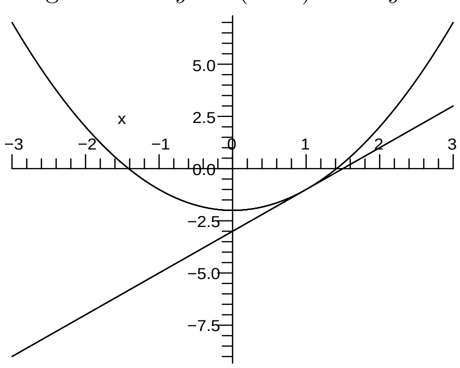
Chapter 2

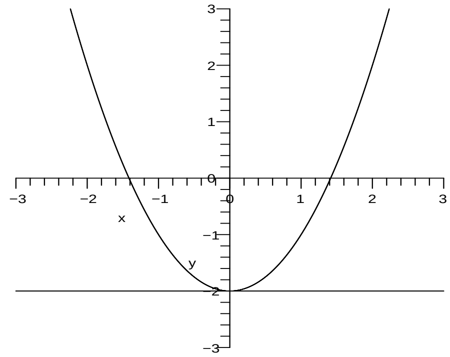
Differentiation

2.1 Tangent Line and Velocity

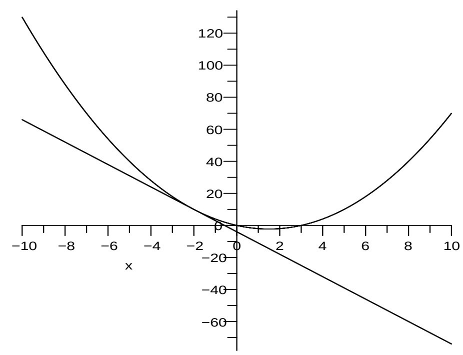
1. Slope is
$\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$
$= \lim_{h\to 0} \frac{(1+h)^2-2-(-1)}{h}$
$= \lim_{h\to 0} \frac{h^2+2h}{h}$
$= \lim_{h\to 0} (h+2) = 2.$
Tangent line is $y = 2(x-1)-1$ or $y = 2x-3$.



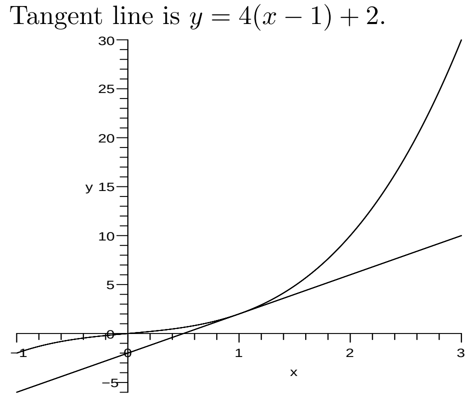
2. Slope is
$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$
$= \lim_{h\to 0} \frac{h^2}{h} = 0.$
Tangent line is $y = -2$.



3. Slope is
$\lim_{h\to 0} \frac{f(-2+h)-f(-2)}{h}$
$= \lim_{h\to 0} \frac{(-2+h)^2-3(-2+h)-(10)}{h}$
$= \lim_{h\to 0} \frac{-7h+h^2}{h} = -7.$
Tangent line is $y = -7(x+2)+10$



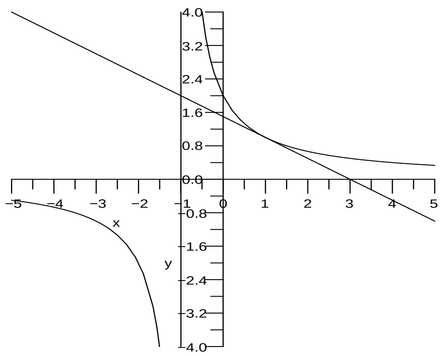
4. Slope is
$\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$
$= \lim_{h\to 0} \frac{(1+3h+3h^2+h^3)+(1+h)-2}{h}$
$= \lim_{h\to 0} \frac{4h+3h^2+h^3}{h} = \lim_{h\to 0} 4+3h+h^2 = 4.$
Tangent line is $y = 4(x-1)+2$.



5. Slope is

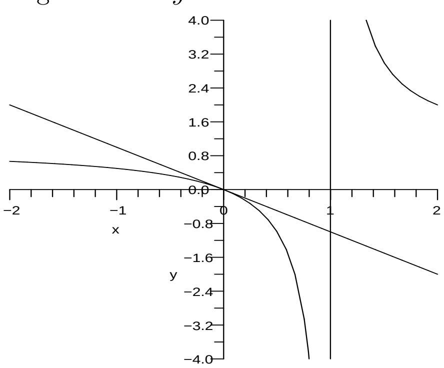
$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$
$$=\lim_{h\to 0}\frac{\frac{2}{(1+h)+1}-\frac{2}{1+1}}{h}$$
$$=\lim_{h\to 0}\frac{\frac{2}{2+h}-1}{h}=\lim_{h\to 0}\frac{\left(\frac{2-(2+h)}{2+h}\right)}{h}$$
$$=\lim_{h\to 0}\frac{\left(\frac{-h}{2+h}\right)}{h}=\lim_{h\to 0}\frac{-1}{2+h}=-\frac{1}{2}.$$

Tangent line is $y=-\frac{1}{2}(x-1)+1$ or $y=-\frac{x}{2}+\frac{3}{2}$.



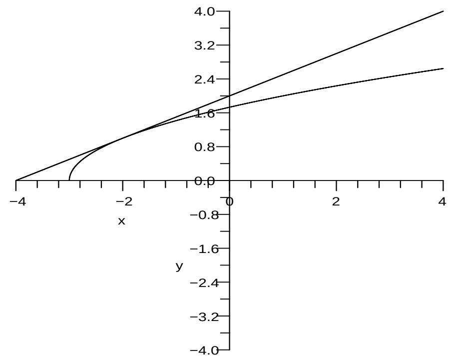
6. Slope is
$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$
$$=\lim_{h\to 0}\frac{\frac{h}{h-1}-0}{h}$$
$$=\lim_{h\to 0}\frac{1}{h-1}=-1$$

Tangent line is $y=-x$.



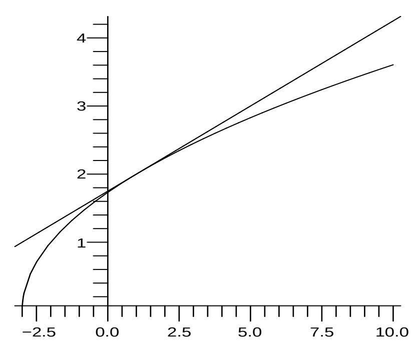
7. Slope is
$$\lim_{h\to 0}\frac{f(-2+h)-f(-2)}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{(-2+h)+3}-1}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{h+1}-1}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{h+1}-1}{h}\cdot\frac{\sqrt{h+1}+1}{\sqrt{h+1}+1}$$
$$=\lim_{h\to 0}\frac{(h+1)-1}{h(\sqrt{h+1}+1)}$$
$$=\lim_{h\to 0}\frac{1}{\sqrt{h+1}+1}=\frac{1}{2}.$$

Tangent line is $y=\frac{1}{2}(x+2)+1$ or $y=\frac{1}{2}x+2$.



8. Slope is
$$\lim_{h\to 0}\frac{f(1+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{(1+h)+3}-\sqrt{1+3}}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{h+4}-2}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{h+4}-2}{h}\cdot\frac{\sqrt{h+4}+2}{\sqrt{h+4}+2}$$
$$=\lim_{h\to 0}\frac{h+4-4}{h}\cdot\frac{1}{\sqrt{h+4}+2}$$
$$=\lim_{h\to 0}\frac{1}{\sqrt{h+4}+2}=\frac{1}{4}.$$

Tangent line is $y=\frac{1}{4}(x-1)+2$.



9. $f(x)=x^3-x$

No.	Points $(x,\,y)$	Slope
(a)	(1,0) and (2,6)	6
(b)	(2,6) and (3,24)	18
(c)	(1.5,1.875) and (2,6)	8.25
(d)	(2,6) and (2.5,13.125)	14.25
(e)	(1.9,4.959) and (2,6)	10.41
(f)	(2,6) and (2.1,7.161)	11.61

(g) Slope seems to be approximately 11.

10. $f(x) = \sqrt{x^2+1}$

No.	Points (x, y)	Slope
(a)	(1,1.414) and (2,2.236)	0.504
(b)	(2,2.236) and (3,3.162)	0.926
(c)	(1.5,1.803) and (2,2.236)	0.867
(d)	(2,2.236) and (2.5,2.269)	0.913
(e)	(1.9,2.147) and (2,2.236)	0.89
(f)	(2,2.236) and (2.1,2.325)	0.899

(g) Slope seems to be approximately 0.89.

11. $f(x) = \dfrac{x-1}{x+1}$

No.	Points (x, y)	Slope
(a)	(1,0) and (2,0.33)	0.33
(b)	(2,0.33) and (3,0.5)	0.17
(c)	(1.5,0.2) and (2,0.33)	0.26
(d)	(2,0.33) and (2.5,0.43)	0.2
(e)	(1.9,0.31) and (2,0.33)	0.2
(f)	(2,0.33) and (2.1,0.35)	0.2

(g) Slope seems to be approximately 0.2.

12. $f(x) = e^x$

No.	Points (x, y)	Slope
(a)	(1,2.718) and (2,7.389)	4.671
(b)	(2,7.389) and (3,20.085)	12.696
(c)	(1.5,4.481) and (2,7.389)	5.814
(d)	(2,7.389) and (2.5,12.182)	9.586
(e)	(1.9,6.686) and (2,7.389)	7.03
(f)	(2,7.389) and (2.1,8.166)	7.77

(g) Slope seems to be approximately 7.4

13. C, B, A, D. At the point labeled C, the slope is very steep and negative. At the point B, the slope is zero and at the point A, the slope is just more than zero. The slope of the line tangent to the point D is large and positive.

14. In order of increasing slope: D (large negative), C (small negative), B (small positive), and A (large positive).

15. (a) Velocity at time $t = 1$ is,

$$\lim_{h\to0}\frac{s(1+h)-s(1)}{h}$$
$$= \lim_{h\to0}\frac{-4.9(1+h)^2+5-(0.1)}{h}$$
$$= \lim_{h\to0}\frac{-4.9(1+2h+h^2)+5-(0.1)}{h}$$
$$= \lim_{h\to0}\frac{-9.8h-4.9h^2}{h}$$
$$= \lim_{h\to0}\frac{h(-9.8-4.9h)}{h} = -9.8.$$

(b) Velocity at time $t = 2$ is,

$$\lim_{h\to0}\frac{s(2+h)-s(2)}{h}$$
$$= \lim_{h\to0}\frac{-4.9(2+h)^2+5-(-14.6)}{h}$$
$$= \lim_{h\to0}\frac{-4.9(4+4h+h^2)+5-(-14.6)}{h}$$
$$= \lim_{h\to0}\frac{-19.6h-4.9h^2}{h}$$
$$= \lim_{h\to0}\frac{h(-19.6-4.9h)}{h} = -19.6$$

16. (a) Velocity at time $t = 0$ is,

$$\lim_{h\to0}\frac{s(0+h)-s(0)}{h}$$
$$= \lim_{h\to0}\frac{4h-4.9h^2}{h}$$
$$= \lim_{h\to0}\frac{h(4-4.9h)}{h}$$
$$= 4 - \lim_{h\to0}4.9h = 4.$$

(b) Velocity at time $t = 1$ is,

$$\lim_{h\to0}\frac{s(1+h)-s(1)}{h}$$
$$= \lim_{h\to0}\frac{4(1+h)-4.9(1+h)^2-(-0.9)}{h}$$
$$= \lim_{h\to0}\frac{4+4h-4.9-9.8h-4.9h^2+0.9}{h}$$
$$= \lim_{h\to0}\frac{-5.8h-4.9h^2}{h}$$
$$= \lim_{h\to0}\frac{h(-5.8-4.9h)}{h} = -5.8$$

17. (a) Velocity at time $t = 0$ is,

$$\lim_{h\to0}\frac{s(0+h)-s(0)}{h}$$
$$= \lim_{h\to0}\frac{\sqrt{h+16}-4}{h}\cdot\frac{\sqrt{h+16}+4}{\sqrt{h+16}+4}$$
$$= \lim_{h\to0}\frac{(h+16)-16}{h(\sqrt{h+16}+4)}$$
$$= \lim_{h\to0}\frac{1}{\sqrt{h+16}+4} = \frac{1}{8}$$

(b) Velocity at time $t = 2$ is,

$$\lim_{h\to0}\frac{s(2+h)-s(2)}{h}$$
$$= \lim_{h\to0}\frac{\sqrt{18+h}-\sqrt{18}}{h}$$

Multiplying by $\dfrac{\sqrt{h+18}+\sqrt{18}}{\sqrt{h+18}+\sqrt{18}}$ gives

$$= \lim_{h\to0}\frac{(h+18)-18}{h(\sqrt{h+18}+\sqrt{18})}$$
$$= \lim_{h\to0}\frac{1}{\sqrt{h+18}+\sqrt{18}} = \frac{1}{2\sqrt{18}}$$

18. (a) Velocity at time $t = 2$ is,

$$\lim_{h\to0}\frac{s(2+h)-s(2)}{h}$$
$$= \lim_{h\to0}\frac{\frac{4}{(2+h)}-2}{h} = \lim_{h\to0}\frac{\frac{4-4-2h}{(2+h)}}{h}$$

$$= \lim_{h\to 0} \frac{-2h}{h(2+h)} = \lim_{h\to 0} \frac{-2}{2+h} = -1.$$

(b) Velocity at time $t = 4$ is,

$$\lim_{h\to 0} \frac{s(4+h) - s(4)}{h} = \lim_{h\to 0} \frac{\frac{4}{(4+h)} - 1}{h}$$
$$= \lim_{h\to 0} \frac{\frac{4-1(4+h)}{(4+h)}}{h} = \lim_{h\to 0} \frac{\frac{4-4-h}{(4+h)}}{h}$$
$$= \lim_{h\to 0} \frac{-h}{h(4+h)} = \lim_{h\to 0} \frac{-1}{4+h} = -\frac{1}{4}$$

19. **(a)** Points: (0, 10) and (2, 74)
Average velocity: $\dfrac{74-10}{2} = 32$

(b) Second point: (1, 26)
Average velocity: $\dfrac{74-26}{1} = 48$

(c) Second point: (1.9, 67.76)
Average velocity: $\dfrac{74-67.76}{0.1} = 62.4$

(d) Second point: (1.99, 73.3616)
Average velocity: $\dfrac{74-73.3616}{0.01} = 63.84$

(e) The instantaneous velocity seems to be 64.

20. **(a)** Points: (0, 0) and (2, 26)
Average velocity: $\dfrac{26-0}{2-0} = 13$

(b) Second point: (1, 4)
Average velocity: $\dfrac{26-4}{2-1} = 22$

(c) Second point: (1.9, 22.477)
Average velocity: $\dfrac{26-22.477}{2-1.9} = 35.23$

(d) Second point: (1.99, 25.6318)
Average velocity:
$\dfrac{26-25.6318}{2-1.99} = 36.8203$

(e) The instantaneous velocity seems to be approaching 37.

21. **(a)** Points: (0, 0) and $(2, \sqrt{20})$
Average velocity: $\dfrac{\sqrt{20}-0}{2-0} = 2.236068$

(b) Second point: (1, 3)
Average velocity: $\dfrac{\sqrt{20}-3}{2-1} = 1.472136$

(c) Second point: $(1.9, \sqrt{18.81})$
Average velocity:
$\dfrac{\sqrt{20}-\sqrt{18.81}}{2-1.9} = 1.3508627$

(d) Second point: $(1.99, \sqrt{19.8801})$
Average velocity:
$\dfrac{\sqrt{20}-\sqrt{19.88}}{2-1.99} = 1.3425375$

(e) One might conjecture that these numbers are approaching 1.34. The exact limit is $\dfrac{6}{\sqrt{20}} \approx 1.341641$.

22. **(a)** Points: (0, −2.7279) and (2, 0)
Average velocity:
$\dfrac{0-(-2.7279)}{2-0} = 1.3639$

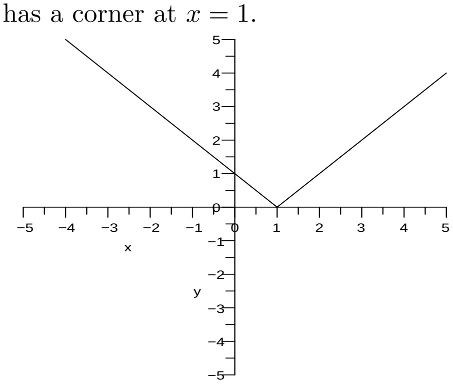
(b) Second point: (1, −2.5244)
Average velocity:
$\dfrac{0-(-2.5244)}{2-1} = 2.5244$

(c) Second point: (1.9, −0.2995)
Average velocity:
$\dfrac{0-(-0.2995)}{2-1.9} = 2.995$

(d) Second point: (1.99, −0.03)
Average velocity: $\dfrac{0-(-0.03)}{2-1.99} = 3$

(e) The instantaneous velocity seems to be 3.

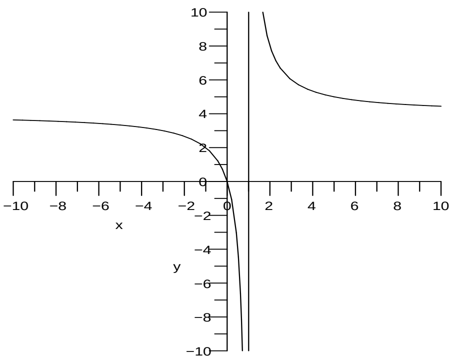
23. A graph makes it apparent that this function has a corner at $x = 1$.



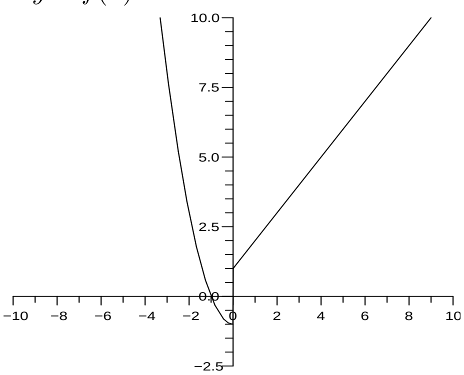
Numerical evidence suggests that,
$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = 1$$
while $\displaystyle\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = -1.$
Since these are not equal, there is no tangent

line.

24. Tangent line does not exist at $x = 1$ because the function is not defined there.



25. From the graph it is clear that, curve is not continuous at $x = 0$ therefore tangent line at $y = f(x)$ at $x = 0$ does not exist.



Also,

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h}$$
$$= \lim_{h\to 0^-} \frac{h^2-1-(-1)}{h}$$
$$= \lim_{h\to 0^-} \frac{h^2}{h} = \lim_{h\to 0^-} h = 0$$

Similarly,

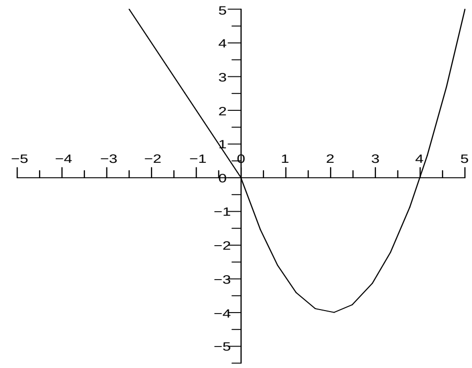
$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$$
$$= \lim_{h\to 0^+} \frac{h+1-(1)}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1.$$

Numerical evidence suggest that,

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h}$$
$$\neq \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}.$$

Therefore tangent line does not exist at $x = 0$.

26. From the graph it is clear that, the curve of $y = f(x)$ is not smooth at $x = 0$ therefore tangent line at $x = 0$ does not exist.



Also,

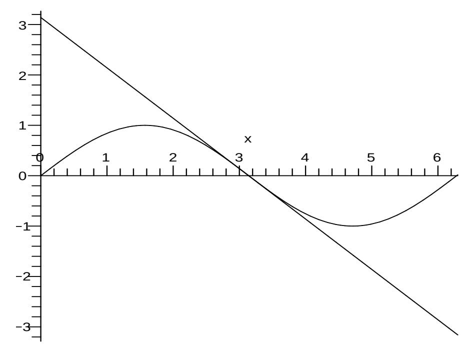
$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{-2h}{h} = -2$$
$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} (h-4) = -4.$$

Numerical evidence suggest that,

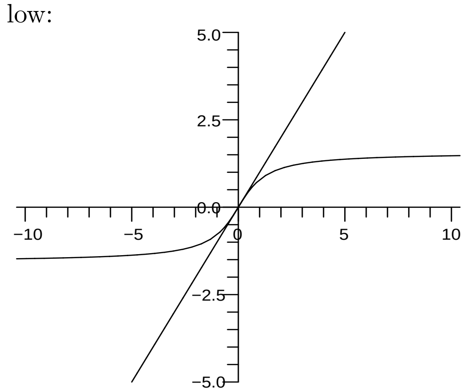
$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h}$$
$$\neq \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}.$$

Therefore tangent line does not exist at $x = 0$.

27. Tangent line at $x = \pi$ to $y = \sin x$ as below:

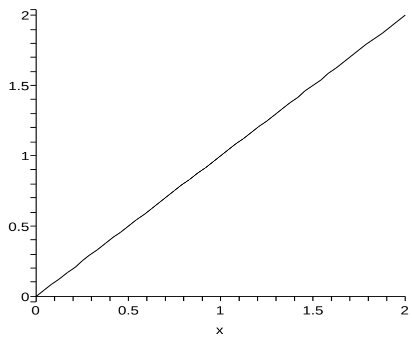


28. Tangent line at $x = 0$ to $y = \tan^{-1} x$ as below:



29. Since the graph has a corner at $x = 0$, tangent line does not exist.

30. The tangent line overlays the line:



31. **(a)** $\dfrac{f(4)-f(2)}{2} = 21{,}034$

Since $\frac{f(b)-f(a)}{b-a}$ is the average rate of change of function f between a and b. The expression tells us that the average rate of change of f between $a = 2$ to $b = 4$ is 21,034. That is the average rate of change in the bank balance between Jan. 1, 2002 and Jan. 1, 2004 was 21,034 ($ per year).

(b) $2\,[f\,(4) - f\,(3.5)] = 25{,}036$

Note that $2[f(4) - f(3.5)] = f(4) - f(3.5)/2$. The expression says that the average rate of change in balance between July 1, 2003 and Jan. 1, 2004 was 25,036 ($ per year).

(c) $\displaystyle\lim_{h\to 0} \frac{f(4+h)-f(4)}{h} = 30{,}000$

The expression says that the instantaneous rate of change in the balance on Jan. 1, 2004 was 30,000 ($ per year).

32. **(a)** $\dfrac{f(40)-f(38)}{2} = -2103$

Since $\frac{f(b)-f(a)}{b-a}$ is the average rate of change of function between a and b. The expression tells us that the average rate of change of f between $a = 38$ to $b = 40$ is -2103. That is the average rate of depreciation between 38 and 40 thousand miles.

(b) $f(40) - f(39) = -2040$

The expression says that the average rate of depreciation between 39 and 40 thousand miles is -2040.

(c) $\displaystyle\lim_{h\to 0} \frac{f(40+h)-f(40)}{h} = -2000$

The expression says that the instantaneous rate of depreciation in the value of the car when it has 40 thousand miles is -2000.

33. $v_{avg} = \dfrac{f(s)-f(r)}{s-r}$

$$
\begin{aligned}
v_{avg} &= \frac{f(s)-f(r)}{s-r} \\
&= \frac{as^2+bs+c-(ar^2+br+c)}{s-r} \\
&= \frac{a(s^2-r^2)+b(s-r)}{s-r} \\
&= \frac{a(s+r)(s-r)+b(s-r)}{s-r} \\
&= a(s+r)+b
\end{aligned}
$$

Let $v(r)$ be the velocity at $t = r$. We have, $v(r) =$

$$
\begin{aligned}
&\lim_{h\to 0} \frac{f(r+h)-f(r)}{h} \\
&= \lim_{h\to 0} \frac{a(r+h)^2+b(r+h)+c-(ar^2+bh+c)}{h} \\
&= \lim_{h\to 0} \frac{a(r^2+2rh+h^2)+bh-ar^2}{h} \\
&= \lim_{h\to 0} \frac{h(2ar+ah+b)}{h} \\
&= \lim_{h\to 0} (2ar+ah+b) = 2ar + \ b
\end{aligned}
$$

So, $v(r) = 2ar + b$.

The same argument shows that $v(s) = 2as + b$.

Finally

$$
\begin{aligned}
\frac{v(r)+v(s)}{2} &= \frac{(2ar+b)+(2as+b)}{2} \\
&= \frac{2a(s+r)+2b}{2} \\
&= a(s+r)+b = v_{avg}
\end{aligned}
$$

34. $f(t) = t^3 - t$ works with $r = 0$, $s = 2$. The average velocity between r and s is, $\dfrac{6-0}{2-0} = 3$.

The instantaneous velocity at r is

$$\lim_{h\to 0} \frac{(0+h)^3-(0+h)-0}{h} = 0$$

and the instantaneous velocity at s is,

$$
\begin{aligned}
&\lim_{h\to 0} \frac{(2+h)^3-(2+h)-6}{h} \\
&= \lim_{h\to 0} \frac{8+12h+6h^2+h^3-2-h-6}{h} \\
&= \lim_{h\to 0} 11+6h+h^2 = 11
\end{aligned}
$$

so, the average between the instantaneous velocities is 5.5.

35. **(a)** $y = x^3 + 3x + 1$

$y' = 3x^2 + 3$

Since the slope needed to be 5, $y' = 5$.

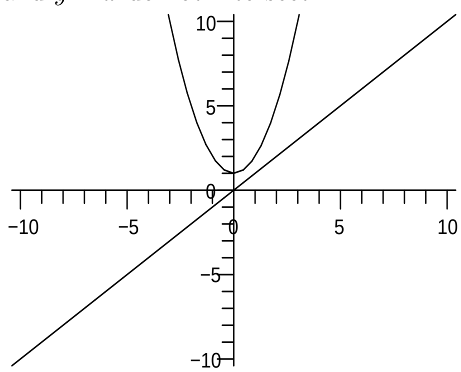
$3x^2 + 3 = 5$

$\Rightarrow 3x^2 = 5 - 3$

$\Rightarrow x^2 = \frac{2}{3}$
$\Rightarrow x = \pm\sqrt{\frac{2}{3}}$
Therefore, slope of tangent line at $x = \sqrt{\frac{2}{3}}$ and $x = -\sqrt{\frac{2}{3}}$ to $y = x^3 + 3x + 1$ equals 5.

(b) Since the slope needed to be 1, $y' = 1$. $3x^2 + 3 = 1$ which has no real roots. Therefore slope of tangent line to $y = x^3 + 3x + 1$ cannot equals 1.

36. (a) From the graph it is clear that $y = x^2+1$ and $y = x$ do not intersect.



(b) $y = x^2 + 1$ and $y = x$
$y = x^2 + 1 \Rightarrow y' = 2x$
$y = x \Rightarrow y' = 1$
For, $y = x^2 + 1$
$y' = 2x = 1.$
$2x = 1$
$\Rightarrow x = \frac{1}{2}$
Therefore, tangent line at $x = \frac{1}{2}$ to $y = x^2+1$ is parallel to the tangent lines to $y = x$.

37. (a) $y = x^3 + 3x + 1$
$\lim_{h\to 0} \frac{f(1+h) - f(x)}{h}$
$= \lim_{h\to 0} \frac{(1+h)^3 + 3(1+h) + 1 - 5}{h}$
$= \lim_{h\to 0} \frac{(1+3h+3h^2+h^3)+(3+3h)+1-5}{h}$
$= \lim_{h\to 0} \frac{6h + 3h^2 + h^3}{h}$
$= \lim_{h\to 0} \frac{h\left(6 + 3h + h^2\right)}{h} = 6$
The point correponding to $x = 1$ is $(1, 5)$. So, line with slope 6 through point (1,5) has equation $y = 6\,(x-1)+5$ or $y = 6x - 1$.

(b) From part (a) we have, equation of tangent line is $y = 6x - 1$.
Given that $y = x^3 + 3x + 1$.
Therefore, we write
$x^3 + 3x + 1 = 6x - 1$
$x^3 - 3x + 2 = 0$
$(x-1)\left(x^2 + x - 2\right) = 0$
$(x-1)(x-1)\,(x+2) = 0$
$(x-1)^2\,(x+2) = 0.$
Therefore, tangent line intersects $y = x^3 + 3x + 1$ at more then one point that is at $x = 1$ and $x = -2$.

(c) $y = x^2 + 1$
$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$
$= \lim_{h\to 0} \frac{(c+h)^2 + 1 - \left(c^2 + 1\right)}{h}$
$= \lim_{h\to 0} \frac{(c^2 + 2ch + h^2) + 1 - \left(c^2 + 1\right)}{h}$
$= \lim_{h\to 0} \frac{c^2 + 2ch + h^2 + 1 - c^2 - 1}{h}$
$= \lim_{h\to 0} \frac{2ch + h^2}{h}$
$= \lim_{h\to 0} \frac{h\,(2c + h)}{h} = 2c$
The point correponding to $x = c$ is $\left(c,\, c^2 + 1\right)$. So, line with slope $2c$ through point $\left(c, c^2 + 1\right)$ has equation $y = 2c\,(x-c)+c^2+1$ or $y = 2cx - c^2 + 1$.
Given that $y = x^2 + 1$
Therefore,
$x^2 + 1 = 2cx - c^2 + 1$
$x^2 - 2cx + c^2 = 0$
$(x-c)^2 = 0.$
Therefore, tangent line intersects $y = x^2 + 1$ only at one point that is at $x = c$.

38. Let $x = h + a$. Then $h = x - a$ and clearly $\frac{f(a+h) - f(a)}{h} = \frac{f(x) - f(a)}{x - a}$.
It is also clear that, $x \to a$ if and only if $h \to 0$. Therefore, if one of the two limits exists, then so does the other and
$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$.

39. The slope of the tangent line at $p = 1$ is approximately
$\frac{-20 - 0}{2 - 0} = -10$
which means that at $p = 1$ the freezing temperature of water decreases by 10 degrees Celsius per 1 atm increase in pressure. The slope of the tangent line at $p = 3$ is approximately

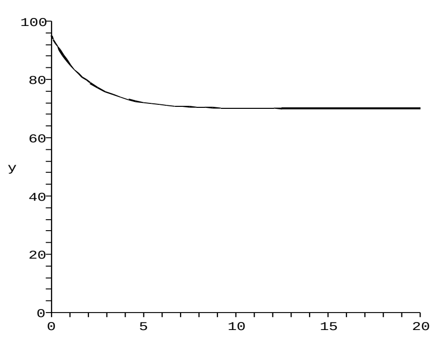
$$\frac{-11-(-20)}{4-2} = 4.5$$
which means that at $p = 3$ the freezing temperature of water increases by 4.5 degrees Celsius per 1 atm increase in pressure.

