Contents

Chapter 9

Sequences and Infinite Series

9.1 An Overview

9.1.1 A *sequence* is an ordered list of numbers $a_1, a_2, a_3, \dots$, often written $\{a_1, a_2, \dots\}$ or $\{a_n\}$. For example, the natural numbers $\{1, 2, 3, ...\}$ are a sequence where $a_n = n$ for every n.

9.1.2 $a_1 = \frac{1}{1} = 1$; $a_2 = \frac{1}{2}$; $a_3 = \frac{1}{3}$; $a_4 = \frac{1}{4}$; $a_5 = \frac{1}{5}$.

9.1.3 $a_1 = 1$ (given); $a_2 = 1 \cdot a_1 = 1$; $a_3 = 2 \cdot a_2 = 2$; $a_4 = 3 \cdot a_3 = 6$; $a_5 = 4 \cdot a_4 = 24$.

9.1.4 A *finite sum* is the sum of a finite number of items, for example the sum of a finite number of terms of a sequence.

9.1.5 An *infinite series* is an infinite sum of numbers. Thus if $\{a_n\}$ is a sequence, then $a_1+a_2+\cdots = \sum_{k=1}^{\infty} a_k$, is an infinite series. For example, if $a_k = \frac{1}{k}$, then $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k}$ is an infinite series.

9.1.6 $S_1 = \sum_{k=1}^{1} k = 1$; $S_2 = \sum_{k=1}^{2} k = 1 + 2 = 3$; $S_3 = \sum_{k=1}^{3} k = 1 + 2 + 3 = 6$; $S_4 = \sum_{k=1}^{4} k = 1 + 2 + 3 + 4 = 10$.

9.1.7 $S_1 = \sum_{k=1}^{1} k^2 = 1$; $S_2 = \sum_{k=1}^{2} k^2 = 1 + 4 = 5$; $S_3 = \sum_{k=1}^{3} k^2 = 1 + 4 + 9 = 14$; $S_4 = \sum_{k=1}^{4} k^2 = 1 + 4 + 9 + 16 = 30$.

9.1.8 $S_1 = \sum_{k=1}^{1} \frac{1}{k} = \frac{1}{1} = 1$; $S_2 = \sum_{k=1}^{2} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2}$; $S_3 = \sum_{k=1}^{3} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$; $S_4 = \sum_{k=1}^{4} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$.

9.1.9 $a_1 = \dfrac{1}{10}; a_2 = \dfrac{1}{100}; a_3 = \dfrac{1}{1000}; a_4 = \dfrac{1}{10000}$.

9.1.10 $a_1 = 3(1) + 1 = 4$. $a_2 = 3(2) + 1 = 7$, $a_3 = 3(3) + 1 = 10$, $a_4 = 3(4) + 1 = 13$.

9.1.11 $a_1 = \frac{-1}{2}$, $a_2 = \frac{1}{2^2} = \frac{1}{4}$. $a_3 = \frac{-2}{2^3} = \frac{-1}{8}$, $a_4 = \frac{1}{2^4} = \frac{1}{16}$.

9.1.12 $a_1 = 2 - 1 = 1$. $a_2 = 2 + 1 = 3$, $a_3 = 2 - 1 = 1$, $a_4 = 2 + 1 = 3$.

9.1.13 $a_1 = \frac{2^2}{2+1} = \frac{4}{3}$. $a_2 = \frac{2^3}{2^2+1} = \frac{8}{5}$. $a_3 = \frac{2^4}{2^3+1} = \frac{16}{9}$. $a_4 = \frac{2^5}{2^4+1} = \frac{32}{17}$.

9.1.14 $a_1 = 1 + \frac{1}{1} = 2$; $a_2 = 2 + \frac{1}{2} = \frac{5}{2}$; $a_3 = 3 + \frac{1}{3} = \frac{10}{3}$; $a_4 = 4 + \frac{1}{4} = \frac{17}{4}$.

9.1.15 $a_1 = 1 + \sin(\pi/2) = 2$; $a_2 = 1 + \sin(2\pi/2) = 1 + \sin(\pi) = 1$; $a_3 = 1 + \sin(3\pi/2) = 0$; $a_4 = 1 + \sin(4\pi/2) = 1 + \sin(2\pi) = 1$.

9.1.16 $a_1 = 2 \cdot 1^2 - 3 \cdot 1 + 1 = 0$; $a_2 = 2 \cdot 2^2 - 3 \cdot 2 + 1 = 3$; $a_3 = 2 \cdot 3^2 - 3 \cdot 3 + 1 = 10$; $a_4 = 2 \cdot 4^2 - 3 \cdot 4 + 1 = 21$.

9.1.17 $a_1 = 2$, $a_2 = 2(2) = 4$, $a_3 = 2(4) = 8$, $a_4 = 2(8) = 16$.

9.1.18 $a_1 = 32$, $a_2 = 32/2 = 16$, $a_3 = 16/2 = 8$, $a_4 = 8/2 = 4$.

9.1.19 $a_1 = 10$ (given); $a_2 = 3 \cdot a_1 - 12 = 30 - 12 = 18$; $a_3 = 3 \cdot a_2 - 12 = 54 - 12 = 42$; $a_4 = 3 \cdot a_3 - 12 = 126 - 12 = 114$.

9.1.20 $a_1 = 1$ (given); $a_2 = a_1^2 - 1 = 0$; $a_3 = a_2^2 - 1 = -1$; $a_4 = a_3^2 - 1 = 0$.

9.1.21 $a_1 = 0$ (given); $a_2 = 3 \cdot a_1^2 + 1 + 1 = 2$; $a_3 = 3 \cdot a_2^2 + 2 + 1 = 15$; $a_4 = 3 \cdot a_3^2 + 3 + 1 = 679$.

9.1.22 $a_0 = 1$ (given); $a_1 = 1$ (given); $a_2 = a_1 + a_0 = 2$; $a_3 = a_2 + a_1 = 3$; $a_4 = a_3 + a_2 = 5$.

9.1.23

a. $\frac{1}{32}$, $\frac{1}{64}$.

b. $a_1 = 1$; $a_{n+1} = \frac{a_n}{2}$.

c. $a_n = \frac{1}{2^{n-1}}$.

9.1.24

a. -6, 7.

b. $a_1 = 1$; $a_{n+1} = (-1)^n(|a_n| + 1)$.

c. $a_n = (-1)^{n+1}n$.

9.1.25

a. -5, 5.

b. $a_1 = -5$, $a_{n+1} = -a_n$.

c. $a_n = (-1)^n 5$.

9.1.26

a. 14, 17.

b. $a_1 = 2$; $a_{n+1} = a_n + 3$.

c. $a_n = -1 + 3n$.

9.1.27

a. 32, 64.

b. $a_1 = 1$; $a_{n+1} = 2a_n$.

c. $a_n = 2^{n-1}$.

9.1.28

a. 36, 49.

b. $a_1 = 1$; $a_{n+1} = (\sqrt{a_n} + 1)^2$.

c. $a_n = n^2$.

9.1.29

a. 243, 729.

b. $a_1 = 1$; $a_{n+1} = 3a_n$.

c. $a_n = 3^{n-1}$.

9.1.30

a. 2, 1.

b. $a_1 = 64$; $a_{n+1} = \frac{a_n}{2}$.

c. $a_n = \frac{64}{2^{n-1}}$.

9.1.31 $a_1 = 9$, $a_2 = 99$, $a_3 = 999$, $a_4 = 9999$. This sequence diverges, because the terms get larger without bound.

9.1.32 $a_1 = 2$, $a_2 = 17$, $a_3 = 82$, $a_4 = 257$. This sequence diverges, because the terms get larger without bound.

9.1.33 $a_1 = \frac{1}{10}$, $a_2 = \frac{1}{100}$, $a_3 = \frac{1}{1000}$, $a_4 = \frac{1}{10{,}000}$. This sequence converges to zero.

9.1.34 $a_1 = 1/2$, $a_2 = 1/4$, $a_3 = 1/8$, $a_4 = 1/16$. This sequence converges to zero.

9.1.35 $a_1 = -1$, $a_2 = \frac{1}{2}$, $a_3 = -\frac{1}{3}$, $a_4 = \frac{1}{4}$. This sequence converges to 0 because each term is smaller in absolute value than the preceding term and they get arbitrarily close to zero.

9.1.36 $a_1 = 0.9$, $a_2 = 0.99$, $a_3 = 0.999$, $a_4 = .9999$. This sequence converges to 1.

9.1.37 $a_1 = 1+1 = 2$, $a_2 = 1+1 = 2$, $a_3 = 2$, $a_4 = 2$. This constant sequence converges to 2.

9.1.38 $a_1 = 1 - \frac{1}{2} \cdot \frac{2}{3} = \frac{2}{3}$. Similarly, $a_2 = a_3 = a_4 = \frac{2}{3}$. This constant sequences converges to $\frac{2}{3}$.

9.1.39 $a_0 = 100$, $a_1 = 0.5 \cdot 100 + 50 = 100$, $a_2 = 0.5 \cdot 100 + 50 = 100$, $a_3 = 0.5 \cdot 100 + 50 = 100$, $a_4 = 0.5 \cdot 100 + 50 = 100$. This constant sequence converges to 100.

9.1.40 $a_1 = 0 - 1 = -1$. $a_2 = -10 - 1 = -11$, $a_3 = -110 - 1 = -111$, $a_4 = -1110 - 1 = -1111$. This sequence diverges.

9.1.41

n	1	2	3	4	4	6	7	8	9	10
a_n	0.4637	0.2450	0.1244	0.0624	0.0312	0.0156	0.0078	0.0039	0.0020	0.0010

This sequence appears to converge to 0.

9.1.42

n	1	2	3	4	5	6	7	8	9	10
a_n	3.1396	3.1406	3.1409	3.1411	3.1412	3.1413	3.1413	3.1413	3.1414	3.1414

This sequence appears to converge to π.

9.1.43

n	1	2	3	4	5	6	7	8	9	10
a_n	0	2	6	12	20	30	42	56	72	90

This sequence appears to diverge.

9.1.44

n	1	2	3	4	5	6	7	8	9	10
a_n	9.9	9.95	9.9667	9.975	9.98	9.9833	9.9857	9.9875	9.9889	9.99

This sequence appears to converge to 10.

9.1.45

n	2	3	4	5	6	7	8	9	10	11
a_n	0.3333	0.5000	0.6000	0.6667	0.7143	0.7500	0.7778	0.8000	0.81818	0.8333

This sequence appears to converge to 1.

9.1.46

n	1	2	3	4	5	6	7	8	9	10	11
a_n	0.9589	0.9896	0.9974	0.9993	0.9998	1.000	1.000	1.0000	1.000	1.000	1.000

This sequence converges to 1.

9.1.47

a. 2.5, 2.25, 2.125, 2.0625.

b. The limit is 2.

9.1.48

a. 1.33333, 1.125, 1.06667, 1.04167.

b. The limit is 1.

9.1.49

n	0	1	2	3	4	5	6	7	8	9	10
a_n	3	3.5000	3.7500	3.8750	3.9375	3.9688	3.9844	3.9922	3.9961	3.9980	3.9990

This sequence converges to 4.

9.1.50

n	0	1	2	3	4	5	6	7	8	9
a_n	1	-2.75	-3.6875	-3.9219	-3.9805	-3.9951	-3.9988	-3.9997	-3.9999	-4.00

This sequence converges to -4.

9.1.51

n	0	1	2	3	4	5	6	7	8	9	10
a_n	0	1	3	7	15	31	63	127	255	511	1023

This sequence diverges.

9.1.52

n	0	1	2	3	4	5	6	7	8	9	10
a_n	32	16	8	4	2	1	.5	.25	.125	.0625	.03125

This sequence converges to 0.

9.1.53

n	0	1	2	3	4	5	6	7	8	9
a_n	1000	18.811	5.1686	4.1367	4.0169	4.0021	4.0003	4.0000	4.0000	4.0000

This sequence converges to 4.

9.1.54

n	0	1	2	3	4	5	6	7	8	9	10
a_n	1	1.4212	1.5538	1.5981	1.6119	1.6161	1.6174	1.6179	1.6180	1.6180	1.6180

This sequence converges to $\frac{1+\sqrt{5}}{2} \approx 1.6180339$.

9.1.55

a. 20, 10, 5, 2.5.

b. $h_n = 20(0.5)^n$.

9.1.56

a. 10, 9, 8.1, 7.29.

b. $h_n = 10(0.9)^n$.

9.1.57

a. 30, 7.5, 1.875, 0.46875.

b. $h_n = 30(0.25)^n$.

9.1.58

a. 20, 15, 11.25, 8.4375

b. $h_n = 20(0.75)^n$.

9.1.59 $S_1 = 0.3$, $S_2 = 0.33$, $S_3 = 0.333$, $S_4 = 0.3333$. It appears that the infinite series has a value of $0.3333\ldots = \frac{1}{3}$.

9.1.60 $S_1 = 0.6$, $S_2 = 0.66$, $S_3 = 0.666$, $S_4 = 0.6666$. It appears that the infinite series has a value of $0.6666\ldots = \frac{2}{3}$.

9.1.61 $S_1 = 4$, $S_2 = 4.9$, $S_3 = 4.99$, $S_4 = 4.999$. The infinite series has a value of $4.999\cdots = 5$.

9.1.62 $S_1 = 1$, $S_2 = \frac{3}{2} = 1.5$, $S_3 = \frac{7}{4} = 1.75$, $S_4 = \frac{15}{8} = 1.875$. The infinite series has a value of 2.

9.1.63

a. $S_1 = \frac{2}{3}$, $S_2 = \frac{4}{5}$, $S_3 = \frac{6}{7}$, $S_4 = \frac{8}{9}$.

b. It appears that $S_n = \frac{2n}{2n+1}$.

c. The series has a value of 1 (the partial sums converge to 1).

9.1.64

a. $S_1 = \frac{1}{2}$, $S_2 = \frac{3}{4}$, $S_3 = \frac{7}{8}$, $S_4 = \frac{15}{16}$.

b. $S_n = 1 - \frac{1}{2^n}$.

c. The partial sums converge to 1, so that is the value of the series.

9.1.65

a. $S_1 = \frac{1}{3}$, $S_2 = \frac{2}{5}$, $S_3 = \frac{3}{7}$, $S_4 = \frac{4}{9}$.

b. $S_n = \frac{n}{2n+1}$.

c. The partial sums converge to $\frac{1}{2}$, which is the value of the series.

9.1.66

a. $S_1 = \frac{2}{3}$, $S_2 = \frac{8}{9}$, $S_3 = \frac{26}{27}$, $S_4 = \frac{80}{81}$.

b. $S_n = 1 - \frac{1}{3^n}$.

c. The partial sums converge to 1, which is the value of the series.

9.1.67

a. True. For example, $S_2 = 1 + 2 = 3$, and $S_4 = a_1 + a_2 + a_3 + a_4 = 1 + 2 + 3 + 4 = 10$.

b. False. For example, $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \cdots$ where $a_n = 1 - \frac{1}{2^n}$ converges to 1, but each term is greater than the previous one.

c. True. In order for the partial sums to converge, they must get closer and closer together. In order for this to happen, the difference between successive partial sums, which is just the value of a_n, must approach zero.

9.1.68 The height at the n^{th} bounce is given by the recurrence $h_n = r \cdot h_{n-1}$; an explicit form for this sequence is $h_n = h_0 \cdot r^n$. The distance traveled by the ball during the n^{th} bounce is thus $2h_n = 2h_0 \cdot r^n$, so that $S_n = \sum_{i=0}^{n} 2h_0 \cdot r^n$.

a. Here $h_0 = 20$, $r = 0.5$, so $S_0 = 40$, $S_1 = 40 + 40 \cdot 0.5 = 60$, $S_2 = S_1 + 40 \cdot (0.5)^2 = 70$, $S_3 = S_2 + 40 \cdot (0.5)^3 = 75$, $S_4 = S_3 + 40 \cdot (0.5)^4 = 77.5$

b.

n	0	1	2	3	4	5
a_n	40	60	70	75	77.5	78.75
n	6	7	8	9	10	11
a_n	79.375	79.6875	79.8438	79.9219	79.9609	79.9805
n	12	13	14	15	16	17
a_n	79.9902	79.9951	79.9976	79.9988	79.9994	79.9997
n	18	19	20	21	22	23
a_n	79.9998	79.9999	79.9999	80.0000	80.0000	80.0000

The sequence converges to 80.

9.1.69 Using the work from the previous problem:

a. Here $h_0 = 20$, $r = 0.75$, so $S_0 = 40$, $S_1 = 40 + 40 \cdot 0.75 = 70$, $S_2 = S_1 + 40 \cdot (0.75)^2 = 92.5$, $S_3 = S_2 + 40 \cdot (0.75)^3 = 109.375$, $S_4 = S_3 + 40 \cdot (0.75)^4 = 122.03125$

b.

n	0	1	2	3	4	5
a_n	40	70	92.5	109.375	122.0313	131.5234
n	6	7	8	9	10	11
a_n	138.6426	143.9819	147.9865	150.9898	153.2424	154.9318
n	12	13	14	15	16	17
a_n	156.1988	157.1491	157.8618	158.3964	158.7973	159.0980
n	18	19	20	21	22	23
a_n	159.3235	159.4926	159.6195	159.715	159.786	159.839

The sequence converges to 160.

9.1.70

a. $s_1 = -1$, $s_2 = 0$, $s_3 = -1$, $s_4 = 0$.

b. The limit does not exist.

9.1.71

a. 0.9, 0.99, 0.999, .9999.

b. The limit is 1.

9.1.72

a. 1.5, 3.75, 7.125, 12.1875.

b. The limit does not exist.

9.1.73

a. $\frac{1}{3}, \frac{4}{9}, \frac{13}{27}, \frac{40}{81}$.

b. The limit is 1/2.

9.1.74

a. 1, 3, 6, 10.

b. The limit does not exist.

9.1.75

a. $-1, 0, -1, 0$.

b. The limit does not exist.

9.1.76

a. $-1, 1, -2, 2$.

b. The limit does not exist.

9.1.77

a. $\frac{3}{10} = 0.3$, $\frac{33}{100} = 0.33$, $\frac{333}{1000} = 0.333$, $\frac{3333}{10000} = 0.3333$.

b. The limit is 1/3.

9.1.78

a. $p_0 = 250$, $p_1 = 250 \cdot 1.03 = 258$, $p_2 = 250 \cdot 1.03^2 = 265$, $p_3 = 250 \cdot 1.03^3 = 273$, $p_4 = 250 \cdot 1.03^4 = 281$.

b. The initial population is 250, so that $p_0 = 250$. Then $p_n = 250 \cdot (1.03)^n$, because the population increases by 3 percent each month.

c. $p_{n+1} = p_n \cdot 1.03$.

d. The population increases without bound.

9.1.79

a. $M_0 = 20,\ \ M_1 = 20 \cdot 0.5 = 10,\ \ M_2 = 20 \cdot 0.5^2 = 5,\ M_3 = 20 \cdot 0.5^3 = 2.5,\ \ M_4 = 20 \cdot 0.5^4 = 1.25$

b. $M_n = 20 \cdot 0.5^n$.

c. The initial mass is $M_0 = 20$. We are given that 50% of the mass is gone after each decade, so that $M_{n+1} = 0.5 \cdot M_n,\ n \geq 0$.

d. The amount of material goes to 0.

