f}\Pi\left(\frac{f}{1000}\right) = 5\Pi\left(\frac{f}{1000}\right)e^{-j0.0025\pi f}$$

Now

$$\Pi\left(\frac{f}{1000}\right)e^{-j0.0025\pi f} \overset{\Im}{\longleftrightarrow} 1000\text{sinc}\left[1000\left(t-0.00125\right)\right]$$

The output of the LP filter is therefore given by

$$y(t) = 5\cos\left(400\pi t - \pi/2\right) + 5000\text{sinc}\left[1000\left(t-0.00125\right)\right]$$

d. $x(t) = 5\cos(800\pi t) + 2\delta(t)$

Solution:

Since $|H(400)| = 5$ and $\measuredangle H(400) = -\pi$, the output of the system for an input $5\cos(800\pi t)$ is $25\cos(800\pi t - \pi)$. The response of the system to $\delta(t)$ is $h(t)$. The impulse response of the ideal LP filter is $10000\text{sinc}\left[2000(t - 0.00125)\right]$. Combining

$$
\begin{aligned}
y(t) &= 25\cos(800\pi t - \pi) + 2h(t) \\
&= 25\cos(800\pi t - \pi) + 20\times 10^3 \text{sinc}\left[2000(t - 0.00125)\right]
\end{aligned}
$$

2.24 The frequency response of an ideal HP filter is given by

$$
H(f) = \begin{cases} 4, & |f| > 20 \text{ Hz}, \\ 0, & |f| < 20 \text{ Hz} \end{cases}
$$

Determine the output signal $y(t)$ for the input

a. $x(t) = 5 + 2\cos(50\pi t - \pi/2) - \cos(75\pi t + \pi/4)$

Solution:

$$y(t) = 8\cos(50\pi t - \pi/2) - 4\cos(75\pi t + \pi/4)$$

b. $x(t) = \cos(20\pi t - 3\pi/4) + 3\cos(100\pi t + \pi/4)$

Solution:

$$y(t) = 12\cos(100\pi t + \pi/4)$$

2.25 The frequency response of an ideal BP filter is given by

$$
H(f) = \begin{cases} 2e^{-j0.0005\pi f}, & 900 < |f| < 1000 \text{ Hz}, \\ 0, & \text{otherwise} \end{cases}
$$

Determine the output signal $y(t)$ for the input

a. $x(t) = 2\cos(1850\pi t - \pi/2) - \cos(1900\pi t + \pi/4)$

Solution:

$|H(925)| = 2$ and $|H(950)| = 2$.

$\measuredangle H(f) = -0.0005\pi f$. Therefore,

$$\measuredangle H(925) = -0.0005\pi f = -0.0005\pi \times 925 = -0.462\pi$$
$$\measuredangle H(950) = -0.0005\pi f = -0.0005\pi \times 950 = -0.475\pi$$

$$y(t) = 4\cos\left(1850\pi t - \pi/2 - 0.462\pi\right) - 2\cos\left(1900\pi t + \pi/4 - 0.475\pi\right)$$

b. $x(t) = \mathrm{sinc}\left(60t\right)\cos(1900\pi t)$

Solution:

$$\begin{aligned} X(f) &= \frac{1}{60}\Pi\left(\frac{f}{60}\right)\otimes\frac{1}{2}\left[\delta(f-950)+\delta(f+950)\right] \\ &= \frac{1}{120}\left[\Pi\left(\frac{f-950}{60}\right)+\Pi\left(\frac{f+950}{60}\right)\right] \end{aligned}$$

$$\begin{aligned} Y(f) &= H(f)X(f) \\ &= 2e^{-j0.0005\pi f}\times\frac{1}{120}\times\left[\Pi\left(\frac{f-950}{60}\right)+\Pi\left(\frac{f+950}{60}\right)\right] \end{aligned}$$

Now

$$\frac{1}{60}\times\Pi\left(\frac{f-950}{60}\right)\xleftrightarrow{\Im}\mathrm{sinc}\left(60t\right)e^{j1900\pi t}$$
$$e^{-j0.0005\pi f}\,\frac{1}{60}\times\Pi\left(\frac{f-950}{60}\right)\xleftrightarrow{\Im}\mathrm{sinc}\left[60\left(t-0.00025\right)\right]e^{j1900\pi\left(t-0.00025\right)}$$