40. The slope of the tangent line at $v = 50$ is approximately $\frac{47-28}{60-40} = .95$.
This means that at an initial speed of 50mph the range of the soccer kick increases by .95 yards per 1 mph increase in initial speed.

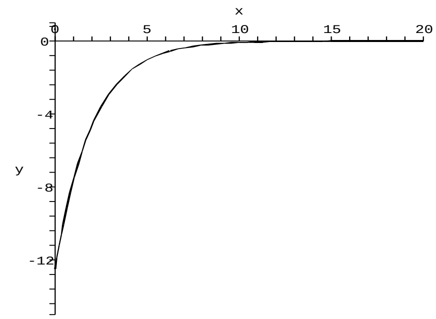
41. The hiker reached the top at the highest point on the graph (about1.75 hours). The hiker was going the fastest on the way up at about 1.5 hours. The hiker was going the fastest on the way down at the point where the tangent line has the least (i.e most negative) slope, at about 4 hours at the end of the hike. Where the graph is level the hiker was either resting or walking on flat ground.

42. The tank is the fullest at the first spike (at around 8 A.M.). The tank is the emptiest just before this at the lowest dip (around 7 A.M.). The tank is filling up the fastest where the graph has the steepest positive slope (in between 7 and 8 A.M.). The tank is emptying the fastest just after 8 A.M. where the graph has the steepest negative slope. The level portions most likely represent night when waterusage is at a minimum.

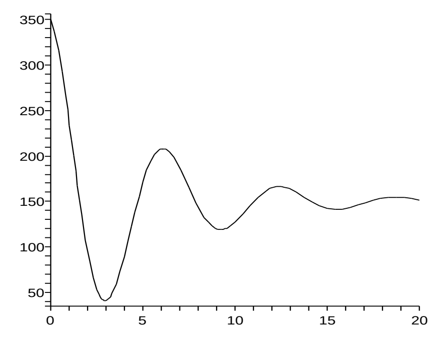
43. A possible graph of the temperature with respect to time:



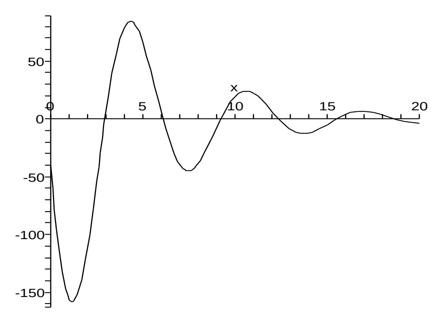
Graph of the rate of change of the temperature:



44. Possible graph of bungee-jumpers height:



A graph of the bungee-jumper s velocity:



2.2 The Derivative

1. Using (2.1):
$$f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$
$$= \lim_{h\to 0}\frac{3(1+h)+1-(4)}{h}$$
$$= \lim_{h\to 0}\frac{3h}{h} = \lim_{h\to 0} 3 = 3$$
Using (2.2):
$$\lim_{b\to 1}\frac{f(b)-f(1)}{b-1}$$
$$= \lim_{b\to 1}\frac{3b+1-(3+1)}{b-1}$$

$$= \lim_{b\to 1} \frac{3b-3}{b-1} = \lim_{b\to 1} \frac{3(b-1)}{b-1} = \lim_{b\to 1} 3 = 3$$

2. Using (2.1):

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{3(1+h)^2+1-4}{h} = \lim_{h\to 0} \frac{6h+3h^2}{h} = \lim_{h\to 0} 6+3h = 6$$

Using (2.2):

$$f'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{(3x^2+1)-4}{x-1} = \lim_{x\to 1} \frac{3(x-1)(x+1)}{x-1} = \lim_{x\to 1} 3(x+1) = 6$$

3. Using (2.1): Since

$$\frac{f(1+h)-f(1)}{h} = \frac{\sqrt{3(1+h)+1}-2}{h} = \frac{\sqrt{4+3h}-2}{h}\cdot\frac{\sqrt{4+3h}+2}{\sqrt{4+3h}+2} = \frac{4+3h-4}{h(\sqrt{4+3h}+2)} = \frac{3h}{h(\sqrt{4+3h}+2)} = \frac{3}{\sqrt{4+3h}+2} = \frac{3}{\sqrt{4+3h}+2},$$

we have

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{3}{\sqrt{4+3h}+2} = \frac{3}{\sqrt{4+3(0)}+2} = \frac{3}{4}$$

Using (2.2): Since

$$\frac{f(b)-f(1)}{b-1} = \frac{\sqrt{3b+1}-2}{b-1} = \frac{(\sqrt{3b+1}-2)(\sqrt{3b+1}+2)}{(b-1)(\sqrt{3b+1}+2)} = \frac{(3b+1)-4}{(b-1)\sqrt{3b+1}+2} = \frac{3(b-1)}{(b-1)\sqrt{3b+1}+2} = \frac{3}{\sqrt{3b+1}+2},$$

we have

$$f'(1) = \lim_{b\to 1} \frac{f(b)-f(1)}{b-1} = \lim_{b\to 1} \frac{3}{\sqrt{3b+1}+2} = \frac{3}{\sqrt{4}+2} = \frac{3}{4}$$

4. Using (2.1):

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\frac{3}{(2+h)+1}-1}{h} = \lim_{h\to 0} \frac{\frac{3}{3+h}-\frac{3+h}{3+h}}{h} = \lim_{h\to 0} \frac{\frac{-h}{3+h}}{h} = \lim_{h\to 0} \frac{-1}{3+h} = -\frac{1}{3}$$

Using (2.2):

$$f'(2) = \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2} \frac{\frac{3}{x+1}-1}{x-2} = \lim_{x\to 2} \frac{\frac{3}{x+1}-\frac{x+1}{x+1}}{x-2} = \lim_{x\to 2} \frac{\frac{-(x-2)}{x+1}}{x-2} = \lim_{x\to 2} \frac{-1}{x+1} = -\frac{1}{3}$$

5.
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{3(x+h)^2+1-(3(x)^2+1)}{h} = \lim_{h\to 0} \frac{3x^2+6xh+3h^2+1-(3x^2+1)}{h} = \lim_{h\to 0} \frac{6xh+3h^2}{h} = \lim_{h\to 0} (6x+3h) = 6x$$

6.
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2-2(x+h)+1-f(x)}{h} = \lim_{h\to 0} \frac{2xh+h^2-2h}{h} = \lim_{h\to 0} \frac{h(2x+h-2)}{h} = 2x-2$$

7.
$$\lim_{b\to x} \frac{f\,(b)-f\,(x)}{b-x} = \lim_{b\to x} \frac{b^3+2b-1-\left(x^3+2x-1\right)}{b-x}$$

$$= \lim_{b\to x} \frac{(b-x)\left(b^2+bx+x^2+2\right)}{b-x}$$
$$= \lim_{b\to x} b^2+bx+x^2+2$$
$$= 3x^2+2$$

8. $f'(x) =$
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0} \frac{(x+h)^4-2(x+h)^2+1-f(x)}{h}$$
$$= \lim_{h\to 0} \left[4x^3+6x^2h+4xh^2+h^3-4x-2h\right]$$
$$= 4x^3-4x$$

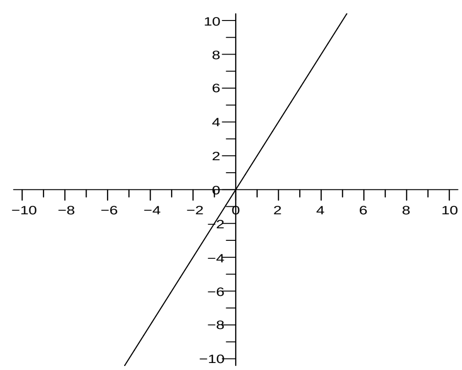
9. $\lim_{b\to x} \frac{f(b)-f(x)}{b-x}$
$$= \lim_{b\to x} \frac{\frac{3}{b+1}-\frac{3}{x+1}}{b-x}$$
$$= \lim_{b\to x} \frac{\frac{3(x+1)-3(b+1)}{(b+1)(x+1)}}{b-x}$$
$$= \lim_{b\to x} \frac{-3(b-x)}{(b+1)(x+1)(b-x)}$$
$$= \lim_{b\to x} \frac{-3}{(b+1)(x+1)}$$
$$= \frac{-3}{(x+1)^2}$$

10. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
$$= \lim_{h\to 0} \frac{\frac{2}{2(x+h)-1}-\frac{2}{2x-1}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{2(2x-1)-2(2x+2h-1)}{(2x+2h-1)(2x-1)}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{-4h}{(2x+2h-1)(2x-1)}}{h}$$
$$= \lim_{h\to 0} \frac{-4}{(2x+2h-1)(2x-1)}$$
$$= \frac{-4}{(2x-1)^2}$$

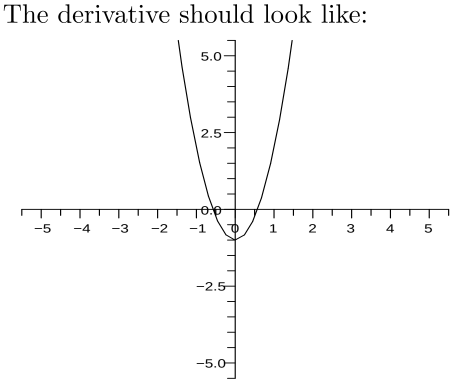
11. $f(t) = \sqrt{3t+1}$
$$f'(t) = \lim_{b\to t} \frac{f(b)-f(t)}{(b-t)}$$
$$= \lim_{b\to t} \frac{\sqrt{3b+1}-\sqrt{3t+1}}{(b-t)}$$
Multiplying by $\frac{\sqrt{3b+1}+\sqrt{3t+1}}{\sqrt{3b+1}+\sqrt{3t+1}}$ gives
$$f'(t) = \lim_{b\to t} \frac{(3b+1)-(3t+1)}{(b-t)\left(\sqrt{3b+1}+\sqrt{3t+1}\right)}$$
$$= \lim_{b\to t} \frac{3(b-t)}{(b-t)\left(\sqrt{3b+1}+\sqrt{3t+1}\right)}$$
$$= \lim_{b\to t} \frac{3}{\sqrt{3b+1}+\sqrt{3t+1}}$$
$$= \frac{3}{2\sqrt{3t+1}}$$

12. $f(t) = \sqrt{2t+4}$
$$f'(t) = \lim_{b\to t} \frac{f(b)-f(t)}{(b-t)}$$
$$= \lim_{b\to t} \frac{\sqrt{2b+4}-\sqrt{2t+4}}{(b-t)}$$
Multiplying by $\frac{\sqrt{2b+4}+\sqrt{2t+4}}{\sqrt{2b+4}+\sqrt{2t+4}}$ gives
$$f'(t) = \lim_{b\to t} \frac{(2b+4)-(2t+4)}{(b-t)\left(\sqrt{2b+4}+\sqrt{2t+4}\right)}$$
$$= \lim_{b\to t} \frac{2(b-t)}{(b-t)\left(\sqrt{2b+4}+\sqrt{2t+4}\right)}$$
$$= \lim_{b\to t} \frac{2}{\sqrt{2b+4}+\sqrt{2t+4}}$$
$$= \frac{2}{2\sqrt{2t+4}} = \frac{1}{\sqrt{2t+4}}$$

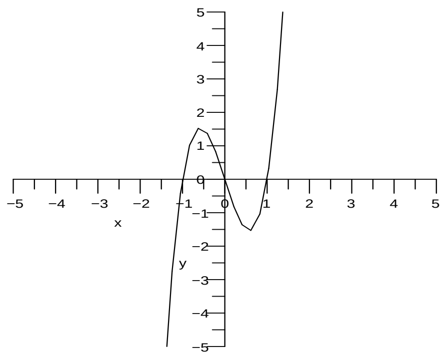
13. **(a)** The derivative should look like:



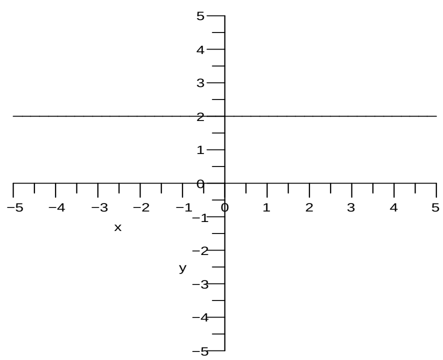
(b) The derivative should look like:



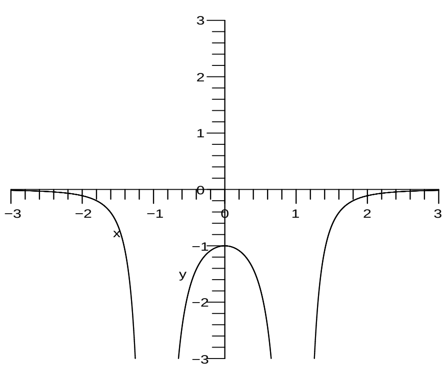
14. (a) The derivative should look like:



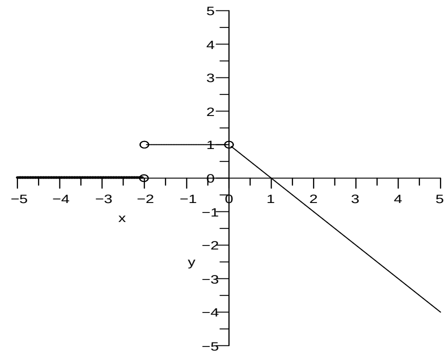
(b) The derivative should look like:



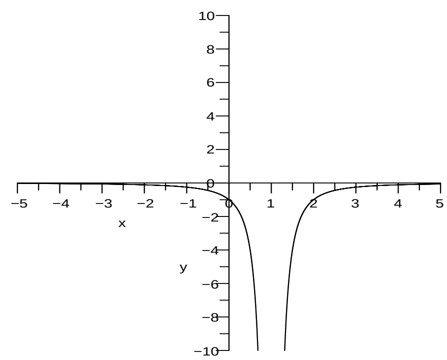
15. (a) The derivative should look like:



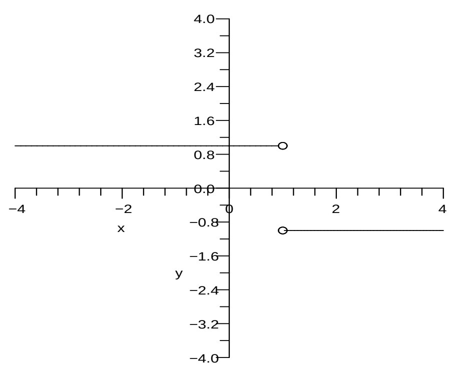
(b) The derivative should look like:



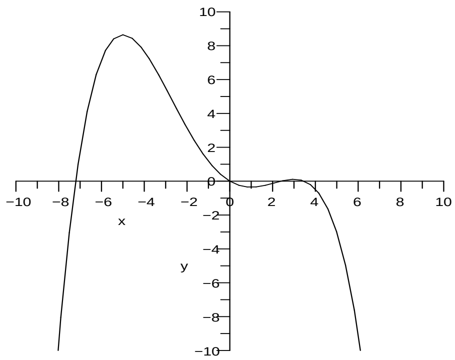
16. (a) The derivative should look like:



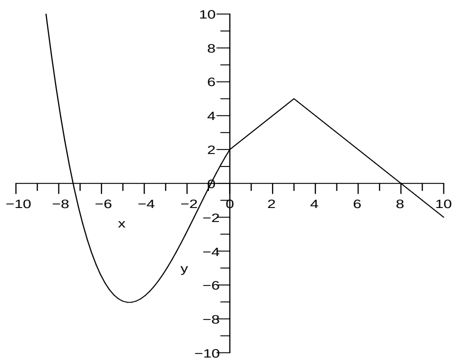
(b) The derivative should look like:



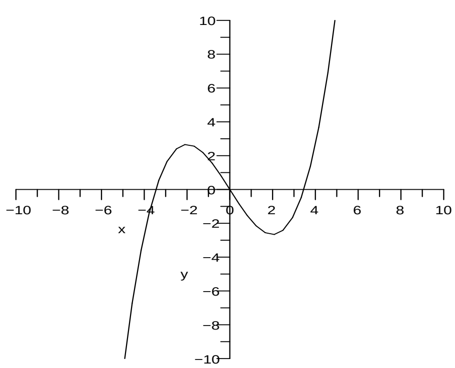
17. (a) The function should look like:



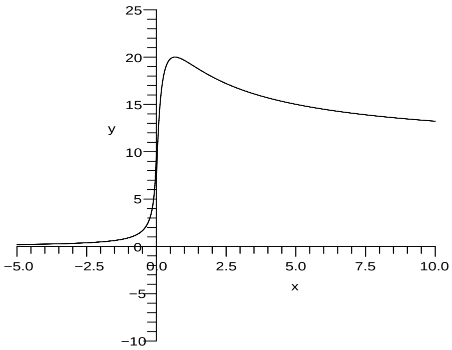
(b) The function should look like:



18. **(a)** The function should look like:



(b) The function should look like:



19. The left-hand derivative is

$$\begin{aligned} D_{-}f(0) &= \lim_{h\to 0^-} \frac{f(h)-f(0)}{h} \\ &= \lim_{h\to 0^-} \frac{2h+1-1}{h} = 2 \end{aligned}$$

The right-hand derivative is

$$\begin{aligned} D_{+}f(0) &= \lim_{h\to 0^+} \frac{f(h)-f(0)}{h} \\ &= \lim_{h\to 0^+} \frac{3h+1-1}{h} = 3 \end{aligned}$$

Since the one-sided limits do not agree ($2 \neq 3$), $f'(0)$ does not exist.

20. The left-hand derivative is

$$\begin{aligned} \mathrm{D}_{-}f(0) &= \lim_{h\to 0^+} \frac{f(h)-f(0)}{h} \\ &= \lim_{h\to 0^-} \frac{0-0}{h} = 0 \end{aligned}$$

The right-hand derivative is

$$\begin{aligned} \mathrm{D}_{+}f(0) &= \lim_{h\to 0^-} \frac{f(h)-f(0)}{h} \\ &= \lim_{h\to 0^+} \frac{2h}{h} = 2 \end{aligned}$$

Since the one-sided limits do not agree ($0 \neq 2$), $f'(0)$ does not exist.

21. The left-hand derivative is

$$\begin{aligned} D_{-}f(0) &= \lim_{h\to 0^-} \frac{f(h)-f(0)}{h} \\ &= \lim_{h\to 0^-} \frac{h^2-0}{h} = 0 \end{aligned}$$

The right-hand derivative is

$$D_+f(0) = \lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h^3-0}{h} = 0$$

Since the one-sided limits are same $(0 = 0)$, $f'(0)$ exist.

22. The left-hand derivative is

$$\mathrm{D}_-f(0) = \lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^-} \frac{2h}{h} = 2$$

The right-hand derivative is

$$\mathrm{D}_+f(0) = \lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h^2+2h}{h} = \lim_{h\to 0^+} \frac{h(h+2)}{h} = \lim_{h\to 0^+} h+2 = 2$$

Since the one-sided limits are same $(2 = 2)$, $f'(0)$ exist.

23. $f(x) = \dfrac{x}{\sqrt{x^2+1}}$

x	$f(x)$	$\frac{f(x)-f(1)}{x-1}$
1.1	0.7399401	0.3283329
1.01	0.7106159	0.3509150
1.001	0.7074601	0.3532884
1.0001	0.7071421	0.3535268
1.00001	0.7071103	0.3535507

The evidence of this table strongly suggests that the difference quotients (essentially) indistinguishable from the values (themselves) 0.353. If true, this would mean that $f'(1) \approx 0.353$.

24. $f(x) = xe^{x^2}$

x	$f(x)$	$\frac{f(x)-f(2)}{x-2}$
1.1	172.7658734	635.6957329
1.01	114.2323717	503.6071639
1.001	109.6888867	492.5866054
1.0001	109.2454504	491.5034872
1.00001	109.201214	491.3953621
1.000001	109.1967915	491.3845515
1.0000001	109.1963492	491.3834702
1.00000001	109.1963050	491.3833622

The evidence of this table strongly suggests that the difference quotients (essentially) indistinguishable from the values (themselves) 491.383. If true, this would mean that $f'(2) \approx 491.383$.

25. $f(x) = \cos 3x$

x	$f(x)$	$\frac{f(x)-f(0)}{x-0}$
0.1	0.9553365	−0.4466351
0.01	0.9995500	−0.0449966
0.001	0.9999955	−0.0045000
0.0001	1.0000000	−0.0004500
0.00001	1.0000000	−0.0000450

The evidence of this table strongly suggests that the difference quotients (essentially) indistinguishable from the values (themselves) 0. If true, this would mean that $f'(0) \approx 0$.

26. $f(x) = \ln 3x$

x	$f(x)$	$\frac{f(x)-f(2)}{x-2}$
2.1	1.8405496	0.4879016
2.01	1.7967470	0.4987542
2.001	1.7922593	0.4998757
2.0001	1.7918095	0.4999875
2.00001	1.7917645	0.4999988
2.000001	1.7917600	0.4999999
2.0000001	1.7917595	0.5000000

The evidence of this table strongly suggests that the difference quotients (essentially) indistinguishable from the values (themselves) 0.5. If true, this would mean that $f'(2) \approx 0.5$.

27. Compute average velocities:

Time Interval	Average Velocity
(1.7, 2.0)	9.0
(1.8, 2.0)	9.5
(1.8, 2.0)	10.0
(2.0, 2.1)	10.0
(2.0, 2.2)	9.5
(2.0, 2.3)	9.0

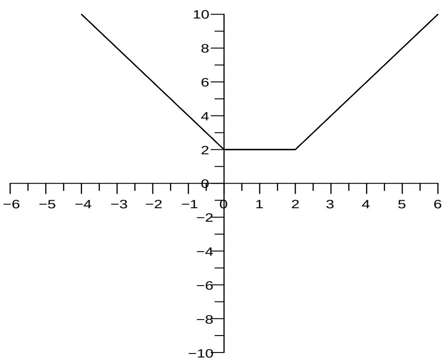
Our best estimate of velocity at $t = 2$ is 10.

28. Compute average velocities:

Time Interval	Average Velocity
(1.7, 2.0)	8
(1.8, 2.0)	8.5
(1.8, 2.0)	9.0
(2.0, 2.1)	8.0
(2.0, 2.2)	8.0
(2.0, 2.3)	7.67

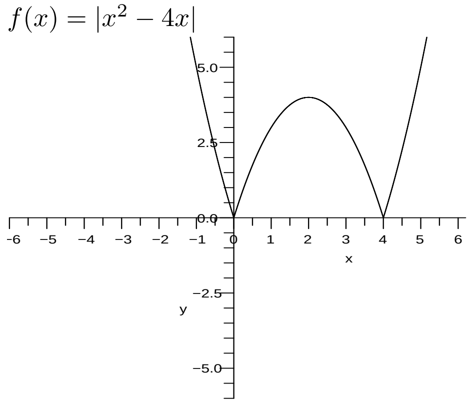
A velocity of between 8 and 9 seems to be a good guess.

29. **(a)** $f(x) = |x| + |x - 2|$



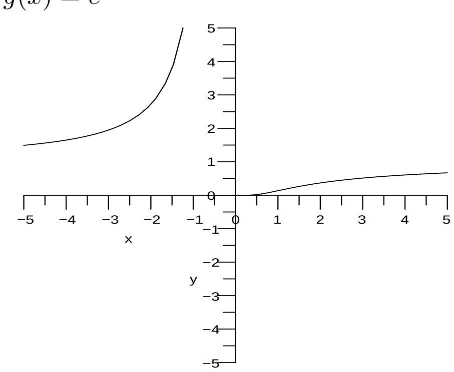
$f(x)$ is not differentiable at $x = 0$ and $x = 2$.

(b) $f(x) = |x^2 - 4x|$



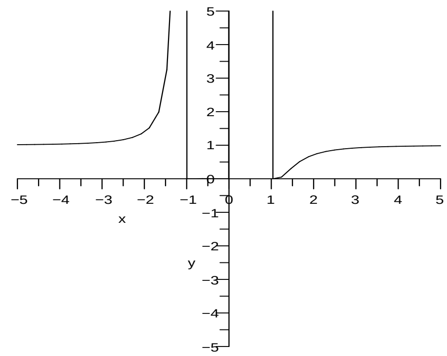
$f(x)$ is not differentiable at $x = 0$ and $x = 4$.

30. **(a)** $g(x) = e^{-2/x}$



$g(x)$ is not differentiable at $x = 0$.

(b) $g(x) = e^{-2/(x^3 - x)}$



$g(x)$ is not differentiable at $x = 0$ and $x = \pm 1$.

31. $\lim\limits_{h \to 0} \dfrac{(0+h)^p - 0^p}{h} = \lim\limits_{h \to 0} \dfrac{h^p}{h} = \lim\limits_{h \to 0} h^{p-1}$

The last limit does not exist when $p < 1$, equals 1 when $p = 1$ and is 0 when $p > 1$. Thus $f(0)$ exists when $p \geq 1$.

32. $f(x) = \begin{cases} x^2 + 2x & x < 0 \\ ax + b & x \geq 0 \end{cases}$

For $h < 0$, $f(h) = h^2 + 2h$, $f(0) = b$

$$D_-f(0) = \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{h^2 + 2h - b}{h}$$

For f to be differentiable $D_-f(0)$ must exist.

$D_-f(0)$ exists if and only if $b = 0$.

Substituting $b = 0$, we get

$$D_-f(0) = \lim_{h \to 0^-} \frac{h^2 + 2h}{h} = \lim_{h \to 0^-} (h + 2) = 2$$

For $h > 0$, $f(h) = ah + b$, $f(0) = b$

$$D_+f(0) = \lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{ah + b - b}{h} = \lim_{h \to 0^+} \frac{ah}{h} = a$$

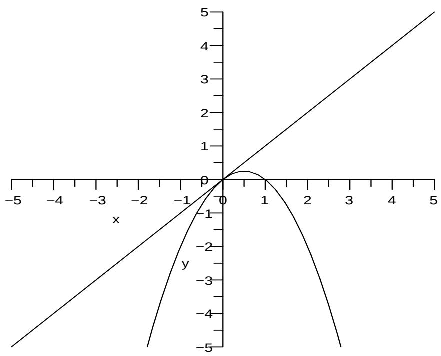
$D_+f(0) = 2$ if and only if $a = 2$.

33. Let $f(x) = -1 - x^2$ then for all, we have $f(x) \leq x$. But at $x = -1$, we find $f(-1) = -2$ and

$$f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \to 0} \frac{-1 - (-1+h)^2 - (-2)}{h} = \lim_{h \to 0} \frac{1 - (1 - 2h + h^2)}{h} = \lim_{h \to 0} \frac{2h - h^2}{h} = \lim_{h \to 0} (2 - h) = 2.$$

So, $f'(x)$ is not always less than 1.

34. This is not always true. For example the function $f(x) = -x^2 + x$ satisfies the hypotheses but $f'(x) > 1$ for all $x < 0$ as the following graph shows.



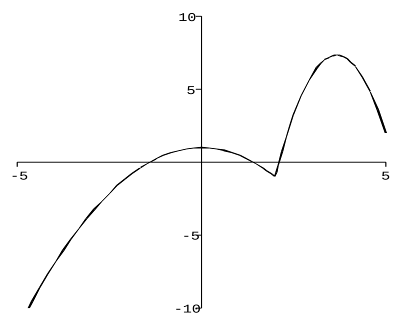
35. $\lim_{x \to a} \dfrac{[f(x)]^2 - [f(a)]^2}{x^2 - a^2}$
$= \lim_{x \to a} \dfrac{[f(x) - f(a)]\,[f(x) + f(a)]}{(x-a)(x+a)}$
$= \left[\lim_{x \to a} \dfrac{[f(x) - f(a)]}{(x-a)}\right]\left[\lim_{x \to a} \dfrac{[f(x) + f(a)]}{(x+a)}\right]$
$= f'(a)\dfrac{2f(a)}{2a} = \dfrac{f(a)f'(a)}{a}$

36. Let $u = ch$ so $h = \dfrac{u}{c}$. Then we have
$\lim_{h \to 0} \dfrac{f(a+ch) - f(a)}{h}$
$= \lim_{\frac{u}{c} \to 0} \dfrac{f(a+u) - f(a)}{\frac{u}{c}}$
$= \lim_{u \to 0} \dfrac{f(a+u) - f(a)}{\frac{u}{c}}$
$= \lim_{u \to 0} c\left(\dfrac{f(a+u) - f(a)}{u}\right)$
$= c\lim_{u \to 0} \dfrac{f(a+u) - f(a)}{u}$
$= cf'(a)$

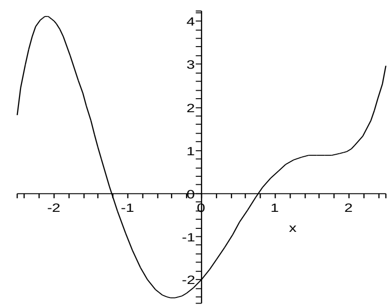
37. Because the curve appears to be bending upward, the slopes of thesecant lines (based at $x = 1$ and with upper endpoint beyond 1) will increase with the upper endpoint. This has also the effect that any one of these slopes is greater than the actual derivative. Therefore $f'(1) < \dfrac{f(1.5) - f(1)}{0.5} < \dfrac{f(2) - f(1)}{1}$. As to where $f(1)$ fits in this list it seems necessary to read the graph and come up with estimates of $f(1)$ about 4, and $f(2)$ about 7. That would put the third number in the above list at about 3, comfortably less than $f(1)$.

38. Note that $f(0) - f(-1)$ is the slope of the secant line from $x = -1$ to $x = 0$ (about), and that $\dfrac{f(0) - f(-0.5)}{0.5}$ is the slope of the secant line from $x = -0.5$ to $x = 0$ (about-0.5). $f(0) = 3$ and $f'(0) = 0$. In increasing order, we have $f(0) - f(-1)$, $\dfrac{f(0) - f(-0.5)}{0.5}$ $f'(0)$, and $f'(0)$.

39. One possible such graph:



40. One possible such graph:



41. $\dfrac{d}{dx}\left(x^2\right) = 2x = 2x^1$
$\dfrac{d}{dx}\left(x^3\right) = 3x^2 4$
$\dfrac{d}{dx}\left(x^4\right) = 4x^3$
In general
$\dfrac{d}{dx}\left(x^n\right) = nx^{n-1}$

42. $\dfrac{d}{dx}\left(x^n\right) = nx^{n-1}$
$\sqrt{x} = x^{1/2}$
$\dfrac{d}{dx}\left(\sqrt{x}\right) = \dfrac{d}{dx}\left(x^{1/2}\right) = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$
$\dfrac{1}{x} = x^{-1}$
$\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \dfrac{d}{dx}\left(x^{-1}\right) = -1x^{-2} = \dfrac{-1}{x^2}$

43. We estimate the derivative at $x = 2.5$ as follows $\dfrac{1.62 - 1.11}{2.7 - 2.39} = \dfrac{0.51}{0.31} = 1.64516$.
For every increase of 1 meter in height of

serving point, there is an increase of 1.64516° in margin of error.

44. We estimate the derivative at $x = 2.854$ as follows
$$\frac{2.12 - 1.62}{3 - 2.7} = \frac{0.5}{0.3} = 1.66666.$$
For every increase of 1 meter in height of serving point, there is an increase of 1.66666°in margin of error.