9.1.80

a. $c_0 = 100,\ \ c_1 = 103,\ \ c_2 = 106.09,\ \ c_3 = 109.27,\ \ c_4 = 112.55$.

b. $c_n = 100(1.03)^n,\ nge0$.

c. We are given that $c_0 = 100$ (where year 0 is 1984); because it increases by 3% per year, $c_{n+1} = 1.03 \cdot c_n$.

d. The sequence diverges.

9.1.81

a. $d_0 = 200,\ \ d_1 = 200 \cdot .95 = 190,\ \ d_2 = 200 \cdot .95^2 = 180.5,\ d_3 = 200 \cdot .95^3 = 171.475,\ \ d_4 = 200 \cdot .95^4 = 162.90125$.

b. $d_n = 200(0.95)^n,\ n \geq 0$.

c. We are given $d_0 = 200$; because 5% of the drug is washed out every hour, that means that 95% of the preceding amount is left every hour, so that $d_{n+1} = 0.95 \cdot d_n$.

d. The sequence converges to 0.

9.1.82

a. Using the recurrence $a_{n+1} = \frac{1}{2}\left(a_n + \frac{10}{a_n}\right)$, we build a table:

n	0	1	2	3	4	5
a_n	10	5.5	3.659090909	3.196005081	3.162455622	3.162277665

The true value is $\sqrt{10} \approx 3.162277660$, so the sequence converges with an error of less than 0.01 after only 4 iterations, and is within 0.0001 after only 5 iterations.

b. The recurrence is now $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$

n	0	1	2	3	4	5	6
a_n	2	1.5	1.416666667	1.414215686	1.414213562	1.414213562	1.414213562

The true value is $\sqrt{2} \approx 1.414213562$, so the sequence converges with an error of less than 0.01 after 2 iterations, and is within 0.0001 after only 3 iterations.

9.2 Sequences

9.2.1 There are many examples; one is $a_n = \frac{1}{n}$. This sequence is nonincreasing (in fact, it is decreasing) and has a limit of 0.

9.2.2 Again there are many examples; one is $a_n = \ln(n)$. It is increasing, and has no limit.

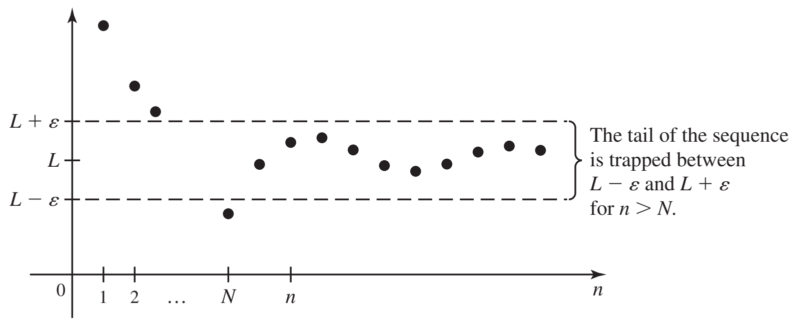
9.2.3 There are many examples; one is $a_n = \frac{1}{n}$. This sequence is nonincreasing (in fact, it is decreasing), is bounded above by 1 and below by 0, and has a limit of 0.

9.2.4 For example, $a_n = (-1)^n \dfrac{n-1}{n}$. $|a_n| < 1$, so it is bounded, but the odd terms approach -1 while the even terms approach 1. Thus the sequence does not have a limit.

9.2.5 $\{r^n\}$ converges for $-1 < r \le 1$. It diverges for all other values of r (see Theorem 9.3).

9.2.6 By Theorem 9.1, if we can find a function $f(x)$ such that $f(n) = a_n$ for all positive integers n, then if $\lim_{x\to\infty} f(x)$ exists and is equal to L, we then have $\lim_{n\to\infty} a_n$ exists and is also equal to L. This means that we can apply function-oriented limit methods such as L'Hôpital's rule to determine limits of sequences.

9.2.7 A sequence a_n converges to l if, given any $\epsilon > 0$, there exists a positive integer N, such that whenever $n > N$, $|a_n - L| < \varepsilon$.



9.2.8 The definition of the limit of a sequence involves only the behavior of the n^{th} term of a sequence as n gets large (see the Definition of Limit of a Sequence). Thus suppose a_n, b_n differ in only finitely many terms, and that M is large enough so that $a_n = b_n$ for $n > M$. Suppose a_n has limit L. Then for $\varepsilon > 0$, if N is such that $|a_n - L| < \varepsilon$ for $n > N$, first increase N if required so that $N > M$ as well. Then we also have $|b_n - L| < \varepsilon$ for $n > N$. Thus a_n and b_n have the same limit. A similar argument applies if a_n has no limit.

9.2.9 Divide numerator and denominator by n^4 to get $\lim_{n\to\infty} \frac{1/n}{1+\frac{1}{n^4}} = 0$.

9.2.10 Divide numerator and denominator by n^{12} to get $\lim_{n\to\infty} \frac{1}{3+\frac{4}{n^{12}}} = \frac{1}{3}$.

9.2.11 Divide numerator and denominator by n^3 to get $\lim_{n\to\infty} \frac{3-n^{-3}}{2+n^{-3}} = \frac{3}{2}$.

9.2.12 Divide numerator and denominator by e^n to get $\lim_{n\to\infty} \frac{2+(1/e^n)}{1} = 2$.

9.2.13 Divide numerator and denominator by 3^n to get $\lim_{n\to\infty} \frac{3+(1/3^{n-1})}{1} = 3$.

9.2.14 Divide numerator by k and denominator by $k = \sqrt{k^2}$ to get $\lim_{k\to\infty} \frac{1}{\sqrt{9+(1/k^2)}} = \frac{1}{3}$.

9.2.15 $\lim_{n\to\infty} \tan^{-1}(n) = \frac{\pi}{2}$.

9.2.16 $\lim_{n\to\infty} \csc^{-1}(n) = \lim_{n\to\infty} \sin^{-1}(1/n) = \sin^{-1}(0) = 0$.

9.2.17 Because $\lim_{n\to\infty} \tan^{-1}(n) = \frac{\pi}{2}$, $\lim_{n\to\infty} \frac{\tan^{-1}(n)}{n} = 0$.

9.2.18 Let $y = n^{2/n}$. Then $\ln y = \frac{2\ln n}{n}$. By L'Hôpital's rule we have $\lim_{x\to\infty} \frac{2\ln x}{x} = \lim_{x\to\infty} \frac{2}{x} = 0$, so $\lim_{n\to\infty} n^{2/n} = e^0 = 1$.

9.2.19 Find the limit of the logarithm of the expression, which is $n\ln\left(1+\frac{2}{n}\right)$. Using L'Hôpital's rule: $\lim_{n\to\infty} n\ln\left(1+\frac{2}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1+\frac{2}{n}\right)}{1/n} = \lim_{n\to\infty} \frac{\frac{1}{1+(2/n)}\left(\frac{-2}{n^2}\right)}{-1/n^2} = \lim_{n\to\infty} \frac{2}{1+(2/n)} = 2$. Thus the limit of the original expression is e^2.

9.2.20 Take the logarithm of the expression and use L'Hôpital's rule: $\lim_{n\to\infty} n\ln\left(\frac{n}{n+5}\right) = \lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+5}\right)}{1/n} = \lim_{n\to\infty} \frac{\frac{n+5}{n}\cdot\frac{5}{(n+5)^2}}{-1/n^2} = \lim_{n\to\infty} -\frac{5n^2(n+5)}{n(n+5)^2} = -\lim_{n\to\infty} \frac{5n^3+25n^2}{n^3+10n^2+25n}$. To find this limit, divide numerator and denominator by n^3 to get $\lim_{n\to\infty} -\frac{5+25n^{-1}}{1+10n^{-1}+25n^{-2}} = -5$. Thus, the original limit is e^{-5}.

9.2.21 Take the logarithm of the expression and use L'Hôpital's rule:

$$\lim_{n\to\infty} \frac{n}{2}\ln\left(1+\frac{1}{2n}\right) = \lim_{n\to\infty} \frac{\ln(1+(1/2n))}{2/n} = \lim_{n\to\infty} \frac{\frac{1}{1+(1/2n)}\cdot\frac{-1}{2n^2}}{-2/n^2} = \lim_{n\to\infty} \frac{1}{4(1+(1/2n))} = \frac{1}{4}.$$

Thus the original limit is $e^{1/4}$.

9.2.22 Find the limit of the logarithm of the expression, which is $3n\ln\left(1+\frac{4}{n}\right)$. Using L'Hôpital's rule: $\lim_{n\to\infty} 3n\ln\left(1+\frac{4}{n}\right) = \lim_{n\to\infty} \frac{3\ln\left(1+\frac{4}{n}\right)}{1/n} = \lim_{n\to\infty} \frac{\frac{1}{1+(4/n)}\left(\frac{-12}{n^2}\right)}{-1/n^2} = \lim_{n\to\infty} \frac{12}{1+(4/n)} = 12$. Thus the limit of the original expression is e^{12}.

9.2.23 Using L'Hôpital's rule: $\lim_{n\to\infty} \frac{n}{e^n+3n} = \lim_{n\to\infty} \frac{1}{e^n+3} = 0$.

9.2.24 $\ln(1/n) = -\ln n$, so this is $\lim_{n\to\infty} \frac{-\ln n}{n}$. By L'Hôpital's rule, we have $\lim_{n\to\infty} \frac{-\ln n}{n} = -\lim_{n\to\infty} \frac{1}{n} = 0$.

9.2.25 Taking logs, we have $\lim_{n\to\infty} \frac{1}{n}\ln(1/n) = \lim_{n\to\infty} -\frac{\ln n}{n} = \lim_{n\to\infty} \frac{-1}{n} = 0$ by L'Hôpital's rule. Thus the original sequence has limit $e^0 = 1$.

9.2.26 Find the limit of the logarithm of the expression, which is $n\ln\left(1-\frac{4}{n}\right)$, using L'Hôpital's rule: $\lim_{n\to\infty} n\ln\left(1-\frac{4}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1-\frac{4}{n}\right)}{1/n} = \lim_{n\to\infty} \frac{\frac{1}{1-(4/n)}\left(\frac{4}{n^2}\right)}{-1/n^2} = \lim_{n\to\infty} \frac{-4}{1-(4/n)} = -4$. Thus the limit of the original expression is e^{-4}.

9.2.27 Except for a finite number of terms, this sequence is just $a_n = ne^{-n}$, so it has the same limit as this sequence. Note that $\lim_{n\to\infty} \frac{n}{e^n} = \lim_{n\to\infty} \frac{1}{e^n} = 0$, by L'Hôpital's rule.

9.2.28 $\ln(n^3+1) - \ln(3n^3+10n) = \ln\left(\frac{n^3+1}{3n^3+10n}\right) = \ln\left(\frac{1+n^{-3}}{3+10n^{-2}}\right)$, so the limit is $\ln(1/3) = -\ln 3$.

9.2.29 $\ln(\sin(1/n)) + \ln n = \ln(n\sin(1/n)) = \ln\left(\frac{\sin(1/n)}{1/n}\right)$. As $n\to\infty$, $\sin(1/n)/(1/n)\to 1$, so the limit of the original sequence is $\ln 1 = 0$.

9.2.30 Using L'Hôpital's rule:

$$\lim_{n\to\infty} n(1-\cos(1/n)) = \lim_{n\to\infty} \frac{1-\cos(1/n)}{1/n} = \lim_{n\to\infty} \frac{-\sin(1/n)(-1/n^2)}{-1/n^2} = -\sin(0) = 0.$$

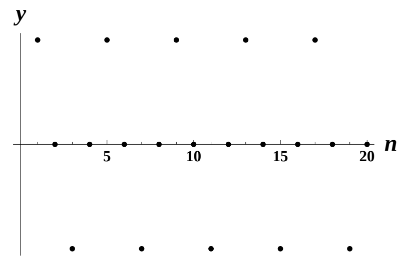
9.2.31 $\lim_{n\to\infty} n\sin(6/n) = \lim_{n\to\infty} \frac{\sin(6/n)}{1/n} = \lim_{n\to\infty} \frac{\frac{-6\cos(6/n)}{n^2}}{(-1/n^2)} = \lim_{n\to\infty} 6\cos(6/n) = 6$.

9.2.32 Because $\frac{-1}{n} \le \frac{(-1)^n}{n} \le \frac{1}{n}$, and because both $\frac{-1}{n}$ and $\frac{1}{n}$ have limit 0 as $n\to\infty$, the limit of the given sequence is also 0 by the Squeeze Theorem.

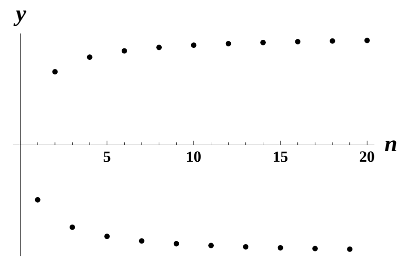
9.2.33 The terms with odd-numbered subscripts have the form $-\frac{n}{n+1}$, so they approach -1, while the terms with even-numbered subscripts have the form $\frac{n}{n+1}$ so they approach 1. Thus, the sequence has no limit.

9.2.34 Because $\frac{-n^2}{2n^3+n} \le \frac{(-1)^{n+1}n^2}{2n^3+n} \le \frac{n^2}{2n^3+n}$, and because both $\frac{-n^2}{2n^3+n}$ and $\frac{n^2}{2n^3+n}$ have limit 0 as $n\to\infty$, the limit of the given sequence is also 0 by the Squeeze Theorem. Note that $\lim_{n\to\infty} \frac{n^2}{2n^3+n} = \lim_{n\to\infty} \frac{1/n}{2+1/n^2} = \frac{0}{2} = 0$.

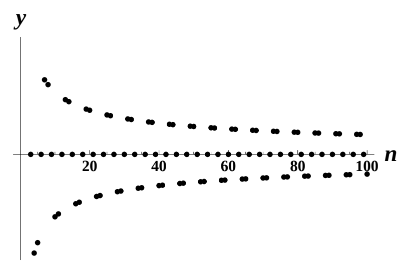
9.2.35 When n is an integer, $\sin\left(\frac{n\pi}{2}\right)$ oscillates between the values ± 1 and 0, so this sequence does not converge.



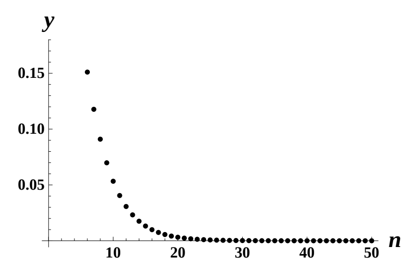
9.2.36 The even terms form a sequence $b_{2n} = \frac{2n}{2n+1}$, which converges to 1 (e.g. by L'Hôpital's rule); the odd terms form the sequence $\frac{-n}{n+1}$, which converges to -1. Thus the sequence as a whole does not converge.



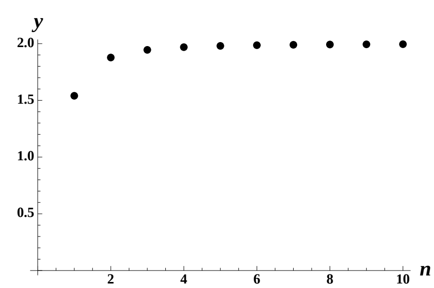
9.2.37 The numerator is bounded in absolute value by 1, while the denominator goes to ∞, so the limit of this sequence is 0.



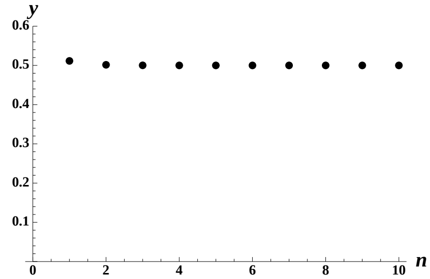
9.2.38 The reciprocal of this sequence is $b_n = \frac{1}{a_n} = 1 + \left(\frac{4}{3}\right)^n$, which increases without bound as $n \to \infty$. Thus a_n converges to zero.



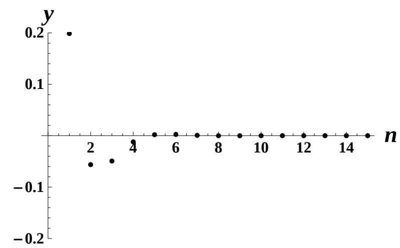
9.2.39 $\lim_{n\to\infty} (1 + \cos(1/n)) = 1 + \cos(0) = 2.$



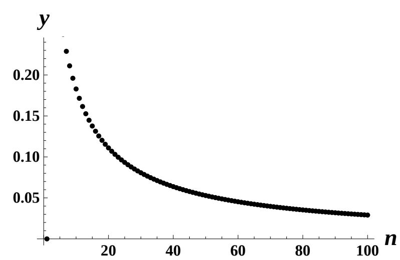
9.2.40 By L'Hôpital's rule we have: $\lim_{n\to\infty} \frac{e^{-n}}{2\sin(e^{-n})} = \lim_{n\to\infty} \frac{-e^{-n}}{2\cos(e^{-n})(-e^{-n})} = \frac{1}{2\cos(0)} = \frac{1}{2}$.



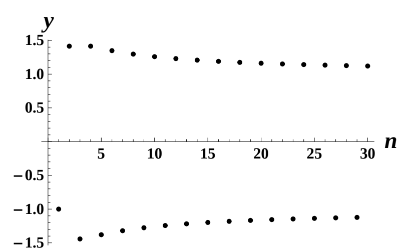
9.2.41 This is the sequence $\frac{\cos n}{e^n}$; the numerator is bounded in absolute value by 1 and the denominator increases without bound, so the limit is zero.



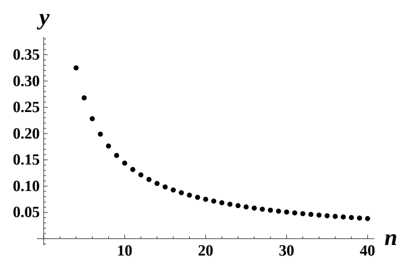
9.2.42 Using L'Hôpital's rule, we have $\lim_{n\to\infty} \frac{\ln n}{n^{1.1}} = \lim_{n\to\infty} \frac{1/n}{(1.1)n^{.1}} = \lim_{n\to\infty} \frac{1}{(1.1)n^{1.1}} = 0$.



9.2.43 Ignoring the factor of $(-1)^n$ for the moment, we see, taking logs, that $\lim_{n\to\infty} \frac{\ln n}{n} = 0$, so that $\lim_{n\to\infty} \sqrt[n]{n} = e^0 = 1$. Taking the sign into account, the odd terms converge to -1 while the even terms converge to 1. Thus the sequence does not converge.



9.2.44 $\lim_{n\to\infty} \frac{n\pi}{2n+2} = \frac{\pi}{2}$, using L'Hôpital's rule. Thus the sequence converges to $\cot(\pi/2) = 0$.



9.2.45 Because $0.2 < 1$, this sequence converges to 0. Because $0.2 > 0$, the convergence is monotone.

9.2.46 Because $1.2 > 1$, this sequence diverges monotonically to ∞.

9.2.47 Because $|-0.7| < 1$, the sequence converges to 0; because $-0.7 < 0$, it does not do so monotonically. The sequence converges by oscillation.