Similarly

$$\frac{1}{60}\times\Pi\left(\frac{f+950}{60}\right)\xleftrightarrow{\Im}\mathrm{sinc}\left(60t\right)e^{-j1900\pi t}$$
$$e^{-j0.0005\pi f}\,\frac{1}{60}\times\Pi\left(\frac{f+950}{60}\right)\xleftrightarrow{\Im}\mathrm{sinc}\left[60\left(t-0.00025\right)\right]e^{-j1900\pi\left(t-0.00025\right)}$$

Therefore,

$$\begin{aligned} y(t) &= \mathrm{sinc}\left[60\left(t-0.00025\right)\right]\left[e^{j1900\pi\left(t-0.00025\right)}+e^{-j1900\pi\left(t-0.00025\right)}\right] \\ &= 2\mathrm{sinc}\left[60\left(t-0.00025\right)\right]\cos\left[1900\pi\left(t-0.00025\right)\right] \end{aligned}$$

c. $x(t) = \text{sinc}^2(30t)\cos(1900\pi t)$

Solution:

$$\begin{aligned} X(f) &= \frac{1}{30}\Lambda\left(\frac{f}{60}\right)\otimes\frac{1}{2}\left[\delta(f-950)+\delta(f+950)\right] \\ &= \frac{1}{60}\left[\Lambda\left(\frac{f-950}{60}\right)+\Lambda\left(\frac{f+950}{60}\right)\right] \end{aligned}$$

$$\begin{aligned} Y(f) &= H(f)X(f) \\ &= 2e^{-j0.0005\pi f}\times\frac{1}{60}\left[\Lambda\left(\frac{f-950}{60}\right)+\Lambda\left(\frac{f+950}{60}\right)\right] \end{aligned}$$

Now

$$\frac{1}{30}\times\Lambda\left(\frac{f-950}{60}\right)\xleftrightarrow{\Im}\text{sinc}^2(30t)e^{j1900\pi t}$$

$$e^{-j0.0005\pi f}\,\frac{1}{30}\times\Lambda\left(\frac{f-950}{60}\right)\xleftrightarrow{\Im}\text{sinc}^2\left[30(t-0.00025)\right]e^{j1900\pi(t-0.00025)}$$

Similarly

$$\frac{1}{30}\times\Pi\left(\frac{f+950}{60}\right)\xleftrightarrow{\Im}\text{sinc}^2(30t)e^{-j1900\pi t}$$

$$e^{-j0.0005\pi f}\,\frac{1}{30}\times\Pi\left(\frac{f+950}{60}\right)\xleftrightarrow{\Im}\text{sinc}^2\left[30(t-0.00025)\right]e^{-j1900\pi(t-0.00025)}$$

Therefore,

$$\begin{aligned} y(t) &= \text{sinc}^2\left[30(t-0.00025)\right]\left[e^{j1900\pi(t-0.00025)}+e^{-j1900\pi(t-0.00025)}\right] \\ &= 2\text{sinc}^2\left[30(t-0.00025)\right]\cos\left[1900\pi(t-0.00025)\right] \end{aligned}$$

2.26 The signal $2e^{-2t}u(t)$ is input to an ideal LP filter with passband edge frequency equal to 5 Hz. Find the energy density spectrum of the output of the filter. Calculate the energy of the input signal and the output signal.

Solution:

$$E_x = \int_{-\infty}^{\infty} \left|2e^{-2t}u(t)\right|^2 dt = \int_0^{\infty} 4e^{-4t} dt = 4\frac{e^{-4t}}{-4}\bigg|_0^{\infty} = 1$$

$$x(t) = 2e^{-2t}u(t) \overset{\Im}{\longleftrightarrow} \frac{2}{2 + j2\pi f}$$

The energy density spectrum of the output $y(t)$, $|Y(f)|^2$, is related to the **energy** density spectrum of the input $x(t)$

$$|Y(f)|^2 = |H(f)|^2 |X(f)|^2 = \left|\Pi\left(\frac{f}{10}\right)\right|^2 \left|\frac{2}{2 + j2\pi f}\right|^2$$

$$E_y = \int_{-\infty}^{\infty} |y(t)|^2 dt = \int_{-\infty}^{\infty} |Y(f)|^2 df = \int_{-\infty}^{\infty} \left|\Pi\left(\frac{f}{10}\right)\right|^2 \left|\frac{2}{2 + j2\pi f}\right|^2 df$$
$$= \int_{-5}^{5} \left|\frac{1}{1 + j\pi f}\right|^2 df = 2\int_0^5 \frac{1}{1 + \pi^2 f^2} df$$

Making change of variables $\pi f = u \Rightarrow df = \frac{1}{\pi} du$, we obtain

$$E_y = \frac{2}{\pi}\int_0^{5\pi} \frac{1}{1 + u^2} du = \frac{2}{\pi}\tan^{-1}(u)\Big|_0^{5\pi} = \frac{2}{\pi}(1.507) = 0.9594$$

Thus output of the LP filter contains 96% of the input signal energy.