45. We compile the rate of change in ton-MPG over each of the four two-year intervals for which data is given:

Intervals	Rate of change
(1992,1994)	0.4
(1994,1996)	0.4
(1996,1998)	0.4
(1998,2000)	0.2

These rates of change are measured in Ton-MPG per year. Either the first or second (they happen to agree) could used as an estimate for the one-year "1994"while only the is a promising estimate for the one-year interval "2000". The mere that all these numbers are positive that efficiency is improving, the last number being smaller to suggest that the rate of improvement is slipping.

46. The average rate of change from 1992 to 1994 is 0.05, and from 1994 to 1996 is 0.1, so a good estimate of the rate of change in 1994 is 0.75. The average rate of change from 1998 to 2000 is -0.2, and this is a good for the rate of change in 2000. Comparing to exercise 35, we see that the MPG per ton increased, but the actual MPG for vehicles decreased. The weight of vehicles must have increased, if the weight remained then the actual MPG would have increased.

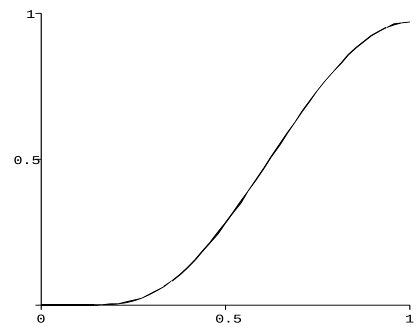
47. **(a)** meters per second
(b) items per dollar

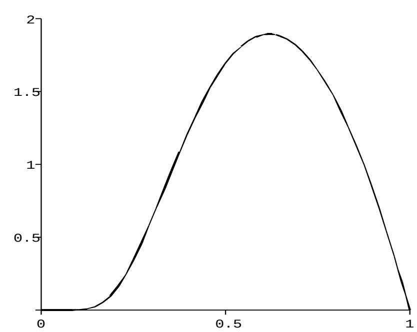
48. $c'(t)$ will represent the rate of change in amount of chemical and will be measured in grams per minute. $p'(x)$ will represent the rate of change of mass and will be measured in kg per meter.

49. If $f'(t) < 0$, the function is negatively sloped and decreasing, meaning the stock is losing value with the passing of time. This may be the basis for selling the stock if the current trend is expected to be a long term one.

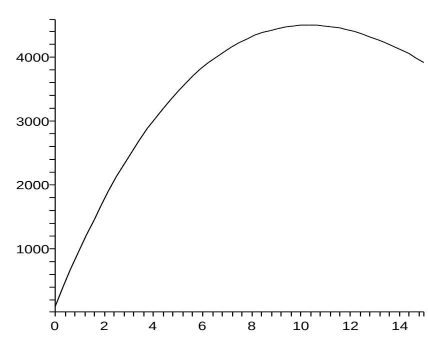
50. You should buy the stock with value $g(t)$. It is cheaper because $f(t) > g(t)$, and growing faster because $f'(t) < g'(t)$ (or possibly declining more slowly).

51. The following sketches are consistent with the hypotheses of infection $I'(t)$ rate rising, peaking and returning to zero. We started with the derivative (infection rate) and had to think backwards to construct the function $I(t)$. One can see in $I(t)$ the slope increasing up to the time of peak infection rate thereafter the slope decreasing but not the values. They merely level off.

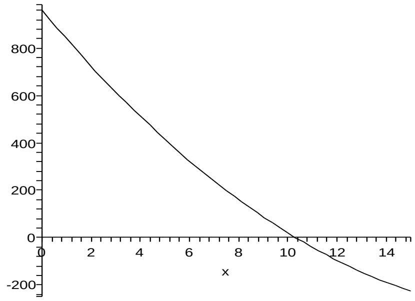




52. One possible graph of the population $P(t)$:



Graph of $P'(t)$:



53. Because the curve appears to be bending upward the slopes of the secant lines (based at $x = 1$ and with upper endpoint beyond 1) will increase with the upper endpoint. This has also the effect that any one of these slopes is greater than the actual actual derivative. Therefore

$$f'(1) < \frac{f(1.5) - f(1)}{0.5} < \frac{f(2) - f(1)}{1}.$$

As to where $f(1)$ fits in this list it seems necessary to read the graph and come up with estimates of $f(1)$ about 4, and $f(2)$ about 7. That would put the third number in the above list at about 3 comfortablyless than $f(1)$.

54. $f(t) = \begin{cases} 0.1t \text{ if } 0 < t \leq 2 \cdot 10^4 \\ 2 \cdot 10^3 + (t - 2 \cdot 10^4)0.16 \text{ if } t > 2 \cdot 10^4 \end{cases}$
This is another example of a piecewise linear function (this one is continuous), and although not differentiable at the income $x = 20000$, elsewhere we have
$f'(x) = \begin{cases} 0.1 & 0 < t < 20000 \\ 0.16 & t > 20000 \end{cases}$

2.3 Computation of Derivatives: The Power Rule

1. $$\begin{aligned} f'(x) &= \frac{d}{dx}(x^3) - \frac{d}{dx}(2x) + \frac{d}{dx}(1) \\ &= 3x^2 - 2\frac{d}{dx}(x) + 0 \\ &= 3x^2 - 2(1) \\ &= 3x^2 - 2 \end{aligned}$$

2. $f'(x) = 9x^8 - 15x^4 + 8x - 4$

3. $$\begin{aligned} f'(t) &= \frac{d}{dt}(3t^3) - \frac{d}{dt}\left(2\sqrt{t}\right) \\ &= 3\frac{d}{dt}(t^3) - 2\frac{d}{dt}\left(t^{1/2}\right) \\ &= 3(3t^2) - 2\left(\frac{1}{2}t^{-1/2}\right) \\ &= 9t^2 - \frac{1}{\sqrt{t}} \end{aligned}$$

4. $$\begin{aligned} f(s) &= 5s^{1/2} - 4s^2 + 3 \\ f'(s) &= \frac{5}{2}s^{-1/2} - 8s \\ &= \frac{5}{2\sqrt{s}} - 8s \end{aligned}$$

5. $$\begin{aligned} f'(w) &= \frac{d}{dw}\left(\frac{3}{w}\right) - \frac{d}{dw}(8w) + \frac{d}{dw}(1) \\ &= 3\frac{d}{dw}\left(w^{-1}\right) - 8\frac{d}{dw}(w) + 0 \\ &= 3\left(-w^{-2}\right) - 8(1) \\ &= -\frac{3}{w^2} - 8 \end{aligned}$$

6. $$\begin{aligned} f'(y) &= \frac{d}{dy}\left(\frac{2}{y^4}\right) - \frac{d}{dy}\left(y^3\right) + \frac{d}{dy}(2) \\ &= 2\frac{d}{dy}\left(y^{-4}\right) - \frac{d}{dy}\left(y^3\right) + 0 \\ &= 2\left(-4y^{-5}\right) - 3\left(y^2\right) \\ &= -\frac{8}{y^5} - 3y^2 \end{aligned}$$

7. $$\begin{aligned} h'(x) &= \frac{d}{dx}\left(\frac{10}{\sqrt[3]{x}}\right) - \frac{d}{dx}(2x) + \frac{d}{dx}(\pi) \\ &= 10\frac{d}{dx}\left(x^{-1/3}\right) - 2\frac{d}{dx}(x) + 0 \\ &= 10\left(-\frac{1}{3}x^{-4/3}\right) - 2 \\ &= -\frac{10}{3x\sqrt[3]{x}} - 2 \end{aligned}$$

8. $$\begin{aligned} h'(x) &= \frac{d}{dx}(12x) - \frac{d}{dx}\left(x^2\right) - \frac{d}{dx}\left(\frac{3}{\sqrt[3]{x^2}}\right) \\ &= 12\frac{d}{dx}(x) - \frac{d}{dx}\left(x^2\right) - 3\frac{d}{dx}\left(x^{-2/3}\right) \\ &= 12 - 2x - 3\left(-\frac{2}{3}x^{-5/3}\right) \\ &= 12 - 2x + \frac{2}{x\sqrt[3]{x^2}} \end{aligned}$$

9. $f'(s) = \frac{d}{ds}2s^{3/2} - \frac{d}{ds}\left(3s^{-1/3}\right)$

$= 2\frac{d}{ds}\left(s^{3/2}\right) - 3\frac{d}{ds}\left(s^{-1/3}\right)$

$= 2\left(\frac{3}{2}s^{1/2}\right) - 3\left(-\frac{1}{3}s^{-4/3}\right)$

$= 3s^{1/2} + s^{-4/3}$

$= 3\sqrt{s} + \frac{1}{\sqrt[3]{s^4}}$

10. $f'(t) = 3\pi t^{\pi-1} - 2.6t^{0.3}$

11. $f(x) = \frac{3x^2 - 3x + 1}{2x}$

$= \frac{3x^2}{2x} - \frac{3x}{2x} + \frac{1}{2x}$

$= \frac{3}{2}x - \frac{3}{2} + \frac{1}{2}x^{-1}$

$f'(x) = \frac{d}{dx}\left(\frac{3}{2}x\right) - \frac{d}{dx}\left(\frac{3}{2}\right) + \frac{d}{dx}\left(\frac{1}{2}x^{-1}\right)$

$= \frac{3}{2}\frac{d}{dx}(x) - 0 + \frac{1}{2}\frac{d}{dx}(x^{-1})$

$= \frac{3}{2}(1) + \frac{1}{2}(-1x^{-2})$

$= \frac{3}{2} - \frac{1}{2x^2}$

12. $f(x) = \frac{4x^2 - x + 3}{\sqrt{x}}$

$= 4x^{3/2} - x^{1/2} + 3x^{-1/2}$

$f'(x) = 6x^{1/2} - \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{-3/2}$

13. $f(x) = x(3x^2 - \sqrt{x})$

$= 3x^3 - x^{3/2}$

$f'(x) = 3\frac{d}{dx}(x^3) - \frac{d}{dx}\left(x^{3/2}\right)$

$= 3(3x^2) - \left(\frac{3}{2}x^{1/2}\right)$

$= 9x^2 - \frac{3}{2}\sqrt{x}$

14. $f(x) = 3x^3 + 3x^2 - 4x - 4,$

$f'(x) = 9x^2 + 6x - 4$

15. $f'(t) = \frac{d}{dt}\left(t^4 + 3t^2 - 2\right)$

$= 4t^3 + 6t$

$f''(t) = \frac{d}{dt}\left(4t^3 + 6t\right)$

$= 12t^2 + 6$

16. $f(t) = 4t^2 - 12 + \frac{4}{t^2} = 4t^2 - 12 + 4t^{-2}$

$f'(t) = \frac{d}{dt}(4t^2 - 12 + 4t^{-2})$

$= 8t^2 - 0 + 4(-2t^{-3}) = 8t^2 - 8t^{-3}$

$f''(t) = \frac{d}{dt}(8t - 8t^{-3}) = 8 - 8(-3t^{-4})$

$= 8 + 24t^{-4}$

$f'''(t) = \frac{d}{dt}(8 + 24t^{-4}) = 0 + 24(-4t^{-5})$

$= -96t^{-5} = -\frac{96}{t^5}$

17. $f(x) = 2x^4 - 3x^{-1/2}$

$\frac{df}{dx} = 8x^3 + \frac{3}{2}x^{-3/2}$

$\frac{d^2f}{dx^2} = 24x^2 - \frac{9}{4}x^{-5/2}$

18. $f(x) = x^6 - \sqrt{x} = x^6 - x^{1/2}$

$\frac{df}{dx} = \frac{d}{dx}\left(x^6 - x^{1/2}\right) = 6x^5 - \frac{1}{2}x^{-1/2}$

$\frac{d^2f}{dx^2} = \frac{d}{dx}\left(6x^5 - \frac{1}{2}x^{-1/2}\right)$

$= 30x^4 - \frac{1}{2}\left(-\frac{1}{2}x^{-3/2}\right)$

$= 30x^4 + \frac{1}{4}x^{-3/2}$

19. $f'(x) = \frac{d}{dx}\left(x^4 + 3x^2 - \frac{2}{\sqrt{x}}\right)$

$= 4x^3 + 6x + x^{-3/2}$

$f''(x) = \frac{d}{dx}\left(4x^3 + 6x + x^{-3/2}\right)$

$= 12x^2 + 6 - \frac{3}{2}x^{-5/2}$

$f'''(x) = \frac{d}{dx}\left(12x^2 + 6 - \frac{3}{2}x^{-5/2}\right)$

$= 24x + \frac{15}{4}x^{-7/2}$

$f^4(x) = \frac{d}{dx}\left(24x + \frac{15}{4}x^{-7/2}\right)$

$= 24 - \frac{105}{8}x^{-9/2}$

20. $f'(x) = 10x^9 - 12x^3 + 2$
$f''(x) = 90x^8 - 36x^2$
$f'''(x) = 720x^7 - 72x$
$f^{(4)}(x) = 5040x^6 - 72$
$f^{(5)}(x) = 30240x^5$

21. $s(t) = -16t^2 + 40t + 10$
$v(t) = s'(t) = -32t + 40$
$a(t) = v'(t) = s''(t) = -32$

22. $s(t) = -4.9t^2 + 12t - 3$
$v(t) = s'(t) = -9.8t + 12$
$a(t) = v'(t) = s''(t) = -9.8$

23. $s(t) = \sqrt{t} + 2t^2 = t^{1/2} + 2t^2$
$v(t) = s'(t) = \dfrac{1}{2}t^{-1/2} + 4t$
$a(t) = v'(t) = s''(t) = -\dfrac{1}{4}t^{-3/2} + 4$

24. $s(t) = 10 - 10t^{-1}$ $v(t) = s'(t) = 10t^{-2}$
$a(t) = s''(t) = -20t^{-3}$

25. $h\,(t) = -16t^2 + 40t + 5$
$v\,(t) = h'\,(t) = -32t + 40$
$a\,(t) = v'\,(t) = h''\,(t) = -32$

(a) At time $t_0 = 1$
$v\,(1) = 8$, object is going up.
$a\,(1) = -32$, speed is decreasing.

(b) At time $t_0 = 2$
$v\,(2) = -24$, object is going down.
$a\,(2) = -32$, speed is increasing.

26. $h\,(t) = 10t^2 - 24t$
$v\,(t) = h'\,(t) = 20t - 24$
$a\,(t) = v'\,(t) = h''\,(t) = 20$

(a) At time $t_0 = 2$
$v\,(2) = 16$, object is going up.
$a\,(2) = 20$, speed is increasing.

(b) At time $t_0 = 1$
$v\,(1) = -4$, object is going down.
$a\,(1) = 20$, speed is decreasing.

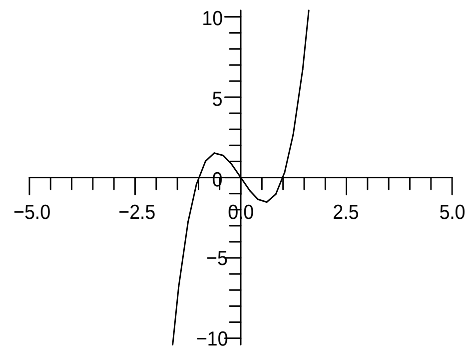
27. $f(x) = x^2 - 2,\ a = 2,\ f(2) = 2,$
$f'(x) = 2x,\ f'(2) = 4$
The equation of the tangent line is
$y = 4(x-2) + 2$ or $y = 4x - 6$.

28. $f(2) = 1,\ f'(x) = 2x - 2,\ f'(2) = 2$
Line through with slope 2 is
$y = 2(x-2) + 1$.

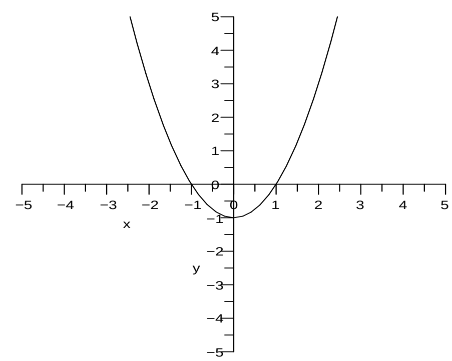
29. $f(x) = 4\sqrt{x} - 2x,\ a = 4$
$f(4) = 4\sqrt{4} - 2(4) = 0.$
$f'(x) = \dfrac{d}{dx}\left(4x^{1/2} - 2x\right)$
$\quad = 2x^{-1/2} - 2 = \dfrac{2}{\sqrt{x}} - 2$
$f'(4) = 1 - 2 = -1$
The equation of the tangent line is
$y = -1\,(x-4) + 0$ or $y = -x + 4$.

30. $f\,(x) = 3\sqrt{x} + 4,\ a = 2$
$f\,(2) = 3\sqrt{2} + 4$
$f'\,(x) = \dfrac{d}{dx}\left(3x^{1/2} + 4\right) = \dfrac{3}{2}x^{-1/2} = \dfrac{3}{2\sqrt{x}}$
$f'\,(2) = \dfrac{3}{2\sqrt{2}}$
The equation of tangent line through $\left(2,\ 3\sqrt{2} + 4\right)$ with slope $\dfrac{3}{2\sqrt{2}}$ is
$y = \dfrac{3}{2\sqrt{2}}\,(x-2) + 3\sqrt{2} + 4.$

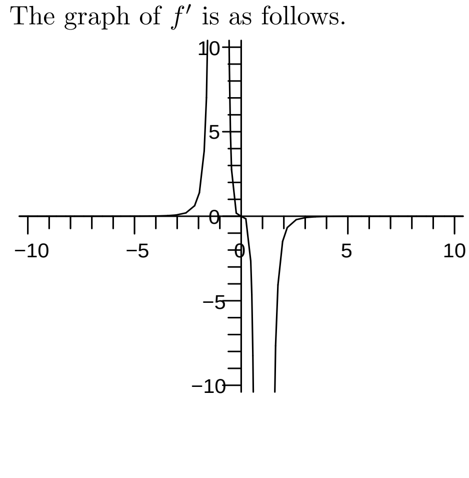
31. **(a)** The graph of f' is as follows:



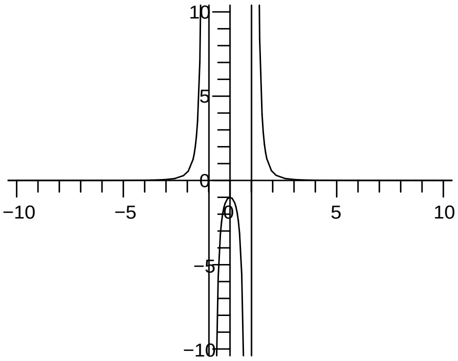
The graph of f'' is as follows.



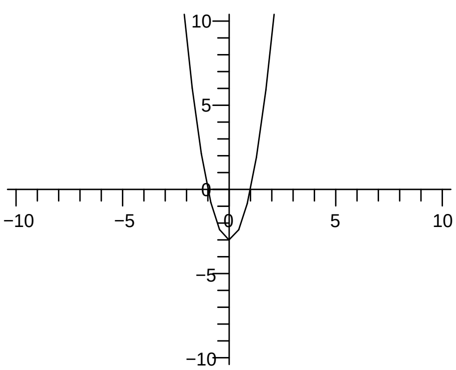
(b) The graph of f' is as follows.



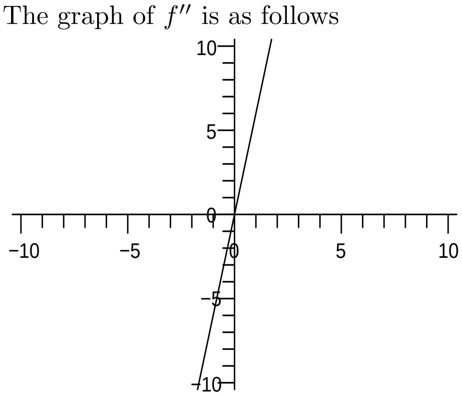
The graph of f'' is as follows.



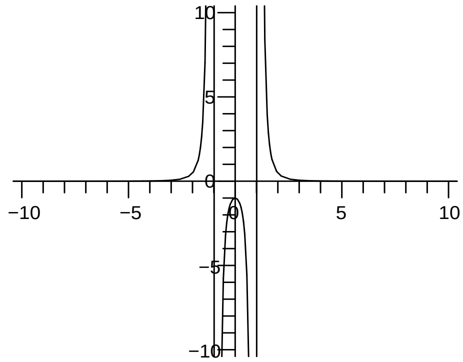
32. **(a)** The graph of f' is as follows.



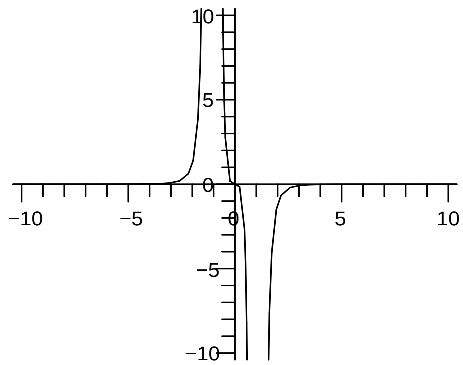
The graph of f'' is as follows



(b) The graph of f' is as follows.

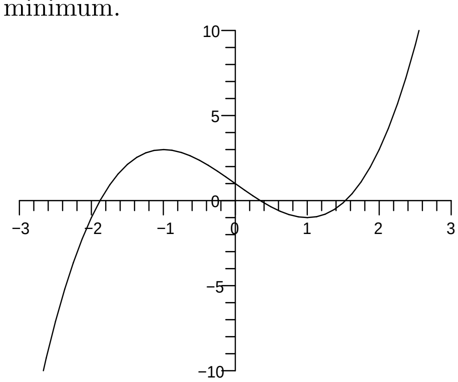


The graph of f'' is as follows.



33. **(a)** $f(x) = x^3 - 3x + 1$
$f'(x) = 3x^2 - 3$
The tangent line to $y = f(x)$ is horizontal when
$f'(x) = 0$
$\Rightarrow 3x^2 - 3 = 0$
$\Rightarrow 3(x^2 - 1) = 0$
$\Rightarrow 3(x + 1)(x - 1) = 0$
$x = -1$ or $x = 1$.

(b) The graph shows that the first is a relative maximum, the second is a relative minimum.

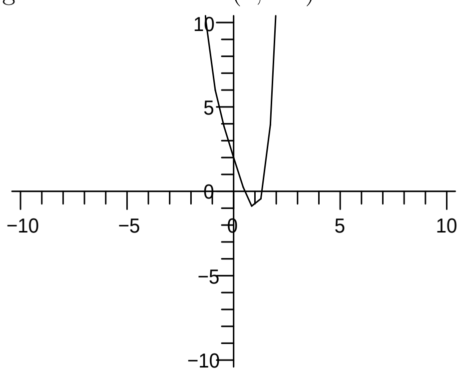


(c) Now to determine the value(s) of x for

which the tangent line to $y = f(x)$ intersects the axis at 45^o angle that is when
$f'(x) = 1$.
$3x^2 - 3 = 1$
$(x^2 - 1) = \frac{1}{3}$
$x^2 = \frac{4}{3}$
$x = \pm\frac{2}{\sqrt{3}}$

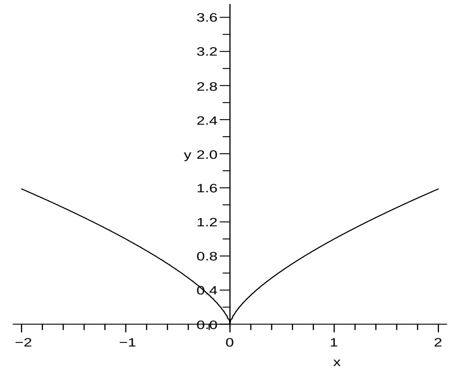
34. **(a)** Now to determine the value(s) of x for which the tangent line to $y = f(x)$ intersects the axis at 45^o angle that is when
$f'(x) = 1$
$3x^2 - 3 = 1$
$3(x^2 - 1) = 1$
$(x^2 - 1) = \dfrac{1}{3}$
$x^2 = \dfrac{4}{3}$
$x = \pm\dfrac{2}{\sqrt{3}}$

(b) The graph shows that the function has global minimum at $(1, -1)$



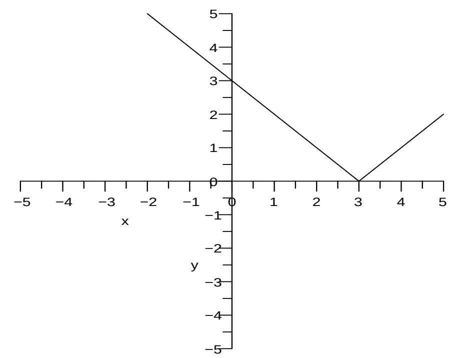
(c) Now to determine the value (s) of for which the tangent line to $y = f(x)$ intersects the axis at 45^o angle that is when
$f'(x) = 1$
$4x^3 - 4 = 1$
$(x^3 - 1) = \dfrac{1}{4}$
$x^3 = \dfrac{5}{4} = \left(\dfrac{5}{2}\right)^{1/3}$

35. **(a)** $f(x) = x^{2/3}$
$f'(x) = \dfrac{2}{3}x^{-1/3} = \dfrac{2}{3\sqrt[3]{x}}$
The slope of the tangent line to $y = f(x)$ does not exist where the derivative is undefined, which is only when x=0.



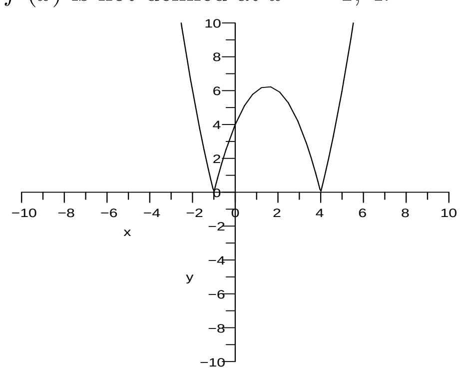
The graphical significance of this point is that there is vertical tangent here.

(b) $f(x) = |x - 3|$
$f'(x) = \begin{cases} 1 \text{ when } x > 3 \\ -1 \text{ when } x < 3 \end{cases}$
$f'(x)$ is not defined at $x = 3$.



Though the graph of function is continuous at $x = 3$ tangent line does not exist as at this point there is sharp corner.

(c) $f(x) = |x^2 - 3x - 4|$
$f'(x) = \begin{cases} 2x - 3 \text{ when } x > 4 \text{ or } x < -1 \\ -2x + 3 \text{ when } -1 < x < 4 \end{cases}$
$f'(x)$ is not defined at $x = -1, 4$.



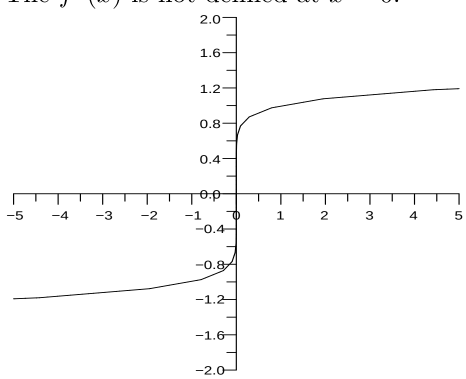
The graph shows that the function has global minima at $(-1, 0)$ and$(4, 0)$. The function has relative maximum at

$\left(\dfrac{3}{2}, \dfrac{25}{4}\right)$.

36. **(a)** $f(x) = x^{1/9}$

$f'(x) = \dfrac{1}{9}x^{-8/9} = \dfrac{1}{9x^{8/9}}$

The $f'(x)$ is not defined at $x = 0$.

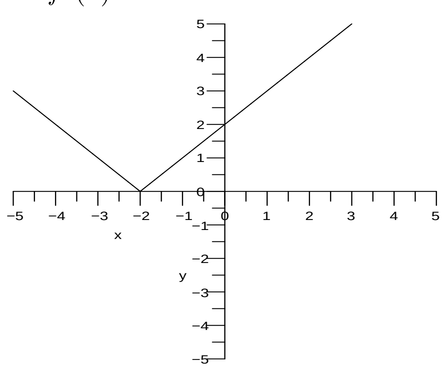


The graphical significance of this point is that there is vertical tangent here.

(b) $f(x) = |x+2|$

$f'(x) = \begin{cases} 1 \text{ when } x > -2 \\ -1 \text{ when } x < -2 \end{cases}$

The $f'(x)$ is not defined at $x = -2$.

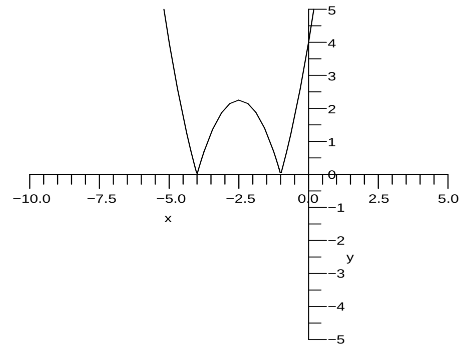


Though the graph of function is continuous at $x = -2$, tangent line does not exist as at this point there is sharp corner.

(c) $f(x) = \left|x^2 + 5x + 4\right| = |(x+4)(x+1)|$

$f'(x) = \begin{cases} 2x + 5 \text{ when } x < -4 \text{ or } x > -1 \\ -2x - 5 \text{ when } -4 < x < -1 \end{cases}$

The $f'(x)$ is not defined at $x = -4, -1$.



The graph shows that the function has global minima at $(-4, 0)$ and $(-1, 0)$. The function has relative maxima at $(-2.5, 2.25)$.

37. **(a)** $y = x^3 - 3x + 1$

$y' = 3x^2 - 3 = 3\left(x^2 - 1\right)$

The tangent line to $y = f(x)$ intersects the x-axis at a 45° angle when

$f'(x) = 1$

$\Leftrightarrow 3\left(x^2 - 1\right) = 1$

$\Leftrightarrow x^2 = 1 + \dfrac{1}{3}$

$\Leftrightarrow x = \dfrac{2}{\sqrt{3}}$ or $x = -\dfrac{2}{\sqrt{3}}$

(b) The tangent line to $y = f(x)$ intersects the x-axis at a 30° angle when

$f'(x) = \dfrac{1}{\sqrt{3}}$.