9.2.48 Because $|-1.01| > 1$, the sequence diverges; because $-1.01 < 0$, the divergence is not monotone.

9.2.49 Because $1.00001 > 1$, the sequence diverges; because $1.00001 > 0$, the divergence is monotone.

9.2.50 This is the sequence $\left(\frac{2}{3}\right)^n$; because $0 < \frac{2}{3} < 1$, the sequence converges monotonically to zero.

9.2.51 Because $|-2.5| > 1$, the sequence diverges; because $-2.5 < 0$, the divergence is not monotone. The sequence diverges by oscillation.

9.2.52 $|-0.003| < 1$, so the sequence converges to zero; because $-.003 < 0$, the convergence is not monotone.

9.2.53 Because $-1 \le \cos(n) \le 1$, we have $\frac{-1}{n} \le \frac{\cos(n)}{n} \le \frac{1}{n}$. Because both $\frac{-1}{n}$ and $\frac{1}{n}$ have limit 0 as $n \to \infty$, the given sequence does as well.

9.2.54 Because $-1 \le \sin(6n) \le 1$, we have $\frac{-1}{5n} \le \frac{\sin(6n)}{5n} \le \frac{1}{5n}$. Because both $\frac{-1}{5n}$ and $\frac{1}{5n}$ have limit 0 as $n \to \infty$, the given sequence does as well.

9.2.55 Because $-1 \le \sin n \le 1$ for all n, the given sequence satisfies $\frac{-1}{2^n} \le \frac{\sin n}{2^n} \le \frac{1}{2^n}$, and because both $\pm\frac{1}{2^n} \to 0$ as $n \to \infty$, the given sequence converges to zero as well by the Squeeze Theorem.

9.2.56 Because $-1 \le \cos(n\pi/2) \le 1$ for all n, we have $\frac{-1}{\sqrt{n}} \le \frac{\cos(n\pi/2)}{\sqrt{n}} \le \frac{1}{\sqrt{n}}$ and because both $\pm\frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$, the given sequence converges to 0 as well by the Squeeze Theorem.

9.2.57 $\tan^{-1}$ takes values between $-\pi/2$ and $\pi/2$, so the numerator is always between $-\pi$ and π. Thus $\frac{-\pi}{n^3+4} \le \frac{2\tan^{-1} n}{n^3+4} \le \frac{\pi}{n^3+4}$, and by the Squeeze Theorem, the given sequence converges to zero.

9.2.58 This sequence diverges. To see this, call the given sequence a_n, and assume it converges to limit L. Then because the sequence $b_n = \frac{n}{n+1}$ converges to 1, the sequence $c_n = \frac{a_n}{b_n}$ would converge to L as well. But $c_n = \sin^3 n$ doesn't converge, so the given sequence doesn't converge either.

9.2.59

a. After the n^{th} dose is given, the amount of drug in the bloodstream is $d_n = 0.5 \cdot d_{n-1} + 80$, because the half-life is one day. The initial condition is $d_1 = 80$.

b. The limit of this sequence is 160 mg.

c. Let $L = \lim_{n\to\infty} d_n$. Then from the recurrence relation, we have $d_n = 0.5 \cdot d_{n-1} + 80$, and thus $\lim_{n\to\infty} d_n = 0.5 \cdot \lim_{n\to\infty} d_{n-1} + 80$, so $L = 0.5 \cdot L + 80$, and therefore $L = 160$.

9.2.60

a.

$$\begin{aligned}
B_0 &= \$20,000\\
B_1 &= 1.005 \cdot B_0 - \$200 = \$19,900\\
B_2 &= 1.005 \cdot B_1 - \$200 = \$19,799.50\\
B_3 &= 1.005 \cdot B_2 - \$200 = \$19,698.50\\
B_4 &= 1.005 \cdot B_3 - \$200 = \$19,596.99\\
B_5 &= 1.005 \cdot B_4 - \$200 = \$19,494.97
\end{aligned}$$

b. $B_n = 1.005 \cdot B_{n-1} - \$200$

c. Using a calculator or computer program, B_n becomes negative after the 139^{th} payment, so 139 months or almost 11 years.

9.2.61

a.

$$\begin{aligned}
&B_0 = 0\\
&B_1 = 1.0075 \cdot B_0 + \$100 = \$100\\
&B_2 = 1.0075 \cdot B_1 + \$100 = \$200.75\\
&B_3 = 1.0075 \cdot B_2 + \$100 = \$302.26\\
&B_4 = 1.0075 \cdot B_3 + \$100 = \$404.52\\
&B_5 = 1.0075 \cdot B_4 + \$100 = \$507.56
\end{aligned}$$

b. $B_n = 1.0075 \cdot B_{n-1} + \$100$.

c. Using a calculator or computer program, $B_n > \$5,000$ during the 43^{rd} month.

9.2.62

a. Let D_n be the *total number* of liters of alcohol in the mixture after the n^{th} replacement. At the next step, 2 liters of the 100 liters is removed, thus leaving $0.98 \cdot D_n$ liters of alcohol, and then $0.1 \cdot 2 = 0.2$ liters of alcohol are added. Thus $D_n = 0.98 \cdot D_{n-1} + 0.2$. Now, $C_n = D_n/100$, so we obtain a recurrence relation for C_n by dividing this equation by 100: $C_n = 0.98 \cdot C_{n-1} + 0.002$.

$$\begin{aligned}
&C_0 = 0.4\\
&C_1 = 0.98 \cdot 0.4 + 0.002 = 0.394\\
&C_2 = 0.98 \cdot C_1 + 0.002 = 0.38812\\
&C_3 = 0.98 \cdot C_2 + 0.002 = 0.38236\\
&C_4 = 0.98 \cdot C_3 + 0.002 = 0.37671\\
&C_5 = 0.98 \cdot C_4 + 0.002 = 0.37118
\end{aligned}$$

The rounding is done to five decimal places.

b. Using a calculator or a computer program, $C_n < 0.15$ after the 89^{th} replacement.

c. If the limit of C_n is L, then taking the limit of both sides of the recurrence equation yields $L = 0.98L + 0.002$, so $.02L = .002$, and $L = .1 = 10\%$.

9.2.63 Because $n! \ll n^n$ by Theorem 9.6, we have $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

9.2.64 $\{3^n\} \ll \{n!\}$ because $\{b^n\} \ll \{n!\}$ in Theorem 9.6. Thus, $\lim_{n\to\infty} \frac{3^n}{n!} = 0$.

9.2.65 Theorem 9.6 indicates that $\ln^q n \ll n^p$, so $\ln^{20} n \ll n^{10}$, so $\lim_{n\to\infty} \frac{n^{10}}{\ln^{20} n} = \infty$.

9.2.66 Theorem 9.6 indicates that $\ln^q n \ll n^p$, so $\ln^{1000} n \ll n^{10}$, so $\lim_{n\to\infty} \frac{n^{10}}{\ln^{1000} n} = \infty$.

9.2.67 By Theorem 9.6, $n^p \ll b^n$, so $n^{1000} \ll 2^n$, and thus $\lim_{n\to\infty} \frac{n^{1000}}{2^n} = 0$.

9.2.68 Note that $e^{1/10} = \sqrt[10]{e} \approx 1.1$. Let $r = \frac{e^{1/10}}{2}$ and note that $0 < r < 1$. Thus $\lim_{n\to\infty} \frac{e^{n/10}}{2^n} = \lim_{n\to\infty} r^n = 0$.

9.2.69 Let $\varepsilon > 0$ be given and let N be an integer with $N > \frac{1}{\varepsilon}$. Then if $n > N$, we have $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} < \varepsilon$.

9.2.70 Let $\varepsilon > 0$ be given. We wish to find N such that $|(1/n^2) - 0| < \varepsilon$ if $n > N$. This means that $\left|\frac{1}{n^2} - 0\right| = \frac{1}{n^2} < \varepsilon$. So choose N such that $\frac{1}{N^2} < \varepsilon$, so that $N^2 > \frac{1}{\varepsilon}$, and then $N > \frac{1}{\sqrt{\varepsilon}}$. This shows that such an N always exists for each ε and thus that the limit is zero.

9.2.71 Let $\varepsilon > 0$ be given. We wish to find N such that for $n > N$, $\left|\frac{3n^2}{4n^2+1} - \frac{3}{4}\right| = \left|\frac{-3}{4(4n^2+1)}\right| = \frac{3}{4(4n^2+1)} < \varepsilon$. But this means that $3 < 4\varepsilon(4n^2 + 1)$, or $16\varepsilon n^2 + (4\varepsilon - 3) > 0$. Solving the quadratic, we get $n > \frac{1}{4}\sqrt{\frac{3}{\varepsilon} - 4}$, provided $\varepsilon < 3/4$. So let $N = \frac{1}{4}\sqrt{\frac{3}{\varepsilon}}$ if $\epsilon < 3/4$ and let $N = 1$ otherwise.

9.2.72 Let $\varepsilon > 0$ be given. We wish to find N such that for $n > N$, $|b^{-n} - 0| = b^{-n} < \varepsilon$, so that $-n \ln b < \ln \varepsilon$. So choose N to be any integer greater than $-\frac{\ln \varepsilon}{\ln b}$.

9.2.73 Let $\varepsilon > 0$ be given. We wish to find N such that for $n > N$, $\left|\frac{cn}{bn+1} - \frac{c}{b}\right| = \left|\frac{-c}{b(bn+1)}\right| = \frac{c}{b(bn+1)} < \varepsilon$. But this means that $\varepsilon b^2 n + (b\varepsilon - c) > 0$, so that $N > \frac{c}{b^2\varepsilon}$ will work.

9.2.74 Let $\varepsilon > 0$ be given. We wish to find N such that for $n > N$, $\left|\frac{n}{n^2+1} - 0\right| = \frac{n}{n^2+1} < \varepsilon$. Thus we want $n < \varepsilon(n^2 + 1)$, or $\varepsilon n^2 - n + \varepsilon > 0$. Whenever n is larger than the larger of the two roots of this quadratic, the desired inequality will hold. The roots of the quadratic are $\frac{1 \pm \sqrt{1-4\varepsilon^2}}{2\varepsilon}$, so we choose N to be any integer greater than $\frac{1+\sqrt{1-4\varepsilon^2}}{2\varepsilon}$.

9.2.75

a. True. See Theorem 9.2 part 4.

b. False. For example, if $a_n = e^n$ and $b_n = 1/n$, then $\lim_{n\to\infty} a_n b_n = \infty$.

c. True. The definition of the limit of a sequence involves only the behavior of the n^{th} term of a sequence as n gets large (see the Definition of Limit of a Sequence). Thus suppose a_n, b_n differ in only finitely many terms, and that M is large enough so that $a_n = b_n$ for $n > M$. Suppose a_n has limit L. Then for $\varepsilon > 0$, if N is such that $|a_n - L| < \varepsilon$ for $n > N$, first increase N if required so that $N > M$ as well. Then we also have $|b_n - L| < \varepsilon$ for $n > N$. Thus a_n and b_n have the same limit. A similar argument applies if a_n has no limit.

d. True. Note that a_n converges to zero. Intuitively, the nonzero terms of b_n are those of a_n, which converge to zero. More formally, given ϵ, choose N_1 such that for $n > N_1$, $a_n < \epsilon$. Let $N = 2N_1 + 1$. Then for $n > N$, consider b_n. If n is even, then $b_n = 0$ so certainly $b_n < \epsilon$. If n is odd, then $b_n = a_{(n-1)/2}$, and $(n-1)/2 > ((2N_1 + 1) - 1)/2 = N_1$ so that $a_{(n-1)/2} < \epsilon$. Thus b_n converges to zero as well.

e. False. If $\{a_n\}$ happens to converge to zero, the statement is true. But consider for example $a_n = 2 + \frac{1}{n}$. Then $\lim_{n\to\infty} a_n = 2$, but $(-1)^n a_n$ does not converge (it oscillates between positive and negative values increasingly close to ± 2).

f. True. Suppose $\{0.000001 a_n\}$ converged to L, and let $\epsilon > 0$ be given. Choose N such that for $n > N$, $|0.000001 a_n - L| < \epsilon \cdot 0.000001$. Dividing through by 0.000001, we get that for $n > N$, $|a_n - 1000000L| < \epsilon$, so that a_n converges as well (to $1000000L$).

9.2.76 $\{2n - 3\}_{n=3}^{\infty}$.

9.2.77 $\{(n-2)^2 + 6(n-2) - 9\}_{n=3}^{\infty} = \{n^2 + 2n - 17\}_{n=3}^{\infty}$.

9.2.78 If $f(t) = \int_1^t x^{-2}\,dx$, then $\lim_{t\to\infty} f(t) = \lim_{n\to\infty} a_n$. But $\lim_{t\to\infty} f(t) = \int_1^{\infty} x^{-2}\,dx = \lim_{b\to\infty} \left[\left.\frac{-1}{x}\right|_1^b\right] = \lim_{b\to\infty} \left(\frac{-1}{b} + 1\right) = 1$.

9.2.79 Evaluate the limit of each term separately: $\lim_{n\to\infty} \frac{75^{n-1}}{99^n} = \frac{1}{99} \lim_{n\to\infty} \left(\frac{75}{99}\right)^{n-1} = 0$, while $\frac{-5^n}{8^n} \le \frac{5^n \sin n}{8^n} \le \frac{5^n}{8^n}$, so by the Squeeze Theorem, this second term converges to 0 as well. Thus the sum of the terms converges to zero.

9.2.80 Because $\lim_{n\to\infty} \frac{10n}{10n+4} = 1$, and because the inverse tangent function is continuous, the given sequence has limit $\tan^{-1}(1) = \pi/4$.

9.2.81 Because $\lim_{n\to\infty} 0.99^n = 0$, and because cosine is continuous, the first term converges to $\cos 0 = 1$. The limit of the second term is $\lim_{n\to\infty} \frac{7^n+9^n}{63^n} = \lim_{n\to\infty} \left(\frac{7}{63}\right)^n + \lim_{n\to\infty} \left(\frac{9}{63}\right)^n = 0$. Thus the sum converges to 1.

9.2.82 Dividing the numerator and denominator by $n!$, gives $a_n = \frac{(4^n/n!)+5}{1+(2^n/n!)}$. By Theorem 9.6, we have $4^n \ll n!$ and $2^n \ll n!$. Thus, $\lim_{n\to\infty} a_n = \frac{0+5}{1+0} = 5$.

9.2.83 Dividing the numerator and denominator by 6^n gives $a_n = \frac{1+(1/2)^n}{1+(n^{100}/6^n)}$. By Theorem 9.6 $n^{100} \ll 6^n$. Thus $\lim_{n\to\infty} a_n = \frac{1+0}{1+0} = 1$.

9.2.84 Dividing the numerator and denominator by n^8 gives $a_n = \frac{1+(1/n)}{(1/n)+\ln n}$. Because $1 + (1/n) \to 1$ as $n \to \infty$ and $(1/n) + \ln n \to \infty$ as $n \to \infty$, we have $\lim_{n\to\infty} a_n = 0$.

9.2.85 We can write $a_n = \frac{(7/5)^n}{n^7}$. Theorem 9.6 indicates that $n^7 \ll b^n$ for $b > 1$, so $\lim_{n\to\infty} a_n = \infty$.

9.2.86 A graph shows that the sequence appears to converge. Assuming that it does, let its limit be L. Then $\lim_{n\to\infty} a_{n+1} = \frac{1}{2} \lim_{n\to\infty} a_n + 2$, so $L = \frac{1}{2}L + 2$, and thus $\frac{1}{2}L = 2$, so $L = 4$.

9.2.87 A graph shows that the sequence appears to converge. Let its supposed limit be L, then $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}(2a_n(1-a_n)) = 2(\lim_{n\to\infty} a_n)(1-\lim_{n\to\infty} a_n)$, so $L = 2L(1-L) = 2L-2L^2$, and thus $2L^2-L = 0$, so $L = 0, \frac{1}{2}$. Thus the limit appears to be either 0 or 1/2; with the given initial condition, doing a few iterations by hand confirms that the sequence converges to 1/2: $a_0 = 0.3$; $a_1 = 2 \cdot 0.3 \cdot 0.7 = .42$; $a_2 = 2 \cdot 0.42 \cdot 0.58 = 0.4872$.

9.2.88 A graph shows that the sequence appears to converge, and to a value other than zero; let its limit be L. Then $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n + \frac{2}{a_n}) = \frac{1}{2} \lim_{n\to\infty} a_n + \frac{1}{\lim_{n\to\infty} a_n}$, so $L = \frac{1}{2}L + \frac{1}{L}$, and therefore $L^2 = \frac{1}{2}L^2 + 1$. So $L^2 = 2$, and thus $L = \sqrt{2}$.

9.2.89 Computing three terms gives $a_0 = 0.5, a_1 = 4 \cdot .5 \cdot 0.5 = 1, a_2 = 4 \cdot 1 \cdot (1-1) = 0$. All successive terms are obviously zero, so the sequence converges to 0.

9.2.90 A graph shows that the sequence appears to converge. Let its limit be L. Then $\lim_{n\to\infty} a_{n+1} = \sqrt{2 + \lim_{n\to\infty} a_n}$, so $L = \sqrt{2+L}$. Thus we have $L^2 = 2 + L$, so $L^2 - L - 2 = 0$, and thus $L = -1, 2$. A square root can never be negative, so this sequence must converge to 2.

9.2.91 For $b = 2$, $2^3 > 3!$ but $16 = 2^4 < 4! = 24$, so the crossover point is $n = 4$. For e, $e^5 \approx 148.41 > 5! = 120$ while $e^6 \approx 403.4 < 6! = 720$, so the crossover point is $n = 6$. For 10, $24! \approx 6.2 \times 10^{23} < 10^{24}$, while $25! \approx 1.55 \times 10^{25} > 10^{25}$, so the crossover point is $n = 25$.

9.2.92

a. Rounded to the nearest fish, the populations are

$$\begin{aligned}
F_0 &= 4000 \\
F_1 &= 1.015F_0 - 80 \approx 3980 \\
F_2 &= 1.015F_1 - 80 \approx 3960 \\
F_3 &= 1.015F_2 - 80 \approx 3939 \\
F_4 &= 1.015F_3 - 80 \approx 3918 \\
F_5 &= 1.015F_4 - 80 \approx 3897
\end{aligned}$$

b. $F_n = 1.015F_{n-1} - 80$

c. The population decreases and eventually reaches zero.

d. With an initial population of 5500 fish, the population increases without bound.

e. If the initial population is less than 5333 fish, the population will decline to zero. This is essentially because for a population of less than 5333, the natural increase of 1.5% does not make up for the loss of 80 fish.

9.2.93

a. The profits for each of the first ten days, in dollars are:

n	0	1	2	3	4	5	6	7	8	9	10
h_n	130.00	130.75	131.40	131.95	132.40	132.75	133.00	133.15	133.20	133.15	133.00

b. The profit on an item is revenue minus cost. The total cost of keeping the hippo for n days is $.45n$, and the revenue for selling the hippo on the n^{th} day is $(200 + 5n) \cdot (.65 - .01n)$, because the hippo gains 5 pounds per day but is worth a penny less per pound each day. Thus the total profit on the n^{th} day is $h_n = (200 + 5n) \cdot (.65 - .01n) - .45n = 130 + 0.8n - 0.05n^2$. The maximum profit occurs when $-.1n + .8 = 0$, which occurs when $n = 8$. The maximum profit is achieved by selling the hippo on the 8^{th} day.