2.27 Calculate and sketch the power spectral density of the following signals. Calculate the normalized average power of the signal in each case.

a. $x(t) = \underbrace{2\cos(1000\pi t - \pi/2)}_{x_1(t)} - \underbrace{\cos(1850\pi t + \pi/4)}_{x_2(t)}$

Solution:

$$\mathcal{R}_x(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t-\tau)dt = \mathcal{R}_{x_1}(\tau) + \mathcal{R}_{x_2}(\tau) - \mathcal{R}_{x_1x_2}(\tau) - \mathcal{R}_{x_2x_1}(\tau)$$

Now

$$\mathcal{R}_{x_1}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x_1(t)x_1(t-\tau)dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} 2\cos\left(1000\pi t - \frac{\pi}{2}\right)2\cos\left[1000\pi\left(t-\tau\right)-\frac{\pi}{2}\right]dt$$

$$= 2\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\left\{\cos\left(1000\pi\tau\right)+\cos\left[1000\pi\left(2t-\tau\right)-\pi\right]\right\}dt$$

$$= 2\cos\left(1000\pi\tau\right)$$

The second term is zero because it integrates a sinusoidal function over a period. Similarly,

$$\mathcal{R}_{x_2}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x_2(t)x_2(t-\tau)dt = \frac{1}{2}\cos\left(1850\pi\tau\right)$$

The cross-correlation term

$$\mathcal{R}_{x_1x_2}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x_1(t)x_2(t-\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} 2\cos\left(1000\pi t-\pi/2\right)\cos\left[1850\pi\left(t-\tau\right)+\pi/4\right]dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\left\{\cos\left(850\pi t-1850\pi\tau+3\pi/4\right)+\sin\left(2850\pi t-1850\pi\tau-\pi/4\right)\right\}dt$$

is zero because it integrates a sinusoidal function over a period in each case. Similarly, it can be shown that all other cross-correlation terms are zero. Therefore,

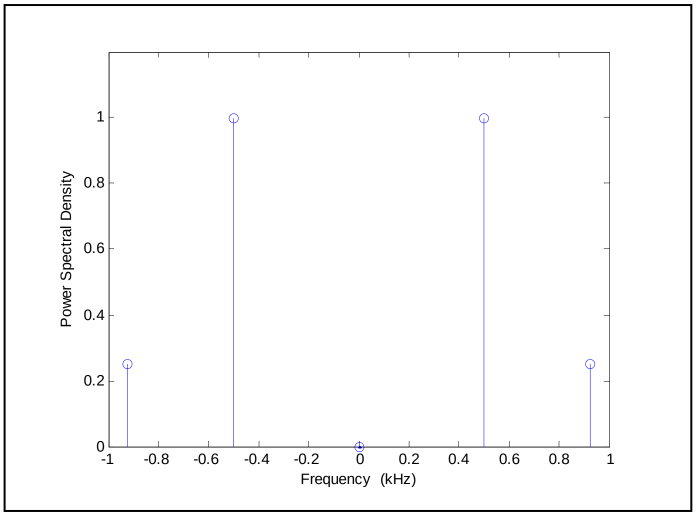
$$\mathcal{R}_{x}(\tau) = \mathcal{R}_{x_1}(\tau)+\mathcal{R}_{x_2}(\tau) = 2\cos\left(1000\pi\tau\right)+\frac{1}{2}\cos\left(1850\pi\tau\right)$$

$$\mathcal{G}_x(f) = \Im\left\{\mathcal{R}_x(\tau)\right\} = \Im\left\{2\cos\left(1000\pi\tau\right)+\frac{1}{2}\cos\left(1850\pi\tau\right)\right\}$$

$$= \left[\delta(f-500)+\delta(f+500)\right]+\frac{1}{4}\left[\delta(f-925)+\delta(f+925)\right]$$