$\Leftrightarrow 3\left(x^2 - 1\right) = \dfrac{1}{\sqrt{3}}$

$\Leftrightarrow x^2 = 1 + \dfrac{1}{3\sqrt{3}}$

$\Leftrightarrow x = \left(1 + \dfrac{1}{3\sqrt{3}}\right)^{1/2}$ or

$x = -\left(1 + \dfrac{1}{3\sqrt{3}}\right)^{1/2}$

38. Answers depend on CAS.

39. $f(x) = ax^2 + bx + c,\ f(0) = c$

$f'(x) = 2ax + b,\ f'(0) = b$

$f''(x) = 2a,\ f''(0) = 2a$

Given $f''(0) = 3$, we learn $2a = 3$, or $a = 3/2$. Given $f'(0) = 2$ we learn $2 = b$, and given $f(0) = -2$, we learn $c = -2$. In the end

$f(x) = ax^2 + bx + c = \dfrac{3}{2}x^2 + 2x - 2$

40. **(a)** $f(x) = \sqrt{x} = x^{1/2}$

$f'(x) = \dfrac{1}{2}x^{-1/2}$

$f''(x) = \dfrac{1}{2}\left(-\dfrac{1}{2}\right)x^{-3/2}$

$f'''(x) = \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)x^{-5/2}$

$f^{(n)}(x)$

$= (-1)^{n-1}\frac{1.3.5...(2n-3)}{2^n}x^{-(2n-1)/2}$

$= (-1)^{n-1}.\frac{1.2.3...(2n-2)}{2^n.2.4...(2n-2)}x^{-(2n-1)/2}$

$= (-1)^{n-1}.\frac{(2n-2)!}{2^{2n-1}(n-1)!}x^{-(2n-1)/2}$

(b) $f'(x) = -2x^{-3}$
$f''(x) = 6x^{-4}$
$f'''(x) = -24x^{-5}$
The pattern is
$f^{(n)}(x) = (-1)^n(n+1)!x^{-n-2}$

41. For $y = \frac{1}{x}$, we have $y' = -\frac{1}{x^2}$. Thus, the slope of the tangent line at $x = a$ is $-\frac{1}{a^2}$.
When $a = 1$, the slope of the tangent line at (1, 1) is -1, and the equation of the tangent line is $y = -x + 2$. The tangent line intersects the axes at (0, 2) and (2, 0). Thus, the area of the triangle is $\frac{1}{2}(2)(2) = 2$.
When $a = 2$, the slope of the tangent line at $\left(2, \frac{1}{2}\right)$ is $-\frac{1}{4}$, and the equation of the tangent line is $y = -\frac{1}{4}x + 2$. The tangent line intersects the axes at (0, 1) and (4, 0). Thus, the area of the triangle is $\frac{1}{2}(4)(1) = 2$.
In general, the equation of the tangent line is $y = -\left(\frac{1}{a^2}\right)x + \frac{2}{a}$. The tangent line intersects the axes at $(0, \frac{2}{a})$ and $(2a, 0)$. Thus, the area of the triangle is $\frac{1}{2}(2a)\left(\frac{2}{a}\right) = 2$.

42. For $y = \frac{1}{x^2} = x^{-2}$, we have
$f'(x) = -2x^{-3} = -\frac{2}{x^3}$
Thus, the slope of the tangent line at
$x = a$ is $-\frac{2}{a^3}$.
When $a = 1$, the slope of the tangent line at (1, 1) is -2, and the equation of the tangent line is $y = -2x + 3$. The tangent line intersects the axes at (0, 3) and $\left(\frac{3}{2}, 0\right)$. Thus the area of the triangle is $\frac{1}{2}(3)(\frac{3}{2}) = \frac{9}{4}$.
When $a = 2$, the slope of the tangent line at $\left(2, \frac{1}{4}\right)$ is $-\frac{1}{4}$, and the equation of the tangent line is
$y = -\frac{1}{4}x + \frac{3}{4}$. The tangent line intersects the axes at $\left(0, \frac{3}{4}\right)$ and (3, 0). Thus the area of the triangle is $\frac{1}{2}(\frac{3}{4})(3) = \frac{9}{8}$.
Since $\frac{9}{4} \neq \frac{9}{8}$, we see that the result for exercise 41 does not hold here.

43. **(a)** $g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$

$= \lim_{h\to 0}\frac{1}{h}\left[\max_{a\le t\le x+h} f(t) - \max_{a\le t\le x} f(t)\right]$

$= \lim_{h\to 0}\frac{1}{h}[f(x+h) - f(x)]$

$= f'(x)$

(b) $g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$

$= \lim_{h\to 0}\frac{1}{h}\left[\max_{a\le t\le x+h} f(t) - \max_{a\le t\le x} f(t)\right]$

$= \lim_{h\to 0}\frac{1}{h}[f(a) - f(a)] = 0$

44. **(a)** $g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$

$= \lim_{h\to 0}\frac{1}{h}\left[\min_{a\le t\le x+h} f(t) - \min_{a\le t\le x} f(t)\right]$

$= \lim_{h\to 0}\frac{1}{h}[f(a) - f(a)] = 0$

(b) $g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$

$= \lim_{h\to 0}\frac{1}{h}\left[\min_{a\le t\le x+h} f(t) - \min_{a\le t\le x} f(t)\right]$

$= \lim_{h\to 0}\frac{1}{h}[f(x+h) - f(x)]$

$= f'(x)$

45. Try $f(x) = cx^4$ for some constant c. Then $f'(x) = 4cx^3$ so c must be 1. One possible answer is $f(x) = x^4$.

46. Try $f(x) = cx^5$ for some constant c. Then $f'(x) = 5cx^4$ so c must be 1. One possible answer is $f(x) = x^5$.

47. $f'(x) = \sqrt{x} = x^{1/2}$
$f(x) = \frac{2}{3}x^{3/2}$ is one possible function

48. If $f'(x) = x^{-2}$, then $f(x) = -x^{-1}$ is one possible function.

49. $w(b) = cb^{3/2}$
$w'(b) = \frac{3c}{2}b^{1/2} = \frac{3c\sqrt{b}}{2}$

$w'(b) > 1$ when
$\frac{3c\sqrt{b}}{2} > 1,$
$\sqrt{b} > \frac{2}{3c}$
$b > \frac{4}{9c^2}$
Since c is constant, when b is large enough b will be greater than $\frac{4}{9c^2}$. After this point, when b increases by 1 *unit*, the leg width w is increasing by more than 1 *unit*, so that leg width is increasing faster than body length. This puts a limitation on the size of land animals since, eventually, the body will not be long enough to accomodate the width of the legs.

50. World Record Times Mens Track

Dist.	*Time*	*Ave.*	$f(d)$
400	43.18	9.26	9.25
800	101.11	7.91	8.17
1000	131.96	7.58	7.86
1500	206.00	7.28	9.25
2000	284.79	7.02	6.95

Here, distance is in meters, time is in seconds and hence average in metersper second.
The function $f(d)$ is quite close to predicting the average speed of worldrecord pace.
$v'(d)$ represents the rate of change in average speed over d meters per meter. $v'(d)$ tells us how much $v(d)$ would change if d changed to $d + 1$.

51. We can approximate
$f'(2000) \approx \frac{9039.5 - 8690.7}{2001 - 1999} = 174.4$. This is the rate of change of the GDP in billions of dollars per year.
To approximate $f''(2000)$, we first estimate
$f'(1999) \approx \frac{9016.8 - 8347.3}{2000 - 1998} = 334.75$
and $f'(1998) \approx \frac{8690.7 - 8004.5}{1999 - 1997} = 343.1$
Since these values are decreasing, $f''(2000)$ is negative. We estimate
$f''(2000) \approx \frac{174.4 - 334.75}{2000 - 1999} = -160.35$
This represents the rate of change of the rate of change of the GDP over time. In 2000, the GDP is increasing by a rate of 174.4 billion dollars per year, but this increase is decreasing by a rate of 160.35 billion dollars-per-year per year.

52. $f'(2000)$ can be approximated by the average rate of change from 1995 to 2000.
$f'(2000) \approx \frac{4619 - 4353}{2000 - 1995} = 53.2$
This is the rate of change of weight of SUVs over time. In 2000the weight of SUVs is increasing by 53.2 pounds per year.
Similarly approximate $f'(1995) \approx 32.8$ and $f'(1990) \approx 26.8$ The second derivative is definitely positive. We can approximate
$f''(2000) \approx \frac{53.2 - 32.8}{2000 - 1995} = 4.08.$
This is the rate of change in the rate of change of the weight of SUVs. Notonly are SUVs getting heavier at a rate of 53.2 pounds per year, this rateis itself increasing at a rate of about 4 pounds-per-year per year.

53. Newton's Law states that force equals mass times acceleration. That is, if $F(t)$ is the driving force at time t, then $m \cdot f''(t) = m \cdot a(t) = F(t)$ in which m is the mass, appropriately unitized. The third derivative of the distance function is then
$f'''(t) = a'(t) = \frac{1}{m}F'(t).$
It is both the derivative of the acceleration and directly proportional to the rate of change in force. Thus an abrupt change in acceleration or "jerk"is the direct consequence of an abrupt changein force.

2.4 The Product and Quotient Rules

1. $f(x) = (x^2 + 3)(x^3 - 3x + 1)$
$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^2+3)\cdot(x^3-3x+1) \\ &\quad + (x^2+3)\cdot\frac{d}{dx}(x^3-3x+1) \\ &= (2x)(x^3-3x+1) \\ &\quad + (x^2+3)(3x^2-3) \end{aligned}$$

2. $f(x) = (x^3 - 2x^2 + 5)(x^4 - 3x^2 + 2)$
$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^3-2x^2+5)(x^4-3x^2+2) \\ &\quad + (x^3-2x^2+5)\frac{d}{dx}(x^4-3x^2+2) \\ &= (3x^2-4x)(x^4-3x^2+2) \\ &\quad + (x^3-2x^2+5)(4x^3-6x) \end{aligned}$$

3. $f(x) = (\sqrt{x} + 3x)\left(5x^2 - \frac{3}{x}\right)$
$$= (x^{1/2} + 3x)(5x^2 - 3x^{-1})$$

$$f'(x) = \left(\frac{1}{2}x^{-1/2} + 3\right)(5x^2 - 3x^{-1}) + (x^{1/2} + 3x)(10x + 3x^{-2})$$

4. $f(x) = (x^{3/2} - 4x)(x^4 - 3x^{-2} + 2)$

$$f'(x) = \frac{d}{dx}(x^{3/2} - 4x)(x^4 - 3x^{-2} + 2) + (x^{3/2} - 4x)\frac{d}{dx}(x^4 - 3x^{-2} + 2)$$
$$= (\frac{3}{2}x^{1/2} - 4)(x^4 - 3x^{-2} + 2) + (x^{3/2} - 4x)(4x^3 + 6x^{-3})$$

5. $g(t) = \dfrac{3t - 2}{5t + 1}$

$$g'(t) = \frac{\left((5t+1)\frac{d}{dt}(3t-2)\right) - \left((3t-2)\frac{d}{dt}(5t+1)\right)}{(5t+1)^2}$$
$$= \frac{3(5t+1) - 5(3t-2)}{(5t+1)^2}$$
$$= \frac{15t + 3 - 15t + 10}{(5t+1)^2} = \frac{13}{(5t+1)^2}$$

6. $g(t) = \dfrac{t^2 + 2t + 5}{t^2 - 5t + 1}$

$$g'(t) = \frac{\left((t^2-5t+1)\frac{d}{dt}(t^2+2t+5)\right) - \left((t^2+2t+5)\frac{d}{dt}(t^2-5t+1)\right)}{(t^2-5t+1)^2}$$
$$= \frac{(t^2 - 5t + 1)(2t + 2) - (t^2 + 2t + 5)(2t - 5)}{(t^2 - 5t + 1)^2}$$

7. $f(x) = \dfrac{3x - 6\sqrt{x}}{5x^2 - 2} = \dfrac{3(x - 2x^{1/2})}{5x^2 - 2}$

$$f'(x) = 3\frac{\left((5x^2-2)\frac{d}{dx}(x-2x^{1/2}) - (x-2x^{1/2})\frac{d}{dx}(5x^2-2)\right)}{(5x^2-2)^2}$$
$$= 3\frac{\left((5x^2 - 2)(1 - x^{-1/2}) - (x - 2x^{1/2})(10x)\right)}{(5x^2 - 2)^2}$$

8. $f(x) = \dfrac{6x - 2x^{-1}}{x^2 + x^{1/2}}$

$$f'(x) = \frac{(x^2+x^{1/2})\frac{d}{dx}(6x-2x^{-1}) - (6x-2x^{-1})\frac{d}{dx}(x^2+x^{1/2})}{(x^2+x^{1/2})^2}$$
$$= \frac{(x^2+x^{1/2})(6+2x^{-2}) - (6x-2x^{-1})(2x+\frac{1}{2}x^{-1/2})}{(x^2+x^{1/2})^2}$$

9. $f(u) = \dfrac{(u+1)(u-2)}{u^2 - 5u + 1} = \dfrac{u^2 - u - 2}{u^2 - 5u + 1}$

$$f'(u) = \frac{\left((u^2-5u+1)\frac{d}{du}(u^2-u-2)\right) - \left((u^2-u-2)\frac{d}{du}(u^2-5u+1)\right)}{(u^2-5u+1)^2}$$
$$= \frac{(u^2-5u+1)(2u-1) - (u^2-u-2)(2u-5)}{(u^2-5u+1)^2}$$
$$= \frac{2u^3 - 10u^2 + 2u - u^2 + 5u - 1 - 2u^3 + 2u^2 + 4u + 5u^2 - 5u - 10}{(u^2-5u+1)^2}$$
$$= \frac{-4u^2 + 6u - 11}{(u^2 - 5u + 1)^2}$$

10. $f(u) = \dfrac{(2u)(u+3)}{u^2 + 1} = \dfrac{2u^2 + 6u}{u^2 + 1}$

$$f'(u) = \frac{\left((u^2+1)\frac{d}{du}(2u^2+6u)\right) - \left((2u^2+6u)\frac{d}{du}(u^2+1)\right)}{(u^2+1)^2}$$
$$= \frac{(u^2 + 1)(4u + 6) - (2u^2 + 6u)(2u)}{(u^2 + 1)^2}$$
$$= \frac{4u^3 + 6u^2 + 4u + 6 - 4u^3 - 12u^2}{(u^2 + 1)^2}$$
$$= \frac{-6u^2 + 4u + 6}{(u^2 + 1)^2}$$
$$= \frac{2(-3u^2 + 2u + 3)}{(u^2 + 1)^2}$$

11. We do not recommend treating this one as a quotient, but advise preliminary simplification.

$$f(x) = \frac{x^2 + 3x - 2}{\sqrt{x}}$$
$$= \frac{x^2}{\sqrt{x}} + \frac{3x}{\sqrt{x}} - \frac{2}{\sqrt{x}}$$
$$= x^{3/2} + 3x^{1/2} - 2x^{-1/2}$$
$$f'(x) = \frac{3}{2}x^{1/2} + \frac{3}{2}x^{-1/2} + x^{-3/2}$$

12. $f(x) = \dfrac{x^2 - 2x}{x^2 + 5x}$

$$f'(x) = \frac{(x^2 + 5x)\frac{d}{dx}(x^2 - 2x) - (x^2 - 2x)\frac{d}{dx}(x^2 + 5x)}{(x^2 + 5x)^2}$$
$$= \frac{(x^2 + 5x)(2x - 2) - (x^2 - 2x)(2x + 5)}{(x^2 + 5x)^2}$$

13. We simplify instead of using the product rule.

$$h(t) = t(\sqrt[3]{t} + 3) = t^{4/3} + 3t$$
$$h'(t) = \frac{4}{3}\sqrt[3]{t} + 3$$

14. $h(t) = \dfrac{t^2}{3} + \dfrac{5}{t^2} = \dfrac{1}{3}t^2 + 5t^{-2}$

$$h'(t) = \frac{2}{3}t - 10t^{-3}$$

15. $f(x) = (x^2 - 1)\dfrac{x^3 + 3x^2}{x^2 + 2}$

$$f'(x) = \frac{d}{dx}(x^2 - 1)\cdot(\frac{x^3 + 3x^2}{x^2 + 2}) + (x^2 - 1)\cdot\frac{d}{dx}(\frac{x^3 + 3x^2}{x^2 + 2})$$

We have

$$\frac{d}{dx}(\frac{x^3 + 3x^2}{x^2 + 2}) = \frac{(x^2 + 2)\frac{d}{dx}(x^3 + 3x^2) - (x^3 + 3x^2)\frac{d}{dx}(x^2 + 2)}{(x^2 + 2)^2}$$
$$= \frac{(x^2 + 2)\cdot(3x^2 + 6x) - (x^3 + 3x^2)\cdot(2x)}{(x^2 + 2)^2}$$
$$= \frac{3x^4 + 6x^2 + 6x^3 + 12x - (2x^4 + 6x^3)}{(x^2 + 2)^2}$$

$= \dfrac{x^4+6x^2+12x}{(x^2+2)^2}$

So, $f'(x) =$

$(2x)\cdot(\dfrac{x^3+3x^2}{x^2+2}) + (x^2-1)\cdot\dfrac{x^4+6x^2+12x}{(x^2+2)^2}$

16. $f(x) = \dfrac{(x+2)(x-1)(x+1)}{x(x+1)}$

$= \dfrac{x^2+x-2}{x}$

$= x+1-2x^{-1}$

$f'(x) = 1+2x^{-2}$

17. $f(x) = (x^2+2x)(x^4+x^2+1)$

$$f'(x) = \left[\frac{d}{dx}(x^2+2x)\right](x^4+x^2+1) + \left[\frac{d}{dx}(x^4+x^2+1)\right](x^2+2x)$$
$$= (2x+2)(x^4+x^2+1) + (4x^3+2x)(x^2+2x)$$

At $x=a=0$, we get:

$f(0)=0$

$f'(0)=2$

Threfore, the line with slope 2 and passing through the point (0, 0) has equation $y=2x$.

18. $f(x) = (x^3+x+1)(3x^2+2x-1)$

$$f'(x) = \left[\frac{d}{dx}(x^3+x+1)\right](3x^2+2x-1) + \left[\frac{d}{dx}(3x^2+2x-1)\right](x^3+x+1)$$
$$= (3x^2+1)(3x^2+2x-1) + (6x+2)(x^3+x+1)$$

At $x=a=1$, we get:

$f(1)=12$

$f'(1) = (3+1)(3+2-1)+(6+2)(1+1+1) = 40$

Therfore, the line with slope 40 and passing through the point (1, 12) has equation $y = 40\,(x-1)+12$.

19. $f(x) = \dfrac{x+1}{x+2}$

By The Quotient Rule, we have

$f'(x)$

$$= \frac{\left((x+2)\frac{d}{dx}(x+1)\right) - \left((x+1)\frac{d}{dx}(x+2)\right)}{(x+2)^2}$$
$$= \frac{(x+2)-(x+1)}{(x+2)^2} = \frac{1}{(x+2)^2}.$$

At $x=a=0$,

$f(0) = \dfrac{0+1}{0+2} = \dfrac{1}{2}$

$f'(0) = \dfrac{1}{4}$.

The line with slope $\frac{1}{4}$ and passing through the point $\left(0,\ \frac{1}{2}\right)$ has equation $y = \frac{1}{4}x + \frac{1}{2}$.

20. $f(x) = \dfrac{x+3}{x^2+1}$

By The Quotient Rule, we have

$f'(x) =$

$$\frac{\left((x^2+1)\frac{d}{dx}(x+3)\right) - \left((x+3)\frac{d}{dx}(x^2+1)\right)}{(x^2+1)^2}$$
$$= \frac{(x^2+1)-(x+3)(2x)}{(x^2+1)^2}$$
$$= \frac{(x^2+1)-(2x^2+6x)}{(x^2+1)^2}$$
$$= \frac{x^2+1-2x^2-6x}{(x^2+1)^2}$$
$$= \frac{-x^2-6x+1}{(x^2+1)^2}.$$

At $x=a=1$,

$f(1) = \frac{1+3}{1^2+1} = 2$

$f'(1) = \dfrac{-1-6+1}{(1+1)^2} = -\dfrac{6}{4} = -\dfrac{3}{2}$.

The line with slope $-\frac{3}{2}$ and passing through the point (1, 2) has equation $y = -\frac{3}{2}(x-1)+2$.

21. $h(x) = f(x)g(x)$

$h'(x) = f'(x)g(x) + g'(x)f(x)$

(a) At $x=a=0$,

$h(0) = f(0)g(0) = (-1)(3) = -3$

$h'(0) = f'(0)g(0) + g'(0)f(0)$

$= (-1)(3) + (-1)(-1) = -2.$

So, the equation of the tangent line is $y=-2x-3$.

(b) At $x=a=1$,

$h(1) = f(1)g(1) = (-2)(1) = -2$

$h'(1) = f'(1)g(1) + g'(1)f(1)$

$= (3)(1) + (-2)(-2) = 7.$

So, the equation of the tangent line is $y = 7(x-1)-2$ or $y=7x-9$.

22. $h(x) = \dfrac{f(x)}{g(x)}$

$h'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

(a) At $x=a=1$,

$h(1) = \dfrac{f(1)}{g(1)} = -\dfrac{2}{1} = -2$

$$h'(1) = \frac{g(1)f'(1) - f(1)g'(1)}{(g(1))^2}$$
$$= \frac{(1)(3) - (-2)(-2)}{1^2}$$
$$= \frac{3-4}{1} = -1.$$

So, the equation of the tangent line is $y = -(x-1) - 2$.

(b) At $x = a = 0$,
$h(0) = \frac{f(0)}{g(0)} = -\frac{1}{3}$

$$h'(0) = \frac{g(0)f'(0) - f(0)g'(0)}{(g(0))^2}$$
$$= \frac{(-1)(3) - (-1)(-1)}{(3)^2}$$
$$= \frac{-3-1}{9}$$
$$= -\frac{4}{9}.$$

So, the equation of the tangent line is $y = -\frac{4}{9}x - \frac{1}{3}$.

23. $h(x) = x^2 f(x)$
$h'(x) = 2xf(x) + x^2 f'(x)$

(a) At $x = a = 1$,
$h(1) = 1^2 f(1) = -2$
$h'(1) = 2 \times 1 \times f(1) + 1^2 f'(1)$
$= (2)(-2) + (3) = -4 + 3 = -1.$
So, the equation of the tangent is $y = -1(x-1) - 2$ or $y = -x - 1$.

(b) At $x = a = 0$,
$h(0) = 0^2 f(0) = 0$
$h'(0) = 2 \times 0 \times f(0) + 0^2 f'(0) = 0.$
So, the equation of the tangent is $y = 0$.

24. $h(x) = \frac{x^2}{g(x)}$

$$h'(x) = \frac{2xg(x) - x^2 g'(x)}{(g(x))^2}$$

(a) At $x = a = 1$,

$$h(1) = \frac{1^2}{g(1)} = \frac{1}{1} = 1$$
$$h'(1) = \frac{2 \times 1 \times g(1) - 1^2 g'(1)}{(g(1))^2}$$
$$= \frac{(2)(1)(1) - (1)(-2)}{1^2}$$
$$= \frac{2+2}{1} = 4.$$

So, the equation of tangent line is $y = 4(x-1) + 1$.

(b) At $x = a = 0$,

$$h(0) = \frac{0^2}{g(0)} = \frac{0}{3} = 0$$
$$h'(0) = \frac{2 \times 0 \times g(0) - 0^2 g'(0)}{(g(0))^2} = 0.$$

So, the equation of the tangent line is $y = 0$.

25. The rate at which the quantity Q changes is Q'. Since the amount is said to be "decreasing at a rate of 4%" we have to ask "4% of what?" The answer in this type of context is usually 4% of itself. In other words, $Q' = -0.04Q$.
As for P, the 3% rate of increase would translate as $P' = 0.03P$. By the product rule with $R = PQ$, we have:
$R' = (PQ)' = P'Q + PQ'$
$= (0.03P)Q + P(-0.04Q)$
$= -(0.01)PQ = (-0.01)R.$
In other words, revenue is decreasing at a rate of 1%.

26. Revenue will be constant when the derivative is 0. Substituting, $Q' = -0.04Q$ and, $P' = aP$ into the expression for R' gives,
$R' = -0.04QP + aQP$
$R' = (-0.04 + a)QP.$
This is zero when $a = 0.04$, so price must increase by 4%.

27. $R' = Q'P + QP'$
At a certain moment of time (call it t_0) we are given $P(t_0) = 20$ (\$/item), $Q(t_0) = 20,000$(items)
$P'(t_0) = 1.25$ (\$/item/year)
$Q'(t_0) = 2,000$ (item/year)
$R'(t_0) = 2,000(20) + (20,000)1.25$
$R'(t_0) = 65,000$ (\$/year) .
So, revenue is increasing by \$65,000/year at the time t_0.

28. We are given $P = \$14$, $Q = 12,000$ and $Q' = 1,200$. We want $R' = \$20,000$. Substituting these values in to the expression for R' (see exercise 25) yields:
$20,000 = 1200 \cdot 14 + 12,000 \cdot P'$
Solve to get $P' = 0.27$ dollars per year.

29. If $u(m) = \frac{82.5m - 6.75}{m + 0.15}$ then using the quotient rule,

$$\frac{du}{dm} = \frac{(m+0.15)(82.5) - (82.5m - 6.75)1}{(m+0.15)^2}$$
$$= \frac{19.125}{(m+0.15)^2}$$

which is clearly positive. It seems to be saying that initial ball speed is an increasing function of the mass of the bat. Meanwhile,
$u'(1) = \frac{19.125}{1.15^2} \approx 14.46$
$u'(1.2) = \frac{19.125}{1.35^2} \approx 10.49,$
which suggests that the rate at which this speed is increasing is decreasing.

30. $$u'(M) = \frac{(M+1.05)\frac{d}{dM}(86.625-45M)}{(M+1.05)^2} - \frac{\frac{d}{dM}(M+1.05)(86.625-45M)}{(M+1.05)^2} = \frac{(-45M-47.25)-(86.625-45M)}{(M+1.05)^2} = \frac{-133.875}{(M+1.05)^2}$$
This quantity is negative. In baseball terms, as the mass of the baseball increases, the initial velocity decreases.

31. If $u(m) = \frac{14.11}{m+0.05} = \frac{282.2}{20m+1}$, then
$$\frac{du}{dm} = \frac{(20m+1)\cdot 0 - 282.2(20)}{(20m+1)^2} = \frac{-5644}{(20m+1)^2}.$$
This is clearly negative, which means that impact speed of the ball is a decreasing function of the weight of the club. It appears that the explanation may have to do with the stated fact that the speed of the club is inversely proportional to its mass. Although the lesson of Example 4.6 was that a heavier club makes for greater ball velocity, that was assuming a fixed club speed, quite a different assumption from this problem.

32. $u'(v) = \frac{0.2822}{0.217} \approx 1.3$. The initial speed of the ball increases 1.3 times more than the increase in club speed.

33. $$\frac{d}{dx}[f(x)g(x)h(x)] = \frac{d}{dx}[(f(x)g(x))\,h(x)]$$
$$= (f(x)g(x))\,h'(x) + h(x)\frac{d}{dx}(f(x)g(x))$$
$$= (f(x)g(x))\,h'(x) + h(x)\,(f(x)g'(x) + g(x)f'(x))$$
$$= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$
In the general case of a product of n functions, the derivative will have n terms to be added, each term a product of all but one of the functions multiplied by the derivative of the missing function.

34. The derivative of $g(x)^{-1} = \frac{1}{g(x)}$ is
$$\frac{d}{dx}\left[g(x)^{-1}\right] = \frac{g(x)\frac{d}{dx}(1) - (1)\frac{d}{dx}g(x)}{g(x)^2} = -\frac{g'(x)}{g(x)^2} = -g'(x)(g(x))^{-2}$$
as claimed. The derivative of $f(x)(g(x))^{-1}$ is then $f'(x)(g(x))^{-1} + f(x)(-g'(x)(g(x))^{-2})$.

35. $$f'(x) = \left[\frac{d}{dx}(x^{2/3})\right](x^2-2)(x^3-x+1) + x^{2/3}\left[\frac{d}{dx}(x^2-2)\right](x^3-x+1) + x^{2/3}(x^2-2)\frac{d}{dx}(x^3-x+1)$$
$$= \frac{2}{3}x^{-1/3}(x^2-2)(x^3-x+1) + x^{2/3}(2x)(x^3-x+1) + x^{2/3}(x^2-2)(3x^2-1)$$

36. $$f'(x) = 1(x^3-2x+1)(3-2/x) + (x+4)(3x^2-2)(3-2/x) + (x+4)(x^3-2x+1)(2/x^2)$$

37. $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{hg(h)-0}{h} = \lim_{h\to 0}\frac{hg(h)}{h} = \lim_{h\to 0} g(h) = g(0)$$
Since, g is continuous at $x = 0$. When $g(x) = |x|$, $g(x)$ is continuous but not differentiable at $x = 0$. We have
$$f(x) = x|x| = \begin{cases} -x^2 & x < 0 \\ x^2 & x \geq 0 \end{cases}$$
This is differentiable at $x = 0$.

38. $f(x) = (x-a)\,g(x)$
$$f'(x) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{(a+h-a)\,g(a+h)}{h} = \lim_{h\to 0}\frac{hg(a+h)}{h} = \lim_{h\to 0} g(a+h) = g(a)$$
As g is continuous at $x = a$, hence $f(x)$ is differentiable.

39. $f(x) = \dfrac{x}{x^2+1}$

$$f'(x) = \frac{\left(x^2+1\right) - x\,(2x)}{\left(x^2+1\right)^2}$$

$$= \frac{x^2+1-2x^2}{\left(x^2+1\right)^2} = \frac{-x^2+1}{\left(x^2+1\right)^2}$$

$f''(x)$

$$= \frac{\left(x^2+1\right)^2(-2x) - \left(-x^2+1\right)2\left(x^2+1\right)(2x)}{\left(x^2+1\right)^4}$$

$$= \frac{\left(x^2+1\right)(-2x) - \left(-x^2+1\right)(4x)}{\left(x^2+1\right)^3}$$

$$= \frac{-2x^3-2x+4x^3-4x}{\left(x^2+1\right)^3} = \frac{2x^3-6x}{\left(x^2+1\right)^3}$$

At maxima or minima of f', we have $f''(x) = 0$. So, $2x^3 - 6x = 0$

$$2x\left(x^2-3\right) = 0$$

$$2x = 0\ ,\ x^2 - 3 = 0$$

$$x = 0,\ x = \pm\sqrt{3}$$

$$f'(0) = \frac{-0^2+1}{\left(0^2+1\right)^2} = 1$$

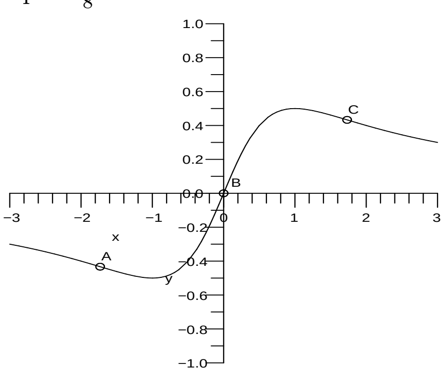
$$f'\left(\pm\sqrt{3}\right) = \frac{-\left(\pm\sqrt{3}\right)^2+1}{\left(\left(\pm\sqrt{3}\right)^2+1\right)^2}$$

$$= \frac{-3+1}{(3+1)^2} = -\frac{2}{16} = -\frac{1}{8}$$

Therefore, $-\dfrac{1}{8} \le m = f'(x) \le 1$.

So, the function f has maximum slope $m = 1$ at $x = 0$ and minimum slope $m = -\dfrac{1}{8}$ at $x = \pm\sqrt{3}$.

In the graph of $f(x)$ in below, the point $B(0,\,0)$ has maximum slope 1 and the points $A(-\sqrt{3},\,-\frac{\sqrt{3}}{4})$, $C(\sqrt{3},\,\frac{\sqrt{3}}{4})$ have minimum slope $-\frac{1}{8}$.



