Instructor's Solutions Manual
to accompany
Classical Electromagnetism

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Preface

This instructor's Solutions Manual contains my worked out solutions for all the problems in the text "Classical Electromagnetism". I have tried to be complete in each answer, and to include all the steps leading to a sraightforward solution. As much as possible I have written down the solutions as I would in doing them on the blackboard, including enough explanation to make each step reasonable.

When a problem requires a particular equation from the book, I have included a reference enclosed in square brackets, as, for instance, [Eq. (1.24)]. References to equations in the solutions manual are given without the square brackets. If I use an equation without a reference to the text, that is an indication that I have expected the students to know that equation without going back to the text. References to Sections in the text are also given in square brackets. For a number of problems, I have included a **Note** that adds an explanatory extension of the solution or relates the solution to material in the text. These notes are always preceded by the word **Note** in bold type. Most of the problems in the text relate directly to the material covered. They are meant to extend and deepen the students knowledge of EM, and to achieve utility with the necessary mathematics. The notes I have added to some problems are meant to emphasize the connection with the text.

The solutions in this manual represent my teaching philosophy in doing the problems. You may have a different approach for some problems. In that case, just use my solutions as one way of doing things. I have tried to keep virtuoso problems, or problems that are used to learn new things, out of the text. I expect that professors will add some of their own problems, in line with their specialties or geared to their student's interests. I don't have solutions for those problems.

This is an "Instructors Solutions Manual". As much as possible, I do not think it would be helpful to students to make it available to them in toto. My feeling is that struggling through the problems, at least once, is an effective tool for absorbing the course material, and also the math techniques needed throughout physics. I have tried to keep the problems in the text at a level that will challenge, but not discourage, the students. In the text, I try to lead the students through the material, but the problems are meant to help them see how to work things out for themselves.

I hope that the text, along with this instructor's solution manual, will make the teaching of EM more pleasureable and rewarding for you and your students.

Jerrold Franklin

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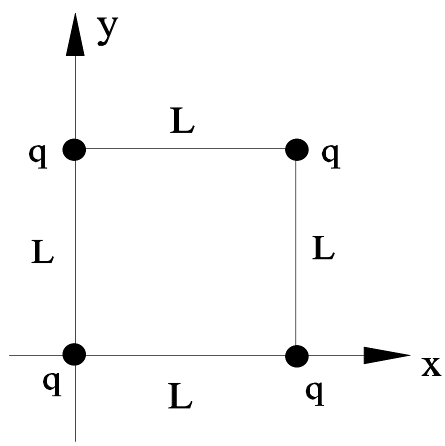
Chapter 1

Foundations of Electrostatics

Problem 1.1:



(a) The force on the upper right hand charge due to the other three charges at the corners of a square with sides of length L is

$$\begin{aligned} \mathbf{F} &= \frac{q^2}{L^2}\hat{\mathbf{i}} + \frac{q^2}{L^2}\hat{\mathbf{j}} + \frac{q^2}{2L^2}\left(\frac{\hat{\mathbf{i}}+\hat{\mathbf{j}}}{\sqrt{2}}\right) \\ &= q^2\left[\frac{1+2\sqrt{2}}{2\sqrt{2}L^2}\right](\hat{\mathbf{i}}+\hat{\mathbf{j}}). \end{aligned} \tag{1.1}$$

The magnitude of this force on any of the four charges is

$$F = q^2(1+2\sqrt{2})/2L^2. \tag{1.2}$$

(b) If the four charges are released from rest, we can write for the acceleration of any of the charges

$$\frac{F}{m} = a = \frac{dv}{dt} = \frac{dv}{dL'}\frac{dL'}{dt}, \tag{1.3}$$

where L' is the side length of the square at any time during the motion. The center of the square remains fixed, and the distance, r, of a charge from the center is related to L' by $L' = \sqrt{2}r$, so

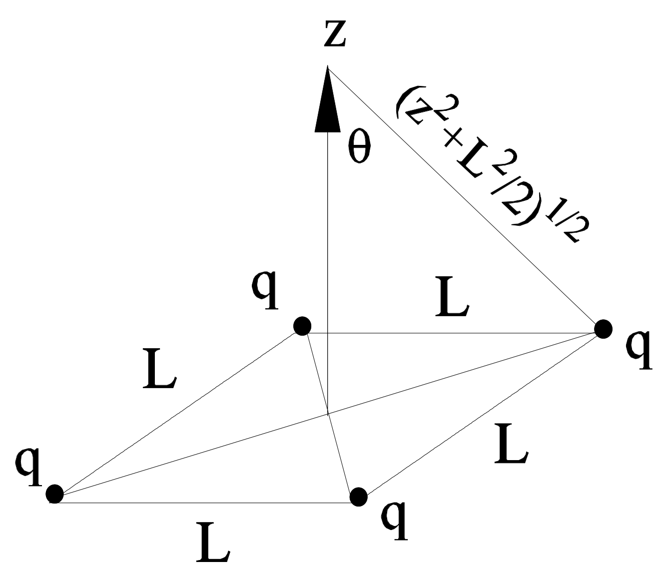
$$\frac{dL'}{dt} = \sqrt{2}\frac{dr}{dt} = \sqrt{2}v. \quad \text{Then,} \ a = \sqrt{2}v\frac{dv}{dL'}, \tag{1.4}$$