9.2.94

a. $x_0 = 7$, $x_1 = 6$, $x_2 = 6.5 = \frac{13}{2}$, $x_3 = 6.25$, $x_4 = 6.375 = \frac{51}{8}$, $x_5 = 6.3125 = \frac{101}{16}$, $x_6 = 6.34375 = \frac{203}{32}$.

b. For the formula given in the problem, we have $x_0 = \frac{19}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^0 = 7$, $x_1 = \frac{19}{3} + \frac{2}{3} \cdot \frac{-1}{2} = \frac{19}{3} - \frac{1}{3} = 6$, so that the formula holds for $n = 0, 1$. Now assume the formula holds for all integers $\leq k$; then

$$\begin{aligned}
x_{k+1} = \frac{1}{2}(x_k + x_{k-1}) &= \frac{1}{2}\left(\frac{19}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^k + \frac{19}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^{k-1}\right) \\
&= \frac{1}{2}\left(\frac{38}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^{k-1}\left(-\frac{1}{2} + 1\right)\right) \\
&= \frac{1}{2}\left(\frac{38}{3} + 4 \cdot \frac{2}{3}\left(-\frac{1}{2}\right)^{k+1} \cdot \frac{1}{2}\right) \\
&= \frac{1}{2}\left(\frac{38}{3} + 2 \cdot \frac{2}{3}\left(-\frac{1}{2}\right)^{k+1}\right) \\
&= \frac{19}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^{k+1}.
\end{aligned}$$

c. As $n \to \infty$, $(-1/2)^n \to 0$, so that the limit is $19/3$, or $6\ 1/3$.

9.2.95 The approximate first few values of this sequence are:

n	0	1	2	3	4	5	6
c_n	.7071	.6325	.6136	.6088	.6076	.6074	.6073

The value of the constant appears to be around 0.607.

9.2.96 We first prove that d_n is bounded by 200. If $d_n \le 200$, then $d_{n+1} = 0.5 \cdot d_n + 100 \le 0.5 \cdot 200 + 100 \le 200$. Because $d_0 = 100 < 200$, all d_n are at most 200. Thus the sequence is bounded. To see that it is monotone, look at

$$d_n - d_{n-1} = 0.5 \cdot d_{n-1} + 100 - d_{n-1} = 100 - 0.5 d_{n-1}.$$

But we know that $d_{n-1} \le 200$, so that $100 - 0.5 d_{n-1} \ge 0$. Thus $d_n \ge d_{n-1}$ and the sequence is nondecreasing.

9.2.97

a. If we "cut off" the expression after n square roots, we get a_n from the recurrence given. We can thus *define* the infinite expression to be the limit of a_n as $n \to \infty$.

b. $a_0 = 1$, $a_1 = \sqrt{2}$, $a_2 = \sqrt{1+\sqrt{2}} \approx 1.5538$, $a_3 \approx 1.598$, $a_4 \approx 1.6118$, and $a_5 \approx 1.6161$.

c. $a_{10} \approx 1.618$, which differs from $\frac{1+\sqrt{5}}{2} \approx 1.61803394$ by less than .001.

d. Assume $\lim\limits_{n\to\infty} a_n = L$. Then $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \sqrt{1+a_n} = \sqrt{1 + \lim\limits_{n\to\infty} a_n}$, so $L = \sqrt{1+L}$, and thus $L^2 = 1 + L$. Therefore we have $L^2 - L - 1 = 0$, so $L = \frac{1 \pm \sqrt{5}}{2}$.
Because clearly the limit is positive, it must be the positive square root.

e. Letting $a_{n+1} = \sqrt{p + \sqrt{a_n}}$ with $a_0 = p$ and assuming a limit exists we have $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \sqrt{p + a_n}$ $= \sqrt{p + \lim\limits_{n\to\infty} a_n}$, so $L = \sqrt{p+L}$, and thus $L^2 = p + L$. Therefore, $L^2 - L - p = 0$, so $L = \frac{1 \pm \sqrt{1+4p}}{2}$, and because we know that L is positive, we have $L = \frac{1+\sqrt{4p+1}}{2}$. The limit exists for all positive p.

9.2.98 Note that $1 - \frac{1}{i} = \frac{i-1}{i}$, so that the product is $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdots$, so that $a_n = \frac{1}{n}$ for $n \ge 2$. The sequence $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ has limit zero.

9.2.99

a. Define a_n as given in the problem statement. Then we can *define* the value of the continued fraction to be $\lim\limits_{n\to\infty} a_n$.

b. $a_0 = 1$, $a_1 = 1 + \frac{1}{a_0} = 2$, $a_2 = 1 + \frac{1}{a_1} = \frac{3}{2} = 1.5$, $a_3 = 1 + \frac{1}{a_2} = \frac{5}{3} \approx 1.67$, $a_4 = 1 + \frac{1}{a_3} = \frac{8}{5} \approx 1.6$, $a_5 = 1 + \frac{1}{a_4} = \frac{13}{8} \approx 1.625$.

c. From the list above, the values of the sequence alternately decrease and increase, so we would expect that the limit is somewhere between 1.6 and 1.625.

d. Assume that the limit is equal to L. Then from $a_{n+1} = 1 + \frac{1}{a_n}$, we have $\lim\limits_{n\to\infty} a_{n+1} = 1 + \frac{1}{\lim\limits_{n\to\infty} a_n}$, so $L = 1 + \frac{1}{L}$, and thus $L^2 - L - 1 = 0$. Therefore, $L = \frac{1 \pm \sqrt{5}}{2}$, and because L is clearly positive, it must be equal to $\frac{1+\sqrt{5}}{2} \approx 1.618$.

e. Here $a_0 = a$ and $a_{n+1} = a + \frac{b}{a_n}$. Assuming that $\lim\limits_{n\to\infty} a_n = L$ we have $L = a + \frac{b}{L}$, so $L^2 = aL + b$, and thus $L^2 - aL - b = 0$. Therefore, $L = \frac{a \pm \sqrt{a^2+4b}}{2}$, and because $L > 0$ we have $L = \frac{a+\sqrt{a^2+4b}}{2}$.

9.2.100

a. Experimenting with recurrence (2) one sees that for $0 < p \le 1$ the sequence converges to 1, while for $p > 1$ the sequence diverges to ∞.

b. With recurrence (1), in addition to converging for $p < 1$ it also converges for values of p less than approximately 1.445. Here is a table of approximate values for different values of p:

p	1.1	1.2	1.3	1.4	1.44	1.444
$\lim\limits_{n\to\infty} a_n$	1.111	1.258	1.471	1.887	2.394	2.586

9.2.101

a. $f_0 = f_1 = 1, f_2 = 2, f_3 = 3, f_4 = 5, f_5 = 8, f_6 = 13, f_7 = 21, f_8 = 34, f_9 = 55, f_{10} = 89.$

b. The sequence is clearly not bounded.

c. $\frac{f_{10}}{f_9} \approx 1.61818$

d. We use induction. Note that $\frac{1}{\sqrt{5}}\left(\varphi + \frac{1}{\varphi}\right) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} + \frac{2}{1+\sqrt{5}}\right) = \frac{1}{\sqrt{5}}\left(\frac{1+2\sqrt{5}+5+4}{2(1+\sqrt{5})}\right) = 1 = f_1$. Also note that $\frac{1}{\sqrt{5}}\left(\varphi^2 - \frac{1}{\varphi^2}\right) = \frac{1}{\sqrt{5}}\left(\frac{3+\sqrt{5}}{2} - \frac{2}{3+\sqrt{5}}\right) = \frac{1}{\sqrt{5}}\left(\frac{9+6\sqrt{5}+5-4}{2(3+\sqrt{5})}\right) = 1 = f_2$. Now note that

$$\begin{aligned} f_{n-1} + f_{n-2} &= \frac{1}{\sqrt{5}}(\varphi^{n-1} - (-1)^{n-1}\varphi^{1-n} + \varphi^{n-2} - (-1)^{n-2}\varphi^{2-n}) \\ &= \frac{1}{\sqrt{5}}((\varphi^{n-1} + \varphi^{n-2}) - (-1)^n(\varphi^{2-n} - \varphi^{1-n})). \end{aligned}$$

Now, note that $\varphi - 1 = \frac{1}{\varphi}$, so that

$$\varphi^{n-1} + \varphi^{n-2} = \varphi^{n-1}\left(1 + \frac{1}{\varphi}\right) = \varphi^{n-1}(\varphi) = \varphi^n$$

and

$$\varphi^{2-n} - \varphi^{1-n} = \varphi^{-n}(\varphi^2 - \varphi) = \varphi^{-n}(\varphi(\varphi - 1)) = \varphi^{-n}$$

Making these substitutions, we get

$$f_{n-1} + f_{n-2} = \frac{1}{\sqrt{5}}(\varphi^n - (-1)^n\varphi^{-n}) = f_n$$

9.2.102

a. We show that the arithmetic mean of any two positive numbers exceeds their geometric mean. Let a, $b > 0$; then $\frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}(a - 2\sqrt{ab} + b) = \frac{1}{2}(\sqrt{a} - \sqrt{b})^2 > 0$. Because in addition $a_0 > b_0$, we have $a_n > b_n$ for all n.

b. To see that $\{a_n\}$ is decreasing, note that

$$a_{n+1} = \frac{a_n + b_n}{2} < \frac{a_n + a_n}{2} = a_n.$$

Similarly,

$$b_{n+1} = \sqrt{a_n b_n} > \sqrt{b_n b_n} = b_n,$$

so that $\{b_n\}$ is increasing.

c. $\{a_n\}$ is monotone and nonincreasing by part (b), and bounded below by part (a) (it is bounded below by any of the b_n), so it converges by the monotone convergence theorem. Similarly, $\{b_n\}$ is monotone and nondecreasing by part (b) and bounded above by part (a), so it too converges.

d.

$$a_{n+1} - b_{n+1} = \frac{a_n + b_n}{2} - \sqrt{a_n b_n} = \frac{1}{2}(a_n - 2\sqrt{a_n b_n} + b_n) < \frac{1}{2}(a_n + b_n),$$

because $\sqrt{a_n b_n} \geq 0$. Thus the difference between a_n and b_n gets arbitrarily small, so the difference between their limits is arbitrarily small, so is zero. Thus $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$.

e. The AGM of 12 and 20 is approximately 15.745; Gauss' constant is $\frac{1}{\text{AGM}(1,\sqrt{2})} \approx 0.8346$.

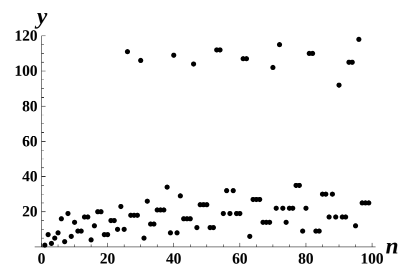
9.2.103

a.

$$\begin{array}{rl} 2: & 1 \\ 3: & 10,\ 5,\ 16,\ 8,\ 4,\ 2,\ 1 \\ 4: & 2,\ 1 \\ 5: & 16,\ 8,\ 4,\ 2,\ 1 \\ 6: & 3,\ 10,\ 5,\ 16,\ 8,\ 4,\ 2,\ 1 \\ 7: & 22,\ 11,\ 34,\ 17,\ 52,\ 26,\ 13,\ 40,\ 20,\ 10,\ 5,\ 16,\ 8,\ 4,\ 2,\ 1 \\ 8: & 4,\ 2,\ 1 \\ 9: & 28,\ 14,\ 7,\ 22,\ 11,\ 34,\ 17,\ 52,\ 26,\ 13,\ 40,\ 20,\ 10,\ 5,\ 16,\ 8,\ 4,\ 2,\ 1 \\ 10: & 5,\ 16,\ 8,\ 4,\ 2,\ 1 \end{array}$$

b. From the above, $H_2 = 1, H_3 = 7$, and $H_4 = 2$.

c. This plot is for $1 \le n \le 100$. Like hailstones, the numbers in the sequence a_n rise and fall but eventually crash to the earth. The conjecture appears to be true.



9.2.104 $\{a_n\} \ll \{b_n\}$ means that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. But $\lim_{n\to\infty} \frac{ca_n}{db_n} = \frac{c}{d} \lim_{n\to\infty} \frac{a_n}{b_n} = 0$, so that $\{ca_n\} \ll \{db_n\}$.

9.3 Infinite Series

9.3.1 A geometric series is a series in which the ratio of successive terms in the underlying sequence is a constant. Thus a geometric series has the form $\sum ar^k$ where r is the constant. One example is $3+6+12+24+48+\cdots$ in which $a = 3$ and $r = 2$.

9.3.2 A geometric sum is the sum of a finite number of terms which have a constant ratio; a geometric series is the sum of an infinite number of such terms.

9.3.3 The ratio is the common ratio between successive terms in the sum.

9.3.4 Yes, because there are only a finite number of terms.

9.3.5 No. For example, the geometric series with $a_n = 3 \cdot 2^n$ does not have a finite sum.

9.3.6 The series converges if and only if $|r| < 1$.

9.3.7 $S = 1 \cdot \dfrac{1-3^9}{1-3} = \dfrac{19682}{2} = 9841.$

9.3.8 $S = 1 \cdot \dfrac{1-(1/4)^{11}}{1-(1/4)} = \dfrac{4^{11}-1}{3 \cdot 4^{10}} = \dfrac{4194303}{3 \cdot 1048576} = \dfrac{1398101}{1048576} \approx 1.333.$

9.3.9 $S = 1 \cdot \dfrac{1-(4/25)^{21}}{1-4/25} = \dfrac{25^{21}-4^{21}}{25^{21}-4 \cdot 25^{20}} \approx 1.1905.$

9.3.10 $S = 16 \cdot \frac{1-2^9}{1-2} = 511 \cdot 16 = 8176.$

9.3.11 $S = 1 \cdot \frac{1-(-3/4)^{10}}{1+3/4} = \frac{4^{10}-3^{10}}{4^{10}+3\cdot 4^9} = \frac{141361}{262144} \approx 0.5392.$

9.3.12 $S = (-2.5) \cdot \frac{1-(-2.5)^5}{1+2.5} = -70.46875.$

9.3.13 $S = 1 \cdot \frac{1-\pi^7}{1-\pi} = \frac{\pi^7-1}{\pi-1} \approx 1409.84.$

9.3.14 $S = \frac{4}{7} \cdot \frac{1-(4/7)^{10}}{3/7} = \frac{375235564}{282475249} \approx 1.328.$

9.3.15 $S = 1 \cdot \frac{1-(-1)^{21}}{2} = 1.$

9.3.16 $\frac{65}{27}.$

9.3.17 $\frac{1093}{2916}.$

9.3.18 $\frac{1}{5}\left(\frac{1-(3/5)^6}{1-3/5}\right) = \frac{7448}{15625}.$

9.3.19 $\frac{1}{1-1/4} = \frac{4}{3}.$

9.3.20 $\frac{1}{1-3/5} = \frac{5}{2}.$

9.3.21 $\frac{1}{1-0.9} = 10.$

9.3.22 $\frac{1}{1-2/7} = \frac{7}{5}.$

9.3.23 Divergent, because $r > 1$.

9.3.24 $\frac{1}{1-1/\pi} = \frac{\pi}{\pi-1}.$

9.3.25 $\frac{e^{-2}}{1-e^{-2}} = \frac{1}{e^2-1}.$

9.3.26 $\frac{5/4}{1-1/2} = \frac{5}{2}.$

9.3.27 $\frac{2^{-3}}{1-2^{-3}} = \frac{1}{7}.$

9.3.28 $\frac{3\cdot 4^3/7^3}{1-4/7} = \frac{64}{49}.$

9.3.29 $\frac{1/625}{1-1/5} = \frac{1}{500}.$

9.3.30 Note that this is the same as $\sum_{i=0}^{\infty}\left(\frac{3}{4}\right)^k$. Then $S = \frac{1}{1-3/4} = 4.$

9.3.31 $\frac{1}{1-e/\pi} = \frac{\pi}{\pi-e}$. (Note that $e < \pi$, so $r < 1$ for this series.)

9.3.32 $\frac{1/16}{1-3/4} = \frac{1}{4}.$

9.3.33 $\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k 5^{6-k} = 5^6\sum_{k=0}^{\infty}\left(\frac{1}{20}\right)^k = 5^6 \cdot \frac{1}{1-1/20} = \frac{5^6\cdot 20}{19} = \frac{312500}{19}.$

9.3.34 $\frac{3^6/8^6}{1-(3/8)^3} = \frac{729}{248320}$

9.3.35 $\frac{1}{1+9/10} = \frac{10}{19}.$

9.3.36 $\frac{-2/3}{1+2/3} = \frac{-2}{5}.$

9.3.37 $3 \cdot \frac{1}{1+1/\pi} = \frac{3\pi}{\pi+1}.$

9.3.38 $\displaystyle\sum_{k=1}^{\infty}\left(\frac{-1}{e}\right)^k = \frac{-1/e}{1+1/e} = \frac{-1}{e+1}$.

9.3.39 $\dfrac{0.15^2}{1.15} = \dfrac{9}{460} \approx 0.0196$.

9.3.40 $\dfrac{-3/8^3}{1+1/8^3} = \dfrac{-1}{171}$.

9.3.41

a. $0.\overline{3} = 0.333\ldots = \sum_{k=1}^{\infty} 3(0.1)^k$.

b. The limit of the sequence of partial sums is $1/3$.

9.3.42

a. $0.\overline{6} = 0.666\ldots = \sum_{k=1}^{\infty} 6(0.1)^k$.

b. The limit of the sequence of partial sums is $2/3$.

9.3.43

a. $0.\overline{1} = 0.111\ldots = \sum_{k=1}^{\infty}(0.1)^k$.

b. The limit of the sequence of partial sums is $1/9$.

9.3.44

a. $0.\overline{5} = 0.555\ldots = \sum_{k=1}^{\infty} 5(0.1)^k$.

b. The limit of the sequence of partial sums is $5/9$.

9.3.45

a. $0.\overline{09} = 0.0909\ldots = \sum_{k=1}^{\infty} 9(0.01)^k$.

b. The limit of the sequence of partial sums is $1/11$.

9.3.46

a. $0.\overline{27} = 0.272727\ldots = \sum_{k=1}^{\infty} 27(0.01)^k$.

b. The limit of the sequence of partial sums is $3/11$.

9.3.47

a. $0.\overline{037} = 0.037037037\ldots = \sum_{k=1}^{\infty} 37(0.001)^k$.

b. The limit of the sequence of partial sums is $37/999 = 1/27$.

9.3.48

a. $0.\overline{027} = 0.027027027\ldots = \sum_{k=1}^{\infty} 27(0.001)^k$

b. The limit of the sequence of partial sums is $27/999 = 1/37$.

9.3.49 $0.\overline{12} = 0.121212\ldots = \displaystyle\sum_{k=0}^{\infty} .12\cdot 10^{-2k} = \frac{.12}{1-1/100} = \frac{12}{99} = \frac{4}{33}$.

9.3.50 $1.\overline{25} = 1.252525\ldots = 1 + \displaystyle\sum_{k=0}^{\infty} .25\cdot 10^{-2k} = 1 + \frac{.25}{1-1/100} = 1 + \frac{25}{99} = \frac{124}{99}$.