40. $f(x) = \dfrac{x}{\sqrt{x^2+1}}$

$$f'(x) = \frac{\left(\sqrt{x^2+1}\right) - x\left(\frac{1}{2\sqrt{x^2+1}} \times 2x\right)}{\left(x^2+1\right)}$$

$$= \frac{\left(\sqrt{x^2+1}\right) - \frac{x^2}{\sqrt{x^2+1}}}{\left(x^2+1\right)}$$

$$= \frac{x^2+1-x^2}{\left(x^2+1\right)^{\frac{3}{2}}} = \left(x^2+1\right)^{-\frac{3}{2}}$$

Since $x^2+1 > 0$, $m > 0$.

$$f''(x) = -\frac{3}{2}\left(x^2+1\right)^{-\frac{5}{2}}(2x)$$

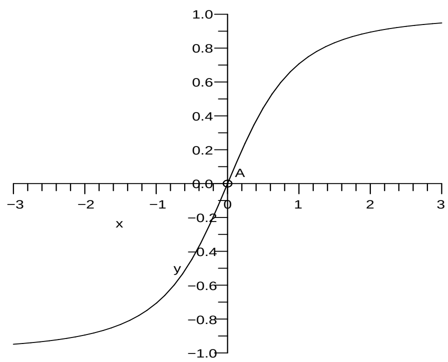
$$= -3x\left(x^2+1\right)^{-\frac{5}{2}} = -3x$$

For maxima or minima of $f''(x)$, we have $f''(x) = 0$. So, $x = 0$

$f'(0) = \left(0^2+1\right)^{-\frac{3}{2}} = 1$

Therefore $0 < m = f'(x) \le 1$

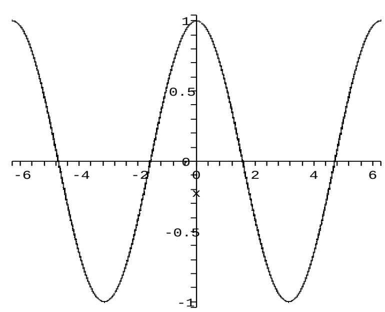
In the graph of $f(x)$ in below, the point $A(0,\,0)$ has maximum slope 1.



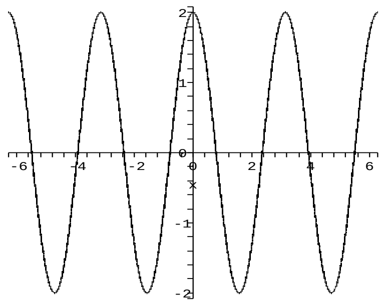
41. Answers depend on CAS.

42. For any constant k, the derivative of $\sin kx$ is $k\cos kx$.

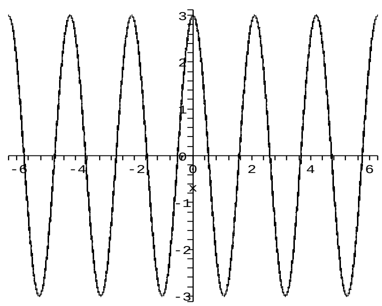
Graph of $\frac{d}{dx}\sin x$:



Graph of $\frac{d}{dx}\sin 2x$



Graph of $\frac{d}{dx}\sin 3x$



43. CAS answers may vary.

44. The function $f(x)$ simplifies to $f(x) = 2x$, so $f'(x) = 2$. CAS answers vary, but should simplify to 2.

45. If $F(x) = f(x)g(x)$, then
$F'(x) = f'(x)g(x) + f(x)g'(x)$ and
$$\begin{aligned} F''(x) &= f''(x)g(x) + f'(x)g'(x) \\ &\quad + f'(x)g'(x) + f(x)g''(x) \\ &= f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x) \\ F'''(x) &= f'''(x)g(x) + f''(x)g'(x) \\ &\quad + 2f''(x)g'(x) + 2f'(x)g''(x) \\ &\quad + f'(x)g''(x) + f(x)g'''(x) \\ &= f'''(x)g(x) + 3f''(x)g'(x) \\ &\quad + 3f'(x)g''(x) + f(x)g'''(x). \end{aligned}$$
One can see obvious parallels to the binomial coefficients as they come from Pascal's Triangle:
$(a+b)^2 = a^2 + 2ab + b^2$
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.
On this basis, one could correctly predict the pattern of the fourth or any higher derivative.

46. $F^{(4)}(x) = f^{(4)}g + 4f'''g' + 6f''g'' + 4f'g''' + fg^{(4)}$

47. If $g(x) = [f(x)]^2 = f(x)f(x)$, then
$g'(x) = f'(x)f(x) + f(x)f'(x) = 2f(x)f'(x)$.

48. $g(x) = f(x)[f(x)]^2$, so
$$\begin{aligned} g'(x) &= f'(x)[f(x)]^2 + f(x)(2f(x)f'(x)) \\ &= 3[f(x)]^2 f'(x) \end{aligned}$$
The derivative of $[f(x)]^n$ is $n[f(x)]^{n-1}f'(x)$.

49. $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 1$. Without any activator there is no enzyme. With unlimited amount of activator, the amount of enzyme approaches 1.
$$f(x) = \frac{x^{2.7}}{1+x^{2.7}}$$
$$\begin{aligned} f'(x) &= \frac{\left(1+x^{2.7}\right)(2.7)\,x^{1.7} - (2.7)\,x^{2.7}x^{1.7}}{\left(1+x^{2.7}\right)^2} \\ &= \frac{2.7x^{1.7}}{\left(1+x^{2.7}\right)^2} \end{aligned}$$
The fact that $0 < f(x) < 1$ when $x > 0$ suggest to us that f may be a kind of concentration ratio or percentage of presence of the allosteric enzymes in some systems. If so, the derivetive would be interpreted as rate of change of concentration per unit activator.

50. $\lim_{x\to 0} f(x) = 1$ and $\lim_{x\to\infty} f(x) = 0$. Without any inhibitor the amount of enzyme approaches 1. With unlimited amount of inhibitor, the amount of enzyme approaches 0. $f'(x) = -\dfrac{2.7x^{1.7}}{\left(1+x^{2.7}\right)^2}$
For positive x, f' is negative. Increase in the amount of inhibitor leads to a decrease in the amount of enzyme.

51. **(a)** $r = \dfrac{1}{\dfrac{0.55}{c} + \dfrac{0.45}{h}} = \left[\dfrac{0.55}{c} + \dfrac{0.45}{h}\right]^{-1}$
$$\begin{aligned} \frac{d}{dc}(r) &= \frac{d}{dc}\left[\frac{0.55}{c} + \frac{0.45}{h}\right]^{-1} \\ &= \frac{-1}{\left[\frac{0.55}{c} + \frac{0.45}{h}\right]^2}\frac{d}{dc}\left[\frac{0.55}{c} + \frac{0.45}{h}\right] \\ &= \frac{0.55}{c^2\left[\frac{0.55}{c} + \frac{0.45}{h}\right]^2} \end{aligned}$$
Therefore, from the above equation we can say that $\dfrac{dr}{dc} > 0$, for every c.

(b) Similarly, $\dfrac{dr}{dh} = \dfrac{0.45}{h^2\left[\frac{0.55}{c} + \frac{0.45}{h}\right]^2}$.
Hence, from the above equation we can say that $\dfrac{dr}{dh} > 0$, for every h.

(c) $r = \dfrac{1}{\frac{0.55}{c} + \frac{0.45}{h}}$
When $c = h$, we get

$$r = \frac{1}{\frac{0.55}{h} + \frac{0.45}{h}} = h = c.$$

(d) If $c < h$

$$r = \frac{ch}{0.55h + 0.45c}$$

$$\frac{r}{c} = \frac{h}{0.55h + 0.45c} > \frac{h}{0.55h + 0.45h} = 1$$

So, $r > c$. And

$$r = \frac{ch}{0.55h + 0.45c}$$

$$\frac{r}{h} = \frac{c}{0.55h + 0.45c} < \frac{c}{0.55c + 0.45c} = 1$$

So, $r < h$ and hence $c < r < h$.

Now, r is an increasing function and $h = c$, we have $r = f(h) = c$. Hence for any value of h greater than c, we have the corresponding value of r greater than c.

$$\frac{dr}{dh} = \frac{0.55}{c^2\left[\frac{0.55}{c} + \frac{0.45}{h}\right]^2}$$

$$= \frac{0.45c^2}{(0.55h + 0.45c)^2} < \frac{0.45c^2}{c^2} = 0.45$$

Also, from part (b),

$$\frac{dr}{dh} = \frac{0.45c^2}{(0.55h + 0.45c)^2}$$

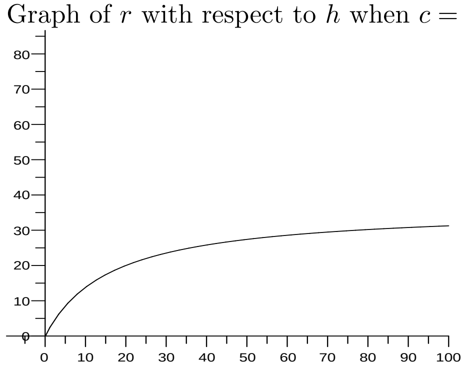
and from part (d),

$$\frac{r}{h} = \frac{c}{0.55h + 0.45c}$$

$$\Rightarrow 0.45\left(\frac{r}{h}\right)^2 < 0.45$$

$$\Rightarrow r < h$$

Graph of r with respect to h when $c = 20$:



When c is constant, r remain stable for large h.

2.5 The Chain Rule

1. $f(x) = (x^3 - 1)^2$

Using the chain rule:

$f'(x) = 2(x^3 - 1)(3x^2) = 6x^2(x^3 - 1)$

Using the product rule:

$f(x) = (x^3 - 1)(x^3 - 1)$

$$\begin{aligned} f'(x) &= (3x^2)(x^3 - 1) + (x^3 - 1)(3x^2) \\ &= 2(3x^2)(x^3 - 1) \\ &= 6x^2(x^3 - 1) \end{aligned}$$

Using preliminary multiplication:

$f(x) = x^6 - 2x^3 + 1$

$$\begin{aligned} f'(x) &= 6x^5 - 6x^2 \\ &= 6x^2(x^3 - 1) \end{aligned}$$

2. $f(x) = (x^2 + 2x + 1)(x^2 + 2x + 1)$

Using the product rule:

$$\begin{aligned} f'(x) = &(2x + 2)(x^2 + 2x + 1) \\ &+ (x^2 + 2x + 1)(2x + 2) \end{aligned}$$

Using the chain rule:

$f'(x) = 2(x^2 + 2x + 1)(2x + 2)$

3. $f(x) = (x^2 + 1)^3$

Using the chain rule:

$f'(x) = 3(x^2 + 1)^2 \cdot 2x$

Using preliminary multiplication:

$f(x) = x^6 + 3x^4 + 3x^2 + 1$

$f'(x) = 6x^5 + 12x^3 + 6x$

4. $f(x) = (2x + 1)^4$

Using preliminary multiplication:

$f(x) = 16x^4 + 32x^3 + 24x^2 + 8x + 1$

$f'(x) = 64x^3 + 96x^2 + 48x + 8.$

Using the chain rule:

$f'(x) = 4(2x + 1)^3(2) = 8(2x + 1)^3$

5. **(a)** By the chain rule:

$$\begin{aligned} f'(x) &= 3\left(x^3 - x\right)^2 \frac{d}{dx}\left(x^3 - x\right) \\ &= 3\left(x^3 - x\right)^2\left(3x^2 - 1\right) \end{aligned}$$

(b) By the chain rule:

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x^2 + 4}} \frac{d}{dx}\left(x^2 + 4\right) \\ &= \frac{1}{2\sqrt{x^2 + 4}} \cdot 2x = \frac{x}{\sqrt{x^2 + 4}} \end{aligned}$$

6. **(a)** By the chain rule:

$$\begin{aligned} f'(x) &= 4\left(x^3 + x - 1\right)^3 \frac{d}{dx}\left(x^3 + x - 1\right) \\ &= 4\left(x^3 + x - 1\right)^3\left(3x^2 + 1\right) \end{aligned}$$

(b) By the chain rule:

$$f'(x) = \frac{1}{2\sqrt{4x - \frac{1}{x}}} \frac{d}{dx}\left(4x - \frac{1}{x}\right)$$
$$= \frac{1}{2\sqrt{4x - \frac{1}{x}}}\left(4 + \frac{1}{x^2}\right)$$
$$= \frac{4 + \frac{1}{x^2}}{2\sqrt{4x - \frac{1}{x}}}$$

7. (a) $f(t) = t^5\sqrt{t^3+2}$

By the product rule:

$$f'(t) = 5t^4\sqrt{t^3+2} + t^5\frac{d}{dt}\left(\sqrt{t^3+2}\right)$$

By the chain rule:

$$f'(t) = 5t^4\sqrt{t^3+2} + t^5\frac{1}{2\sqrt{t^3+2}}\frac{d}{dt}\left(t^3+2\right)$$
$$= 5t^4\sqrt{t^3+2} + t^5\frac{1}{2\sqrt{t^3+2}}3t^2$$
$$= \frac{\left(5t^4\sqrt{t^3+2}\right)\left(2\sqrt{t^3+2}\right) + 3t^7}{2\sqrt{t^3+2}}$$
$$= \frac{10t^4\left(t^3+2\right) + 3t^7}{2\sqrt{t^3+2}}$$
$$= \frac{10t^7 + 20t^4 + 3t^7}{2\sqrt{t^3+2}}$$
$$= \frac{13t^7 + 20t^4}{2\sqrt{t^3+2}}$$

(b) $f(t) = \left(t^3+2\right)\sqrt{t}$

By the product rule:

$$f'(t) = 3t^2\sqrt{x} + \left(t^3+2\right)\frac{1}{2\sqrt{t}}$$
$$= \frac{6t^3 + t^3 + 2}{2\sqrt{t}} = \frac{7t^3+2}{2\sqrt{t}}$$

8. (a) $f(t) = \left(t^4+2\right)\sqrt{t^2+1}$

By the product rule:

$$f'(t) = 4t^3\sqrt{t^2+1} + \left(t^4+2\right)\frac{d}{dt}\left(\sqrt{t^2+1}\right)$$

By the chain rule:

$$f'(t) = 4t^3\sqrt{t^2+1} + \left(t^4+2\right)\frac{1}{2\sqrt{t^2+1}}(2t)$$
$$= 4t^3\sqrt{t^2+1} + \frac{t\left(t^4+2\right)}{\sqrt{t^2+1}}$$
$$= \frac{4t^3\left(t^2+1\right) + t\left(t^4+2\right)}{\sqrt{t^2+1}}$$
$$= \frac{4t^5 + 4t^3 + t^5 + 2t}{\sqrt{t^2+1}}$$
$$= \frac{5t^5 + 4t^3 + 2t}{\sqrt{t^2+1}}$$

(b) $f(t) = \sqrt{t}\left(t^{4/3}+3\right)$

By the product rule:

$$f'(t) = \frac{1}{2\sqrt{t}}\left(t^{4/3}+3\right) + \frac{4}{3}t^{1/3}\sqrt{t}$$
$$= \frac{1}{2\sqrt{t}}\left(t^{4/3}+3\right) + \frac{4}{3}t^{1/3}t^{1/2}$$
$$= \frac{1}{2\sqrt{t}}\left(t^{4/3}+3\right) + \frac{4}{3}t^{5/6}$$

9. (a) $f(u) = \frac{u^2+1}{u+4}$

By the quotient rule:

$$f'(u) = \frac{(u+4)(2u) - \left(u^2+1\right)}{(u+4)^2}$$
$$= \frac{2u^2 + 8u - u^2 - 1}{(u+4)^2}$$
$$= \frac{u^2 + 8u - 1}{(u+4)^2}$$

(b) $f(u) = \frac{u^3}{\left(u^2+4\right)^2}$

By the quotient rule:

$$f'(u) = \frac{\left(u^2+4\right)^2\left(3u^2\right) - \left(u^3\right)\frac{d}{du}\left(u^2+4\right)^2}{\left(u^2+4\right)^4}$$

By the chain rule:

$$f'(u) = \frac{\left(u^2+4\right)^2\left(3u^2\right) - 2u^3\left(u^2+4\right)(2u)}{\left(u^2+4\right)^4}$$
$$= \frac{\left(u^2+4\right)\left[3u^2\left(u^2+4\right) - 4u^4\right]}{\left(u^2+4\right)^4}$$
$$= \frac{3u^2\left(u^2+4\right) - 4u^4}{\left(u^2+4\right)^3}$$
$$= \frac{3u^4 + 12u^2 - 4u^4}{\left(u^2+4\right)^3}$$
$$= \frac{12u^2 - u^4}{\left(u^2+4\right)^3} = \frac{u^2\left(12-u^2\right)}{\left(u^2+4\right)^3}$$

10. (a) $f(x) = \frac{x^2-1}{x^2+1}$

By the quotient rule:

$$f'(x) = \frac{\left(x^2+1\right)(2x) - \left(x^2-1\right)(2x)}{\left(x^2+1\right)^2}$$
$$= \frac{(2x)\left(x^2+1-x^2+1\right)}{\left(x^2+1\right)^2}$$
$$= \frac{4x}{\left(x^2+1\right)^2}$$

(b) $f(x) = \dfrac{x^2+4}{\left(x^3\right)^2}$

By the quotient rule:

$$f'(x) = \frac{x^6(2x) - \left(x^2+4\right)\left(6x^5\right)}{\left(x^6\right)^2}$$
$$= \frac{2x^7 - 6x^7 - 24x^5}{x^{12}}$$
$$= \frac{-4x^7 - 24x^5}{x^{12}}$$
$$= -\frac{4x^5\left(x^2+6\right)}{x^{12}}$$
$$= -\frac{4\left(x^2+6\right)}{x^7}$$

11. **(a)** $g(x) = \dfrac{x}{\sqrt{x^2+1}}$

By the quotient rule:

$$g'(x) = \frac{\sqrt{x^2+1} - (x)\frac{d}{dx}\left(\sqrt{x^2+1}\right)}{(x^2+1)}$$

By the chain rule:

$$g'(x) = \frac{\sqrt{x^2+1} - (x)\left(\frac{1}{2\sqrt{x^2+1}}\right)(2x)}{(x^2+1)}$$
$$= \frac{\sqrt{x^2+1} - \frac{x^2}{\sqrt{x^2+1}}}{(x^2+1)}$$
$$= \frac{x^2+1 - x^2}{\sqrt{x^2+1}\,(x^2+1)}$$
$$= \frac{1}{\sqrt{x^2+1}\,(x^2+1)}$$
$$= \frac{1}{(x^2+1)^{3/2}}$$

(b) $g(x) = \sqrt{\dfrac{x}{x^2+1}}$

By the chain rule:

$$g'(x) = \frac{1}{2\sqrt{\frac{x}{x^2+1}}}\frac{d}{dx}\left(\frac{x}{x^2+1}\right)$$

By the quotient rule:

$$g'(x) = \frac{1}{2\sqrt{\frac{x}{x^2+1}}}\left(\frac{\left(x^2+1\right) - x(2x)}{\left(x^2+1\right)^2}\right)$$
$$= \frac{1}{2\sqrt{\frac{x}{x^2+1}}}\left(\frac{x^2+1-2x^2}{\left(x^2+1\right)^2}\right)$$
$$= \frac{1}{2\sqrt{x}}\left(\frac{1-x^2}{\left(x^2+1\right)^{3/2}}\right)$$
$$= \frac{1-x^2}{2\sqrt{x}(x^2+1)^{3/2}}$$

12. **(a)** $g(x) = x^2\sqrt{x+1}$

By the product rule:

$$g'(x) = 2x\sqrt{x+1} + \left(x^2\right)\frac{d}{dx}\left(\sqrt{x+1}\right)$$

By the chain rule:

$$g'(x) = 2x\sqrt{x+1} + \left(x^2\right)\frac{1}{2\sqrt{x+1}}$$
$$= 2x\sqrt{x+1} + \frac{x^2}{2\sqrt{x+1}}$$
$$= \frac{4x(x+1) + x^2}{2\sqrt{x+1}}$$
$$= \frac{4x^2 + 4x + x^2}{2\sqrt{x+1}}$$
$$= \frac{5x^2+4x}{2\sqrt{x+1}}$$

(b) $g(x) = \sqrt{(x^2+1)\left(\sqrt{x}+1\right)^3}$

By the chain rule:

$$g'(x) = \frac{\frac{d}{dx}\left[\left(x^2+1\right)\left(\sqrt{x}+1\right)^3\right]}{2\sqrt{\left(x^2+1\right)\left(\sqrt{x}+1\right)^3}}$$

By the product rule:

$$g'(x) = \frac{\left(2x\left(\sqrt{x}+1\right)^3\right) + \left(x^2+1\right)\frac{d}{dx}\left(\sqrt{x}+1\right)^3}{2\sqrt{\left(x^2+1\right)\left(\sqrt{x}+1\right)^3}}.$$

By the chain rule:

$$g'(x) = \frac{1}{2\sqrt{\left(x^2+1\right)\left(\sqrt{x}+1\right)^3}}\left(2x\left(\sqrt{x}+1\right)^3 + 3\left(x^2+1\right)\left(\sqrt{x}+1\right)^2\frac{1}{2\sqrt{x}}\right)$$

13. **(a)** $h(x) = 6\left(x^2+4\right)^{-1/2}$

By the chain rule:

$$h'(x) = 6 \times \left(\frac{-1}{2}\right)\left(x^2+4\right)^{-3/2}(2x)$$
$$= \frac{-6x}{\left(x^2+4\right)^{3/2}}$$

(b) $h(x) = \dfrac{\sqrt{x^2+4}}{6}$

By the chain rule:

$$h'(x) = \frac{1}{6}\cdot\frac{1}{2\sqrt{x^2+4}}\frac{d}{dx}\left(x^2+4\right)$$
$$= \frac{1}{6}\cdot\frac{1}{2\sqrt{x^2+4}}(2x)$$
$$= \frac{x}{6\sqrt{x^2+4}}$$

14. **(a)** $h(t) = \dfrac{\left(t^3+4\right)^5}{8}$

By the chain rule:

$$h'(t) = \frac{5}{8}\left(t^3+4\right)^4 \frac{d}{dt}\left(t^3+4\right)$$
$$= \frac{5}{8}\left(t^3+4\right)^4 \left(3t^2\right)$$
$$= \frac{15t^2}{8}\left(t^3+4\right)^4$$

(b) $h(t) = 8\left(t^3+4\right)^{-5}$
By the chain rule:
$$h'(t) = 8\times(-5)\left(t^3+4\right)^{-6}\frac{d}{dt}\left(t^3+4\right)$$
$$= -40\left(t^3+4\right)^{-6}\left(3t^2\right)$$
$$= -120t^2\left(t^3+4\right)^{-6}$$

15. **(a)** $f(x) = \left(\sqrt{x^3+2}+2x\right)^{-2}$
By the chain rule:
$f'(x)$
$$= -2\left(\sqrt{x^3+2}+2x\right)^{-3}\tfrac{d}{dx}\left(\sqrt{x^3+2}+2x\right)$$
$$= -2\left(\sqrt{x^3+2}+2x\right)^{-3}\left(\tfrac{3x^2}{2\sqrt{x^3+2}}+2\right)$$
$$= \tfrac{-2}{\left(\sqrt{x^3+2}+2x\right)^3}\left(\tfrac{3x^2+4\sqrt{x^3+2}}{2\sqrt{x^3+2}}\right)$$
$$= -\tfrac{3x^2+4\left(\sqrt{x^3+2}\right)}{\left(\sqrt{x^3+2}+2x\right)^3\sqrt{x^3+2}}$$

(b) $f(x) = \sqrt{x^3+2+2x^{-2}}$
By the chain rule:
$f'(x) =$
$$\frac{1}{2\sqrt{x^3+2+2x^{-2}}}\frac{d}{dx}\left(x^3+2+2x^{-2}\right)$$
$$= \frac{1}{2\sqrt{x^3+2+2x^{-2}}}\left(3x^2-4x^{-3}\right)$$
$$= \frac{3x^2-4x^{-3}}{2\sqrt{x^3+2+2x^{-2}}}$$

16. **(a)** $f(x) = \sqrt{4x^2+(8-x^2)^2}$
By the chain rule:
$$f'(x) = \frac{8x-4x\left(8-x^2\right)}{2\sqrt{4x^2+(8-x^2)^2}}$$
$$= \frac{8x-32x+4x^3}{2\sqrt{4x^2+(8-x^2)^2}}$$
$$= \frac{-24x+4x^3}{2\sqrt{4x^2+(8-x^2)^2}}$$
$$= \frac{2x^3-12x}{\sqrt{4x^2+(8-x^2)^2}}$$

(b) $f(x) = \left(\sqrt{4x^2+8}-x^2\right)^2$
By the chain rule:
$f'(x) =$
$$2\left(\sqrt{4x^2+8}-x^2\right)\tfrac{d}{dx}\left(\sqrt{4x^2+8}-x^2\right)$$
$$= 2\left(\sqrt{4x^2+8}-x^2\right)\left(\tfrac{4x}{\sqrt{4x^2+8}}-2x\right)$$
$$= 2\left(\sqrt{4x^2+8}-x^2\right)\left(\tfrac{4x-2x\sqrt{4x^2+8}}{\sqrt{4x^2+8}}\right)$$
$$= 4\left(\sqrt{4x^2+8}-x^2\right)\left(\tfrac{2x-x\sqrt{4x^2+8}}{\sqrt{4x^2+8}}\right)$$

17. $f(x) = x^3+4x-1$ is a one-to-one function with $f(0) = -1$ and $f'(0) = 4$. Therefore $g(-1) = 0$ and
$$g'(-1) = \frac{1}{f'(g(-1))} = \frac{1}{f'(0)} = \frac{1}{4}.$$

18. $f(x) = x^5+4x-2$ is a one-to-one function with $f(0) = -2$ and $f'(0) = 4$. Therefore $g(-2) = 0$ and
$$g'(-2) = \frac{1}{f'(g(-2))} = \frac{1}{f'(0)} = \frac{1}{4}$$
.

19. $f(x) = x^5+3x^3+x$ is a one-to-one function with $f(1) = 5$ and $f'(1) = 5+9+1 = 15$. Therefore $g(5) = 1$ and
$$g'(5) = \frac{1}{f'(g(5))} = \frac{1}{f'(1)} = \frac{1}{15}.$$

20. $f(x) = x^3+2x+1$ is a one-to-one function with $f(-1) = -2$ and $f'(-1) = 5$. Therefore $g(-2) = -1$ and
$$g'(-2) = \frac{1}{f'(g(-2))} = \frac{1}{f'(-1)} = \frac{1}{5}.$$

21. $f(x) = \sqrt{x^3+2x+4}$ is a one-to-one function and $f(0) = 2$ so $g(2) = 0$. Meanwhile,
$$f'(x) = \frac{1}{2\sqrt{x^3+2x+4}}(3x^2+2)$$
$f'(0) = 1/2$
$$g'(2) = \frac{1}{f'(g(2))} = \frac{1}{f'(0)} = 2.$$

22. $f(x) = \sqrt{x^5+4x^3+3x+1}$ is a one-to-one function and $f(1) = 3$ so $g(3) = 1$. Meanwhile,
$$f'(x) = \frac{5x^4+12x^2+3}{2\sqrt{x^5+4x^3+3x+1}}$$
$$f'(1) = \frac{20}{6} = \frac{10}{3}$$
$$g'(3) = \frac{1}{f'(g(3))} = \frac{1}{f'(1)} = \frac{3}{10}.$$

23. $f(x) = \sqrt[3]{x\sqrt{x^4+2x\sqrt[4]{\frac{8}{x+2}}}}$
Use Chain rule to find the derivative of the function. We can also use Product rule.

24. $f(x) = \dfrac{3x^2 + 2\sqrt{x^3 + \frac{4}{x^4}}}{(x^3-4)\sqrt{x+2}}$
Use Quotient rule to find the derivative of the function. We can also use Chain rule and Product rule.

25. $f(t) = \sqrt{t^2 + \dfrac{4}{t^3}}\left(\dfrac{8t+5}{2t-1}\right)^3$
Use product rule to find the derivative of the function. We can also use chain rule and Quotient rule.

26. $f(t) = \left(3t + \dfrac{4\sqrt{t^2+1}}{t-5}\right)^3$
Use Chain rule to find the derivative of the function. We can also use Quotient rule.

27. $f(x) = \sqrt{x^2+16},\ a = 3,\ f(3) = 5$
$f'(x) = \dfrac{1}{2\sqrt{x^2+16}}(2x) = \dfrac{x}{\sqrt{x^2+16}}$
$f'(3) \ = \dfrac{3}{\sqrt{3^2+16}} = \dfrac{3}{5}$
So, the tangent line is $y = \dfrac{3}{5}(x-3) + 5$ or $y = \dfrac{3}{5}x + \dfrac{16}{5}$.

28. $f(-2) = \dfrac{3}{4}$
$f'(x) = \dfrac{-12x}{(x^2+4)^2}$
$f'(-2) = \dfrac{24}{64} = \dfrac{3}{8}$
The equation of the tangent line is
$y = \dfrac{3}{8}(x+2) + \dfrac{3}{4}$.

29. $s(t) = \sqrt{t^2+8}$
$v(t) = s'(t) = \dfrac{2t}{2\sqrt{t^2+8}} = \dfrac{t}{\sqrt{t^2+8}}\text{m/s}$
$v(2) = \dfrac{2}{\sqrt{12}} = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}\text{m/s}$

30. $s(t) = \dfrac{60t}{\sqrt{t^2+1}}$
$v(t) = \dfrac{\sqrt{t^2+1}(60) - 60t\frac{1}{2\sqrt{t^2+1}}2t}{t^2+1}\text{m/s}$
$v(2) = \dfrac{60\sqrt{5} - \frac{240}{\sqrt{5}}}{5} = \dfrac{12\sqrt{5}}{5}\text{m/s}$

31. $h'(x) = f'(g(x))g'(x)$
$h'(1) = f'(g(1))g'(1) = f'(2)\cdot(-2) = -6$

32. $h'(x) = f'(g(x))g'(x)$
$h'(2) = f'(g(2))g'(2) = f'(3)\cdot(4) = -12$

33. As a temporary device given any f, set $g(x) = f(-x)$. Then by the chain rule,
$$g'(x) = f'(-x)(-1) = -f'(-x).$$
In the even case ($g = f$) this reads $f'(-x) = -f'(x)$ and shows f' is odd.
In the odd case ($g = -f$ and therefore $g' = -f'$), this reads $-f'(x) = -f'(-x)$ or $f'(x) = f'(-x)$ and shows f' is even.

34. To say that $f(x)$ is symmetric about the line $x = a$ is the same as saying that $f(a+x) = f(a-x)$. Taking derivatives (using the chain rule), we have
$\dfrac{d}{dx}f(a+x) = f'(a+x)$
$\dfrac{d}{dx}f(a-x) = f'(a-x)(-1) = -f'(a-x)$.
Thus, $f'(a+x) = -f'(a-x)$ and the graph of $f'(x)$ is symmetric through the point $(a,\, 0)$.