and the squared velocity after a long time is given by

$$V^2 \ = \ \int_L^\infty \sqrt{2}a\, dL' = \left[\frac{q^2(1+2\sqrt{2})}{\sqrt{2}m}\right]\int_L^\infty \frac{dL'}{L'^2} = \frac{q^2(1+2\sqrt{2})}{\sqrt{2}mL}. \tag{1.5}$$

The velocity is the square root of this.

Problem 1.2:



(a) The four point charges q are located at the corners of a square with sides of length L. The distance from each charge to a point z above the square, on the perpendicular axis of the square, is $\sqrt{z^2 + L^2/2}$. The horizontal fields cancel, and the magnitude of the vertical field is given by

$$E_z = E\cos\theta = \frac{4qz}{(z^2 + L^2/2)^{\frac{3}{2}}}. \tag{1.6}$$

(b) For small oscillations, $z << L$, and we can approximate the force on a charge $-q$ as

$$F = -\frac{8\sqrt{2}q^2 z}{L^3} = -kz. \tag{1.7}$$

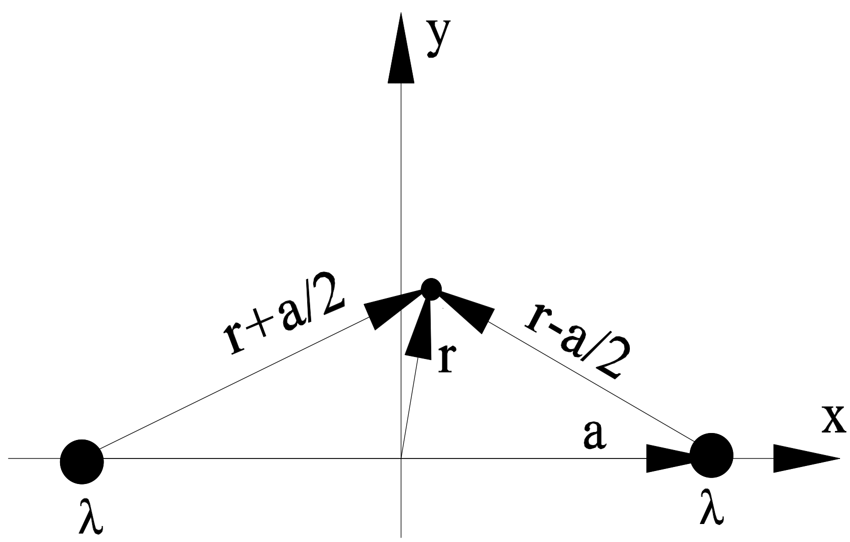
This is a restoring force proportional to the distance with an effective spring constant

$$k = \frac{8\sqrt{2}q^2}{L^3}, \tag{1.8}$$

which leads to simple harmonic motion with period

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\left[\frac{mL^3}{8\sqrt{2}q^2}\right]^{\frac{1}{2}}. \tag{1.9}$$

Problem 1.3:



(a) By symmetry, the electric field of a long straight wire is perpendicular to the wire. Its magnitude a distance r from the wire is given by

$$E_z = \int_{-\infty}^{\infty} \frac{r\lambda dz}{(z^2+r^2)^{3/2}} = \frac{\lambda}{r}\int_{-\pi/2}^{\pi/2}\cos\theta d\theta = \frac{2\lambda}{r}. \tag{1.10}$$

We made the substitution $z = r\tan\theta$ in doing the integral. For the configuration of two wires a distance **a** apart, the electric field is

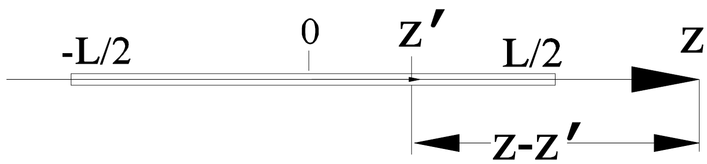
$$\mathbf{E} = \frac{2\lambda(\mathbf{r}-\mathbf{a}/2)}{|\mathbf{r}-\mathbf{a}/2)|^2} + \frac{2\lambda(\mathbf{r}+\mathbf{a}/2)}{|\mathbf{r}+\mathbf{a}/2)|^2}. \tag{1.11}$$