The normalized average power is obtained by using (2.172) as

$$P_x = \int_{-\infty}^{\infty} \mathscr{G}_x(f)df = \int_{-\infty}^{\infty} \left[\delta(f-500)+\delta(f+500)\right]df + \frac{1}{4}\int_{-\infty}^{\infty}\left[\delta(f-925)+\delta(f+925)\right]df$$
$$= 1+1+\frac{1}{4}+\frac{1}{4}=\frac{5}{2}$$



b. $x(t) = \left[1+\sin(200\pi t)\right]\cos(2000\pi t)$

Solution:

$$\begin{aligned} x(t) &= \left[1+\sin(200\pi t)\right]\cos(2000\pi t) \\ &= \cos(2000\pi t)+\sin(200\pi t)\cos(2000\pi t) \\ &= \underbrace{\cos(2000\pi t)}_{x_1(t)} + \underbrace{\frac{1}{2}\sin\left(2200\pi t\right)}_{x_2(t)} \underbrace{-\frac{1}{2}\sin\left(1800\pi t\right)}_{x_3(t)} \end{aligned}$$

Now

$$\mathcal{R}_x(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t-\tau)dt = \mathcal{R}_{x_1}(\tau)+\mathcal{R}_{x_2}(\tau)+\mathcal{R}_{x_3}(\tau)$$

where

$$\mathcal{R}_{x_1}(\tau) = \frac{1}{2}\cos\left(2000\pi\tau\right)$$
$$\mathcal{R}_{x_2}(\tau) = \frac{1}{8}\cos\left(2200\pi\tau\right)$$
$$\mathcal{R}_{x_3}(\tau) = \frac{1}{8}\cos\left(1800\pi\tau\right)$$

Therefore,

$$\mathscr{R}_x(\tau) = \frac{1}{2}\cos(2000\pi\tau) + \frac{1}{8}\cos(2200\pi\tau) + \frac{1}{8}\cos(1800\pi\tau)$$

$$\mathscr{G}_x(f) = \Im\{\mathscr{R}_x(\tau)\} = \Im\left\{\frac{1}{2}\cos(2000\pi\tau) + \frac{1}{8}\cos(2200\pi\tau) + \frac{1}{8}\cos(1800\pi\tau)\right\}$$

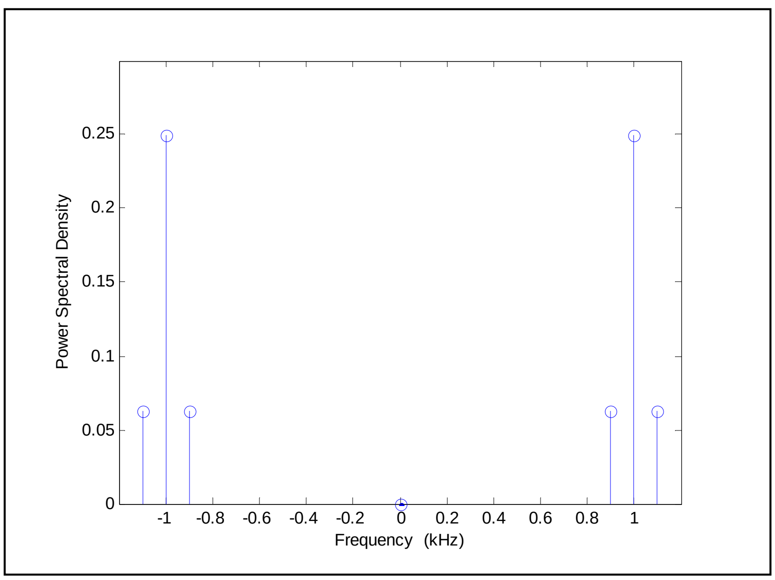
$$= \frac{1}{4}[\delta(f-1000)+\delta(f+1000)] + \frac{1}{16}[\delta(f-1100)+\delta(f+1100)]$$

$$+\frac{1}{16}[\delta(f-900)+\delta(f+900)]$$

The normalized average power is obtained by using (2.172) as

$$P_x = \int_{-\infty}^{\infty} \mathscr{G}_x(f)df = \frac{1}{4}\int_{-\infty}^{\infty}[\delta(f-1000)+\delta(f+1000)]df + \frac{1}{16}\int_{-\infty}^{\infty}[\delta(f-1100)+\delta(f+1100)]df$$