35. **(a)** Chain rule gives,
$$\begin{aligned}\frac{d}{dx}f\left(x^2\right) &= f'\left(x^2\right)\frac{d}{dx}\left(x^2\right)\\ &= f'\left(x^2\right)(2x)\\ &= 2xf'\left(x^2\right).\end{aligned}$$
(b) Chain rule gives,
$$\begin{aligned}\frac{d}{dx}[f(x)]^2 &= 2f(x)\frac{d}{dx}f(x)\\ &= 2f(x)f'(x).\end{aligned}$$
(c) Chain rule gives,
$$\begin{aligned}\frac{d}{dx}f\left(f(x)\right) &= f'\left(f(x)\right)\frac{d}{dx}f(x)\\ &= f'\left(f(x)\right)f'(x).\end{aligned}$$

36. **(a)** Chain rule gives,
$$\begin{aligned}\frac{d}{dx}f\left(\sqrt{x}\right) &= f'\left(\sqrt{x}\right)\frac{d}{dx}\left(\sqrt{x}\right)\\ &= f'\left(\sqrt{x}\right)\frac{1}{2\sqrt{x}}.\end{aligned}$$
(b) Chain rule gives,
$$\begin{aligned}\frac{d}{dx}\left(\sqrt{f(x)}\right) &= \frac{1}{2\sqrt{f(x)}}\frac{d}{dx}f(x)\\ &= \frac{1}{2\sqrt{f(x)}}f'(x).\end{aligned}$$
(c) Chain rule gives,
$\dfrac{d}{dx}\left[f\left(xf(x)\right)\right]$
$= f'\left(xf(x)\right)\dfrac{d}{dx}\left(xf(x)\right)$
and product rule gives
$= f'\left(xf(x)\right)\left(f(x) + x\dfrac{d}{dx}f(x)\right)$
$= f'\left(xf(x)\right)\left(f(x) + xf'(x)\right).$

37. (a) Chain rule gives,

$$\frac{d}{dx}\left[f\left(\frac{1}{x}\right)\right] = f'\left(\frac{1}{x}\right).\frac{d}{dx}\left(\frac{1}{x}\right)$$
$$= f'\left(\frac{1}{x}\right).\left(-\frac{1}{x^2}\right)$$
$$= -\left(\frac{1}{x^2}\right).f'\left(\frac{1}{x}\right).$$

(b) Chain rule gives,

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \left(-\frac{1}{f(x)^2}\right).\frac{d}{dx}f(x)$$
$$= \left(-\frac{1}{f(x)^2}\right).f'(x).$$

(c) Chain rule gives,

$$\frac{d}{dx}\left[f\left(\frac{x}{f(x)}\right)\right]$$
$$= f'\left(\frac{x}{f(x)}\right)\frac{d}{dx}\left(\frac{x}{f(x)}\right)$$

and quotient rule gives,

$$= f'\left(\frac{x}{f(x)}\right)\left(\frac{f(x) - xf'(x)}{[f(x)]^2}\right).$$

38. (a) Chain rule gives,

$$\frac{d}{dx}\left(1 + f(x^2)\right) = f'\left(x^2\right)\frac{d}{dx}\left(x^2\right)$$
$$= f'\left(x^2\right)(2x) = 2xf'\left(x^2\right).$$

(b) Chain rule gives,

$$\frac{d}{dx}[1 + f(x)]^2$$
$$= 2[1 + f(x)]\frac{d}{dx}(1 + f(x))$$
$$= 2[1 + f(x)]f'(x) = 2f'(x)[1 + f(x)].$$

(c) Chain rule gives,

$$\frac{d}{dx}[f(1 + f(x))]$$
$$= f'(1 + f(x)).\frac{d}{dx}(1 + f(x))$$
$$= f'(1 + f(x))f'(x)$$
$$= f'(x)f'(1 + f(x)).$$

39. $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

(a) At $x = 0:$ $g'(0) = 1,\ g(0) = 1,$

$$\frac{d}{dx}f(g(0)) = f'(g(0))g'(0)$$
$$= f'(1)\cdot g'(0) = 3 \times 1 = 3$$

(b) At $x = 1:$ $g'(1)$ does not exist.

So $\frac{d}{dx}f(g(1))$ does not exist.

(c) At $x = 3:$

$g'(3) = 3,\ g(3) = 1$

$$\frac{d}{dx}f(g(3)) = f'(g(3))g'(3)$$
$$= f'(1)\cdot g'(3) = 3 \times 3 = 9$$

40. $\frac{d}{dx}g(f(x)) = g'(f(x))f'(x)$

(a) At $x = 0:$

$f'(0)$ does not exist. So $\frac{d}{dx}g(f(0))$ does not exist.

(b) At $x = 1:$

$f'(1) = 3,\ f(1) = 0,$

$$\frac{d}{dx}g(f(1)) = g'(f(1))f'(1)$$
$$= g'(0)\cdot f'(1) = 1 \times 3 = 3$$

(c) At $x = 3:$

$f'(3) = 0,\ f(3) = 3,$

$$\frac{d}{dx}g(f(3)) = g'(f(3))f'(3)$$
$$= g'(3)\cdot f'(3) = 3 \times 0 = 0$$

41. (a) $f(x) = \sqrt{x^2 + 4}$

By the chain rule:

$$f'(x) = \frac{1}{2\sqrt{x^2+4}}\frac{d}{dx}\left(x^2 + 4\right)$$ By the

$$= \frac{2x}{2\sqrt{x^2+4}}$$
$$= \frac{x}{\sqrt{x^2+4}}$$

quotient rule:

$$f''(x) = \frac{\sqrt{x^2+4} - x\frac{d}{dx}\left(\sqrt{x^2+4}\right)}{(x^2+4)}$$
$$= \frac{\sqrt{x^2+4} - x\frac{2x}{2\sqrt{x^2+4}}}{(x^2+4)}$$
$$= \frac{\sqrt{x^2+4} - \frac{x^2}{\sqrt{x^2+4}}}{(x^2+4)}$$
$$= \frac{x^2 + 4 - x^2}{\sqrt{x^2+4}\,(x^2+4)}$$
$$= \frac{4}{(x^2+4)^{3/2}}$$

(b) $f(t) = 2\left(t^2 + 4\right)^{-1/2}$

By the chain rule:

$$f'(t) = 2\cdot\frac{-1}{2}\left(t^2+4\right)^{-3/2}\frac{d}{dt}\left(t^2+4\right)$$
$$= \frac{-1}{(t^2+4)^{3/2}}(2t)$$
$$= \frac{-2t}{(t^2+4)^{3/2}}$$

By the quotient rule:

$$f''(t) = -2\left[\frac{\left(t^2+4\right)^{3/2} - t\frac{d}{dt}\left(t^2+4\right)^{3/2}}{\left(t^2+4\right)^3}\right]$$
$$= -2\left[\frac{\left(t^2+4\right)^{3/2} - t\left(\frac{3}{2}\right)\left(t^2+4\right)^{1/2}\frac{d}{dt}\left(t^2+4\right)}{\left(t^2+4\right)^3}\right]$$
$$= -2\left[\frac{\left(t^2+4\right)^{3/2} - t\left(\frac{3}{2}\right)\left(t^2+4\right)^{1/2}2t}{\left(t^2+4\right)^3}\right]$$
$$= -2\left[\frac{\left(t^2+4\right)^{3/2} - 3t^2\left(t^2+4\right)^{1/2}}{\left(t^2+4\right)^3}\right]$$
$$= \left[\frac{-2\left(t^2+4\right)^{3/2} + 6t^2\left(t^2+4\right)^{1/2}}{\left(t^2+4\right)^3}\right]$$
$$= \left[\frac{\left(t^2+4\right)^{1/2}\left[-2\left(t^2+4\right)+6t^2\right]}{\left(t^2+4\right)^3}\right]$$
$$= \left[\frac{-2t^2-8+6t^2}{\left(t^2+4\right)^{5/2}}\right]$$
$$= \frac{4t^2-8}{\left(t^2+4\right)^{5/2}} = \frac{4\left(t^2-2\right)}{\left(t^2+4\right)^{5/2}}$$

42. **(a)** By the chain rule:

$$h'(t) = 2\left(t^3+3\right)\frac{d}{dt}\left(t^3+3\right)$$
$$= 2\left(t^3+3\right)\left(3t^2\right) = 6t^5 + 18t^2$$
$$h''(t) = 30t^4 + 36t$$

(b) $g(s) = 3\left(s^2+1\right)^{-2}$

By the chain rule:

$$g'(s) = 3(-2)\left(s^2+1\right)^{-3}\frac{d}{ds}\left(s^2+1\right)$$
$$= -6\left(s^2+1\right)^{-3}(2s)$$
$$= \frac{-12s}{\left(s^2+1\right)^3}$$

By the product and chain rule:

$$g''(s) = \frac{d}{dx}\left(-12s(s^2+1)^{-3}\right)$$
$$= -12\left(\left(s^2+1\right)^{-3} - 6s^2\left(s^2+1\right)^{-4}\right)$$
$$= -12(s^2+1)^{-4}\left(s^2+1-6s^2\right)$$
$$= -\frac{12(1-5s^2)}{(s^2+1)^4}$$

43. **(a)** $f(x) = (x^3-3x^2+2x)^{1/3}$

$$f'(x) = \frac{\frac{d}{dx}\left(x^3-3x^2+2x\right)}{3(x^3-3x^2+2x)^{2/3}}$$
$$= \frac{3x^2-6x+2}{3(x^3-3x^2+2x)^{2/3}}$$

The derivetive of f does not exist at values of x for which

$x^3-3x^2+2x = 0$

$x(x^2-3x+2) = 0$

$x(x-1)(x-2) = 0.$

Thus, the derivative of f does not exist for $x = 0$, 1, and 2. The derivative fails to exist at these points because the tangent lines at these points are vertical.

(b) $f(x) = \sqrt{x^4-3x^3+3x^2-x}$

$$f'(x) = \frac{\frac{d}{dx}\left(x^4-3x^3+3x^2-x\right)}{2\sqrt{x^4-3x^3+3x^2-x}}$$
$$= \frac{4x^3-9x^2+6x-1}{2\sqrt{x^4-3x^3+3x^2-x}}$$

The derivative of f does not exist at values of x for which

$x^4-3x^3+3x^2-x = 0$

$x\left(x^3-3x^2+3x-1\right) = 0$

$x(x-1)^3 = 0.$

Thus, the derivative of x does not exist for $x = 0$ and 1. The derivative fails to exist at these points because the tangent lines at these points are vertical.

44. Multiply numerator and denominator by $g(x+h) - g(x)$.

$$\lim_{h\to 0}\left(\frac{f(g(x+h))-f(g(x))}{h}\right)\left(\frac{g(x+h)-g(x)}{g(x+h)-g(x)}\right)$$

The above step is not well documented and in this step we use the assumption that $g'(x) \neq 0$. Since $g'(x) \neq 0$ implies that $g(x+h)-g(x) \neq 0$ for $h \neq 0$.

45. $f(x) = (x^2+3)^2 \cdot 2x$

Recognizing the "$2x$" as the derivative of x^2+3, we guess $g(x) = c(x^2+3)^3$ where c is some constant.

$g'(x) = 3c(x^2+3)^2 \cdot 2x$

which will be $f(x)$ only if $3c = 1$, so $c = 1/3$, and

$$g(x) = \frac{(x^2+3)^3}{3}.$$

46. A good initial guess is $(x^3+4)^{5/3}$, then adjust the constant to get

$$g(x) = \frac{1}{5}(x^3+4)^{5/3}.$$

47. $f(x) = \dfrac{x}{\sqrt{x^2+1}}$.

Recognizing the "x" as half the derivative of x^2+1, and knowing that differentiation throws the square root into the denominator, we guess $g(x) = c\sqrt{x^2+1}$ where c is some constant and find that

$$g'(x) = \frac{c}{2\sqrt{x^2+1}}(2x)$$

will match $f(x)$ if $c = 1$, so

$$g(x) = \sqrt{x^2+1}.$$

48. A good initial guess is $(x^2+1)^{-1}$, then adjust the constant to get

$$g(x) = -\frac{1}{2}(x^2+1)^{-1}.$$

2.6 Derivatives of Trigonometric Functions

1. $f(x) = 4\sin 3x - x$
$$f'(x) = 4(\cos 3x)(3) - 1 = 12\cos 3x - 1$$

2. $f(x) = 4x^2 - 3\tan 2x$
$$f'(x) = 4(2x) - 3\sec^2(2x)(2) = 8x - 6\sec^2(2x)$$

3. $f(t) = \tan^3 2t - \csc^4 3t$
$$f'(t) = 3\tan^2(2t)\sec^2(2t)(2) - 4\csc^3(3t)[-\csc(3t)\cot(3t)](3) = 6\tan^2(2t)\sec^2(2t) + 12\csc^4(3t)\cot(3t)$$

4. $f(t) = t^2 + 2\cos^2 4t$
$$f'(t) = 2t + 4\cos(4t)[-\sin(4t)](4) = 2t - 16\sin(4t)\cos(4t)$$

5. $f(x) = x\cos 5x^2$
$$f'(x) = (1)\cos 5x^2 + x(-\sin 5x^2)\cdot 10x = \cos 5x^2 - 10x^2\sin 5x^2$$

6. $f(x) = x^2\sec 4x$
$$f'(x) = x^2(\sec 4x\tan 4x)4 + (\sec 4x)2x = 4x^2(\sec 4x\tan 4x) + 2x\sec(4x)$$

7. $f(x) = \dfrac{\sin(x^2)}{x^2}$
$$f'(x) = \frac{x^2\cos(x^2)\cdot 2x - \sin(x^2)\cdot 2x}{x^4} = \frac{2x[x^2\cos(x^2) - \sin(x^2)]}{x^4} = \frac{2[x^2\cos(x^2) - \sin(x^2)]}{x^3}$$

8. $f(x) = \dfrac{x^2}{\csc^4(2x)}$
$$f'(x) = \frac{2x[\csc^4(2x)] - 4x^2[\csc^3(2x)][-\csc(2x)\cot(2x)](2)}{[\csc^4(2x)]^2} = \frac{2x}{\csc^4(2x)} + \frac{8x^2[\csc^4(2x)\cot(2x)]}{[\csc^4(2x)]^2} = \frac{2x}{\csc^4(2x)} + \frac{8x^2\cot(2x)}{\csc^4(2x)} = \frac{2x + 8x^2\cot(2x)}{\csc^4(2x)}$$

9. $f(t) = \sin 3t\sec 3t = \tan 3t$
$$f'(t) = \frac{d}{dt}[\tan(3t)] = \sec^2(3t)(3) = 3\sec^2(3t)$$

10. $f(t) = \sqrt{\cos 5t\sec 5t} = \sqrt{\cos 5t\cdot\frac{1}{(\cos 5t)}} = 1$
$$f'(t) = \frac{d}{dt}(1) = 0$$

11. $f(w) = \dfrac{1}{\sin 4w}$
$$f'(w) = \frac{-1}{(\sin 4w)^2}\cos 4w(4) = \frac{-4\cos 4w}{\sin^2 4w}$$

12. $f(w) = w^2\sec^2 3w$
$$f'(w) = w^2(2\sec 3w)(\sec 3w\tan 3w)(3) + \sec^2(3w)(2w) = 6w^2\sec^2 3x\tan 3w + 2w\sec^2 3w$$

13. $f(x) = 2\sin(2x)\cos(2x)$
$$f'(x) = 2\{\sin(2x)[-\sin(2x)](2) + \cos(2x)[\cos(2x)](2)\} = -4\sin^2(2x) + 4\cos^2(2x) = 4\cos^2(2x) - 4\sin^2(2x)$$

14. $f(x) = 4\sin^2(3x) + 4\cos^2(3x) = 4\left[\sin^2(3x) + \cos^2(3x)\right] = 4$
$$f'(x) = \frac{d}{dx}(4) = 0$$

15. $f(x) = \tan\sqrt{x^2+1}$
$$f'(x) = (\sec^2\sqrt{x^2+1})\cdot\left(\frac{1}{2}\right)(x^2+1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}}\sec^2\sqrt{x^2+1}$$

16. $f(x) = 4x^2\sin x\sec 3x$
$$f'(x) = 8x\sin x\sec 3x + 4x^2[\cos x\sec 3x + \sin x\sec 3x\tan 3x(3)]$$

17. $f(x) = \sin^3\left(\cos\sqrt{x^3+2x^2}\right)$

$$f'(x) = 3\sin^2\left(\cos\sqrt{x^3+2x^2}\right) \cdot \cos\left(\cos\sqrt{x^3+2x^2}\right) \cdot \left(-\sin\sqrt{x^3+2x^2}\right) \cdot \frac{1}{2}\left(x^3+2x^2\right)^{-1/2}\left(3x^2+4x\right)$$
$$= \frac{3}{2}\left(3x^2+4x\right)\left(x^3+2x^2\right)^{-1/2} \cdot \sin^2\left(\cos\sqrt{x^3+2x^2}\right) \cdot \cos\left(\cos\sqrt{x^3+2x^2}\right) \cdot \left(-\sin\sqrt{x^3+2x^2}\right)$$

18. $f(x) = \tan^4\left[\sin^2\left(x^3+2x\right)\right]$

$$f'(x) = 4\left[\tan^3\left(\sin^2\left(x^3+2x\right)\right)\right] \cdot \left[\sec^2\left(\sin^2\left(x^3+2x\right)\right)\right] \cdot \left[2\sin\left(x^3+2x\right)\right] \cdot \left[\cos\left(x^3+2x\right)\right] \cdot \left(3x^2+2\right)$$

19. **(a)** $f(x) = \sin x^2$

$f'(x) = \cos\left(x^2\right) \cdot (2x) = 2x\cos\left(x^2\right)$

(b) $f(x) = \sin^2 x$

$f'(x) = 2\sin x\cos x$

(c) $f(x) = \sin 2x$

$f'(x) = \cos 2x\,(2) = 2\cos 2x$

20. **(a)** $f(x) = \cos\sqrt{x}$

$$f'(x) = \left(-\sin\sqrt{x}\right).\frac{1}{2}(x)^{-1/2} = -\frac{1}{2}(x)^{-1/2}\sin\sqrt{x}$$

(b) $f(x) = \sqrt{\cos x}$

$$f'(x) = \frac{1}{2}(\cos x)^{-1/2}.\left(-\sin x\right) = -\frac{1}{2}\sin x(\cos x)^{-1/2}$$

(c) $f(x) = \cos\left(\frac{1}{2}x\right)$

$$f'(x) = -\sin\left(\frac{1}{2}x\right).\left(\frac{1}{2}\right) = -\frac{1}{2}\sin\left(\frac{1}{2}x\right)$$

21. **(a)** $f(x) = \sin x^2\tan x$

$f'(x) = \sin x^2\left(\sec^2 x\right) + 2x\ \cos x^2\tan x$

(b) $f(x) = \sin^2(\tan x)$

$f'(x) = 2\sin(\tan x)\cdot\cos(\tan x)\cdot\sec^2 x$

(c) $f(x) = \sin\left(\tan^2 x\right)$

$$f'(x) = \left[\cos\left(\tan^2 x\right)\right](2\tan x)\left(\sec^2 x\right) = (2\tan x)\left(\sec^2 x\right)\left[\cos\left(\tan^2 x\right)\right]$$

22. **(a)** $f(x) = \sec x^2\tan x^2$

$$f'(x) = \sec^3\left(x^2\right)(2x) + \tan^2\left(x^2\right)\sec\left(x^2\right)(2x) = 2x\ \sec\ x^2\left[\sec^2 x^2 + \tan^2 x^2\right]$$

(b) $f(x) = \sec^2(\tan x)$

$$f'(x) = 2\sec(\tan x)\left[\sec(\tan x)\,.\tan(\tan x)\right]\left(\sec^2 x\right)$$

(c) $f(x) = \sec\left(\tan^2 x\right)$

$$f'(x) = \left[\sec\left(\tan^2 x\right)\tan\left(\tan^2 x\right)\right]\cdot(2\tan x)\left(\sec^2 x\right) = \left(2\tan x\sec^2 x\right)\cdot\left[\sec\left(\tan^2 x\right)\tan\left(\tan^2 x\right)\right]$$

23. $f\left(\frac{\pi}{8}\right) = \sin\frac{\pi}{2} = 1$

$f'(x) = 4\cos 4x$

$f'\left(\frac{\pi}{8}\right) = 4\cos\frac{\pi}{2} = 0$

So, the equation of the tangent line is

$y = 0\left(x - \frac{\pi}{8}\right) + 1$ *i.e.* $y = 1$.

24. $f(0) = 0$

$f'(x) = 3\sec^2 3x$,

$f'(0) = 3$.

So, the equation of tangent line is $y = 3x$.

25. $f\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^2\cos\left(\frac{\pi}{2}\right) = 0$

$$f'(x) = x^2(-\sin x) + \cos x\,(2x) = -x^2(\sin x) + (2x)\cos x$$

$$f'\left(\frac{\pi}{2}\right) = -\frac{\pi^2}{4}\sin\frac{\pi}{2} + 2\cdot\frac{\pi}{2}\cos\frac{\pi}{2} = -\frac{\pi^2}{4}$$

So, the equation of the tangent line is $y = -\frac{\pi^2}{4}\left(x - \frac{\pi}{2}\right)$.

26. $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

$f'(x) = \sin x + x\cos x$, so $f'\left(\frac{\pi}{2}\right) = 1$.

So, the equation of the tangent line is $y = x$.

27. $s(t) = t^2 - \sin(2t)$, $t_0 = 0$

$v(t) = s'(t) = 2t - 2\cos(2t)$

$v(0) = 0 - 2\cos(0) = 0 - 2 = -2\,\text{ft/s}$

28. $s(t) = 4 + 3\sin t$, $t_0 = \pi$

$v(t) = s'(t) = 3\cos t$

$v(\pi) = -3$ ft/s

29. $s(t) = \dfrac{\cos t}{t}$, $t_0 = \pi$

$v(t) = s'(t) = \frac{-1}{t^2}\cos t + \frac{1}{t}(-\sin t)$

$v(\pi) = -\dfrac{\cos\pi}{\pi^2} - \dfrac{\sin\pi}{\pi}$

$= \dfrac{1}{\pi^2} - \dfrac{1}{\pi}(0) = \dfrac{1}{\pi^2}$ ft/s

30. $s(t) = t\cos(t^2+\pi)$, $t_0 = 0$

$v(t) = s'(t) = \cos(t^2+\pi) - 2t^2\sin(t^2+\pi)$

$v(0) = \cos\pi - 0 = -1$ ft/s.

31. **(a)** $f(t) = 4\sin 3t$. The velocity at time t is $f'(t) = 12\cos 3t$.

(b) The maximum speed is 12.

(c) The maximum speed of 12 occurs when the vertical position is zero.

32. **(a)** The velocity is $f'(t) = 12\cos 3t$ which is 0 when $3t = \frac{k\pi}{2}$ or $t = \frac{k\pi}{6}$ for any odd integer k.

(b) The location of the spring at these times is given (for any odd integer k) by $f\left(k\frac{\pi}{6}\right) = 4\sin\left(3k\frac{\pi}{6}\right) = 4\sin\left(k\frac{\pi}{2}\right) = \pm 4$.

(c) The spring changes directions at the top and bottom.

33. **(a)** $\lim\limits_{x\to 0}\dfrac{\sin 3x}{x} = \lim\limits_{x\to 0}\dfrac{3\sin 3x}{3x}$

$= 3\cdot\lim\limits_{x\to 0}\dfrac{\sin(3x)}{(3x)}$

$= 3\cdot 1 = 3$

(b) $\lim\limits_{t\to 0}\dfrac{\sin t}{4t} = \dfrac{1}{4}\lim\limits_{t\to 0}\dfrac{\sin t}{t} = \dfrac{1}{4}\cdot 1 = \dfrac{1}{4}$

(c) $\lim\limits_{x\to 0}\dfrac{\cos x - 1}{5x} = \dfrac{1}{5}\lim\limits_{x\to 0}\dfrac{\cos x - 1}{x} = 0$

(d) Let $u = x^2$: then $u \to 0$ as $x \to 0$, and

$\lim\limits_{x\to 0}\dfrac{\sin x^2}{x^2} = \lim\limits_{u\to 0}\dfrac{\sin u}{u} = 1$

34. **(a)** $\lim\limits_{t\to 0}\dfrac{2t}{\sin t} = \lim\limits_{t\to 0}\dfrac{2}{\frac{\sin t}{t}} = 2$

(b) Let $u = x^2$: then $u \to 0$ as $x \to 0$, and

$\lim\limits_{x\to 0}\dfrac{\cos x^2 - 1}{x^2} = \lim\limits_{u\to 0}\dfrac{\cos u - 1}{u} = 0$

(c) $\lim\limits_{x\to 0}\dfrac{\sin 6x}{\sin 5x} = \lim\limits_{x\to 0}\dfrac{\frac{6\sin 6x}{6x}}{\frac{5\sin 5x}{5x}} = \dfrac{6}{5}$

(d) $\lim\limits_{x\to 0}\dfrac{\tan 2x}{x} = \lim\limits_{x\to 0}\dfrac{\frac{\sin 2x}{\cos 2x}}{x}$

$= \lim\limits_{x\to 0}\dfrac{2\sin 2x}{2x}\dfrac{1}{\cos 2x} = 2$

35. $f(x) = \sin(2x) = 2^0\sin(2x)$

$f'(x) = 2\cos 2x = 2^1\cos(2x)$

$f''(x) = -4\sin 2x = -2^2\sin(2x)$

$f'''(x) = -8\cos 2x = -2^3\cos(2x)$

$f^{(4)}(x) = 16\sin 2x = 2^4\sin(2x)$

$f^{(75)}(x) = \left(f^{(72)}\right)^{(3)}(x)$

$= \left(f^{(18\cdot 4)}\right)^{(3)}(x)$

$= 2^{72}f'''(x)$

$= 2^{72}\left[-2^3\cos(2x)\right]$

$= -2^{75}\cos(2x)$

$f^{(150)}(x) = \left(f^{(148)}\right)^{(2)}(x)$

$= \left(f^{(37\cdot 4)}\right)^{(2)}(x)$

$= 2^{148}f''(x)$

$= 2^{148}\left[-2^2\sin(2x)\right]$

$= -2^{150}\sin(2x)$

36. $f(x) = \cos(3x) = 3^0\cos(3x)$

$f'(x) = -3\sin 3x = -3^1\sin(3x)$

$f''(x) = -9\cos 3x = -3^2\cos(3x)$

$f'''(x) = 27\sin 3x = 3^3\sin(3x)$

$f^{(4)}(x) = 81\cos 3x = 3^4\cos(3x)$

$f^{(77)}(x) = \left(f^{(76)}\right)^{(1)}(x)$

$= \left(f^{(19\cdot 4)}\right)^{(1)}(x)$

$= 3^{76}f'(x)$

$= 3^{76}\left[-3^1\sin(3x)\right]$

$= -3^{77}\sin(3x)$

$f^{(120)}(x) = \left(f^{(120)}\right)(x)$

$= \left(f^{(30\cdot 4)}\right)(x)$

$= 3^{120}\cos(3x)$

37. Since, $0 \le \sin\theta \le \theta$, we have
$-\theta \le -\sin\theta \le 0$ which implies
$-\theta \le \sin(-\theta) \le 0$,
so for $-\dfrac{\pi}{2} \le \theta \le 0$
we have $\theta \le \sin\theta \le 0$.
We also know that
$\lim\limits_{\theta\to 0^-}\theta = 0 = \lim\limits_{\theta\to 0^-} 0$,
so the Squeeze Theorem implies that
$\lim\limits_{\theta\to 0^-}\sin\theta = 0$.

38. Since $\cos^2\theta + \sin^2\theta = 1$, we have
$\cos\theta = \sqrt{1-\sin^2\theta}$. Then
$\lim\limits_{\theta\to 0}\cos\theta = \lim\limits_{\theta\to 0}\sqrt{1-\sin^2\theta} = \pm 1$.
Since cos θ is a continuous function and cos $0 = 1$, we conclude that $\lim\limits_{\theta\to 0}\cos\theta = 1$

39. If $f(x) = \cos(x)$, then

$$\frac{f(x+h)-f(x)}{h} = \frac{\cos(x+h)-\cos(x)}{h} = \frac{\cos x\cos h - \sin x \sin h - \cos x}{h} = (\cos x)\frac{(\cos h - 1)}{h} - (\sin x)\left(\frac{\sin h}{h}\right).$$

Taking the limit according to lemma 6.1

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = (\cos x)\cdot\lim_{h\to 0}\frac{\cos h - 1}{h} - (\sin x)\cdot\lim_{h\to 0}\frac{\sin h}{h} = \cos x\cdot 0 - \sin x \cdot 1 = -\sin x$$

40. $$\frac{d}{dx}\cot x = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{\sin x(-\sin x) - \cos x\cos x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

$$\frac{d}{dx}\sec x = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{\cos x\cdot 0 - 1(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos x}\left(\frac{1}{\cos x}\right) = \sec x\tan x.$$

$$\frac{d}{dx}\csc x = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{\sin x\cdot 0 - 1\cos x}{\sin^2 x} = -\frac{1}{\sin x}\left(\frac{\cos x}{\sin x}\right) = -\csc x\cot x.$$

41. Answers depend on CAS.

42. Answers depend on CAS.

43. Answers depend on CAS.

44. Answers depend on CAS.

45. **(a)** If $x \neq 0$, then f is continuous by Theorem 4.2 in Section 1.4, and f is differentiable by the Quotient rule (Theorem 4.2 in Section 2.4) Thus, we only need to check $x = 0$. To see that f is continuous at $x = 0$.

$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\sin x}{x} = 1$
(By Lemma 6.3)
Since $\lim_{x\to 0} f(x) = f(0)$, f is continuous at $x = 0$.

To see that f is differentiable at $x = 0$.

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$
$$f'(0) = \lim_{x\to a}\frac{f(x)-f(0)}{x-0}$$
$$f'(0) = \lim_{x\to a}\frac{\frac{\sin x}{x} - 1}{x}$$

In the proof of Lemma 6.3, equation 6.8 was derived:
$1 > \frac{\sin x}{x} > \cos x$.
Thus, $0 > \frac{\sin x}{x} - 1 > \cos x - 1$ and therefore if $x > 0$,
$0 > \frac{\frac{\sin x}{x} - 1}{x} > \frac{\cos x - 1}{x}$
and if $x < 0$,
$0 < \frac{\frac{\sin x}{x} - 1}{x} < \frac{\cos x - 1}{x}$

By lemma 6.4, $\lim_{x\to 0}\frac{\cos x - 1}{x} = 1$.
By applyings squeeze theorem to previous two inequalities, we obtain
$\lim_{x\to 0}\frac{\frac{\sin x}{x} - 1}{x} = 1$ so, $f'(0) = 0$.

(b) From part(a) and quotient rule we have,

$$f'(x) = \begin{cases} 0 & x = 0 \\ \frac{x\cos x - \sin x}{x^2} & x \neq 0 \end{cases}$$

Thus to show that $f'(x)$ is continous, we need only to show that

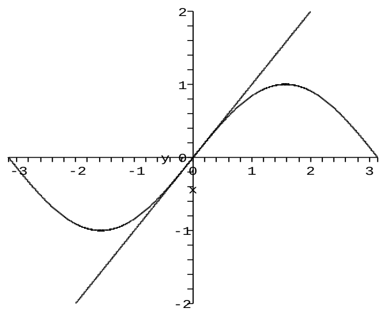
$$\lim_{x\to 0} f'(x) = f'(0) = 0.$$

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0}\frac{x\cos x - \sin x}{x^2} = \lim_{x\to 0}\frac{x\left(\cos x - \frac{\sin x}{x}\right)}{x^2} = \lim_{x\to 0}\frac{\left(\cos x - \frac{\sin x}{x}\right)}{x} = 0$$

Since, $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

46. We use the assumption that x is in radians in Lemma 6.3. The derivative of $\sin x^\circ = \sin(\frac{\pi}{180^\circ}x)$ is $\frac{\pi}{180^\circ}\cos(x^\circ)$. The factor of $\frac{\pi}{180^\circ}$ comes from applying the chain rule.