(b) The electric field in Cartesian coordinates is

$$E_x = \frac{2\lambda(x-a/2)}{(x-a/2)^2+y^2} + \frac{2\lambda(x+a/2)}{(x+a/2)^2+y^2} \tag{1.12}$$

$$E_y = \frac{2\lambda y}{(x-a/2)^2+y^2} + \frac{2\lambda y}{(x+a/2)^2+y^2}. \tag{1.13}$$

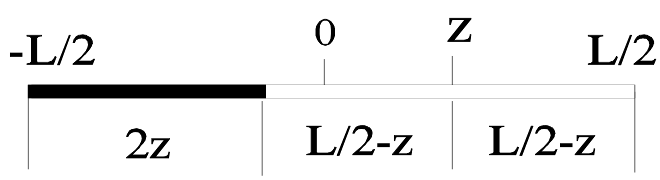
Problem 1.4:



(a) The field on the axis of the uniformly charged wire, a distance z from the center of the wire, is given for $z > L/2$ by

$$
\begin{aligned}
E_z &= \frac{Q}{L}\int_{-L/2}^{+L/2}\frac{dz'}{(z-z')^2} \\
&= \frac{Q}{L}\left[\frac{1}{z-L/2}-\frac{1}{z+L/2}\right] \\
&= \frac{Q}{z^2-L^2/4}.
\end{aligned}
\tag{1.14}
$$

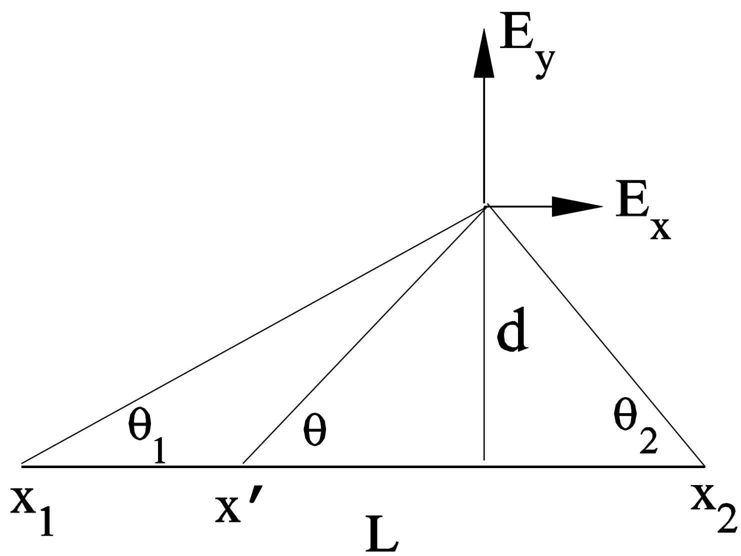
(b) For $-L/2 < z < L/2$, the point z is a distance $(L/2 - z)$ from the end of the wire. The wire can be thought of as two parts.



The part of the wire from $z' = z - (L/2 - z) = 2z - L/2$ to $z' = 2L$ is symmetric about the point z. This means that the field due to that portion of the wire will cancel. The remaining part of the wire has a length $L' = L - 2(L/2 - z) = 2z$ and a charge $Q' = 2zQ/L$. The midpoint of this part of the wire is at $z_0 = (2z - L/2 - L/2)/2 = z - L/2$. Thus the electric field from this part of the wire is

$$
\begin{aligned}
E_z &= \frac{Q'}{[(z-z_0)^2 - L'^2/4]} \\
&= \frac{2Qz}{L[(z-z+L/2)^2 - z^2]} \\
&= \frac{2Qz}{L[L^2/4 - z^2]}.
\end{aligned}
\tag{1.15}
$$

Problem 1.5:



The parallel component of the electric field a distance d from a uniformly charged straight wire of length L is given by the integral

$$E_x = \frac{Q}{L}\int_{x_1}^{x_2} \frac{dx' \cos\theta}{r^2}, \tag{1.16}$$

where x_1 and x_2 are the two endpoints of the wire. Let

$$\begin{aligned} x' &= -d \cot\theta \\ dx' &= d \csc^2\theta d\theta \\ r &= d \csc\theta. \end{aligned} \tag{1.17}$$

Then

$$E_x = \frac{Q}{Ld}\int_{\theta_1}^{\pi-\theta_2} \cos\theta\, d\theta = \frac{Q}{Ld}(\sin\theta_2 - \sin\theta_1), \tag{1.18}$$

where θ_1 and θ_2 are the angles shown.
For the perpendicular component of **E**, the same substitution for z, leads to

$$E_y = \frac{Q}{Ld}\int_{\theta_1}^{\pi-\theta_2} \sin\theta\, d\theta = \frac{Q}{Ld}(\cos\theta_2 + \cos\theta_1). \tag{1.19}$$

Problem 1.6:

(a) Every point on the uniformly charged ring is the same distance from a point a distance z along the axis of the ring, and a line from any point on the ring makes the same angle θ with the z-axis. Thus the electric field at z is

$$E_z = \frac{Q\cos\theta}{r^2} = \frac{Qz}{(z^2+R^2)^{\frac{3}{2}}}. \tag{1.20}$$

(b) The disk has a surface charge density $\sigma = Q/\pi R^2$. It can be considered as a collection of rings, each of radius r' with a charge

$$dq = 2\pi r'\sigma dr' = \frac{2Qr'dr'}{R^2}. \tag{1.21}$$

The electric field a distance z along the axis of the disk is given as (using part a)

$$\begin{aligned} E_z &= \int_0^R \frac{2Qr'}{R^2}\frac{zdr'}{(z^2+r'^2)^{\frac{3}{2}}} \\ &= \frac{2Q}{R^2}\left[1-\frac{z}{\sqrt{z^2+R^2}}\right]. \end{aligned} \tag{1.22}$$

(c) (**i**) For $z = 0_+$ (just above the disk),

$$E_z = \frac{2Q}{R^2} = 2\pi\sigma. \tag{1.23}$$

(**ii**) For $z >> R$, we write E as

$$E_z = \frac{2Q}{R^2}\left[1-\left(1+\frac{R^2}{z^2}\right)^{-\frac{1}{2}}\right]. \tag{1.24}$$

Using the binomial theorem, we get

$$\begin{aligned} E_z &= \frac{2Q}{R^2}\left[1-\left(1-\frac{R^2}{2z^2}+\dots\right)\right]. \\ &\simeq \frac{Q}{z^2}. \end{aligned} \tag{1.25}$$

This limit, equal to the field of a point charge Q, can be used as a check on the original result.

Problem 1.7:

(a) Each charge q, at the corner of a square with sides of length L is a distance $\sqrt{z^2 + L^2/2}$ from a point z along the perpendicular axis of the square. (See the figure in Prob. 1.2.)

Thus the potential at the point z is
$$\phi = \frac{4q}{\sqrt{z^2 + L^2/2}}. \tag{1.26}$$

(b) The electric field is
$$E_z = -\partial_z \phi = \frac{4qz}{(z^2 + L^2/2)^{\frac{3}{2}}}, \qquad \text{as in Prob. 1.2.} \tag{1.27}$$

Problem 1.8:

(a) The potential on the axis of the uniformly charged wire, a distance z from the center of the wire, is given for $z > L/2$ by
$$\phi = \frac{Q}{L}\int_{-L/2}^{+L/2} \frac{dz'}{z - z'} = \frac{Q}{L}\ln\left(\frac{z + L/2}{z - L/2}\right). \tag{1.28}$$

(b) The electric field is
$$\begin{aligned} E_z &= -\partial_z \phi = \frac{Q}{L}\left[\frac{1}{z - L/2} - \frac{1}{z + L/2}\right] \\ &= \frac{Q}{z^2 - L^2/4}, \qquad \text{as in Prob. 1.4}(a). \end{aligned} \tag{1.29}$$

Problem 1.9:

(a) Every point on the uniformly charged ring is the same distance $\sqrt{z^2 + R^2}$ from a point a distance z along the axis of the ring, so the potential at z is
$$\phi = \frac{Q}{\sqrt{z^2 + R^2}}. \tag{1.30}$$

(b) The uniformly charged disk of radius R has a surface charge density $\sigma = Q/\pi R^2$. It can be considered as a collection of rings, each of radius r' with a charge
$$dq = 2\pi r' \sigma dr' = \frac{2Qr'dr'}{R^2}. \tag{1.31}$$

The potential a distance z along the axis of the disk is given as (using part a)
$$\phi = \frac{2Q}{R^2}\int_0^R \frac{r'dr'}{\sqrt{z^2 + r'^2}} = \frac{2Q}{R^2}\left[\sqrt{z^2 + R^2} - z\right]. \tag{1.32}$$

(c) The electric field of the ring is
$$E_z = -\partial_z\left[\frac{Q}{\sqrt{z^2 + R^2}}\right] = \frac{Qz}{(z^2 + R^2)^{\frac{3}{2}}}, \qquad \text{as in Prob. 1.6}(a). \tag{1.33}$$

The electric field of the disk is
$$E_z = -\frac{2Q}{R^2}\partial_z\left[\sqrt{z^2 + R^2} - z\right] = \frac{2Q}{R^2}\left[1 - \frac{z}{\sqrt{z^2 + R^2}}\right], \qquad \text{as in Prob. 1.6}(b). \tag{1.34}$$

Problem 1.10:

(a) A uniformly charged spherical shell has a surface charge density

$$\sigma = \frac{Q}{4\pi R^2}. \tag{1.35}$$

The spherical surface can be thought of as composed of rings of radius $r = R\sin\theta$, where θ is the angle from the z axis. Each ring has a charge

$$dq == 2\pi r\sigma R\, d\theta = \frac{1}{2}Q\sin\theta\, d\theta. \tag{1.36}$$

The distance from the plane of a ring to the point d is $z = d - R\cos\theta$. Thus the potential at the point d from the center of the sphere is

$$\begin{aligned}
\phi &= \int \frac{dq}{\sqrt{z^2+r^2}} = \frac{Q}{2}\int_0^\pi \frac{\sin\theta\, d\theta}{\sqrt{(d-R\cos\theta)^2 + R^2\sin^2\theta}} \\
&= \frac{Q}{2}\int_0^\pi \frac{\sin\theta\, d\theta}{\sqrt{d^2+R^2-2Rd\cos\theta}} = \frac{Q}{2}\left[\frac{\sqrt{d^2+R^2-2Rd\cos\theta}}{-Rd}\right]_0^\pi \\
&= \frac{Q}{2Rd}[(d-R)-(d+R)] = \frac{Q}{d}, \quad d \geq R.
\end{aligned} \tag{1.37}$$

Note that, for $d \leq R$,

$$\phi = \frac{Q}{2Rd}[(R-d)-(d+R)] = \frac{Q}{R}, \quad d \leq R. \tag{1.38}$$

For d outside the sphere, the potential is the same as that of a point charge. For d inside the sphere, the potential is constant, and hence the electric field is zero.

(b) The potential at a distance $d \geq R$ from the center of a uniformly charged spherical shell is $\Delta q/d$, where Δq is the charge on the shell. The potential does not depend on the radius R of the shell, as long as $d \geq R$. A uniformly charged sphere can be considered a collection of the uniformly charged shells, so that the potential due to the uniformly charged sphere will be $\phi = Q/d$, where Q is the net charge on the sphere. (Q is the sum of all the Δq on each spherical shell.)
Note that the charged sphere need not be uniformly charged as long as its charge distribution is spherically symmetric.

(c) Since the potentials in (a) and (b) are the same as the potential of a point charge, the electric fields are the same as the electric field of a point charge:

$$\mathbf{E} = -\boldsymbol{\nabla}\left(\frac{Q}{r}\right) = \frac{Q\hat{\mathbf{r}}}{r^2}, \quad r \geq R. \tag{1.39}$$

Problem 1.11:

(a) The long straight wire has radius R and uniform charge density ρ. By symmetry, the electric field is perpendicular to the axis of the wire. We consider a Gaussian cylinder of length L and radius r, concentric with the wire. Then, for $r \leq R$, Gauss's law becomes

$$\begin{aligned} \oint \mathbf{E}\cdot\mathbf{dA} &= 4\pi Q \\ 2\pi rLE &= 4\pi^2 r^2 \rho L \\ E &= 2\pi\rho r, \quad r \leq R. \end{aligned} \tag{1.40}$$

For $r \geq R$, Gauss's law is

$$\begin{aligned} \oint \mathbf{E}\cdot\mathbf{dA} &= 4\pi Q \\ 2\pi rLE &= 4\pi^2 R^2 \rho L \\ E &= \frac{2\pi\rho R^2}{r}, \quad r \geq R. \end{aligned} \tag{1.41}$$

(b) We have to pick some radius for which $\phi = 0$. We choose $\phi(R) = 0$, so that the surface of the wire is at zero potential. Then, for $r \leq R$,

$$\phi(r) = -\int_R^r \mathbf{E}\cdot\mathbf{dr} = -\int_R^r 2\pi\rho r dr = \pi\rho(R^2 - r^2). \tag{1.42}$$

For $r \geq R$,

$$\phi(r) = -\int_R^r \mathbf{E}\cdot\mathbf{dr} = -\int_R^r \frac{2\pi R^2 \rho dr}{r} = 2\pi\rho R^2 \ln(R/r). \tag{1.43}$$

Problem 1.12:

(a) By symmetry, the electric field of the uniformly charged hollow sphere is in the radial direction. We consider a Gaussian sphere of radius r, concentric with the hollow sphere. Then, for $r \leq R$, Gauss's law becomes

$$\begin{aligned} \oint \mathbf{E}\cdot\mathbf{dA} &= 4\pi Q \\ \pi r^2 E &= 0 \\ E &= 0. \end{aligned} \tag{1.44}$$

For $r \geq R$, Gauss's law is

$$\begin{aligned} \oint \mathbf{E}\cdot\mathbf{dA} &= 4\pi Q \\ \pi r^2 E &= 4\pi Q \\ E &= \frac{Q}{r^2}. \end{aligned} \tag{1.45}$$