$$+\frac{1}{16}\int_{-\infty}^{\infty}[\delta(f-900)+\delta(f+900)]df = \frac{1}{2}+\frac{1}{8}+\frac{1}{8}=\frac{3}{4}$$



c. $x(t) = \cos^2(200\pi t)\sin(1800\pi t)$

Solution:

$$x(t) = \frac{1}{2}\left[1 + \cos(400\pi t)\right]\sin(1800\pi t) = \frac{1}{2}\sin(1800\pi t) + \frac{1}{2}\sin(1800\pi t)\cos(400\pi t)$$
$$= \frac{1}{2}\sin(1800\pi t) + \frac{1}{4}\sin(1400\pi t) + \frac{1}{4}\sin(2200\pi t)$$

Now

$$\mathcal{R}_x(\tau) = \mathcal{R}_{x_1}(\tau) + \mathcal{R}_{x_2}(\tau) + \mathcal{R}_{x_3}(\tau)$$

where

$$\mathcal{R}_{x_1}(\tau) = \frac{1}{8}\cos(1800\pi\tau)$$

$$\mathcal{R}_{x_2}(\tau) = \frac{1}{32}\cos(1400\pi\tau)$$

$$\mathcal{R}_{x_3}(\tau) = \frac{1}{32}\cos(2200\pi\tau)$$

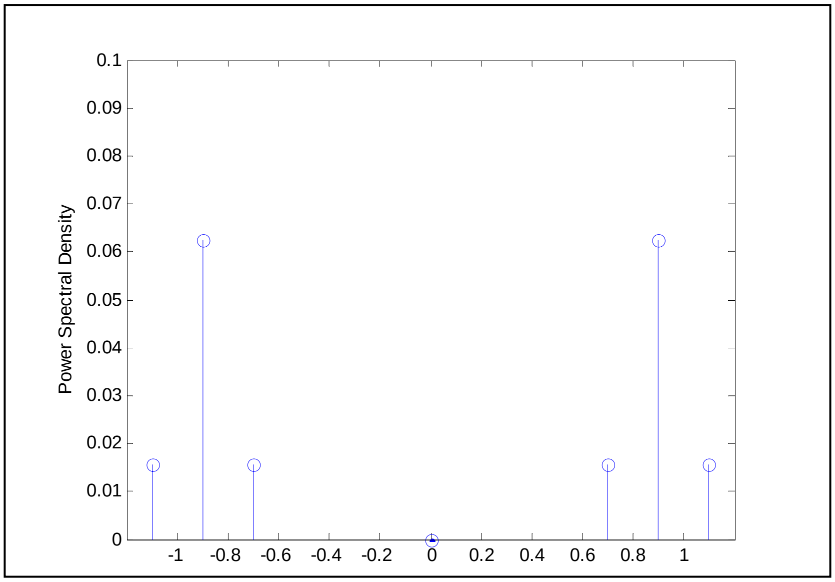
Therefore,

$$\mathcal{R}_x(\tau) = \frac{1}{8}\cos(1800\pi\tau) + \frac{1}{32}\cos(1400\pi\tau) + \frac{1}{32}\cos(2200\pi\tau)$$

$$\mathcal{G}_x(f) = \Im\{\mathcal{R}_x(\tau)\} = \Im\left\{\frac{1}{8}\cos(1800\pi\tau) + \frac{1}{32}\cos(1400\pi\tau) + \frac{1}{32}\cos(2200\pi\tau)\right\}$$
$$= \frac{1}{16}\left[\delta(f-900) + \delta(f+900)\right] + \frac{1}{64}\left[\delta(f-700) + \delta(f+700)\right]$$
$$+ \frac{1}{64}\left[\delta(f-1100) + \delta(f+1100)\right]$$

The normalized average power is obtained by using (2.172) as

$$P_x = \int_{-\infty}^{\infty} \mathcal{G}_x(f)df = \frac{1}{16}\int_{-\infty}^{\infty}\left[\delta(f-900) + \delta(f+900)\right]df + \frac{1}{64}\int_{-\infty}^{\infty}\left[\delta(f-700) + \delta(f+700)\right]df$$
$$+ \frac{1}{64}\int_{-\infty}^{\infty}\left[\delta(f-1100) + \delta(f+1100)\right]df = \frac{1}{8} + \frac{1}{32} + \frac{1}{32} = \frac{3}{16}$$


Power Spectral Density
0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1