47. The Sketch: $y = x$ and $y = \sin x$



It is not possible visually to either detect or rule out intersections near $x = 0$ (other than zero itself).

We have that $f'(x) = \cos x$, which is less than 1 for $0 < x < 1$. If $\sin x \geq x$ for some x in the interval (0, 1), then there would be a point on the graph of $y = \sin x$ which lies above the line $y = x$, but then (since $\sin x$ is continuous) the slope of the tangent line of $\sin x$ would have to be greater than 1 or equal to at some point in that interval, contradicting $f'(x) < 1$. Since $\sin x < x$ for $0 < x < 1$, we have for $-\sin x > -x$ for $0 < x < 1$. Then $-\sin x = \sin(-x)$ so $\sin(-x) > -x$ for $0 < x < 1$, which is the same as saying $\sin x > x$ for $-1 < x < 0$.

Since $-1 \leq \sin x \leq 1$, the only interval on which $y = \sin x$ might intersect $y = x$ is [-1, 1]. We know they intersect at $x = 0$ and we just showed that they do not intersect on the intervals (-1, 0) and (0, 1). So the only other points they might intersect are $x = \pm 1$, but we know that $\sin(\pm 1) \neq \pm 1$, so these graphs intersect only at $x = 0$.

48. $0 < k \leq 1$ produces one intersection. For $1 < k < 7.8$ (roughly) there are exactly three intersections. For $k \approx 7.8$ there are 5 intersections. For $k > 7.8$ there are 7 or more intersections.

2.7 Derivatives of Exponential and Logarithmic Functions

1. $f'(x) = 3x^2.e^x + x^3.e^x = x^2e^x(x+3)$

2. $f'(x) = 2e^{2x}\cos 4x + e^{2x}(-\sin 4x)4$

3. $f(t) = t + 2^t$
$f'(t) = 1 + 2^t \log 2$

4. $f(t) = t4^{3t}$
$f'(t) = 4^{3t} + t4^{3t}(\ln 4)\, 3 = 4^{3t}(1 + 3t\ln 4)$

5. $f'(x) = 2e^{4x+1}(4) = 8e^{4x+1}$

6. $f'(x) = e^{-x}$, so $f'(x) = -e^{-x}$.

7. $h(x) = (\frac{1}{3})^{x^2}$

$$\begin{aligned} h'(x) &= \ln(\tfrac{1}{3}) \cdot 2x \cdot (\tfrac{1}{3})^{x^2} \\ &= 2x \cdot \ln(\tfrac{1}{3}) \cdot (\tfrac{1}{3})^{x^2} \\ &= -2x \cdot \ln(3) \cdot (\tfrac{1}{3})^{x^2} \end{aligned}$$

8. $h(x) = 4^{-x^2}$

$$\begin{aligned} h'(x) &= 4^{-x^2} \cdot \ln(4) \cdot (-2x) \\ &= -2x \cdot 4^{-x^2} \cdot \ln(4) \end{aligned}$$

9. $f(u) = e^{u^2+4u}$

$$\begin{aligned} f'(u) &= e^{u^2+4u} \cdot \frac{d}{du}(u^2 + 4u) \\ &= e^{u^2+4u} \cdot (2u + 4) \end{aligned}$$

10. $f(x) = 3e^{\tan x}$

$$\begin{aligned} f'(x) &= 3e^{\tan x} \cdot \frac{d}{dx}(\tan x) \\ &= 3e^{\tan x}\sec^2 x \end{aligned}$$

11. $f(w) = \dfrac{e^{4w}}{w}$

$$\begin{aligned} f'(w) &= \frac{w \cdot 4e^{4w} - e^{4w} \cdot 1}{w^2} \\ &= \frac{e^{4w}(4w - 1)}{w^2} \end{aligned}$$

12. $f(w) = \dfrac{w}{e^{6w}}$

$$\begin{aligned} f'(w) &= \frac{1 \cdot e^{6w} - w \cdot e^{6w} \cdot 6}{(e^{6w})^2} \\ &= \frac{e^{6w} - 6we^{6w}}{(e^{6w})^2} = \frac{(1 - 6w)}{e^{6w}} \end{aligned}$$

13. $f'(x) = \dfrac{1}{2x}.(2) = \dfrac{1}{x}$

14. $f(x) = \dfrac{1}{2}\ln 8 + \dfrac{1}{2}\ln x$

$$f'(x) = \frac{1}{2x}$$

15. $f(t) = \ln(t^3 + 3t)$

$$\begin{aligned} f'(t) &= \frac{1}{t^3 + 3t} \cdot (3t + 3) \\ &= \frac{3t^2 + 3}{t^3 + 3t} = \frac{3\left(t^2 + 1\right)}{t\left(t^2 + 3\right)} \end{aligned}$$

16. $f(t) = t^3 \ln(t)$

$$f'(t) = 3t^2 \cdot \ln(t) + t^3 \cdot \frac{1}{t} = 3t^2 \ln(t) + t^2$$

17. $g(x) = \ln(\cos x)$

$$g'(x) = \frac{1}{\cos x}(-\sin x) = -\tan x$$

18. $g(x) = \cos x \ln(x^2+1)$

$$g'(x) = \ln(x^2+1) \cdot (-\sin x) + \frac{2x \cos x}{x^2+1} = \frac{2x\cos x}{x^2+1} - \sin x \cdot \ln(x^2+1)$$

19. **(a)** $f(x) = \sin(\ln x^2)$

$$f'(x) = \cos(\ln x^2) \cdot \frac{2x}{x^2} = \frac{2\cos(\ln x^2)}{x}$$

(b) $g(t) = \ln(\sin t^2)$

$$g'(t) = \frac{1}{\sin t^2} \cdot \cos t^2 \cdot 2t = \frac{\cos t^2 \cdot 2t}{\sin t^2} = 2t\cot(t^2)$$

20. **(a)** $f(x) = \dfrac{\sqrt{\ln x}}{x}$

$$f'(x) = \frac{x \cdot \frac{1}{2}(\ln x)^{-\frac{1}{2}} \cdot \frac{1}{x} - (\ln x)^{\frac{1}{2}} \cdot 1}{x^2} = \frac{\frac{1}{2\sqrt{\ln x}} - \sqrt{\ln x}}{x^2} = \frac{1 - 2\ln x}{2x^2\sqrt{\ln x}}$$

(b) $g(t) = \dfrac{\ln\sqrt{t}}{t}$

$$g'(t) = \frac{t \cdot \frac{1}{2\sqrt{t}} \cdot t^{-\frac{1}{2}} - \ln\sqrt{t}}{t^2} = \frac{\frac{1}{2} - \ln\sqrt{t}}{t^2} = \frac{1 - 2 \cdot \ln\sqrt{t}}{2t^2}$$

21. **(a)** $h(x) = e^x \cdot \ln x$

$$h'(x) = e^x \cdot \frac{1}{x} + \ln x \cdot e^x$$

(b) $f(x) = e^{\ln x}$

$$f'(x) = e^{\ln x} \cdot \frac{1}{x}$$

22. **(a)** $h(x) = 2^{e^x}$

$$h'(x) = 2^{e^x} \cdot e^x \cdot \ln 2$$

(b) $f(x) = \dfrac{e^x}{2^x}$

$$f'(x) = \frac{2^x \cdot e^x - e^x \cdot 2^x \cdot \ln 2}{(2^x)^2} = \frac{e^x(1 - \ln 2)}{2^x}$$

23. **(a)** $f(x) = \ln(\sin x)$

$$f'(x) = \frac{1}{\sin x} \cdot \cos x = \cot x$$

(b) $f(t) = \ln(\sec t + \tan t)$

$$f'(t) = \frac{\sec t \tan t + \sec^2 t}{\sec t + \tan t} = \sec t$$

24. **(a)** $f(x) = \sqrt[3]{e^{2x} \cdot x^3}$

$$f'(x) = \frac{1}{3}\left(e^{2x} \cdot x^3\right)^{-\frac{2}{3}} \cdot \left(3x^2 e^{2x} + 2x^3 e^{2x}\right) = \frac{x^2 \cdot e^{2x} \cdot (3+2x)}{3(e^{2x} \cdot x^3)^{\frac{2}{3}}}$$

(b) $f(w) = \sqrt[3]{e^{2w} + w^3}$

$$f'(w) = \frac{1}{3}(e^{2w} + w^3)^{-\frac{2}{3}} \cdot (2e^{2w} + 3w^2)$$

25. $f(x) = 3e^{x^2}$
$f(1) = 3e^{1^2} = 3e$
$f'(x) = 3e^{x^2} 2x$
$f'(1) = 3e^{1^2} 2(1) = 6e$
So, the equation of the tangent line is,
$y = 6e(x-1) + 3e$.

26. $f(x) = 3^{x^e}$
$f(1) = 3^{1^e} = 3$
$f'(x) = 3^{x^e} \ln 3 \cdot ex^{(e-1)}$
$f'(1) = 3\ln 3 \cdot e$
So, the equation of the tangent line is,
$y = 3\ln 3 \cdot e(x-1) + 3$.

27. $f(1) = 0$
$f'(x) = 2x\ln x + x^2 . \dfrac{1}{x} = 2x\ln x + x$
$f'(1) = 2 \cdot 1 \ln 1 + 1 = 2 \cdot 0 + 1 = 1$
So the equation of tangent line is
$y = 1(x-1) + 0$ or $y = x - 1$.

28. $f(x) = 2\ln x^3$
$f'(x) = \dfrac{2}{x^3} \cdot 3x^2 = \dfrac{6}{x}$
Slope $= f'(x)$ at $x = 1$.
Slope $m = \dfrac{6}{1} = 6$.
Equation of the line passing through (x_1, y_1) with slope m is $y - y_1 = m(x - x_1)$.
At $x_1 = 1$, $y_1 = f(1) = 2.\ln 1^3 = 0$.
Therefore equation is $y - 0 = 6 \cdot (x-1)$ or $y = 6x - 6$.

29. **(a)** $f(x) = xe^{-2x}$
Given that, the tangent line to $f(x)$ is horizontal. Therefore slope is zero.

Hence,
$f'(x) = e^{-2x} - 2xe^{-2x} = 0$
$e^{-2x}(1-2x) = 0$
$x = \dfrac{1}{2}.$

(b) $f(x) = x \cdot e^{-3x}$
Given that, the tangent line to $f(x)$ is horizontal. Therefore slope is zero. Hence,
$f'(x) = x \cdot (-3e^{-3x}) + e^{-3x} = 0.$
$\Rightarrow e^{-3x}(-3x+1) = 0$
$\Rightarrow 3x - 1 = 0$
$\Rightarrow x = \dfrac{1}{3}$

30. **(a)** $f(x) = x^2 \cdot e^{-2x}$
Given that, the tangent line to $f(x)$ is horizontal. Therefore slope is zero. Hence,
$f'(x) = x^2 \cdot (-2e^{-2x}) + 2x \cdot e^{-2x} = 0$
$\Rightarrow -x + 1 = 0$
$\Rightarrow x = 1$

(b) $f(x) = x^2 \cdot e^{-3x}$
Given that, the tangent line to $f(x)$ is horizontal. Therefore slope is zero. Hence,
$f'(x) = x^2 \cdot (-3e^{-3x}) + 2x \cdot e^{-3x} = 0$
$\Rightarrow -3x + 2 = 0$
$\Rightarrow x = \dfrac{2}{3}.$

31. $v'(t) = 100.3^t \ln 3$
$\dfrac{v'(t)}{v(t)} = \dfrac{100.3^t \ln 3}{100.3^t} = \ln 3 \approx 1.10$
So, the percentage change is about 110%

32. $v'(t) = 1004^t (\ln 4)$
$\dfrac{v'(t)}{v(t)} = \ln 4 \approx 1.3863$
The instantaneous percentage rate of change is 138.6%

33. $v(t) = 40e^{0.4t}$
$v'(t) = 40e^{0.4t}(0.4) = 16e^{0.4t}$
$\dfrac{v'(t)}{v(t)} = \dfrac{16e^{0.4t}}{40e^{0.4t}} = 0.4$
The instantaneous percentage rate of change is 40%.

34. $v(t) = 60e^{-0.2t}$
$v'(t) = 60e^{-0.2t}(-0.2) = -12e^{-0.2t}$
$\frac{v'(t)}{v(t)} = -\frac{12e^{-0.2t}}{60e^{-0.2t}} = -0.2$
The instantaneous percentage rate of change is -20%.

35. $p(t) = 200.3^t$
$\ln(p(t)) = \ln(200) + t\ln(3)$
$\frac{p'(t)}{p(t)} = \frac{d}{dt}[\ln(p(t))] = \ln 3 \approx 1.099,$
so the rate of change of population is about 110% per unit of time.

36. The population after t days will be $p(t) = 500.2^{t/4}$. The rate of change is $p'(t) = 500.2^{t/4}(\ln 2)(1/4)$. So the relative rate of change is $\dfrac{\ln 2}{4} \approx 0.1733$. Therefore the percentage rate of change is about 17.3%.

37. $c(t) = \dfrac{6}{2e^{-8t}+1} = 6\left(2e^{-8t}+1\right)^{-1}$
$c'(t) = -6\left(2e^{-8t}+1\right)^{-2}.\left(-16e^{-8t}\right)$
$= \dfrac{96e^{-8t}}{\left(2e^{-8t}+1\right)^2}$
Since $e^{-8t} > 0$ for any t both numerator and denominator are positive,so that $c'(t) > 0$. Then, since $c(t)$ is an increasing function with a limiting value of 6 (as t goes to infinity) the concentration never exceeds (indeed, never reaches) the value of 6.

38. $c'(t) = -10\left(9e^{-10t}+2\right)^{-2}\left(-90e^{-10t}\right)$
$= \dfrac{900e^{-10t}}{\left(9e^{-10t}+2\right)^2}$
Since $e^{-10t} > 0$ for all t, $c'(t) > 0$ for all t, and $c(t)$ is increasing for all t. This forces, $c(t) < \lim\limits_{t\to\infty} c(t) = 5$

39. $f(x) = x^{\sin x}$
$\ln f(x) = \sin x . \ln x$
$\dfrac{f'(x)}{f(x)} = \dfrac{d}{dx}(\sin x . \ln x)$
$= \cos x . \ln x + \dfrac{\sin x}{x}$
$f'(x) = x^{\sin x}\left(\dfrac{x\cos x . \ln x + \sin x}{x}\right)$

40. $f(x) = x^{4-x^2}$
$\ln f(x) = \left(4 - x^2\right)\ln x$
$\dfrac{f'(x)}{f(x)} = -2x\ln x + \left(4-x^2\right)\dfrac{1}{x}$
$f'(x) = x^{4-x^2}\left(-2x\ln x + \left(4-x^2\right)\dfrac{1}{x}\right)$

41. $f(x) = (\sin x)^x$
$\ln f(x) = x . \ln(\sin x)$
$\dfrac{f'(x)}{f(x)} = \dfrac{d}{dx}(x . \ln(\sin x))$
$= \dfrac{x\cos x}{\sin x} + \ln(\sin x)$
$= x\cot x + \ln(\sin x)$

$f'(x) = (\sin x)^x . (x \cot x + \ln(\sin x))$

42. $f(x) = \left(x^2\right)^{4x}$
$\ln f(x) = 8x \ln x$
$\dfrac{f'(x)}{f(x)} = 8 \ln x + 8x\dfrac{1}{x}$
$f'(x) = \left(x^2\right)^{4x}(8 \ln x + 8)$

43. $f(x) = x^{\ln x}$
$\ln f(x) = \ln x . \ln x = \ln^2 x$
$\dfrac{f'(x)}{f(x)} = \dfrac{d}{dx}\left(\ln^2 x\right) = \dfrac{2 \ln x}{x}$
$f'(x) = x^{\ln x}\left[\dfrac{2 \ln x}{x}\right] = 2x^{[(\ln x)-1]\ln x}$

44. $f(x) = x^{\sqrt{x}}$
$\ln f(x) = \sqrt{x} \ln x$
$\dfrac{f'(x)}{f(x)} = \dfrac{1}{2\sqrt{x}} \ln x + \sqrt{x}\dfrac{1}{x}$
$f'(x) = x^{\sqrt{x}}\left(\dfrac{1}{2\sqrt{x}} \ln x + \dfrac{1}{\sqrt{x}}\right)$

45. Let $(a, \ln a)$ be the point of intersection of the tangent line and the graph of $y = f(x)$.
$f(x) = \ln x$
$f'(x) = \dfrac{1}{x}$
$m = f'(a) = \dfrac{1}{a}$
Since the tangent line passes through the origin,the equation of the tangent line is

$$y = mx = \frac{1}{a}x.$$

Since $(a, \ln a)$ is a point on the tangent line $\ln a = \dfrac{1}{a}a = 1$ so, $a = e$.
Second part: Let $(a,\ e^a)$ be the point of intersection of the tangent line and the graph of $y = f(x)$.
$f(x) = e^x$
$f'(x) = e^x$
$m = f'(a) = e^a$
Since the tangent passes through the origin, the equation of the tangent line is

$$y = mx = e^a x.$$

Since (a, e^a) is a point on the tangent line,

$$e^a = e^a a$$

so, $a = 1$. The slope of the tangent line in $y = \ln x$ is $1/e$ while the slope of the tangent line in $y = e^x$ is e.

46. We approximate $\lim\limits_{h \to 0} \dfrac{a^h - 1}{h}$ for $a = 3$.

h	$\dfrac{a^h-1}{h}$
0.01	1.10466919
0.001	1.09921598
0.0001	1.09867264
0.00001	1.09861832
-0.01	1.09259958
-0.001	1.09800903
-0.0001	1.09855194

The limit seems to be approaching approximately 1.0986, which is very close to $\ln 3 \approx 1.09861$

Second part: We approximate $\lim\limits_{h \to 0} \dfrac{a^h - 1}{h}$ for $a = \dfrac{1}{3}$.

h	$\dfrac{a^h-1}{h}$
0.01	-1.09259958
0.001	-1.09800904
0.0001	-1.09855194
0.00001	-1.09860625
-0.01	-1.10466919
-0.001	-1.09921598
-0.0001	-1.09867264

The limit seems to be approaching approximately, -1.0986, which is very close to $\ln \dfrac{1}{3} \approx 1.09861228867$

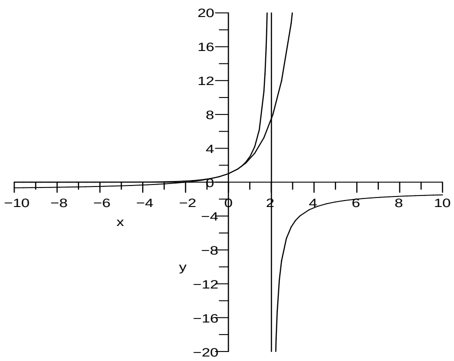
47. Answers depend on CAS.

48. Answers depend on CAS.

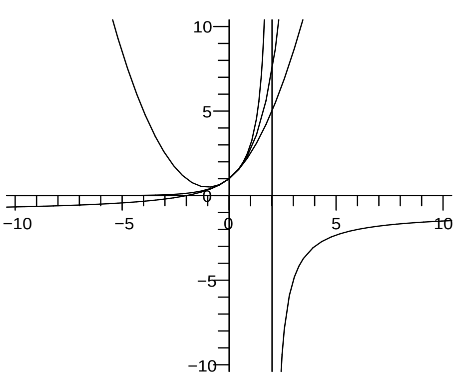
49. $f(x) = \dfrac{a + bx}{1 + cx}$
$f(0) = a$
$f'(x) = \dfrac{b(1 + cx) - (a + bx)c}{(1 + cx)^2} = \dfrac{b - ac}{(1 + cx)^2}$
$f'(0) = b - ac$
$f''(x) = \dfrac{-2c(b - ac)}{(1 + cx)^3}$
$f''(0) = -2c(b - ac)$
Now,
$f(0) = 1 \Rightarrow a = 1.$
$f'(0) = 1 \Rightarrow b - ac = 1 \Rightarrow b - c = 1$
$f''(0) = 1 \Rightarrow -2c(b - ac) = 1$
$\Rightarrow 2c(b - c) = -1$
$\Rightarrow 2c = -1$
$\Rightarrow c = -\dfrac{1}{2}$
So, $a = 1$, $b = 1 + c = \dfrac{1}{2}$, $c = -\dfrac{1}{2}$ and

$f(x) = \dfrac{2+x}{2-x}$.

The graphs of e^x and $\dfrac{2+x}{2-x}$ are as follows:



50. If $g(x) = e^x$, then
$g'(x) = e^x$ and $g''(x) = e^x$ so
$g(0) = g'(0) = g''(0) = 1$
If $f(x) = a + bx + cx^2$, then $f(0) = a$,
$f'(x) = b + 2cx$
$f'(0) = b$
$f''(x) = 2c$
$f''(0) = 2c$
$1 = g(0) = f(0) = a$ so, $a = 1$
$1 = g'(0) = f'(0) = b$ so, $b = 1$
$1 = g''(0) = f''(0) = 2c$ so, $c = \dfrac{1}{2}$
In summary, $a = 1$, $b = 1$, $c = \dfrac{1}{2}$ and $g(x) = 1 + x + \dfrac{1}{2}x^2$. The graphs of the functions e^x, $1 + x + \dfrac{1}{2}x^2$ and the Pade approximation of e^x, which is $\dfrac{2+x}{2-x}$ are as follows:



51. $f(x) = e^{-x^2/2}$

$$f'(x) = e^{-x^2/2}.(-2x/2)$$
$$= -xe^{-x^2/2}$$
$$f''(x) = -\left[x\left(-xe^{-x^2/2}\right) + 1.e^{-x^2/2}\right]$$
$$= xe^{-x^2/2}\left(x^2 - 1\right)$$
This will be zero only when $x = \pm 1$

52. $f(x) = e^{-x^2/8}$, $f'(x) = (-x/4)e^{-x^2/8}$ and
$$f''(x) = (-1/4)e^{-x^2/8} + \left(x^2/16\right)e^{-x^2/8}$$
$$= e^{-x^2/8}\left((-1/4) + x^2/16\right).$$
This is zero when $x = \pm 2$. The graph is flatter in the middle, but the tails are thicker.

53. It helps immensely to leave the name f as it was in #51 and give a new name g to the new function here, so that

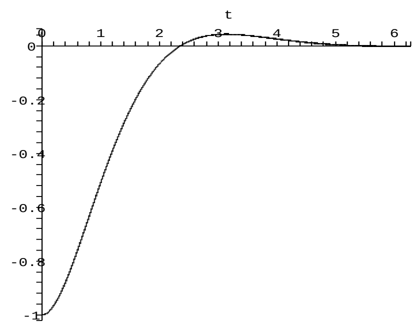
$$g(x) = e^{-(x-m)^2/2c^2} = f(u)$$

in which $u = \dfrac{x-m}{c}$. Then
$$g'(x) = f'(u)\frac{du}{dx} = \frac{f'(u)}{c} = \frac{-uf(u)}{c}$$
$$= \frac{-(x-m)e^{-(x-m)^2/2c^2}}{c^2}$$
$$g''(x) = \frac{d}{dx}\left(\frac{f'(u)}{c}\right) = \frac{f''(u)\frac{du}{dx}}{c}$$
$$= \frac{f''(u)}{c^2} = \frac{\left(u^2-1\right)f(u)}{c^2}$$
$$= \frac{\left((x-m)^2 - c^2\right)e^{-(x-m)^2/2c^2}}{c^4}$$
This will be zero only when, $x = m \pm c$.

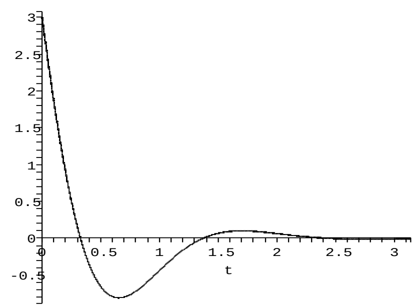
54. $f(x) = e^{-(x-m)^2/2c^2}$
$$f'(x) = \frac{-(x-m)}{c^2}e^{-(x-m)^2/2c^2},$$
and this is equal to zero when $x = m$.

55. $f(t) = e^{-t}\cos t$
$$v(t) = f'(t) = -e^{-t}\cos t + e^{-t}(-\sin t)$$
$$= -e^{-t}(\cos t + \sin t)$$
If the velocity is zero, it is because $\cos t = -\sin t$, so
$$t = \frac{3\pi}{4}, \frac{7\pi}{4}, \ldots, \frac{(3+4n)\pi}{4}, \ldots$$
Position when velocity is zero:
$$f(3\pi/4) = e^{-3\pi/4}\cos(3\pi/4)$$
$$= e^{-3\pi/4}\left(-1/\sqrt{2}\right) \approx -.067020$$
$$f(7\pi/4) = e^{-7\pi/4}\cos(7\pi/4)$$
$$= e^{-7\pi/4}\left(-1/\sqrt{2}\right) \approx .002896$$

Graph of the velocity function:



56. $f'(t) = -2e^{-2t}\sin 3t + 3e^{-2t}\cos 3t$
$= e^{-2t}(-2\sin 3t + 3\cos 3t)$



The velocity of the spring is zero when it is changing direction at the top and bottom of the motion.This occurs when $3\cos 3t = 2\sin 3t$ or $\tan 3t = 3/2$, The *i.e.*, at $t = \dfrac{1}{3}\tan^{-1}(3/2) \approx 0.3276$ position of the spring at this time is approximate.

57. Graphically the maximum velocity seems to occur at,$t = \pi$.

58. Graphically,the maximum velocity seems to occur at $t = 0$; the maximum velocity is not reached on $t \succ 0$.

59. Consider $f(x) = \frac{Ax^n}{\theta^n + x^n}$ for A,n,$\theta > 0$

$$f(x) = \frac{A}{\left(\frac{\theta}{x}\right)^n + 1}$$

$$\ln f(x) = \ln A - \ln\left[\left(\frac{\theta}{x}\right)^n + 1\right]$$

On diffrentiating with respect to x

$$\frac{1}{f(x)} f'(x) = -\frac{1}{\left[\left(\frac{\theta}{x}\right)^n + 1\right]}.n\left(\frac{\theta}{x}\right)^{n-1}.\left(-\frac{\theta}{x^2}\right)$$

$$f'(x) = \frac{An}{\left[\left(\frac{\theta}{x}\right)^n + 1\right]^2}\frac{\left(\frac{\theta}{x}\right)^n}{x}$$

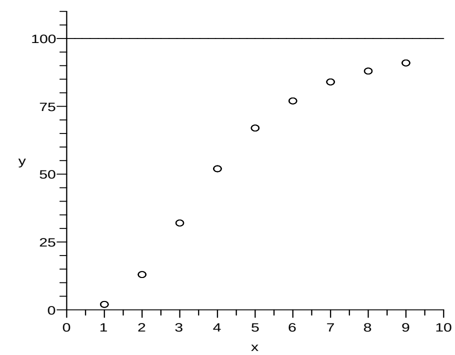
$f'(x) > 0$ if and only if $x > 0$ $(A, n, \theta > 0)$

Also, $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\left[\frac{A}{\left(\frac{\theta}{x}\right)^n + 1}\right] = A$

$$u = \ln\left(\frac{f(x)/A}{1 - f(x)/A}\right)$$
$$= \ln\left(\frac{\frac{1}{\left(\frac{\theta}{x}\right)^n + 1}}{1 - \frac{1}{\left(\frac{\theta}{x}\right)^n + 1}}\right)$$
$$= \ln\left(\frac{1}{\left(\frac{\theta}{x}\right)^n}\right)$$
$$= -n\ln\left(\frac{\theta}{x}\right)$$
$$= -n(\ln\theta - \ln x)$$
$$= -n\ln\theta + n\ln x$$
$$= nv - n\ln\theta$$

Therefore, u is a linear function of v.

Graph of $(x,\, y)$ in below:

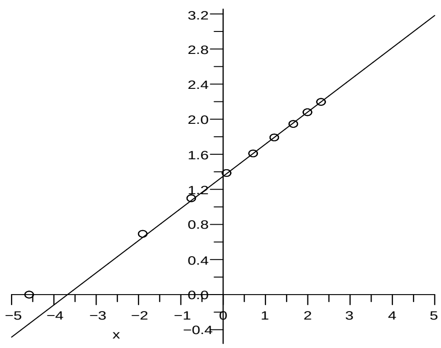


From the graph, we can see that $y = f(x) \to 100$ as $x \to \infty$.

The table gives $(u,\, v)$ values as follows:

x	y	$u = \ln\dfrac{y}{100-y}$	$v = \ln x$
1	2	-3.8918	0
2	13	-1.9009	0.6931
3	32	-.75377	1.098
4	52	.80012	1.3863
5	67	.70818	1.6094
6	77	1.2083	1.7918
7	84	1.6582	1.9459
8	88	1.9924	2.0794
9	91	2.3136	2.1972

The graph of $(u,\, v)$ points are as below which are almost linear.



Comparing the line passing through the points $(u,\, v)$ with $v = \frac{1}{n}u + ln\theta$, we get $\frac{1}{n} = 0.3679$, $\ln\theta = 1.3458$ and hence $n = 2.7174$, $\theta = 3.8413$.

60. Answers very depending on source.Linear growth corrseponds to constant slope. In other words the population changes by the same fixed amount per year. In exponential growth, the size of the change depends on the sizeof the population. The percentage change is the same though from year to year.

2.8 Implicit Differentiation and Inverse Trignometric Fuction

1. Explicitly:

$4y^2 = 8 - x^2$

$y^2 = \dfrac{8 - x^2}{4}$

$y = \pm\dfrac{\sqrt{8 - x^2}}{2}$ (choose plus to fit(2,1))

For $y = \dfrac{\sqrt{8 - x^2}}{2}$,

$y' = \dfrac{1}{2}\dfrac{(-2x)}{2\sqrt{8 - x^2}} = \dfrac{-x}{2\sqrt{8 - x^2}}$,

$y'(2) = \dfrac{-1}{2}$.

Implicitly:

$\dfrac{d}{dx}(x^2 + 4y^2) = \dfrac{d}{dx}(8)$

$2x + 8y \cdot y' = 0$

$y' = -\dfrac{2x}{8y} = -\dfrac{x}{4y}$

At $(2,\, 1)$: $y' = -\dfrac{2}{4} = -\dfrac{1}{2}$

2. Explicitly:

$y = \dfrac{4\sqrt{x}}{x^3 - x^2}$

$$y' = \frac{\left(x^3 - x^2\right)\frac{2}{\sqrt{x}} - 4\sqrt{x}\left(3x^2 - 2x\right)}{\left(x^3 - x^2\right)^2}$$

Implicitly differentiating:

$3x^2y + x^3y' - \dfrac{2}{\sqrt{x}} = 2xy + x^2y'$,

And we solve for y' to get

$y' = \dfrac{2xy + \frac{2}{\sqrt{x}} - 3x^2y}{x^3 - x^2}$.