(b) The electric field for $r \geq R$ is the same as that for a point charge, so the potential is the same as the potential of a point charge:

$$\phi(r) = \frac{Q}{r}. \tag{1.46}$$

For $r \leq R$, $\mathbf{E} = \mathbf{0}$, so the interior of a uniformly charged shell is an equipotential with

$$\phi = \frac{Q}{R}. \tag{1.47}$$

Problem 1.13:

(a) The charge density inside the uniformly charged sphere is

$$\rho = \frac{3Q}{4\pi R^3}. \tag{1.48}$$

By symmetry, the electric field is in the radial direction. We consider a Gaussian sphere of radius r, concentric with the uniformly charged sphere. Then, for $r \leq R$, Gauss's law becomes

$$\begin{aligned} \oint \mathbf{E}\cdot\mathbf{dA} &= 4\pi Q \\ \pi r^2 E &= \left[\frac{3Q}{4\pi R^3}\right]\left[\frac{4\pi r^3}{3}\right] \\ E &= \frac{Qr}{R^3}. \end{aligned} \tag{1.49}$$

For $r \geq R$, Gauss's law is

$$\begin{aligned} \oint \mathbf{E}\cdot\mathbf{dA} &= 4\pi Q \\ \pi r^2 E &= 4\pi Q \\ E &= \frac{Q}{r^2}. \end{aligned} \tag{1.50}$$

(b) The electric field for $r \geq R$ is the same as that for a point charge, so the potential is the same as the potential of a point charge:

$$\phi(r) = \frac{Q}{r}, \quad r \geq R. \tag{1.51}$$

For $r \leq R$,

$$\begin{aligned} \phi(r) - \phi(R) &= -\int_R^r \mathbf{E}\cdot\mathbf{dr} \\ \phi(r) &= \frac{Q}{R} - \frac{Q}{R^3}\int_r^R r dr = \frac{Q}{R}\left[1 - \frac{1}{2R^3}(r^2 - R^2)\right] \\ &= \frac{Q}{2R}(3R^2 - r^2). \end{aligned} \tag{1.52}$$

Problem 1.14:

(a) For $\phi = qe^{-\mu r}/r$, the electric field is

$$\begin{aligned}
\mathbf{E} &= -q\boldsymbol{\nabla}(e^{-\mu r}/r) = -qe^{-\mu r}\boldsymbol{\nabla}\left(\frac{1}{r}\right) - \frac{q}{r}\boldsymbol{\nabla}(e^{-\mu r}) \\
&= -qe^{-\mu r}\left(-\frac{\hat{\mathbf{r}}}{r^2} - \frac{\mu\hat{\mathbf{r}}}{r}\right) = \frac{q\hat{\mathbf{r}}e^{-\mu r}}{r^2}(1+\mu r)\,. \qquad (1.53)
\end{aligned}$$

(b) The charge density is given by

$$\rho = \frac{1}{4\pi}\boldsymbol{\nabla\cdot}\mathbf{E} = \frac{q}{4\pi}\boldsymbol{\nabla\cdot}\left[e^{-\mu r}\left(\frac{\mathbf{r}}{r^3} + \frac{\mu\mathbf{r}}{r^2}\right)\right], \qquad (1.54)$$

where we have written $\mathbf{E}$ in a more convenient form for taking its divergence. We differentiate each term in turn, leading to

$$\begin{aligned}
\rho &= \frac{q}{4\pi}\left\{e^{-\mu r}\left[\boldsymbol{\nabla\cdot}\left(\frac{\mathbf{r}}{r^3}\right) + \boldsymbol{\nabla\cdot}\left(\frac{\mu\mathbf{r}}{r^2}\right)\right] + \left[\frac{\mathbf{r}}{r^3} + \frac{\mu\mathbf{r}}{r^2}\right]\boldsymbol{\cdot\nabla}(e^{-\mu r})\right\} \\
&= \frac{qe^{-\mu r}}{4\pi}\left[4\pi\delta(\mathbf{r}) + \frac{3\mu}{r^2} - \frac{2\mu}{r^2} - \frac{\mu}{r^2} - \frac{\mu^2}{r}\right] \\
&= q\delta(\mathbf{r}) - \frac{\mu^2 qe^{-\mu r}}{4\pi r}. \qquad (1.55)
\end{aligned}$$

Note that above we isolated the term $\boldsymbol{\nabla\cdot}(\mathbf{r}/r^3)$, because we knew that it equals $4\pi\delta(\mathbf{r})$. This correctly accounted for the singular behavior at the origin. The term $q\delta(\mathbf{r})$ corresponds to a point charge q at the origin. The other part of Eq. (1.55) represents a negative charge distribution surrounding the point charge.