9.3.51 $0.\overline{456} = 0.456456456\ldots = \displaystyle\sum_{k=0}^{\infty} .456\cdot 10^{-3k} = \frac{.456}{1-1/1000} = \frac{456}{999} = \frac{152}{333}$.

9.3.52 $1.0\overline{39} = 1.00393939\ldots = 1+\displaystyle\sum_{k=0}^{\infty} .0039\cdot 10^{-2k} = 1+\frac{.0039}{1-1/100} = 1+\frac{.39}{99} = 1+\frac{39}{9900} = \frac{9939}{9900} = \frac{3313}{3300}$.

9.3.53 $0.00\overline{952} = 0.00952952\ldots = \displaystyle\sum_{k=0}^{\infty} .00952\cdot 10^{-3k} = \frac{.00952}{1-1/1000} = \frac{9.52}{999} = \frac{952}{99900} = \frac{238}{24975}$.

9.3.54 $5.12\overline{83} = 5.12838383\ldots = 5.12 + \displaystyle\sum_{k=0}^{\infty} .0083\cdot 10^{-2k} = 5.12 + \frac{.0083}{1-1/100} = \frac{512}{100} + \frac{.83}{99} = \frac{128}{25} + \frac{83}{9900} = \frac{50771}{9900}$.

9.3.55 The second part of each term cancels with the first part of the succeeding term, so $S_n = \frac{1}{1+1} - \frac{1}{n+2} = \frac{n}{2n+4}$, and $\lim\limits_{n\to\infty} \frac{n}{2n+4} = \frac{1}{2}$.

9.3.56 The second part of each term cancels with the first part of the succeeding term, so $S_n = \frac{1}{1+2} - \frac{1}{n+3} = \frac{n}{3n+6}$, and $\lim_{n\to\infty} \frac{n}{3n+9} = \frac{1}{3}$.

9.3.57 $\dfrac{1}{(k+6)(k+7)} = \dfrac{1}{k+6} - \dfrac{1}{k+7}$, so the series given is the same as $\sum_{k=1}^{\infty} \left(\frac{1}{k+6} - \frac{1}{k+7}\right)$. In that series, the second part of each term cancels with the first part of the succeeding term, so $S_n = \frac{1}{1+6} - \frac{1}{n+7}$. Thus $\lim_{n\to\infty} S_n = \frac{1}{7}$.

9.3.58 $\dfrac{1}{(3k+1)(3k+4)} = \dfrac{1}{3}\left(\dfrac{1}{3k+1} - \dfrac{1}{3k+4}\right)$, so the series given can be written $\dfrac{1}{3}\displaystyle\sum_{k=0}^{\infty}\left(\dfrac{1}{3k+1} - \dfrac{1}{3k+4}\right)$. In that series, the second part of each term cancels with the first part of the succeeding term (because $3(k+1)+1 = 3k+4$), so we are left with $S_n = \frac{1}{3}\left(\frac{1}{1} - \frac{1}{3n+4}\right) = \frac{n+1}{3n+4}$ and $\lim_{n\to\infty} \frac{n+1}{3n+4} = \frac{1}{3}$.

9.3.59 Note that $\frac{4}{(4k-3)(4k+1)} = \frac{1}{4k-3} - \frac{1}{4k+1}$. Thus the given series is the same as $\displaystyle\sum_{k=3}^{\infty}\left(\dfrac{1}{4k-3} - \dfrac{1}{4k+1}\right)$. In that series, the second part of each term cancels with the first part of the succeeding term (because $4(k+1)-3 = 4k+1$), so we have $S_n = \frac{1}{9} - \frac{1}{4n+9}$, and thus $\lim_{n\to\infty} S_n = \dfrac{1}{9}$.

9.3.60 Note that $\frac{2}{(2k-1)(2k+1)} = \frac{1}{2k-1} - \frac{1}{2k+1}$. Thus the given series is the same as $\displaystyle\sum_{k=3}^{\infty}\left(\dfrac{1}{2k-1} - \dfrac{1}{2k+1}\right)$. In that series, the second part of each term cancels with the first part of the succeeding term (because $2(k+1)-1 = 2k+1$), so we have $S_n = \frac{1}{5} - \frac{1}{2n+1}$. Thus, $\lim_{n\to\infty} S_n = \dfrac{1}{5}$.

9.3.61 $\ln\left(\dfrac{k+1}{k}\right) = \ln(k+1) - \ln k$, so the series given is the same as $\sum_{k=1}^{\infty}(\ln(k+1) - \ln k)$, in which the first part of each term cancels with the second part of the next term, so we get $S_{n-1} = \ln n - \ln 1 = \ln n$, and thus the series diverges.

9.3.62 Note that $S_n = (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + \cdots + (\sqrt{n+1} - \sqrt{n})$. The second part of each term cancels with the first part of the previous term. Thus, $S_n = \sqrt{n+1} - 1$. and because $\lim_{n\to\infty} \sqrt{n+1} - 1 = \infty$, the series diverges.

9.3.63 $\dfrac{1}{(k+p)(k+p+1)} = \dfrac{1}{k+p} - \dfrac{1}{k+p+1}$, so that $\displaystyle\sum_{k=1}^{\infty} \dfrac{1}{(k+p)(k+p+1)} = \sum_{k=1}^{\infty}\left(\dfrac{1}{k+p} - \dfrac{1}{k+p+1}\right)$ and this series telescopes to give $S_n = \frac{1}{p+1} - \frac{1}{n+p+1} = \frac{n}{n(p+1)+(p+1)^2}$ so that $\lim_{n\to\infty} S_n = \frac{1}{p+1}$.

9.3.64 $\dfrac{1}{(ak+1)(ak+a+1)} = \dfrac{1}{a}\left(\dfrac{1}{ak+1} - \dfrac{1}{ak+a+1}\right)$, so that $\displaystyle\sum_{k=1}^{\infty} \dfrac{1}{(ak+1)(ak+a+1)} = \dfrac{1}{a}\sum_{k=1}^{\infty}\left(\dfrac{1}{ak+1} - \dfrac{1}{ak+a+1}\right)$. This series telescopes - the second term of each summand cancels with the first term of the succeeding summage – so that $S_n = \frac{1}{a}\left(\frac{1}{a+1} - \frac{1}{an+a+1}\right)$, and thus the limit of the sequence is $\frac{1}{a(a+1)}$.

9.3.65 Let $a_n = \dfrac{1}{\sqrt{n+1}} - \dfrac{1}{\sqrt{n+3}}$. Then the second term of a_n cancels with the first term of a_{n+2}, so the series telescopes and $S_n = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{n-1+3}} - \frac{1}{\sqrt{n+3}}$ and thus the sum of the series is the limit of S_n, which is $\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}}$.

9.3.66 The first term of the k^{th} summand is $\sin(\frac{(k+1)\pi}{2k+1})$; the second term of the $(k+1)^{\text{st}}$ summand is $-\sin(\frac{(k+1)\pi}{2(k+1)-1})$; these two are equal except for sign, so they cancel. Thus $S_n = -\sin 0 + \sin(\frac{(n+1)\pi}{2n+1}) = \sin(\frac{(n+1)\pi}{2n+1})$. Because $\frac{(n+1)\pi}{2n+1}$ has limit $\pi/2$ as $n \to \infty$, and because the sine function is continuous, it follows that $\lim_{n\to\infty} S_n$ is $\sin(\frac{\pi}{2}) = 1$.

9.3.67 $16k^2 + 8k - 3 = (4k+3)(4k-1)$, so $\frac{1}{16k^2+8k-3} = \frac{1}{(4k+3)(4k-1)} = \frac{1}{4}\left(\frac{1}{4k-1} - \frac{1}{4k+3}\right)$. Thus the series given is equal to $\displaystyle\frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{1}{4k-1} - \frac{1}{4k+3}\right)$. This series telescopes, so $S_n = \frac{1}{4}\left(-1 - \frac{1}{4n+3}\right)$,so the sum of the series is equal to $\lim_{n\to\infty} S_n = -\frac{1}{4}$.

9.3.68 This series clearly telescopes to give $S_n = -\tan^{-1}(1) + \tan^{-1}(n) = \tan^{-1}(n) - \frac{\pi}{4}$. Then because $\lim_{n\to\infty} \tan^{-1}(n) = \frac{\pi}{2}$, the sum of the series is equal to $\lim_{n\to\infty} S_n = \frac{\pi}{4}$.

9.3.69

a. True. $\left(\frac{\pi}{e}\right)^{-k} = \left(\frac{e}{\pi}\right)^k$; because $e < \pi$, this is a geometric series with ratio less than 1.

b. True. If $\displaystyle\sum_{k=12}^{\infty} a^k = L$, then $\displaystyle\sum_{k=0}^{\infty} a^k = \left(\sum_{k=0}^{11} a^k\right) + L$.

c. False. For example, let $0 < a < 1$ and $b > 1$.

9.3.70 We have $S_n = (\sin^{-1}(1) - \sin^{-1}(1/2)) + (\sin^{-1}(1/2) - \sin^{-1}(1/3)) + \cdots + (\sin^{-1}(1/n) - \sin^{-1}(1/(n+1)))$. Note that the first part of each term cancels the second part of the previous term, so the nth partial sum telescopes to be $\sin^{-1}(1) - \sin^{-1}(1/(n+1))$. Because $\sin^{-1}(1) = \pi/2$ and $\lim_{n\to\infty} \sin^{-1}(1/(n+1)) = \sin^{-1}(0) = 0$, we have $\lim_{n\to\infty} S_n = \dfrac{\pi}{2}$.

9.3.71 This can be written as $\displaystyle\frac{1}{3}\sum_{k=1}^{\infty}\left(\frac{-2}{3}\right)^k$. This is a geometric series with ratio $r = \frac{-2}{3}$ so the sum is $\frac{1}{3} \cdot \frac{-2/3}{1-(-2/3)} = \frac{1}{3} \cdot \frac{-2}{5} = \frac{-2}{15}$.

9.3.72 This can be written as $\displaystyle\frac{1}{e}\sum_{k=1}^{\infty}\left(\frac{\pi}{e}\right)^k$. This is a geometric series with $r = \frac{\pi}{e} > 1$, so the series diverges.

9.3.73 Note that $\frac{\ln((k+1)k^{-1})}{(\ln k)(\ln(k+1))} = \frac{\ln(k+1)}{(\ln k)(\ln(k+1))} - \frac{\ln k}{(\ln k)(\ln(k+1))} = \frac{1}{\ln k} - \frac{1}{\ln(k+1)}$. In the partial sum S_n, the first part of each term cancels the second part of the preceding term, so we have $S_n = \frac{1}{\ln 2} - \frac{1}{\ln(n+2)}$. Thus we have $\lim_{n\to\infty} S_n = \dfrac{1}{\ln 2}$.

9.3.74

a. Because the first part of each term cancels the second part of the previous term, the nth partial sum telescopes to be $S_n = \frac{1}{2} - \frac{1}{2^{n+1}}$. Thus, the sum of the series is $\lim_{n\to\infty} S_n = \dfrac{1}{2}$.

b. Note that $\frac{1}{2^k} - \frac{1}{2^{k+1}} = \frac{2^{k+1}-2^k}{2^k 2^{k+1}} = \frac{1}{2^{k+1}}$. Thus, the original series can be written as $\displaystyle\sum_{k=1}^{\infty}\frac{1}{2^{k+1}}$ which is geometric with $r = 1/2$ and $a = 1/4$, so the sum is $\frac{1/4}{1-1/2} = \frac{1}{2}$.

9.3.75

a. Because the first part of each term cancels the second part of the previous term, the nth partial sum telescopes to be $S_n = \frac{4}{3} - \frac{4}{3^{n+1}}$. Thus, the sum of the series is $\lim_{n\to\infty} S_n = \dfrac{4}{3}$.

b. Note that $\frac{4}{3^k} - \frac{4}{3^{k+1}} = \frac{4\cdot 3^{k+1} - 4\cdot 3^k}{3^k 3^{k+1}} = \frac{8}{3^{k+1}}$. Thus, the original series can be written as $\displaystyle\sum_{k=1}^{\infty} \frac{8}{3^{k+1}}$ which is geometric with $r = 1/3$ and $a = 8/9$, so the sum is $\frac{8/9}{1-1/3} = \frac{8}{9}\cdot\frac{3}{2} = \frac{4}{3}$.

9.3.76 It will take Achilles 1/5 hour to cover the first mile. At this time, the tortoise has gone 1/5 mile more, and it will take Achilles 1/25 hour to reach this new point. At that time, the tortoise has gone another 1/25 of a mile, and it will take Achilles 1/125 hour to reach this point. Adding the times up, we have

$$\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots = \frac{1/5}{1-1/5} = \frac{1}{4},$$

so it will take Achilles 1/4 of an hour (15 minutes) to catch the tortoise.

9.3.77 At the n^{th} stage, there are 2^{n-1} triangles of area $A_n = \frac{1}{8}A_{n-1} = \frac{1}{8^{n-1}}A_1$, so the total area of the triangles formed at the n^{th} stage is $\dfrac{2^{n-1}}{8^{n-1}}A_1 = \left(\dfrac{1}{4}\right)^{n-1} A_1$. Thus the total area under the parabola is

$$\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^{n-1} A_1 = A_1 \sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^{n-1} = A_1\frac{1}{1-1/4} = \frac{4}{3}A_1.$$

9.3.78

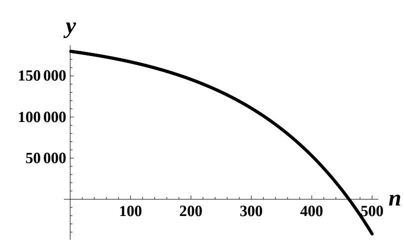
a. Note that $\frac{3^k}{(3^{k+1}-1)(3^k-1)} = \frac{1}{2}\cdot\left(\frac{1}{3^k-1} - \frac{1}{3^{k+1}-1}\right)$. Then

$$\sum_{k=1}^{\infty}\frac{3^k}{(3^{k+1}-1)(3^k-1)} = \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{3^k-1} - \frac{1}{3^{k+1}-1}\right).$$

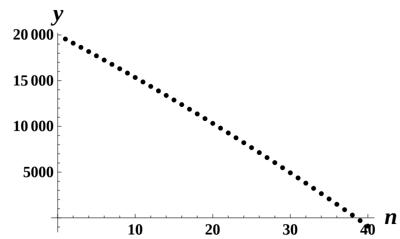
This series telescopes to give $S_n = \frac{1}{2}\left(\frac{1}{3-1} - \frac{1}{3^{n+1}-1}\right)$, so that the sum of the series is $\lim_{n\to\infty} S_n = \frac{1}{4}$.

b. We mimic the above computations. First, $\frac{a^k}{(a^{k+1}-1)(a^k-1)} = \frac{1}{a-1}\cdot\left(\frac{1}{a^k-1} - \frac{1}{a^{k+1}-1}\right)$, so we see that we cannot have $a = 1$, because the fraction would then be undefined. Continuing, we obtain $S_n = \frac{1}{a-1}\left(\frac{1}{a-1} - \frac{1}{a^{n+1}-1}\right)$. Now, $\lim_{n\to\infty}\frac{1}{a^{n+1}-1}$ converges if and only if the denominator grows without bound; this happens if and only if $|a| > 1$. Thus, the original series converges for $|a| > 1$, when it converges to $\frac{1}{(a-1)^2}$. Note that this is valid even for a negative.

9.3.79 It appears that the loan is paid off after about 470 months. Let B_n be the loan balance after n months. Then $B_0 = 180000$ and $B_n = 1.005\cdot B_{n-1} - 1000$. Then $B_n = 1.005\cdot B_{n-1} - 1000 = 1.005(1.005\cdot B_{n-2} - 1000) - 1000 = (1.005)^2\cdot B_{n-2} - 1000(1 + 1.005) = (1.005)^2\cdot(1.005\cdot B_{n-3} - 1000) - 1000(1+1.005) = (1.005)^3\cdot B_{n-3} - 1000(1 + 1.005 + (1.005)^2) = \cdots = (1.005)^n B_0 - 1000(1+1.005+(1.005)^2+\cdots+(1.005)^{n-1}) = (1.005)^n\cdot 180000 - 1000\left(\frac{(1.005)^n-1}{1.005-1}\right)$. Solving this equation for $B_n = 0$ gives $n \approx 461.66$ months, so the loan is paid off after 462 months.



9.3.80 It appears that the loan is paid off after about 38 months. Let B_n be the loan balance after n months. Then $B_0 = 20000$ and $B_n = 1.0075 \cdot B_{n-1} - 60$. Then $B_n = 1.0075 \cdot B_{n-1} - 600 = 1.0075(1.0075 \cdot B_{n-2} - 600) - 600 = (1.0075)^2 \cdot B_{n-2} - 600(1 + 1.0075) = (1.0075)^2(1.0075 \cdot B_{n-3} - 600) - 600(1 + 1.0075) = (1.0075)^3 \cdot B_{n-3} - 600(1 + 1.0075 + (1.0075)^2) = \cdots = (1.0075)^n B_0 - 600(1 + 1.0075 + (1.0075)^2 + \cdots + (1.0075)^{n-1}) = (1.0075)^n \cdot 20000 - 600\left(\frac{(1.0075)^n - 1}{1.0075 - 1}\right)$. Solving this equation for $B_n = 0$ gives $n \approx 38.5$ months, so the loan is paid off after 39 months.



9.3.81 $F_n = (1.015)F_{n-1} - 120 = (1.015)((1.015)F_{n-2} - 120) - 120 = (1.015)((1.015)((1.015)F_{n-3} - 120) - 120) - 120 = \cdots = (1.015)^n(4000) - 120(1 + (1.015) + (1.015)^2 + \cdots + (1.015)^{n-1})$. This is equal to

$$(1.015)^n(4000) - 120\left(\frac{(1.015)^n - 1}{1.015 - 1}\right) = (-4000)(1.015)^n + 8000.$$

The long term population of the fish is 0.

9.3.82 Let A_n be the amount of antibiotic in your blood after n 6-hour periods. Then $A_0 = 200, A_n = 0.5A_{n-1} + 200$. We have $A_n = .5A_{n-1} + 200 = .5(.5A_{n-2} + 200) + 200 = .5(.5(.5A_{n-3} + 200) + 200) + 200 = \cdots = .5^n(200) + 200(1 + .5 + .5^2 + \cdots + .5^{n-1})$. This is equal to

$$.5^n(200) + 200\left(\frac{.5^n - 1}{.5 - 1}\right) = (.5^n)(200 - 400) + 400 = (-200)(.5^n) + 400.$$

The limit of this expression as $n \to \infty$ is 400, so the steady-state amount of antibiotic in your blood is 400 mg.