(c) Gauss's law is

$$\oint \mathbf{E\cdot dA} = 4\pi Q. \qquad (1.56)$$

We choose a sphere of radius R as our Gaussian surface. The integral of $\mathbf{E\cdot dS}$ over the surface of the sphere is

$$\oint \mathbf{E\cdot dA} = 4\pi R^2 E_r(R) = 4\pi qe^{-\mu R}(1+\mu R)\,, \qquad (1.57)$$

where we have taken E_r from Eq. (1.53) The charge within the Gaussian sphere is given (using Eq. (1.55) by

$$Q = \int_{r\le R}\rho d\tau = q - 4\pi\int_0^R \frac{\mu^2 qe^{-\mu r}r^2 dr}{4\pi r} = 4\pi qe^{-\mu R}(1+\mu R)\,, \qquad (1.58)$$

which agrees with Eq. (1.57), and Gauss's law is satisfied. (The last integral above was done using integration by parts.)

Note the importance of the proper treatment of the delta function at the origin.

Problem 1.15:

(a) $\mathbf{F} = (\mathbf{r}\times\mathbf{p})(\mathbf{r}\cdot\mathbf{p})$, with $\mathbf{p}$ a constant vector.

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{F} &= \boldsymbol{\nabla}\times[(\mathbf{r}\times\mathbf{p})(\mathbf{r}\cdot\mathbf{p})] \\
&= (\mathbf{r}\cdot\mathbf{p})[\boldsymbol{\nabla}\times(\mathbf{r}\times\mathbf{p})] - (\mathbf{r}\times\mathbf{p})\times[\boldsymbol{\nabla}(\mathbf{r}\cdot\mathbf{p})] \\
&= (\mathbf{r}\cdot\mathbf{p})[(\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{r} - \mathbf{p}(\boldsymbol{\nabla}\cdot\mathbf{r})] - (\mathbf{r}\times\mathbf{p})\times\mathbf{p} \\
&= (\mathbf{r}\cdot\mathbf{p})(\mathbf{p} - \mathbf{3p}) - \mathbf{p}(\mathbf{r}\cdot\mathbf{p}) + p^2\mathbf{r} \\
&= p^2\mathbf{r} - 3\mathbf{p}(\mathbf{r}\cdot\mathbf{p}). \qquad (1.59)
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{F} &= \boldsymbol{\nabla}\cdot[(\mathbf{r}\times\mathbf{p})(\mathbf{r}\cdot\mathbf{p})] \\
&= (\mathbf{r}\cdot\mathbf{p})[\boldsymbol{\nabla}\cdot(\mathbf{r}\times\mathbf{p})] + (\mathbf{r}\times\mathbf{p})\cdot[\boldsymbol{\nabla}(\mathbf{r}\cdot\mathbf{p})] \\
&= (\mathbf{r}\cdot\mathbf{p})[\mathbf{p}\cdot(\boldsymbol{\nabla}\times\mathbf{r})] + (\mathbf{r}\times\mathbf{p})\cdot\mathbf{p} \\
&= 0 + \mathbf{r}\cdot(\mathbf{p}\times\mathbf{p}) = 0. \qquad (1.60)
\end{aligned}$$

(b) $\mathbf{F} = (\mathbf{r}\cdot\mathbf{p})^2\mathbf{r}$, with $\mathbf{p}$ a constant vector.

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{F} &= \boldsymbol{\nabla}\times[(\mathbf{r}\cdot\mathbf{p})^2\mathbf{r}] \\
&= (\mathbf{r}\cdot\mathbf{p})^2(\boldsymbol{\nabla}\times\mathbf{r}) - \mathbf{r}\times[\boldsymbol{\nabla}(\mathbf{r}\cdot\mathbf{p})^2] \\
&= 0 - \mathbf{r}\times[\mathbf{2}(\mathbf{r}\cdot\mathbf{p})\mathbf{p}] \\
&= -2(\mathbf{r}\cdot\mathbf{p})(\mathbf{r}\times\mathbf{p}) \qquad (1.61)
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{F} &= \boldsymbol{\nabla}\cdot[(\mathbf{r}\cdot\mathbf{p})^2\mathbf{r}] \\
&= (\mathbf{r}\cdot\mathbf{p})^2(\boldsymbol{\nabla}\cdot\mathbf{r}) + \mathbf{r}\cdot[\boldsymbol{\nabla}(\mathbf{r}\cdot\mathbf{p})^2] \\
&= 3(\mathbf{r}\cdot\mathbf{p})^2 + \mathbf{r}\cdot[\mathbf{2}(\mathbf{r}\cdot\mathbf{p})\mathbf{p}] \\
&= 5(\mathbf{r}\cdot\mathbf{p})^2. \qquad (1.62)
\end{aligned}$$

Problem 1.16:

We want to show that $\mathbf{L}\times\mathbf{L}\phi = i\mathbf{L}\phi$, where $\mathbf{L} = -\,i\mathbf{r}\times\boldsymbol{\nabla}$. This corresponds to $-(\mathbf{r}\times\boldsymbol{\nabla})\times(\mathbf{r}\times\boldsymbol{\nabla}\phi) = \mathbf{r}\times\boldsymbol{\nabla}\phi$. We introduce $\mathbf{E} = -\boldsymbol{\nabla}\phi$, and then will show that

$$(\mathbf{r}\times\boldsymbol{\nabla})\times(\mathbf{r}\times\mathbf{E}) = -\mathbf{r}\times\mathbf{E}. \tag{1.63}$$

We use

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b}), \tag{1.64}$$

with

$$\mathbf{r}\times\boldsymbol{\nabla} = \mathbf{a}, \quad \mathbf{r} = \mathbf{b}, \quad \mathbf{E} = \mathbf{c}. \tag{1.65}$$

We take the vector derivative twice, first holding $\mathbf{E}$ constant and then holding $\mathbf{r}$ constant. For $\mathbf{E}$ constant,

$$\begin{aligned}(\mathbf{r}\times\boldsymbol{\nabla})\times(\mathbf{r}\times\mathbf{E}) &= [\mathbf{E}\cdot(\mathbf{r}\times\boldsymbol{\nabla})]\mathbf{r} - \mathbf{E}[(\mathbf{r}\times\boldsymbol{\nabla})\cdot\mathbf{r}] \\ &= [(\mathbf{E}\times\mathbf{r})\cdot\boldsymbol{\nabla})]\mathbf{r} - \mathbf{E}[\mathbf{r}\cdot(\boldsymbol{\nabla}\times\mathbf{r})] \\ &= \mathbf{E}\times\mathbf{r}. \end{aligned} \tag{1.66}$$

For $\mathbf{r}$ constant,

$$\begin{aligned}(\mathbf{r}\times\boldsymbol{\nabla})\times(\mathbf{r}\times\mathbf{E}) &= \mathbf{r}[(\mathbf{r}\times\boldsymbol{\nabla})\cdot\mathbf{E}] - [\mathbf{r}\cdot(\mathbf{r}\times\boldsymbol{\nabla})]\mathbf{E} \\ &= \mathbf{r}[\mathbf{r}\cdot(\boldsymbol{\nabla}\times\mathbf{E})] - [(\mathbf{r}\times\mathbf{r})\cdot\boldsymbol{\nabla}]\mathbf{E} \\ &= \mathbf{0}. \end{aligned} \tag{1.67}$$

Adding the results of Eqs. (1.66) and (1.67) gives

$$(\mathbf{r}\times\boldsymbol{\nabla})\times(\mathbf{r}\times\mathbf{E}) = -\mathbf{r}\times\mathbf{E}, \tag{1.68}$$

which confirms that $\mathbf{L}\times\mathbf{L}\phi = i\mathbf{L}\phi$.

Chapter 2

Further Development of Electrostatics

Problem 2.1:

(a) In the configuration of four point charges q at the four corners of a square of side L, each of the four charges is a distance L from two other charges, and a distance $\sqrt{2}L$ from a third charge. Consequently the potential energy is given by

$$U_0 = \frac{1}{2}\sum_{i,j\neq i}\left[\frac{q_i q_j}{|\mathbf{r_i} - \mathbf{r_j}|}\right] = \frac{4}{2}\left[\frac{2q^2}{L} + \frac{q^2}{\sqrt{2}L}\right] = \frac{\sqrt{2}q^2}{L}(1 + 2\sqrt{2}). \qquad (2.1)$$

(b) The kinetic energy of the four charges (each of mass m) at a long time after their release (so that $L \to \infty$) is given by

$$4\left(\frac{1}{2}mv^2\right) = U_0 = \frac{\sqrt{2}q^2}{L}(1 + 2\sqrt{2}), \qquad (2.2)$$

$$\text{so} \quad v = \left[\frac{q^2(1 + 2\sqrt{2})}{\sqrt{2}mL}\right]^{\frac{1}{2}}, \quad \text{as in Prob. 1.1.} \qquad (2.3)$$

Problem 2.2:

(a) The potential energy of a uniformly charged spherical shell of charge Q and radius R is given by

$$U = \frac{1}{2}\int \phi dq = \frac{1}{2}\left(\frac{Q}{R}\right)Q = \frac{Q^2}{2R}. \qquad (2.4)$$

(b) The potential energy is also given by integrating E^2:

$$\begin{aligned} U &= \frac{1}{8\pi}\int E^2 d\tau = \frac{1}{8\pi}\int_{r\geq R}\frac{Q^2 d\tau}{r^4} \\ &= \frac{Q^2}{8\pi}\int_R^\infty \frac{4\pi r^2 dr}{r^4} = \frac{Q^2}{2R}, \quad \text{the same as in part (a).} \end{aligned} \qquad (2.5)$$