9.3.83 Under the one-child policy, each couple will have one child. Under the one-son policy, we compute the expected number of children as follows: with probability $1/2$ the first child will be a son; with probability $(1/2)^2$, the first child will be a daughter and the second child will be a son; in general, with probability $(1/2)^n$, the first $n - 1$ children will be girls and the n^{th} a boy. Thus the expected number of children is the sum $\displaystyle\sum_{i=1}^{\infty} i \cdot \left(\frac{1}{2}\right)^i$. To evaluate this series, use the following "trick": Let $f(x) = \displaystyle\sum_{i=1}^{\infty} ix^i$. Then $f(x) + \displaystyle\sum_{i=1}^{\infty} x^i = \sum_{i=1}^{\infty}(i+1)x^i$. Now, let

$$g(x) = \sum_{i=1}^{\infty} x^{i+1} = -1 - x + \sum_{i=0}^{\infty} x^i = -1 - x + \frac{1}{1-x}$$

and

$$g'(x) = f(x) + \sum_{i=1}^{\infty} x^i = f(x) - 1 + \sum_{i=0}^{\infty} x^i = f(x) - 1 + \frac{1}{1-x}.$$

Evaluate $g'(x) = -1 - \frac{1}{(1-x)^2}$; then

$$f(x) = 1 - \frac{1}{1-x} - 1 - \frac{1}{(1-x)^2} = \frac{-1 + x + 1}{(1-x)^2} = \frac{x}{(1-x)^2}$$

Finally, evaluate at $x = \frac{1}{2}$ to get $f\left(\frac{1}{2}\right) = \sum_{i=1}^{\infty} i \cdot \left(\frac{1}{2}\right)^i = \frac{1/2}{(1-1/2)^2} = 2$. There will thus be twice as many children under the one-son policy as under the one-child policy.

9.3.84 Let L_n be the amount of light transmitted through the window the n^{th} time the beam hits the second pane. Then the amount of light that was available before the beam went through the pane was $\frac{L_n}{1-p}$, so $\frac{pL_n}{1-p}$ is reflected back to the first pane, and $\frac{p^2 L_n}{1-p}$ is then reflected back to the second pane. Of that, a fraction equal to $1-p$ is transmitted through the window. Thus

$$L_{n+1} = (1-p)\frac{p^2 L_n}{1-p} = p^2 L_n.$$

The amount of light transmitted through the window the first time is $(1-p)^2$. Thus the total amount is

$$\sum_{i=0}^{\infty} p^{2n}(1-p)^2 = \frac{(1-p)^2}{1-p^2} = \frac{1-p}{1+p}.$$

9.3.85 Ignoring the initial drop for the moment, the height after the n^{th} bounce is $10p^n$, so the total time spent in that bounce is $2 \cdot \sqrt{2 \cdot 10p^n/g}$ seconds. The total time before the ball comes to rest (now including the time for the initial drop) is then $\sqrt{20/g} + \sum_{i=1}^{\infty} 2 \cdot \sqrt{2 \cdot 10p^n/g} = \sqrt{\frac{20}{g}} + 2\sqrt{\frac{20}{g}} \sum_{i=1}^{\infty} (\sqrt{p})^n = \sqrt{\frac{20}{g}} + 2\sqrt{\frac{20}{g}}\frac{\sqrt{p}}{1-\sqrt{p}} = \sqrt{\frac{20}{g}}\left(1 + \frac{2\sqrt{p}}{1-\sqrt{p}}\right) = \sqrt{\frac{20}{g}}\left(\frac{1+\sqrt{p}}{1-\sqrt{p}}\right)$ seconds.

9.3.86

a. The fraction of available wealth spent each month is $1-p$, so the amount spent in the n^{th} month is $W(1-p)^n$ (so that all \$$W$ is spent during the first month). The total amount spent is then $\sum_{n=1}^{\infty} W(1-p)^n = \frac{W(1-p)}{1-(1-p)} = W\left(\frac{1-p}{p}\right)$ dollars.

b. As $p \to 1$, the total amount spent approaches 0. This makes sense, because in the limit, if everyone saves all of the money, none will be spent. As $p \to 0$, the total amount spent gets larger and larger. This also makes sense, because almost all of the available money is being respent each month.

9.3.87

a. I_{n+1} is obtained by I_n by dividing each edge into three equal parts, removing the middle part, and adding two parts equal to it. Thus 3 equal parts turn into 4, so $L_{n+1} = \frac{4}{3}L_n$. This is a geometric sequence with a ratio greater than 1, so the n^{th} term grows without bound.

b. As the result of part (a), I_n has $3 \cdot 4^n$ sides of length $\frac{1}{3^n}$; each of those sides turns into an added triangle in I_{n+1} of side length 3^{-n-1}. Thus the added area in I_{n+1} consists of $3 \cdot 4^n$ equilateral triangles with side 3^{-n-1}. The area of an equilateral triangle with side x is $\dfrac{x^2\sqrt{3}}{4}$. Thus $A_{n+1} = A_n + 3 \cdot 4^n \cdot \frac{3^{-2n-2}\sqrt{3}}{4} = A_n + \frac{\sqrt{3}}{12} \cdot \left(\frac{4}{9}\right)^n$, and $A_0 = \frac{\sqrt{3}}{4}$. Thus $A_{n+1} = A_0 + \sum_{i=0}^{n} \frac{\sqrt{3}}{12} \cdot \left(\frac{4}{9}\right)^i$, so that

$$A_\infty = A_0 + \frac{\sqrt{3}}{12}\sum_{i=0}^{\infty}\left(\frac{4}{9}\right)^i = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}\frac{1}{1-4/9} = \frac{\sqrt{3}}{4}(1+\frac{3}{5}) = \frac{2}{5}\sqrt{3}$$

9.3.88

a. $5\displaystyle\sum_{i=1}^{\infty} 10^{-k} = 5\sum_{i=1}^{\infty}\left(\frac{1}{10}\right)^k = 5\left(\frac{1/10}{9/10}\right) = \frac{5}{9}$

b. $54\displaystyle\sum_{i=1}^{\infty} 10^{-2k} = 54\sum_{i=1}^{\infty}\left(\frac{1}{100}\right)^k = 54\left(\frac{1/100}{99/100}\right) = \frac{54}{99}$

c. Suppose $x = 0.n_1n_2\ldots n_pn_1n_2\ldots$. Then we can write this decimal as $n_1n_2\ldots n_p\sum_{i=1}^{\infty}10^{-ip} = n_1n_2\ldots n_p\sum_{i=1}^{\infty}\left(\frac{1}{10^p}\right)^i = n_1n_2\ldots n_p\frac{1/10^p}{(10^p-1)/10^p} = \frac{n_1n_2\ldots n_p}{999\ldots9}$, where here $n_1n_2\ldots n_p$ does not mean multiplication but rather the digits in a decimal number, and where there are p 9's in the denominator.

d. According to part (c), $0.12345678912345678912\ldots = \frac{123456789}{999999999}$

e. Again using part (c), $0.\bar{9} = \frac{9}{9} = 1$.

9.3.89 $|S - S_n| = \left|\sum_{i=n}^{\infty} r^k\right| = \left|\dfrac{r^n}{1-r}\right|$ because the latter sum is simply a geometric series with first term r^n and ratio r.

9.3.90

a. Solve $\frac{0.6^n}{0.4} < 10^{-6}$ for n to get $n = 29$.

b. Solve $\frac{0.15^n}{0.85} < 10^{-6}$ for n to get $n = 8$.

9.3.91

a. Solve $\left|\frac{(-0.8)^n}{1.8}\right| = \frac{0.8^n}{1.8} < 10^{-6}$ for n to get $n = 60$.

b. Solve $\frac{0.2^n}{0.8} < 10^{-6}$ for n to get $n = 9$.

9.3.92

a. Solve $\frac{0.72^n}{0.28} < 10^{-6}$ for n to get $n = 46$.

b. Solve $\left|\frac{(-0.25)^n}{1.25}\right| = \frac{0.25^n}{1.25} < 10^{-6}$ for n to get $n = 10$.

9.3.93

a. Solve $\frac{1/\pi^n}{1-1/\pi} < 10^{-6}$ for n to get $n = 13$.

b. Solve $\frac{1/e^n}{1-1/e} < 10^{-6}$ for n to get $n = 15$.

9.3.94

a. $f(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$; because f is represented by a geometric series, $f(x)$ exists only for $|x| < 1$. Then $f(0) = 1$, $f(0.2) = \frac{1}{0.8} = 1.25$, $f(0.5) = \frac{1}{1-0.5} = 2$. Neither $f(1)$ nor $f(1.5)$ exists.

b. The domain of f is $\{x : |x| < 1\}$.

9.3.95

a. $f(x) = \sum_{k=0}^{\infty}(-1)^k x^k = \frac{1}{1+x}$; because f is a geometric series, $f(x)$ exists only when the ratio, $-x$, is such that $|-x| = |x| < 1$. Then $f(0) = 1$, $f(0.2) = \frac{1}{1.2} = \frac{5}{6}$, $f(0.5) = \frac{1}{1+.05} = \frac{2}{3}$. Neither $f(1)$ nor $f(1.5)$ exists.

b. The domain of f is $\{x : |x| < 1\}$.

9.3.96

a. $f(x) = \sum_{k=0}^{\infty} x^{2k} = \frac{1}{1-x^2}$. f is a geometric series, so $f(x)$ is defined only when the ratio, x^2, is less than 1, which means $|x| < 1$. Then $f(0) = 1$, $f(0.2) = \frac{1}{1-.04} = \frac{25}{24}$, $f(0.5) = \frac{1}{1-0.25} = \frac{4}{3}$. Neither $f(1)$ nor $f(1.5)$ exists.

b. The domain of f is $\{x : |x| < 1\}$.

9.3.97 $f(x)$ is a geometric series with ratio $\frac{1}{1+x}$; thus $f(x)$ converges when $\left|\frac{1}{1+x}\right| < 1$. For $x > -1$, $\left|\dfrac{1}{1+x}\right| = \dfrac{1}{1+x}$ and $\dfrac{1}{1+x} < 1$ when $1 < 1 + x$, $x > 0$. For $x < -1$, $\left|\dfrac{1}{1+x}\right| = \dfrac{1}{-1-x}$, and this is less than 1 when $1 < -1 - x$, i.e. $x < -2$. So $f(x)$ converges for $x > 0$ and for $x < -2$. When $f(x)$ converges, its value is $\frac{1}{1-\frac{1}{1+x}} = \frac{1+x}{x}$, so $f(x) = 3$ when $1 + x = 3x$, $x = \frac{1}{2}$.

9.3.98

a. Clearly for $k < n$, h_k is a leg of a right triangle whose hypotenuse is r_k and whose other leg is formed where the vertical line (in the picture) meets a diameter of the next smaller sphere; thus the other leg of the triangle is r_{k+1}. The Pythagorean theorem then implies that $h_k^2 = r_k^2 - r_{k+1}^2$.

b. The height is $H_n = \sum_{i=1}^n h_i = r_n + \sum_{i=1}^{n-1} \sqrt{r_i^2 - r_{i+1}^2}$ by part (a).

c. From part (b), because $r_i = a^{i-1}$,

$$\begin{aligned}
H_n &= r_n + \sum_{i=1}^{n-1} \sqrt{r_i^2 - r_{i+1}^2} = a^{n-1} + \sum_{i=1}^{n-1} \sqrt{a^{2i-2} - a^{2i}} \\
&= a^{n-1} + \sum_{i=1}^{n-1} a^{i-1}\sqrt{1-a^2} = a^{n-1} + \sqrt{1-a^2}\sum_{i=1}^{n-1} a^{i-1} \\
&= a^{n-1} + \sqrt{1-a^2}\left(\frac{1-a^{n-1}}{1-a}\right)
\end{aligned}$$

d. $\lim\limits_{n\to\infty} H_n = \lim\limits_{n\to\infty} a^{n-1} + \sqrt{1-a^2}\lim\limits_{n\to\infty} \frac{1-a^{n-1}}{1-a} = 0 + \sqrt{1-a^2}\left(\frac{1}{1-a}\right) = \sqrt{\frac{1-a^2}{(1-a)(1-a)}} = \sqrt{\frac{1+a}{1-a}}$.

9.4 The Divergence and Integral Tests

9.4.1 A series may diverge so slowly that no reasonable number of terms may definitively show that it does so.

9.4.2 No. For example, the harmonic serkes $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges although $\frac{1}{k} \to 0$ as $k \to \infty$.

9.4.3 Yes. Either the series and the integral both converge, or both diverge, if the terms are positive and decreasing.

9.4.4 It converges for $p > 1$, and diverges for all other values of p.

9.4.5 For the same values of p as in the previous problem – it converges for $p > 1$, and diverges for all other values of p.

9.4.6 Let S_n be the partial sums. Then $S_{n+1} - S_n = a_{n+1} > 0$ because $a_{n+1} > 0$. Thus the sequence of partial sums is increasing.

9.4.7 The remainder of an infinite series is the error in approximating a convergent infinite series by a finite number of terms.

9.4.8 Yes. Suppose $\sum a_k$ converges to S, and let the sequence of partial sums be $\{S_n\}$. Then for any $\epsilon > 0$ there is some N such that for any $n > N$, $|S - S_n| < \epsilon$. But $|S - S_n|$ is simply the remainder R_n when the series is approximated to n terms. Thus $R_n \to 0$ as $n \to \infty$.

9.4.9 $a_k = \frac{k}{2k+1}$ and $\lim\limits_{k\to\infty} a_k = \frac{1}{2}$, so the series diverges.

9.4.10 $a_k = \frac{k}{k^2+1}$ and $\lim\limits_{k\to\infty} a_k = 0$, so the divergence test is inconclusive.

9.4.11 $a_k = \frac{k}{\ln k}$ and $\lim\limits_{k\to\infty} a_k = \infty$, so the series diverges.

9.4.12 $a_k = \frac{k^2}{2^k}$ and $\lim\limits_{k\to\infty} a_k = 0$, so the divergence test is inconclusive.

9.4.13 $a_k = \frac{1}{1000+k}$ and $\lim\limits_{k\to\infty} a_k = 0$, so the divergence test is inconclusive.

9.4.14 $a_k = \frac{k^3}{k^3+1}$ and $\lim\limits_{k\to\infty} a_k = 1$, so the series diverges.

9.4.15 $a_k = \frac{\sqrt{k}}{\ln^{10} k}$ and $\lim\limits_{k\to\infty} a_k = \infty$, so the series diverges.

9.4.16 $a_k = \frac{\sqrt{k^2+1}}{k}$ and $\lim\limits_{k\to\infty} a_k = 1$, so the series diverges.

9.4.17 $a_k = k^{1/k}$. In order to compute $\lim_{k\to\infty} a_k$, we let $y_k = \ln(a_k) = \frac{\ln k}{k}$. By Theorem 9.6, (or by L'Hôpital's rule) $\lim_{k\to\infty} y_k = 0$, so $\lim_{k\to\infty} a_k = e^0 = 1$. The given series thus diverges.

9.4.18 By Theorem 9.6 $k^3 \ll k!$, so $\lim_{k\to\infty} \frac{k^3}{k!} = 0$. The divergence test is inconclusive.

9.4.19 Let $f(x) = \frac{1}{x\ln x}$. Then $f(x)$ is continuous and decreasing on $(1,\infty)$, because $x\ln x$ is increasing there. Because $\int_1^\infty f(x)\,dx = \infty$, the series diverges.

9.4.20 Let $f(x) = \frac{x}{\sqrt{x^2+4}}$. $f(x)$ is continuous for $x \geq 1$. Note that $f'(x) = \frac{4}{(\sqrt{x^2+4})^3} > 0$. Thus f is increasing, and the conditions of the Integral Test aren't satisfied. The given series diverges by the Divergence Test.

9.4.21 Let $f(x) = x\cdot e^{-2x^2}$. This function is continuous for $x \geq 1$. Its derivative is $e^{-2x^2}(1-4x^2) < 0$ for $x \geq 1$, so $f(x)$ is decreasing. Because $\int_1^\infty x\cdot e^{-2x^2}\,dx = \frac{1}{4e^2}$, the series converges.

9.4.22 Let $f(x) = \frac{1}{\sqrt[3]{x+10}}$. $f(x)$ is obviously continuous and decreasing for $x \geq 1$. Because $\int_1^\infty \frac{1}{\sqrt[3]{x+10}}\,dx = \infty$, the series diverges.

9.4.23 Let $f(x) = \frac{1}{\sqrt{x+8}}$. $f(x)$ is obviously continuous and decreasing for $x \geq 1$. Because $\int_1^\infty \frac{1}{\sqrt{x+8}}\,dx = \infty$, the series diverges.

9.4.24 Let $f(x) = \frac{1}{x(\ln x)^2}$. $f(x)$ is continuous and decreasing for $x \geq 2$. Because $\int_2^\infty f(x)\,dx = \frac{1}{\ln 2}$ the series converges.

9.4.25 Let $f(x) = \frac{x}{e^x}$. $f(x)$ is clearly continuous for $x > 1$, and its derivative, $f'(x) = \frac{e^x - xe^x}{e^{2x}} = (1-x)\frac{e^x}{e^{2x}}$, is negative for $x > 1$ so that $f(x)$ is decreasing. Because $\int_1^\infty f(x)\,dx = 2e^{-1}$, the series converges.

9.4.26 Let $f(x) = \frac{1}{x\cdot\ln x\cdot\ln\ln x}$. $f(x)$ is continuous and decreasing for $x > 3$, and $\int_3^\infty \frac{1}{x\cdot\ln x\cdot\ln\ln x}\,dx = \infty$. The given series therefore diverges.

9.4.27 The integral test does not apply, because the sequence of terms is not decreasing.

9.4.28 $f(x) = \frac{x}{(x^2+1)^3}$ is decreasing and continuous, and $\int_1^\infty \frac{x}{(x^2+1)^3}\,dx = \frac{1}{16}$. Thus, the given series converges.

9.4.29 This is a p-series with $p = 10$, so this series converges.

9.4.30 $\sum_{k=2}^\infty \frac{k^e}{k^\pi} = \sum_{k=2}^\infty \frac{1}{k^{\pi-e}}$. Note that $\pi - e \approx 3.1416 - 2.71828 < 1$, so this series diverges.

9.4.31 $\sum_{k=3}^\infty \frac{1}{(k-2)^4} = \sum_{k=1}^\infty \frac{1}{k^4}$, which is a p-series with $p = 4$, thus convergent.

9.4.32 $\sum_{k=1}^\infty 2k^{-3/2} = 2\sum_{k=1}^\infty \frac{1}{k^{3/2}}$ is a p-series with $p = 3/2$, thus convergent.

9.4.33 $\sum_{k=1}^\infty \frac{1}{\sqrt[3]{k}} = \sum_{k=1}^\infty \frac{1}{k^{1/3}}$ is a p-series with $p = 1/3$, thus divergent.

9.4.34 $\sum_{k=1}^\infty \frac{1}{\sqrt[3]{27k^2}} = \frac{1}{3}\sum_{k=1}^\infty \frac{1}{k^{2/3}}$ is a p-series with $p = 2/3$, thus divergent.

9.4.35

a. The remainder R_n is bounded by $\int_n^\infty \frac{1}{x^6}\,dx = \frac{1}{5n^5}$.

b. We solve $\frac{1}{5n^5} < 10^{-3}$ to get $n = 3$.

c. $L_n = S_n + \int_{n+1}^\infty \frac{1}{x^6}\,dx = S_n + \frac{1}{5(n+1)^5}$, and $U_n = S_n + \int_n^\infty \frac{1}{x^6}\,dx = S_n + \frac{1}{5n^5}$.

d. $S_{10} \approx 1.017341512$, so $L_{10} \approx 1.017341512 + \frac{1}{5\cdot 11^5} \approx 1.017342754$, and $U_{10} \approx 1.017341512 + \frac{1}{5\cdot 10^5} \approx 1.017343512$.

9.4.36

a. The remainder R_n is bounded by $\int_n^\infty \frac{1}{x^8}\,dx = \frac{1}{7n^7}$.

b. We solve $\frac{1}{7n^7} < 10^{-3}$ to obtain $n = 3$.

c. $L_n = S_n + \int_{n+1}^\infty \frac{1}{x^8}\,dx = S_n + \frac{1}{7(n+1)^7}$, and $U_n = S_n + \int_n^\infty \frac{1}{x^8}\,dx = S_n + \frac{1}{7n^7}$.

d. $S_{10} \approx 1.004077346$, so $L_{10} \approx 1.004077346 + \frac{1}{7\cdot 11^7} \approx 1.00408$, and $U_{10} \approx 1.004077346 + \frac{1}{7\cdot 10^7} \approx 1.00408$.

9.4.37

a. The remainder R_n is bounded by $\int_n^\infty \frac{1}{3^x}\,dx = \frac{1}{3^n \ln(3)}$.

b. We solve $\frac{1}{3^n \ln(3)} < 10^{-3}$ to obtain $n = 7$.

c. $L_n = S_n + \int_{n+1}^\infty \frac{1}{3^x}\,dx = S_n + \frac{1}{3^{n+1}\ln(3)}$, and $U_n = S_n + \int_n^\infty \frac{1}{3^x}\,dx = S_n + \frac{1}{3^n \ln(3)}$.

d. $S_{10} \approx 0.4999915325$, so $L_{10} \approx 0.4999915325 + \frac{1}{3^{11}\ln 3} \approx 0.4999966708$, and $U_{10} \approx 0.4999915325 + \frac{1}{3^{10}\ln 3} \approx 0.5000069475$.

9.4.38

a. The remainder R_n is bounded by $\int_n^\infty \frac{1}{x\ln^2 x}\,dx = \frac{1}{\ln n}$.

b. We solve $\frac{1}{\ln n} < 10^{-3}$ to get $n = e^{1000} \approx 10^{434}$.

c. $L_n = S_n + \int_{n+1}^\infty \frac{1}{x\ln^2 x}\,dx = S_n + \frac{1}{\ln(n+1)}$, and $U_n = S_n + \int_n^\infty \frac{1}{x\ln^2 x}\,dx = S_n + \frac{1}{\ln n}$.

d. $S_{10} = \sum_{k=2}^{11} \frac{1}{k\ln^2 k} \approx 1.700396385$, so $L_{10} \approx 1.700396385 + \frac{1}{\ln 12} \approx 2.102825989$, and $U_{10} \approx 1.700396385 + \frac{1}{\ln 11} \approx 2.117428776$.

9.4.39

a. The remainder R_n is bounded by $\int_n^\infty \frac{1}{x^{3/2}}\,dx = 2n^{-1/2}$.

b. We solve $2n^{-1/2} < 10^{-3}$ to get $n > 4 \times 10^6$, so let $n = 4 \times 10^6 + 1$.

c. $L_n = S_n + \int_{n+1}^\infty \frac{1}{x^{3/2}}\,dx = S_n + 2(n+1)^{-1/2}$, and $U_n = S_n + \int_n^\infty \frac{1}{x^{3/2}}\,dx = S_n + 2n^{-1/2}$.

d. $S_{10} = \sum_{k=1}^{10} \frac{1}{k^{3/2}} \approx 1.995336494$, so $L_{10} \approx 1.995336494 + 2 \cdot 11^{-1/2} \approx 2.598359183$, and $U_{10} \approx 1.995336494 + 2 \cdot 10^{-1/2} \approx 2.627792026$.

9.4.40

a. The remainder R_n is bounded by $\int_n^\infty e^{-x}\,dx = e^{-n}$.

b. We solve $e^{-n} < 10^{-3}$ to get $n = 7$.

c. $L_n = S_n + \int_{n+1}^\infty e^{-x}\,dx = S_n + e^{-(n+1)}$, and $U_n = S_n + \int_n^\infty e^{-x}\,dx = S_n + e^{-n}$.

d. $S_{10} = \sum_{k=1}^{10} e^{-k} \approx 0.5819502852$, so $L_{10} \approx 0.5819502852 + e^{-11} \approx 0.5819669869$, and $U_{10} \approx 0.5819502852 + e^{-10} \approx 0.5819956852$.

9.4.41

a. The remainder R_n is bounded by $\int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2}$.

b. We solve $\frac{1}{2n^2} < 10^{-3}$ to get $n = 23$.

c. $L_n = S_n + \int_{n+1}^\infty \frac{1}{x^3}\,dx = S_n + \frac{1}{2(n+1)^2}$, and $U_n = S_n + \int_n^\infty \frac{1}{x^3}\,dx = S_n + \frac{1}{2n^2}$.

d. $S_{10} \approx 1.197531986$, so $L_{10} \approx 1.197531986 + \frac{1}{2\cdot 11^2} \approx 1.201664217$, and $U_{10} \approx 1.197531986 + \frac{1}{2\cdot 10^2} \approx 1.202531986$.

9.4.42

a. The remainder R_n is bounded by $\int_n^\infty xe^{-x^2}\,dx = \frac{1}{2e^{n^2}}$.

b. We solve $\frac{1}{2e^{n^2}} < 10^{-3}$ to get $n = 3$.

c. $L_n = S_n + \int_{n+1}^\infty xe^{-x^2}\,dx = S_n + \frac{1}{2e^{(n+1)^2}}$, and $U_n = S_n + \int_n^\infty xe^{-x^2}\,dx = S_n + \frac{1}{2e^{n^2}}$.

d. $S_{10} \approx 0.4048813986$, so $L_{10} \approx 0.4048813986 + \frac{1}{2e^{11^2}} \approx 0.4048813986$, and $U_{10} \approx 0.4048813986 + \frac{1}{2e^{10^2}} \approx 0.4048813986$.

9.4.43 This is a geometric series with $a = \frac{1}{3}$ and $r = \frac{1}{12}$, so $\sum_{k=1}^\infty \frac{4}{12^k} = \frac{1/3}{1-1/12} = \frac{1/3}{11/12} = \frac{4}{11}$.

9.4.44 This is a geometric series with $a = 3/e^2$ and $r = 1/e$, so $\sum_{k=2}^\infty 3e^{-k} = \frac{3/e^2}{1-(1/e)} = \frac{3/e^2}{(e-1)/e} = \frac{3}{e(e-1)}$.

9.4.45 $\displaystyle\sum_{k=0}^\infty \left(3\left(\frac{2}{5}\right)^k - 2\left(\frac{5}{7}\right)^k\right) = 3\sum_{k=0}^\infty \left(\frac{2}{5}\right)^k - 2\sum_{k=0}^\infty \left(\frac{5}{7}\right)^k = 3\left(\frac{1}{3/5}\right) - 2\left(\frac{1}{2/7}\right) = 5 - 7 = -2.$

9.4.46 $\displaystyle\sum_{k=1}^\infty \left(2\left(\frac{3}{5}\right)^k + 3\left(\frac{4}{9}\right)^k\right) = 2\sum_{k=1}^\infty \left(\frac{3}{5}\right)^k + 3\sum_{k=1}^\infty \left(\frac{4}{9}\right)^k = 2\left(\frac{3/5}{2/5}\right) + 3\left(\frac{4/9}{5/9}\right) = 3 + \frac{12}{5} = \frac{27}{5}.$

9.4.47 $\displaystyle\sum_{k=1}^\infty \left(\frac{1}{3}\left(\frac{5}{6}\right)^k + \frac{3}{5}\left(\frac{7}{9}\right)^k\right) = \frac{1}{3}\sum_{k=1}^\infty \left(\frac{5}{6}\right)^k + \frac{3}{5}\sum_{k=1}^\infty \left(\frac{7}{9}\right)^k = \frac{1}{3}\left(\frac{5/6}{1/6}\right) + \frac{3}{5}\left(\frac{7/9}{2/9}\right) = \frac{5}{3} + \frac{21}{10} = \frac{113}{30}.$

9.4.48 $\displaystyle\sum_{k=0}^\infty \left(\frac{1}{2}(0.2)^k + \frac{3}{2}(0.8)^k\right) = \frac{1}{2}\sum_{k=0}^\infty (0.2)^k + \frac{3}{2}\sum_{k=0}^\infty (0.8)^k = \frac{1}{2}\left(\frac{1}{0.8}\right) + \frac{3}{2}\left(\frac{1}{0.2}\right) = \frac{5}{8} + \frac{15}{2} = \frac{65}{8}.$

9.4.49 $\displaystyle\sum_{k=1}^\infty \left(\left(\frac{1}{6}\right)^k + \left(\frac{1}{3}\right)^{k-1}\right) = \sum_{k=1}^\infty \left(\frac{1}{6}\right)^k + \sum_{k=1}^\infty \left(\frac{1}{3}\right)^{k-1} = \frac{1/6}{5/6} + \frac{1}{2/3} = \frac{17}{10}.$

9.4.50 $\displaystyle\sum_{k=0}^\infty \frac{2-3^k}{6^k} = \sum_{k=0}^\infty \left(\frac{2}{6^k} - \frac{3^k}{6^k}\right) = 2\sum_{k=0}^\infty \left(\frac{1}{6}\right)^k - \sum_{k=0}^\infty \left(\frac{1}{2}\right)^k = 2\left(\frac{1}{5/6}\right) - \frac{1}{1/2} = \frac{2}{5}.$

9.4.51

a. True. The two series differ by a finite amount ($\sum_{k=1}^9 a_k$), so if one converges, so does the other.

b. True. The same argument applies as in part (a).

c. False. If $\sum a_k$ converges, then $a_k \to 0$ as $k \to \infty$, so that $a_k + 0.0001 \to 0.0001$ as $k \to \infty$, so that $\sum(a_k + 0.0001)$ cannot converge.

d. False. Suppose $p = -1.0001$. Then $\sum p^k$ diverges but $p + .0001 = -0.9991$ so that $\sum(p + .0001)^k$ converges.

e. False. Let $p = 1.0005$; then $-p + .001 = -(p - .001) = -.9995$, so that $\sum k^{-p}$ converges (p-series) but $\sum k^{-p+.001}$ diverges.

f. False. Let $a_k = \frac{1}{k}$, the harmonic series.

9.4.52 Diverges by the Divergence Test because $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \sqrt{\frac{k+1}{k}} = 1 \neq 0$.

9.4.53 Converges by the Integral Test because $\int_1^\infty \frac{1}{(3x+1)(3x+4)}\,dx = \int_1^\infty \frac{1}{3(3x+1)} - \frac{1}{3(3x+4)}\,dx = \lim_{b\to\infty} \int_1^b \left(\frac{1}{3(3x+1)} - \frac{1}{3(3x+4)}\right) dx = \lim_{b\to\infty} \frac{1}{9}\left(\ln\left(\frac{3x+1}{3x+4}\right)\right)\Big|_1^b = \lim_{b\to\infty} = \frac{-1}{9} \cdot \ln(4/7) \approx 0.06217 < \infty$.

9.4.54 Converges by the Integral Test because $\int_0^\infty \frac{10}{x^2+9}\,dx = \frac{10}{3}\lim_{b\to\infty}\left(\tan^{-1}(x/3)\big|_0^b\right) = \frac{10}{3}\frac{\pi}{2} \approx 5.236 < \infty$.

9.4.55 Diverges by the Divergence Test because $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k}{\sqrt{k^2+1}} = 1 \neq 0$.

9.4.56 Converges because it is the sum of two geometric series. In fact, $\sum_{k=1}^\infty \frac{2^k+3^k}{4^k} = \sum_{k=1}^\infty (2/4)^k + \sum_{k=1}^\infty (3/4)^k = \frac{1/2}{1-(1/2)} + \frac{3/4}{1-(3/4)} = 1 + 3 = 4$.

9.4.57 Converges by the Integral Test because $\int_2^\infty \frac{4}{x\ln^2 x}\,dx = \lim_{b\to\infty}\left(\frac{-4}{\ln x}\Big|_2^b\right) = \frac{4}{\ln 2} < \infty$.

9.4.58

a. In order for the series to converge, the integral $\int_2^\infty \frac{1}{x(\ln x)^p}\,dx$ must exist. But

$$\int \frac{1}{x(\ln x)^p}\,dx = \frac{1}{1-p}(\ln x)^{1-p},$$

so in order for this improper integral to exist, we must have that $1 - p < 0$ or $p > 1$.

b. The series converges faster for $p = 3$ because the terms of the series get smaller faster.

9.4.59

a. Note that $\int \frac{1}{x\ln x(\ln\ln x)^p}\,dx = \frac{1}{1-p}(\ln\ln x)^{1-p}$, and thus the improper integral with bounds n and ∞ exists only if $p > 1$ because $\ln\ln x > 0$ for $x > e$. So this series converges for $p > 1$.

b. For large values of z, clearly $\sqrt{z} > \ln z$, so that $z > (\ln z)^2$. Write $z = \ln x$; then for large x, $\ln x > (\ln\ln x)^2$; multiplying both sides by $x\ln x$ we have that $x\ln^2 x > x\ln x(\ln\ln x)^2$, so that the first series converges faster because the terms get smaller faster.

9.4.60

a. $\sum \frac{1}{k^{2.5}}$.

b. $\sum \frac{1}{k^{0.75}}$.

c. $\sum \frac{1}{k^{3/2}}$.

9.4.61 Let $S_n = \sum_{k=1}^n \frac{1}{\sqrt{k}}$. Then this looks like a left Riemann sum for the function $y = \frac{1}{\sqrt{x}}$ on $[1, n+1]$. Because each rectangle lies above the curve itself, we see that S_n is bounded below by the integral of $\frac{1}{\sqrt{x}}$ on $[1, n+1]$. Now,

$$\int_1^{n+1} \frac{1}{\sqrt{x}}\, dx = \int_1^{n+1} x^{-1/2}\, dx = 2\sqrt{x}\Big|_1^{n+1} = 2\sqrt{n+1} - 2$$

This integral diverges as $n \to \infty$, so the series does as well by the bound above.

9.4.62 $\sum_{k=1}^\infty (a_k \pm b_k) = \lim_{n\to\infty} \sum_{k=1}^n (a_k \pm b_k) = \lim_{n\to\infty} \left(\sum_{k=1}^n a_k \pm \sum_{k=1}^n b_k\right) = \lim_{n\to\infty} \sum_{k=1}^n a_k \pm \lim_{n\to\infty} \sum_{k=1}^n b_k = A \pm B$.

9.4.63 $\sum_{k=1}^\infty ca_k = \lim\limits_{n\to\infty} \sum_{k=1}^n ca_k = \lim\limits_{n\to\infty} c\sum_{k=1}^n a_k = c \lim\limits_{n\to\infty} \sum_{k=1}^n a_k$, so that one sum diverges if and only if the other one does.

9.4.64 $\displaystyle\sum_{k=2}^\infty \frac{1}{k\ln k}$ diverges by the Integral Test, because $\int_2^\infty \frac{1}{x\ln x} = \lim_{b\to\infty} \left(\ln\ln x\big|_2^b\right) = \infty$.

9.4.65 To approximate the sequence for $\zeta(m)$, note that the remainder R_n after n terms is bounded by

$$\int_n^\infty \frac{1}{x^m}\, dx = \frac{1}{m-1} n^{1-m}.$$

For $m = 3$, if we wish to approximate the value to within 10^{-3}, we must solve $\displaystyle\frac{1}{2}n^{-2} < 10^{-3}$, so that $n = 23$, and $\displaystyle\sum_{k=1}^{23} \frac{1}{k^3} \approx 1.201151926$. The true value is ≈ 1.202056903.

For $m = 5$, if we wish to approximate the value to within 10^{-3}, we must solve $\displaystyle\frac{1}{4}n^{-4} < 10^{-3}$, so that $n = 4$, and $\displaystyle\sum_{k=1}^{4} \frac{1}{k^5} \approx 1.036341789$. The true value is ≈ 1.036927755.

For $m = 7$, if we wish to approximate the value to within 10^{-3}, we must solve $\displaystyle\frac{1}{6}n^{-6} < 10^{-3}$, so that $n = 3$, and $\displaystyle\sum_{k=1}^{3} \frac{1}{k^7} \approx 1.008269747$. The true value is ≈ 1.008349277.

9.4.66

a. Starting with $\cot^2 x < \dfrac{1}{x^2} < 1 + \cot^2 x$, substitute $k\theta$ for x:

$$\cot^2(k\theta) < \frac{1}{k^2\theta^2} < 1 + \cot^2(k\theta),$$

$$\sum_{k=1}^n \cot^2(k\theta) < \sum_{k=1}^n \frac{1}{k^2\theta^2} < \sum_{k=1}^n (1 + \cot^2(k\theta)),$$

$$\sum_{k=1}^n \cot^2(k\theta) < \frac{1}{\theta^2}\sum_{k=1}^n \frac{1}{k^2} < n + \sum_{k=1}^n \cot^2(k\theta).$$

Note that the identity is valid because we are only summing for k up to n, so that $k\theta < \frac{\pi}{2}$.

b. Substitute $\dfrac{n(2n-1)}{3}$ for the sum, using the identity:

$$\frac{n(2n-1)}{3} < \frac{1}{\theta^2}\sum_{k=1}^{n}\frac{1}{k^2} < n + \frac{n(2n-1)}{3},$$
$$\theta^2\frac{n(2n-1)}{3} < \sum_{k=1}^{n}\frac{1}{k^2} < \theta^2\frac{n(2n+2)}{3},$$
$$\frac{n(2n-1)\pi^2}{3(2n+1)^2} < \sum_{k=1}^{n}\frac{1}{k^2} < \frac{n(2n+2)\pi^2}{3(2n+1)^2}.$$

c. By the Squeeze Theorem, if the expressions on either end have equal limits as $n \to \infty$, the expression in the middle does as well, and its limit is the same. The expression on the left is

$$\pi^2\frac{2n^2-n}{12n^2+12n+3} = \pi^2\frac{2-n^{-1}}{12+12n^{-1}+3n^{-2}},$$

which has a limit of $\dfrac{\pi^2}{6}$ as $n \to \infty$. The expression on the right is

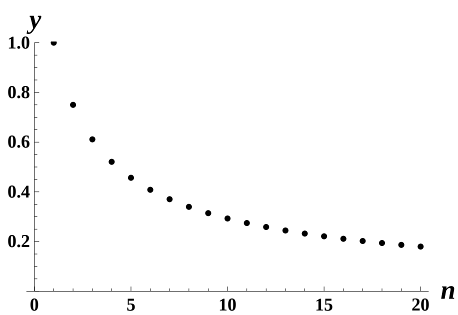
$$\pi^2\frac{2n^2+2n}{12n^2+12n+3} = \pi^2\frac{2+2n^{-1}}{12+12n^{-1}+3n^{-3}},$$

which has the same limit. Thus $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2} = \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$.

9.4.67 $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2} = \sum_{k=1}^{\infty}\frac{1}{(2k)^2} + \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$, splitting the series into even and odd terms. But $\sum_{k=1}^{\infty}\frac{1}{(2k)^2} = \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}$. Thus $\frac{\pi^2}{6} = \frac{1}{4}\frac{\pi^2}{6} + \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$, so that the sum in question is $\frac{3\pi^2}{24} = \frac{\pi^2}{8}$.

9.4.68

a. $\{F_n\}$ is a decreasing sequence because each term in F_n is smaller than the corresponding term in F_{n-1} and thus the sum of terms in F_n is smaller than the sum of terms in F_{n-1}.

b. 

c. It appears that $\displaystyle\lim_{n\to\infty} F_n = 0$.

9.4.69

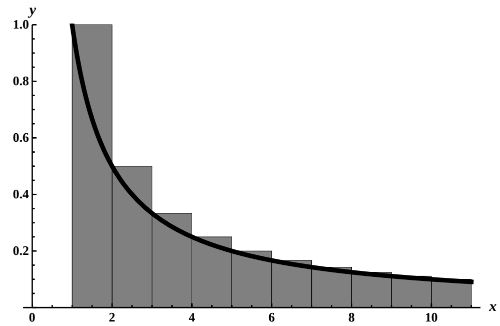
a. $x_1 = \sum_{k=2}^{2}\frac{1}{k} = \frac{1}{2}$, $x_2 = \sum_{k=3}^{4}\frac{1}{k} = \frac{1}{3}+\frac{1}{4} = \frac{7}{12}$, $x_3 = \sum_{k=4}^{6}\frac{1}{k} = \frac{1}{4}+\frac{1}{5}+\frac{1}{6} = \frac{37}{60}$.

b. x_n has n terms. Each term is bounded below by $\frac{1}{2n}$ and bounded above by $\frac{1}{n+1}$. Thus $x_n \ge n\cdot\frac{1}{2n} = \frac{1}{2}$, and $x_n \le n\cdot\frac{1}{n+1} < n\cdot\frac{1}{n} = 1$.

c. The right Riemann sum for $\int_1^2 \frac{dx}{x}$ using n subintervals has n rectangles of width $\frac{1}{n}$; the right edges of those rectangles are at $1+\frac{i}{n} = \frac{n+i}{n}$ for $i = 1, 2, \ldots, n$. The height of such a rectangle is the value of $\frac{1}{x}$ at the right endpoint, which is $\frac{n}{n+i}$. Thus the area of the rectangle is $\frac{1}{n}\cdot\frac{n}{n+i} = \frac{1}{n+i}$. Adding up over all the rectangles gives x_n.

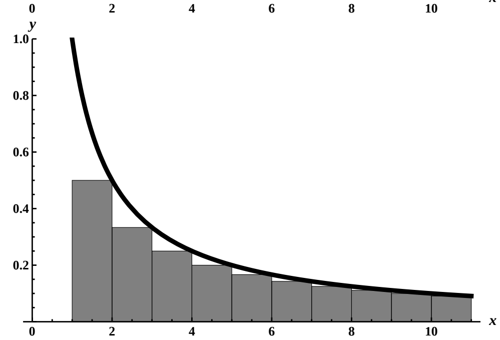
d. The limit $\lim_{n\to\infty} x_n$ is the limit of the right Riemann sum as the width of the rectangles approaches zero. This is precisely $\int_1^2 \frac{dx}{x} = \ln x\Big|_1^2 = \ln 2$.

9.4.70



The first diagram is a left Riemann sum for $f(x) = \frac{1}{x}$ on the interval $[1, 11]$ (we assume $n = 10$ for purposes of drawing a graph). The area under the curve is $\int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1)$, and the sum of the areas of the rectangles is obviously $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Thus

$$\ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$



a. The second diagram is a right Riemann sum for the same function on the same interval. Considering only $[1, n]$, we see that, comparing the area under the curve and the sum of the areas of the rectangles, that

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n$$

Adding 1 to both sides gives the desired inequality.

b. According to part (a), $\ln(n+1) < S_n$ for $n = 1, 2, 3, \ldots,$, so that $E_n = S_n - \ln(n+1) > 0$.

c. Using the second figure above and assuming $n = 9$, the final rectangle corresponds to $\frac{1}{n+1}$, and the area under the curve between $n+1$ and $n+2$ is clearly $\ln(n+2) - \ln(n+1)$.

d. $E_{n+1} - E_n = S_{n+1} - \ln(n+2) - (S_n - \ln(n+1)) = \frac{1}{n+1} - (\ln(n+2) - \ln(n+1))$. But this is positive because of the bound established in part (c).

e. Using part (a), $E_n = S_n - \ln(n+1) < 1 + \ln(n) - \ln(n+1) < 1$.

f. E_n is a monotone (increasing) sequence that is bounded, so it has a limit.

g. The first ten values (E_1 through E_{10}) are

$$.3068528194, .401387711, .447038972, .473895421, .491573864,$$
$$.504089851, .513415601, .520632565, .526383161, .531072981.$$

$E_{1000} \approx 0.576716082$.

h. For $S_n > 10$ we need $10 - 0.5772 = 9.4228 > \ln(n+1)$. Solving for n gives $n \approx 12366.16$, so $n = 12367$.

9.4.71

a. Note that the center of gravity of any stack of dominoes is the average of the locations of their centers. Define the midpoint of the zeroth (top) domino to be $x = 0$, and stack additional dominoes down and to its right (to increasingly positive x-coordinates.) Let $m(n)$ be the x-coordinate of the midpoint of the n^{th} domino. Then in order for the stack not to fall over, the left edge of the n^{th} domino must be placed directly under the center of gravity of dominos 0 through $n-1$, which is $\frac{1}{n}\sum_{i=0}^{n-1} m(i)$, so that $m(n) = 1 + \frac{1}{n}\sum_{i=0}^{n-1} m(i)$. Claim that in fact $m(n) = \sum_{k=1}^{n} \frac{1}{k}$. Use induction. This is certainly true for $n = 1$. Note first that $m(0) = 0$, so we can start the sum at 1 rather than at 0. Now, $m(n) = 1 + \frac{1}{n}\sum_{i=1}^{n-1} m(i) = 1 + \frac{1}{n}\sum_{i=1}^{n-1}\sum_{j=1}^{i} \frac{1}{j}$. Now, 1 appears $n-1$ times in the double sum, 2 appears $n-2$ times, and so forth, so we can rewrite this sum as $m(n) = 1 + \frac{1}{n}\sum_{i=1}^{n-1} \frac{n-i}{i} = 1 + \frac{1}{n}\sum_{i=1}^{n-1}\left(\frac{n}{i} - 1\right) = 1 + \frac{1}{n}\left(n\sum_{i=1}^{n-1}\frac{1}{i} - (n-1)\right) = \sum_{i=1}^{n-1}\frac{1}{i} + 1 - \frac{n-1}{n} = \sum_{i=1}^{n}\frac{1}{i}$, and we are done by induction (noting that the statement is clearly true for $n = 0$, $n = 1$). Thus the maximum overhang is $\sum_{k=2}^{n}\frac{1}{k}$.

b. For an infinite number of dominos, because the overhang is the harmonic series, the distance is potentially infinite.

9.5 The Ratio, Root, and Comparison Tests

9.5.1 Given a series $\sum a_k$ of positive terms, compute $\lim_{k\to\infty} \frac{a_{k+1}}{a_k}$ and call it r. If $0 \le r < 1$, the given series converges. If $r > 1$ (including $r = \infty$), the given series diverges. If $r = 1$, the test is inconclusive.

9.5.2 Given a series $\sum a_k$ of positive terms, compute $\lim_{k\to\infty} \sqrt[k]{a_k}$ and call it r. If $0 \le r < 1$, the given series converges. If $r > 1$ (including $r = \infty$), the given series diverges. If $r = 1$, the test is inconclusive.

9.5.3 Given a series of positive terms $\sum a_k$ that you suspect converges, find a series $\sum b_k$ that you know converges, for which $\lim_{k\to\infty} \frac{a_k}{b_k} = L$ where $L \ge 0$ is a finite number. If you are successful, you will have shown that the series $\sum a_k$ converges.

Given a series of positive terms $\sum a_k$ that you suspect diverges, find a series $\sum b_k$ that you know diverges, for which $\lim_{k\to\infty} \frac{a_k}{b_k} = L$ where $L > 0$ (including the case $L = \infty$). If you are successful, you will have shown that $\sum a_k$ diverges.

9.5.4 The Divergence Test.

9.5.5 The Ratio Test.

9.5.6 The Comparison Test or the Limit Comparison Test.

9.5.7 The difference between successive partial sums is a term in the sequence. Because the terms are positive, differences between successive partial sums are as well, so the sequence of partial sums is increasing.

9.5.8 No. They all determine convergence or divergence by approximating or bounding the series by some other series known to converge or diverge; thus, the actual value of the series cannot be determined.

9.5.9 The ratio between successive terms is $\frac{a_{k+1}}{a_k} = \frac{1}{(k+1)!} \cdot \frac{(k)!}{1} = \frac{1}{k+1}$, which goes to zero as $k \to \infty$, so the given series converges by the Ratio Test.

9.5.10 The ratio between successive terms is $\frac{a_{k+1}}{a_k} = \frac{2^{k+1}}{(k+1)!} \cdot \frac{(k)!}{2^k} = \frac{2}{k+1}$; the limit of this ratio is zero, so the given series converges by the Ratio Test.

9.5.11 The ratio between successive terms is $\frac{a_{k+1}}{a_k} = \frac{(k+1)^2}{4^{(k+1)}} \cdot \frac{4^k}{(k)^2} = \frac{1}{4}\left(\frac{k+1}{k}\right)^2$. The limit is $1/4$ as $k \to \infty$, so the given series converges by the Ratio Test.

9.5.12 The ratio between successive terms is

$$\frac{a_{k+1}}{a_k} = \frac{2^{(k+1)}}{(k+1)^{(k+1)}} \cdot \frac{(k)^k}{2^k} = \frac{2}{k+1}\left(\frac{k}{k+1}\right)^k.$$

Note that $\lim_{k\to\infty}\left(\frac{k}{k+1}\right)^k = \lim_{k\to\infty}(1 + \frac{-1}{k+1})^k = \frac{1}{e}$, so the limit of the ratio is $0 \cdot \frac{1}{e} = 0$, so the given series converges by the Ratio Test.

9.5.13 The ratio between successive terms is $\frac{a_{k+1}}{a_k} = \frac{(k+1)e^{-(k+1)}}{(k)e^{-(k)}} = \frac{k+1}{(k)e}$. The limit of this ratio as $k \to \infty$ is $1/e < 1$, so the given series converges by the Ratio Test.

9.5.14 The ratio between successive terms is $\frac{a_{k+1}}{a_k} = \frac{(k+1)!}{(k+1)^{(k+1)}} \cdot \frac{(k)^k}{(k)!} = \left(\frac{k}{k+1}\right)^k$. This is the reciprocal of $\left(\frac{k+1}{k}\right)^k$ which has limit e as $k \to \infty$, so the limit of the ratio of successive terms is $1/e < 1$, so the given series converges by the Ratio Test.

9.5.15 The ratio between successive terms is $\frac{2^{k+1}}{(k+1)^{99}} \cdot \frac{(k)^{99}}{2^k} = 2\left(\frac{k}{k+1}\right)^{99}$; the limit as $k \to \infty$ is 2, so the given series diverges by the Ratio Test.

9.5.16 The ratio between successive terms is $\frac{(k+1)^6}{(k+1)!} \cdot \frac{(k)!}{(k)^6} = \frac{1}{k+1}\left(\frac{k+1}{k}\right)^6$; the limit as $k \to \infty$ is zero, so the given series converges by the Ratio Test.

9.5.17 The ratio between successive terms is $\frac{((k+1)!)^2}{(2(k+1))!} \cdot \frac{(2k)!}{((k)!)^2} = \frac{(k+1)^2}{(2k+2)(2k+1)}$; the limit as $k \to \infty$ is $1/4$, so the given series converges by the Ratio Test.

9.5.18 The ratio between successive terms is $\frac{(k+1)^4 2^{-(k+1)}}{(k)^4 2^{-k}} = \frac{1}{2}\left(\frac{k+1}{k}\right)^4$; the limit as $k \to \infty$ is $\frac{1}{2}$, so the given series converges by the Ratio Test.

9.5.19 The kth root of the kth term is $\frac{4k^3+k}{9k^3+k+1}$. The limit of this as $k \to \infty$ is $\frac{4}{9} < 1$, so the given series converges by the Root Test.

9.5.20 The kth root of the kth term is $\frac{k+1}{2k}$. The limit of this as $k \to \infty$ is $\frac{1}{2} < 1$, so the given series converges by the Root Test.

9.5.21 The kth root of the kth term is $\frac{k^{2/k}}{2}$. The limit of this as $k \to \infty$ is $\frac{1}{2} < 1$, so the given series converges by the Root Test.

9.5.22 The kth root of the kth term is $\left(1 + \frac{3}{k}\right)^k$. The limit of this as $k \to \infty$ is $= e^3 > 1$, so the given series diverges by the Root Test.

9.5.23 The kth root of the kth term is $\left(\frac{k}{k+1}\right)^{2k}$. The limit of this as $k \to \infty$ is $e^{-2} < 1$, so the given series converges by the Root Test.

9.5.24 The kth root of the kth term is $\frac{1}{\ln(k+1)}$. The limit of this as $k \to \infty$ is 0, so the given series converges by the Root Test.

9.5.25 The kth root of the kth term is$\left(\frac{1}{k^k}\right)$. The limit of this as $k \to \infty$ is 0, so the given series converges by the Root Test.

9.5.26 The kth root of the kth term is $\frac{k^{1/k}}{e}$. The limit of this as $k \to \infty$ is $\frac{1}{e} < 1$, so the given series converges by the Root Test.

9.5.27 $\frac{1}{k^2+4} < \frac{1}{k^2}$, and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, so $\sum_{k=1}^{\infty} \frac{1}{k^2+4}$ converges as well, by the Comparison Test.

9.5.28 Use the Limit Comparison Test with $\left\{\frac{1}{k^2}\right\}$. The ratio of the terms of the two series is $\frac{k^4+k^3-k^2}{k^4+4k^2-3}$ which has limit 1 as $k \to \infty$. Because the comparison series converges, the given series does as well.

9.5.29 Use the Limit Comparison Test with $\left\{\frac{1}{k}\right\}$. The ratio of the terms of the two series is $\frac{k^3-k}{k^3+4}$ which has limit 1 as $k \to \infty$. Because the comparison series diverges, the given series does as well.

9.5.30 Use the Limit Comparison Test with $\left\{\frac{1}{k}\right\}$. The ratio of the terms of the two series is $\frac{0.0001k}{k+4}$ which has limit 0.0001 as $k \to \infty$. Because the comparison series diverges, the given series does as well.

9.5.31 For all k, $\frac{1}{k^{3/2}+1} < \frac{1}{k^{3/2}}$. The series whose terms are $\frac{1}{k^{3/2}}$ is a p-series which converges, so the given series converges as well by the Comparison Test.

9.5.32 Use the Limit Comparison Test with $\{1/k\}$. The ratio of the terms of the two series is $k\sqrt{\frac{k}{k^3+1}} = \sqrt{\frac{k^3}{k^3+1}}$, which has limit 1 as $k \to \infty$. Because the comparison series diverges, the given series does as well.

9.5.33 $\sin(1/k) > 0$ for $k \ge 1$, so we can apply the Comparison Test with $1/k^2$. $\sin(1/k) < 1$, so $\frac{\sin(1/k)}{k^2} < \frac{1}{k^2}$. Because the comparison series converges, the given series converges as well.

9.5.34 Use the Limit Comparison Test with $\{1/3^k\}$. The ratio of the terms of the two series is $\frac{3^k}{3^k-2^k} = \frac{1}{1-\left(\frac{2^k}{3^k}\right)}$, which has limit 1 as $k \to \infty$. Because the comparison series converges, the given series does as well.

9.5.35 Use the Limit Comparison Test with $\{1/k\}$. The ratio of the terms of the two series is $\frac{k}{2k-\sqrt{k}} = \frac{1}{2-1/\sqrt{k}}$, which has limit $1/2$ as $k \to \infty$. Because the comparison series diverges, the given series does as well.

9.5.36 $\frac{1}{k\sqrt{k+2}} < \frac{1}{k\sqrt{k}} = \frac{1}{k^{3/2}}$. Because the series whose terms are $\frac{1}{k^{3/2}}$ is a $p-$series with $p > 1$, it converges. Because the comparison series converges, the given series converges as well.

9.5.37 Use the Limit Comparison Test with $\frac{k^{2/3}}{k^{3/2}}$. The ratio of corresponding terms of the two series is $\frac{\sqrt[3]{k^2+1}}{\sqrt{k^3+1}} \cdot \frac{k^{3/2}}{k^{2/3}} = \frac{\sqrt[3]{k^2+1}}{\sqrt[3]{k^2}} \cdot \frac{\sqrt{k^3}}{\sqrt{k^3+1}}$, which has limit 1 as $k \to \infty$. The comparison series is the series whose terms are $k^{2/3-3/2} = k^{-5/6}$, which is a p-series with $p < 1$, so it, and the given series, both diverge.

9.5.38 For all k, $\frac{1}{(k\ln k)^2} < \frac{1}{k^2}$. Because the series whose terms are $\frac{1}{k^2}$ converges, the given series converges as well.

9.5.39

a. False. For example, let $\{a_k\}$ be all zeros, and $\{b_k\}$ be all 1's.

b. True. This is a result of the Comparison Test.

c. True. Both of these statements follow from the Comparison Test.

9.5.40 Use the Divergence Test: $\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} \left(1 + \frac{-1}{k}\right)^k = \frac{1}{e} \neq 0$, so the given series diverges.

9.5.41 Use the Divergence Test: $\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} \left(1 + \frac{2}{k}\right)^k = e^2 \neq 0$, so the given series diverges.

9.5.42 Use the Root Test: The kth root of the kth term is $\frac{k^2}{2k^2+1}$. The limit of this as $k \to \infty$ is $\frac{1}{2} < 1$, so the given series converges by the Root Test.

9.5.43 Use the Ratio Test: the ratio of successive terms is $\frac{(k+1)^{100}}{(k+2)!} \cdot \frac{(k+1)!}{k^{100}} = \left(\frac{k+1}{k}\right)^{100} \cdot \frac{1}{k+2}$. This has limit $1^{100} \cdot 0 = 0$ as $k \to \infty$, so the given series converges by the Ratio Test.

9.5.44 Use the Comparison Test. Note that $\sin^2 k \le 1$ for all k, so $\frac{\sin^2 k}{k^2} \le \frac{1}{k^2}$ for all k. Because $\sum_{k=1}^\infty \frac{1}{k^2}$ converges, so does the given series.

9.5.45 Use the Root Test. The kth root of the kth term is $(k^{1/k} - 1)^2$, which has limit 0 as $k \to \infty$, so the given series converges by the Root Test.

9.5.46 Use the Limit Comparison Test with the series whose kth term is $\left(\frac{2}{e}\right)^k$. Note that $\lim_{k\to\infty} \frac{2^k}{e^k-1} \cdot \frac{e^k}{2^k} = \lim_{k\to\infty} \frac{e^k}{e^k-1} = 1$. The given series thus converges because $\sum_{k=1}^\infty \left(\frac{2}{e}\right)^k$ converges (which converges because it is a geometric series with $r = \frac{2}{e} < 1$. Note that it is also possible to show convergence with the Ratio Test.