Substitute $x = 2$ into the first expression, and $(x,\, y) = \left(2,\, \sqrt{2}\right)$, into the second to get $y' = -\dfrac{7\sqrt{2}}{4}$.

3. Explicitly:

$y(1 - 3x^2) = \cos x$

$y = \dfrac{\cos x}{1 - 3x^2}$

$$y'(x) = \frac{(1 - 3x^2)(-\sin x) - \cos x(-6x)}{\left(1 - 3x^2\right)^2}$$

$$= \frac{-\sin x + 3x^2\sin x + 6x\cos x}{\left(1 - 3x^2\right)^2}$$

$y'(0) = 0$.

Implicitly:

$\dfrac{d}{dx}(y - 3x^2y) = \dfrac{d}{dx}(\cos x)$

$y' - 3x^2y' - 6xy = -\sin x$

$y'(1 - 3x^2) = 6xy - \sin x$

$y' = \dfrac{6xy - \sin x}{1 - 3x^2}$

At $(0, 1)$: $y' = 0$(again)

4. Explicitly:

$y = -x \pm \sqrt{x^2 - 4}$

At the point $(-2, 2)$, the sign is irrelevant, so we choose

$y = -x + \sqrt{x^2 - 4}$

$y' = -1 + \dfrac{2x}{2\sqrt{x^2 - 4}} = -1 + \dfrac{x}{\sqrt{x^2 - 4}}$

Implicitly differentiating:

$y' + 2y + 2xy' = 0$,

and we solve for y' :

$y' = \dfrac{-2y}{2x + 2y}$

Substitute $x = -2$ in the first expression and $(x, y) = (-2, 2)$ in to the second expression to see that y' is undefined. There is a vertical tangent at this point.

5. $\dfrac{d}{dx}(x^2y^2 + 3y) = \dfrac{d}{dx}(4x)$

$2xy^2 + x^2 2yy' + 3y' = 4$

$y'(2x^2y + 3) = 4 - 2xy^2$

$y' = \dfrac{4 - 2xy^2}{2x^2y + 3}$

6. $3y^3 + 3x(3y^2)y' - 4 = 20yy'$

$(9xy^2-20y)y' = 4-3y^3$
$y' = \dfrac{3y^3-4}{20y-9xy^2}$

7. $\dfrac{d}{dx}(\sqrt{xy}-4y^2) = \dfrac{d}{dx}(12)$
$\dfrac{1}{2\sqrt{xy}}\cdot\dfrac{d}{dx}(xy) - 8y\cdot y' = 0$
$\dfrac{1}{2\sqrt{xy}}.(xy'+y) - 8y\cdot y' = 0$
$(xy'+y) - 16y\cdot y'\sqrt{xy} = 0$
$y'(x-16y\sqrt{xy}) = -y$
$y' = \dfrac{-y}{(x-16y\sqrt{xy})} = \dfrac{y}{16y\sqrt{xy}-x}$

8. $\cos(xy)(y+xy') = 2x$
$y' = \dfrac{2x-y\cos(xy)}{x\cos(xy)}$

9. $x+3 = 4xy+y^3$
$1 = \dfrac{d}{dx}(4xy+y^3) = 4(xy'+y)+3y^2y'$
$1-4y = (4x+3y^2)y'$
$y' = \dfrac{1-4y}{3y^2+4x}$

10. $3x+y^3-\dfrac{4y}{x+2} = 10x^2$
Diffrentiating with respect to x,
$\dfrac{d}{dx}\left(3x+y^3-\dfrac{4y}{x+2}\right) = \dfrac{d}{dx}\left(10x^2\right)$
By the Chain rule and Product rule,
$3+3y^2y' - \left[\dfrac{(x+2)\,4y'-4y}{(x+2)^2}\right] = 20x$
$3(x+2)^2+3y^2y'(x+2)^2$
$-4y'(x+2)+4y = 20x(x+2)^2$
$3y^2y'(x+2)^2-4y'(x+2)$
$= 20x(x+2)^2-3(x+2)^2-4y$
$y'(x+2)\left[3y^2(x+2)-4\right]$
$= (x+2)^2(20x-3)-4y$
$y' = \dfrac{(x+2)^2(20x-3)-4y}{(x+2)\left[3y^2(x+2)-4\right]}$

11. $\dfrac{d}{dx}(e^{x^2y}-e^{y)}) = \dfrac{d}{dx}(x)$
$e^{x^2y}\dfrac{d}{dx}(e^{x^2y}) - e^yy' = 1$
$e^{x^2y}(2xy+x^2y') - e^yy' = 1$
$y'(x^2e^{x^2y}-e^y) = 1-2xye^{x^2y}$
$y' = \dfrac{1-2xye^{x^2}}{(x^2e^{x^2y}-e^y)}$

12. $e^y+xe^yy'-3y'\sin x-3y\cos x = 0$
$y' = \dfrac{3y\cos x-e^y}{xe^y-3\sin x}$

13. $y^2\sqrt{x+y}-4x^2 = y$
Diffrentiating with respect to x,
$\dfrac{d}{dx}\left(y^2\sqrt{x+y}-4x^2\right) = \dfrac{d}{dx}(y)$
By the Chain rule and Product rule,
$\dfrac{d}{dx}\left(y^2\sqrt{x+y}\right) - 4\dfrac{d}{dx}\left(x^2\right) = \dfrac{d}{dx}(y)$
$\left[y^2\left(\dfrac{1}{2\sqrt{x+y}}\right)(1+y')\right]$
$+2yy'\sqrt{x+y}-8x = y'$
$y^2+y^2y'+4yy'(x+y)-16x\sqrt{x+y}$
$= 2y'\sqrt{x+y}$
$y^2y'+4yy'(x+y)-2y'\sqrt{x+y}$
$= 16x\sqrt{x+y}-y^2$
$y'\left[y^2+4y(x+y)-2\sqrt{x+y}\right]$
$= 16x\sqrt{x+y}-y^2$
$y' = \dfrac{16x\sqrt{x+y}-y^2}{y^2+4y(x+y)-2\sqrt{x+y}}$

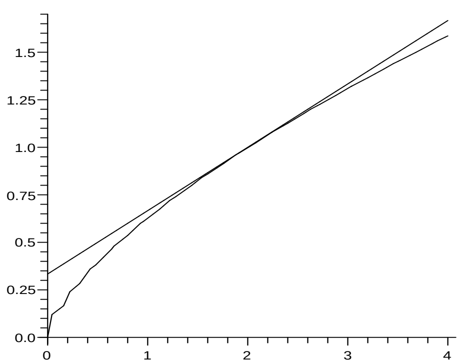
14. $x\cos(x+y)-y^2 = 8$
Diffrentiating with respect to x,
$\dfrac{d}{dx}\left(x\cos(x+y)-y^2\right) = \dfrac{d}{dx}(8)$
By the Chain rule and Product rule,
$\dfrac{d}{dx}(x\cos(x+y)) - \dfrac{d}{dx}\left(y^2\right) = \dfrac{d}{dx}(8)$
$\cos(x+y)-x\sin(x+y)(1+y')-2yy' = 0$
$\cos(x+y)-x\sin(x+y)-x\sin(x+y)\,y'$
$-2yy' = 0$
$y'(-x\sin(x+y)-2y)$
$= x\sin(x+y)-\cos(x+y)$
$y' = \dfrac{x\sin(x+y)-\cos(x+y)}{-x\sin(x+y)-2y}$
$y' = \dfrac{\cos(x+y)-x\sin(x+y)}{x\sin(x+y)+2y}$

15. $e^{4y}-\ln\left(y^2+3\right) = 2x$
Diffrentiating with respect to x,
$\dfrac{d}{dx}\left(e^{4y}-\ln\left(y^2+3\right)\right) = \dfrac{d}{dx}(2x)$
By the Chain rule and Product rule,
$\dfrac{d}{dx}\left(e^{4y}\right) - \dfrac{d}{dx}\left(\ln\left(y^2+3\right)\right) = \dfrac{d}{dx}(2x)$
$e^{4y}(4y') - \dfrac{2yy'}{y^2+3} = 2$
$4e^{4y}\left(y^2+3\right)y'-2yy' = 2\left(y^2+3\right)$
$y'\left(4e^{4y}\left(y^2+3\right)-2y\right) = 2\left(y^2+3\right)$
$y' = \dfrac{2\left(y^2+3\right)}{4e^{4y}(y^2+3)-2y}$

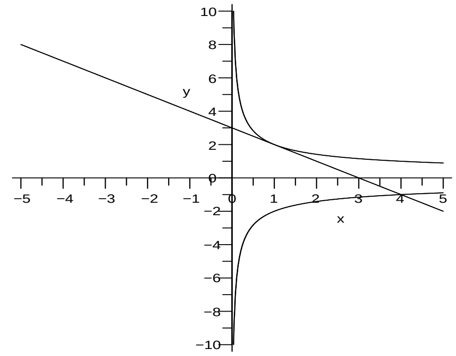
16. $e^{x^2}y-3\sqrt{y^2+2} = x^2+1$
Diffrentiating with respect to x,

$\frac{d}{dx}\left(e^{x^2}y - 3\sqrt{y^2+2}\right) = \frac{d}{dx}\left(x^2+1\right)$
By the Chain rule and Produt rule,
$\frac{d}{dx}\left(e^{x^2}y\right) - 3\frac{d}{dx}\left(\sqrt{y^2+2}\right) = 2x$
$e^{x^2}(2x)\,y + e^{x^2}y' - 3\cdot\frac{2yy'}{2\sqrt{y^2+2}} = 2x$
$2xye^{x^2} + e^{x^2}y' - \frac{3yy'}{\sqrt{y^2+2}} = 2x$
$2xye^{x^2}\sqrt{y^2+2} + e^{x^2}y'\sqrt{y^2+2} - 3yy'$
$= 2x\sqrt{y^2+2}$
$y'\left(e^{x^2}\sqrt{y^2+2} - 3y\right) = 2x\sqrt{y^2+2}$
$-2xye^{x^2}\sqrt{y^2+2}$
$y' = \frac{2x\sqrt{y^2+2}\left(1-ye^{x^2}\right)}{e^{x^2}\sqrt{y^2+2}-3y}$

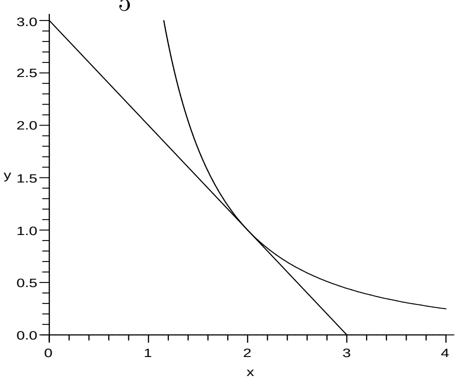
17. Rewrite: $x^2 = 4y^3$
Differentiate by $x : 2x = 12y^2y'$
$y' = \frac{2x}{12y^2}$
At $(2,\,1) : y' = \frac{2}{6\cdot 1^2} = \frac{1}{3}$
The equation of the tangent line is
$y - 1 = \frac{1}{3}(x-2)$ or $y = \frac{1}{3}(x+1)$.



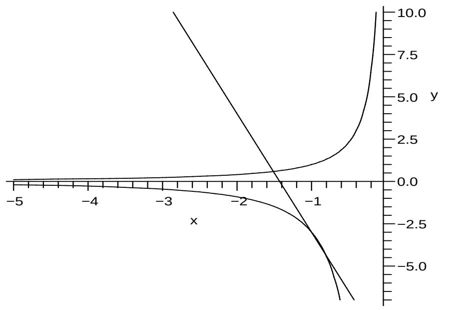
18. $2xy^2 + x^2 2y.y' = 4$, so $y' = \frac{4-2xy^2}{2x^2y}$.
y' at $(1,\,2)$ is -1, and the equation of the line is $y = -1(x-1)+2$.



19. $x^2y^2 = 3y+1$
Diffrentiating with respect to x,
$\frac{d}{dx}\left(x^2y^2\right) = \frac{d}{dx}\left(3y+1\right)$
By using the Product Rule we have,
$2xy^2 + 2yy'x^2 = 3y'$
$y' = \frac{2xy^2}{3-2yx^2}$
At $(2,\,1)$, $y' = -\frac{4}{5}$.
The equation of the tangent line is given by
$y - 1 = -\frac{4}{5}\left(x-2\right)$.



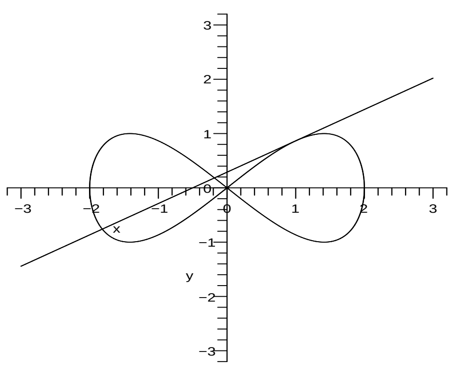
20. $x^3y^2 = -2xy-3$
Diffrentiating with respect to x,
$\frac{d}{dx}\left(x^3y^2\right) = \frac{d}{dx}\left(-2xy-3\right)$
By using Product Rule,
$3x^2y^2 + 2yy'x^3 = -2y - 2y'$
$y'\left(2x^3y+2x\right) = -2y - 3x^2y^2$
$y' = -\frac{2y+3x^2y^2}{2x^3y+2x}$
Substituting $x=-1$ and $y=-3$,
$y'(-1) = -\frac{2(-3)+3(-1)^2(-3)^2}{2(-1)^3(-3)+2(-1)}$
$= -\frac{-6+27}{6-2} = -\frac{21}{4}$
The equation of the tangent line is
$(y+3) = -\frac{21}{4}\left(x+1\right)$.



21. $4y^2 = 4x^2 - x^4$
$8yy' = 8x - 4x^3$
$y' = \dfrac{x(2-x^2)}{2y}.$

The slope of the tangent line at $(1, \frac{\sqrt{3}}{2})$ is
$m = \dfrac{1(2-1^2)}{2.(\frac{\sqrt{3}}{2})} = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}.$

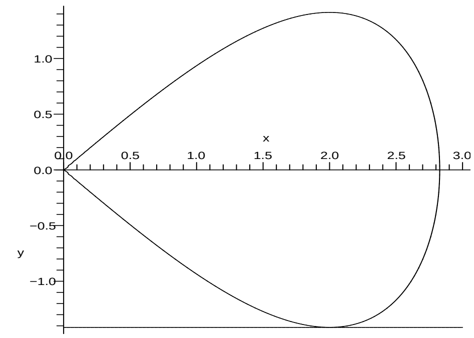
The equation of the tangent line is
$y - \dfrac{\sqrt{3}}{2} = \dfrac{\sqrt{3}}{3}(x-1)$
$y == \dfrac{\sqrt{3}}{3}x + \dfrac{\sqrt{3}}{6}.$



22. $x^4 - 8x^2 = -8y^2$
$4x^3 - 16x = -16yy'$
$y' = \dfrac{-(4x^3-16x)}{16y} = \dfrac{4x(4-x^2)}{16y}$

The slope of the tangent line at $(2, -\sqrt{2})$ is
$m = \dfrac{2(4-2^2)}{4(-\sqrt{2})} = 0.$

The equation of the tangent line is $y = -\sqrt{2}.$



23. $\dfrac{d}{dx}(x^2y^2 + 3x - 4y) = \dfrac{d}{dx}(5)$
$x^2 2yy' + 2xy^2 + 3 - 4y' = 0$
Differentiate both sides of this with respect to x :
$\dfrac{d}{dx}(x^2 2yy' + 2xy^2 + 3 - 4y') = \dfrac{d}{dx}(0)$
$2(2xyy' + x^2(y')^2 + x^2yy'') + 2(2xyy' + y^2)$
$- 4y'' = 0.$
$y' + x^2(y')^2 + x^2yy'' + 2xyy' + y^2 - 2y'' = 0.$
$y' + x^2(y')^2 + y^2 = y''(2 - x^2y)$
$y'' = \dfrac{4xyy' + x^2(y')^2 + y^2}{2 - x^2y}$

24. $\dfrac{d}{dx}(x^{2/3} + y^{2/3}) = \dfrac{d}{dx}(4)$
$\dfrac{2}{3}x^{-1/3} + \dfrac{2}{3}y^{-1/3}y' = 0$
Multiply by $\frac{3}{2}$ and implicitly differentiate again:
$-\dfrac{1}{3}x^{-4/3} - \dfrac{1}{3}y^{-4/3}y'y' + y^{-1/3}y'' = 0$
so
$y'' = \dfrac{x^{-4/3} + y^{-4/3}(y')^2}{3y^{-1/3}}$

25. $\dfrac{d}{dx}(y^2) = \dfrac{d}{dx}(x^3 - 6x + 4\cos y)$
$2yy' = 3x^2 - 6 - 4\sin y.y'.$
Differentiating again with respect to x :
$2yy'' + 2(y')^2 = 6x - 4\left[\sin y.y'' + \cos y.(y')^2\right]$
$yy'' + (y')^2 = 3x - 2\sin y.y'' - 2\cos y.(y')^2$
$y''(y + 2\sin y) = 3x - (2\cos y + 1)(y')^2$
$y'' = \dfrac{3x - (2\cos y + 1)(y')^2}{y + 2\sin y}$

26. $\dfrac{d}{dx}(e^{xy} + 2y - 3x) = \dfrac{d}{dx}(\sin y)$
$e^{xy}(y + xy') + 2y' - 3 = \cos y.y'$
Differentiating again with respect to x :
$e^{xy}(y + xy')^2 + e^{xy}(y' + y' + xy'') + 2y''$
$= -\sin y(y')^2 + \cos y.y''$
and

$$y'' = \frac{e^{xy}(y+xy')^2 + 2e^{xy}y' + \sin y(y')^2}{\cos y - xe^{xy} - 2}$$

27. $(y-1)^2 = 3xy + e^{4y}$
Diffrentiating with respect to x,
$\frac{d}{dx}(y-1)^2 = \frac{d}{dx}\left(3xy + e^{4y}\right)$
By the Chain and Product rule,
$2(y-1)y' = 3y + 3xy' + 4e^{4y}y'$
Diffrentiating with respect to x,
$\frac{d}{dx}[2(y-1)y'] = \frac{d}{dx}\left[3y + 3xy' + 4e^{4y}y'\right]$
By the Chain and Product rule,
$2(y-1)y'' + 2(y')^2$
$= 3y' + 3xy'' + 3y' + 4e^{4y}y'' + 16e^{4y}(y')^2$
$2(y-1)y'' - 3xy'' - 4e^{4y}y''$
$= 6y'(x) + 16e^{4y}(y')^2 - 2(y')^2$
$y''\left[2(y-1) - 3x - 4e^{4y}\right]$
$= 2y'\left(3 + 8e^{4y}y' - y'\right)$
$$y'' = \frac{2y'\left(3 + 8e^{4y}y' - y'\right)}{2(y-1) - 3x - 4e^{4y}}$$

28. $(x+y)^2 - e^{y+1} = 3x$
Diffrentiating with respect to x,
$\frac{d}{dx}\left[(x+y)^2 - e^{y+1}\right] = \frac{d}{dx}(3x)$
By the Chain rule,
$2(x+y)(1+y') - e^{y+1}y' = 3$
Diffrentiating with respect to x,
$\frac{d}{dx}\left[2(x+y)(1+y') - e^{y+1}y'\right] = 0$
By the Chain and Product rule,
$2(x+y)y'' + 2(1+y')^2 - e^{y+1}y''$
$- e^{y+1}(y')^2 = 0$
$y''\left[2(x+y) - e^{y+1}\right]$
$= e^{y+1}(y')^2 - 2(1+y')^2$
$$y'' = \frac{e^{y+1}(y')^2 - 2(1+y')^2}{2(x+y) - e^{y+1}}$$

29. **(a)** $f(x) = \sin^{-1}\left(x^3+1\right)$
Diffrentiating with respect to x,
$f'(x) = \frac{d}{dx}\left[\sin^{-1}\left(x^3+1\right)\right].$
By the Chain rule we get,
$$f'(x) = \frac{1}{\sqrt{1-(x^3+1)^2}}\frac{d}{dx}\left(x^3+1\right)$$
$$= \frac{1}{\sqrt{1-(x^3+1)^2}}\left(3x^2\right)$$
$$= \frac{3x^2}{\sqrt{1-(x^3+1)^2}}.$$

(b) $f(x) = \sin^{-1}\left(\sqrt{x}\right)$
Diffrentiating with respect to x,
$f'(x) = \frac{d}{dx}\left[\sin^{-1}\left(\sqrt{x}\right)\right].$
By the Chain rule, we get
$$f'(x) = \frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}\left(\sqrt{x}\right)$$
$$= \frac{1}{\sqrt{1-x}}\left(\frac{1}{2\sqrt{x}}\right)$$
$$= \frac{1}{2\sqrt{x(1-x)}}$$

30. **(a)** $f(x) = \cos^{-1}\left(x^2+x\right)$
Diffrentiating with respect to x,
$f'(x) = \frac{d}{dx}\left[\cos^{-1}\left(x^2+x\right)\right].$
By using Chain rule,
$$f'(x) = \frac{-1}{\sqrt{1-(x^2+x)^2}}\frac{d}{dx}\left(x^2+x\right)$$
$$= \frac{-(2x+1)}{\sqrt{1-(x^2+x)^2}}$$

(b) $f(x) = \cos^{-1}\left(\frac{2}{x}\right)$
Diffrentiating with respect to x,
$f'(x) = \frac{d}{dx}\left[\cos^{-1}\left(\frac{2}{x}\right)\right]$
By using Chain rule,
$$f'(x) = \frac{-1}{\sqrt{1-\left(\frac{2}{x}\right)^2}}\frac{d}{dx}\left(\frac{2}{x}\right)$$
$$= \frac{-1}{\sqrt{1-\left(\frac{4}{x^2}\right)}}\left(\frac{-2}{x^2}\right)$$
$$= \frac{2}{x\sqrt{x^2-4}}$$

31. **(a)** $f(x) = \tan^{-1}\left(\sqrt{x}\right)$
Diffrentiating with respect to x,
$f'(x) = \frac{d}{dx}\left[\tan^{-1}\left(\sqrt{x}\right)\right].$
By the Chain rule,
$$f'(x) = \frac{1}{1+(\sqrt{x})^2}\frac{d}{dx}\left(\sqrt{x}\right)$$
$$= \frac{1}{(1+x)}\left(\frac{1}{2\sqrt{x}}\right)$$
$$= \frac{1}{2\sqrt{x}(1+x)}$$

(b) $f(x) = \tan^{-1}\left(\frac{1}{x}\right)$
Diffrentiating with respect to x,
$f'(x) = \frac{d}{dx}\left[\tan^{-1}\left(\frac{1}{x}\right)\right].$
By the Chain rule,

$$f'(x) = \frac{1}{1+\left(\frac{1}{x}\right)^2}\frac{d}{dx}\left(\frac{1}{x}\right)$$
$$= \frac{1}{\left(1+\frac{1}{x^2}\right)}\left(\frac{-1}{x^2}\right)$$
$$= \frac{-1}{(x^2+1)}$$

32. **(a)** $f(x) = \sqrt{2+\tan^{-1}x}$
Diffrentiating with respect to x,
$f'(x) = \frac{d}{dx}\left(\sqrt{2+\tan^{-1}x}\right)$.
By the Chain rule,
$$f'(x) = \frac{1}{2\sqrt{2+\tan^{-1}x}}\frac{d}{dx}\left(2+\tan^{-1}x\right)$$
$$= \frac{1}{2\sqrt{2+\tan^{-1}x}}\left(\frac{1}{1+x^2}\right)$$
$$= \frac{1}{2(1+x^2)\sqrt{2+\tan^{-1}x}}$$

(b) $f(x) = e^{\tan^{-1}x}$
Diffrentiating with respect to x,
$f'(x) = \frac{d}{dx}\left(e^{\tan^{-1}x}\right)$.
By the Chain rule,
$$f'(x) = \left(e^{\tan^{-1}x}\right)\left(\frac{1}{1+x^2}\right)$$
$$= \frac{e^{\tan^{-1}x}}{1+x^2}$$

33. **(a)** $f(x) = 4\sec\left(x^4\right)$
Diffrentiating with respect to x,
$f'(x) = \frac{d}{dx}\left(4\sec\left(x^4\right)\right)$
By Chain rule,
$$f'(x) = 4\sec\left(x^4\right)\tan\left(x^4\right)\frac{d}{dx}\left(x^4\right)$$
$$= 4\sec\left(x^4\right)\tan\left(x^4\right)\left(4x^3\right)$$
$$= 16x^3\sec\left(x^4\right)\tan\left(x^4\right)$$

(b) $f(x) = 4\sec^{-1}\left(x^4\right)$
Diffrentiating with respect to x,
$f'(x) = \frac{d}{dx}\left(4\sec^{-1}\left(x^4\right)\right)$.
By Chain rule,
$$f'(x) = 4\frac{1}{x^4\sqrt{\left(x^4\right)^2-1}}\frac{d}{dx}\left(x^4\right)$$
$$= 4\frac{1}{x^4\sqrt{x^8-1}}\left(4x^3\right)$$
$$= \frac{16}{x\sqrt{x^8-1}}$$

34. **(a)** $f(x) = \sin^{-1}\left(\frac{1}{x}\right)$
Diffrentiating with respect to x,
$f'(x) = \frac{d}{dx}\left(\sin^{-1}\left(\frac{1}{x}\right)\right)$.
By the Chain rule,
$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\frac{d}{dx}\left(\frac{1}{x}\right)$$
$$= \frac{x}{\sqrt{x^2-1}}\left(\frac{-1}{x^2}\right)$$
$$= -\frac{1}{x\sqrt{x^2-1}}$$

(b) $f(x) = \mathrm{c}\,sc^{-1}(x)$
Diffrentiating with respect to x,
$f'(x) = \frac{d}{dx}\mathrm{c}\,sc^{-1}(x)$.
By the Chain rule,
$f'(x) = -\frac{1}{x\sqrt{x^2-1}}$.

35. In example 8.6,we are given
$\theta'(d) = \frac{2(-130)}{4+d^2}$
Setting this equal to -3 and solving for d gives $d^2 = 82 \Rightarrow d = 9$ feet. The better can track the ball after they would have to start swinging(when the ball is 30 feet away),but not all the way to home plate.

36. From example 8.6, the rate of angle is
$\theta'(t) = \frac{1}{1+\left[\frac{d(t)}{2}\right]^2}\left(\frac{d'(t)}{2}\right)$
Given a maximum rotational rate of $\theta'(t) = -3$ (radians/second), the distance from the plate at which a player can track the ball can be obtained by solving the equation
$-3 = \frac{2d'(t)}{4+[d(t)]^2}$
for $d(t)$ in terms of $d'(t)$ This leads to
$d(t) = \frac{\sqrt{-6.d'(t)-36}}{3}$
if $d'(t) \le -6$ which may be reasonable since the distance is decreasing as the ball approaches the plate. We get $d(t) = 4$ for $d'(t) = -30$ ft/sec and $d(t) = 9.45$ for $d'(t) = -140$ ft/sec. This would mean a player can track the ball to within 4 feet from the plate in slowpitch, but only to within 9.45 feet from the plate in the major leagues.

37. Suppose that d is the distance from ball to home plate and θ is the angle of gaze Since distance is changing with time, therefore $d = d(t)$. The velocity 130 ft/sec means that $d'(t) = -130$
$\theta(t) = \tan^{-1}\left[\frac{d(t)}{3}\right]$
The rate of change of angle is then

$$\theta'(t) = \frac{1}{1+\left(\frac{d(t)}{3}\right)^2}\frac{d'(t)}{3}$$

$$= \frac{3d'(t)}{9+[d(t)]^2} \text{ radians/second}$$

when $d''(t) = 0$.
The rate of the change is then

$$\theta'(t) = \frac{3(-130)}{9} = -43.33 \text{ radians/sec.}$$

38. Let d is the distance from ball to home plate and θ is the angle of gaze, Since distance is changing with time therefore $d = d(t)$. The velocity 130 ft/sec means that $d'(t) = -130$,

$$\theta(t) = \tan^{-1}\left[\frac{d(t)}{x}\right]$$

The rate of change of angle is then

$$\theta'(t) = \frac{1}{1+\left(\frac{d(t)}{x}\right)^2}\frac{d'(t)}{x}$$

$$= \frac{xd'(t)}{x^2+[d(t)]^2} \text{ radians/second}$$

when $d(t) = 0$,
The rate of the change is then

$$\theta'(t) = \frac{x(-130)}{x^2} \text{radians/second}$$

$$= \frac{-130}{x} = -3 \text{ radians/second}$$

Therefore, $x = \frac{-130}{-3} = 43.33$

39. $\dfrac{d}{dx}(x^2+y^2-3y) = \dfrac{d}{dx}(0)$

$2x + 2y.y' - 3y' = 0$

$y'(2y-3) = -2x$

$y' = \dfrac{2x}{3-2y}$

Horizontal tangents:
From the formula, $y' = 0$ only when $x = 0$. When $x = 0$ we have $0 + y^2 - 3y = 0$. Therefore $y = 0$ and $y = 3$ are the horizontal tangents.
Vertical tangents:
The denominator in y' must be zero.

$3 - 2y = 0$

$y = 1.5$

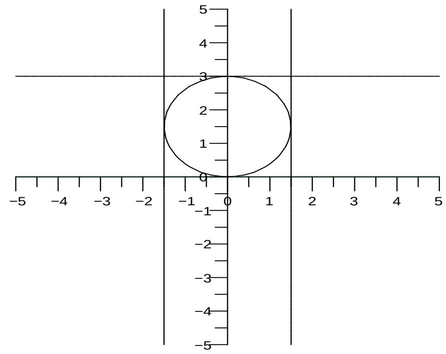
When $y = 1.5$,

$x^2 + (1.5)^2 - 3(1.5) = 0$

$x^2 = 2.25$

$x = \pm 1.5$

$x = \pm 1.5$ are the vertical tangents.



40. $\dfrac{d}{dx}(x^2+y^2-2y) = \dfrac{d}{dx}(3)$

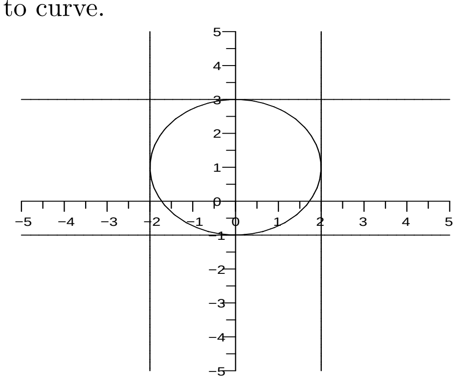
$2x + 2yy' - 2y' = 0$

$x + y'(y-1) = 0$

$y'(y-1) = -x$

$y' = \dfrac{x}{1-y}$

Horizontal tangents:
The curve has horizontal tangents when $y' = 0$ *i.e.* when $x = 0$.
At $x = 0$, $y = \dfrac{2 \pm \sqrt{4-4(-3)}}{2} = \dfrac{2 \pm 4}{2}$ which gives $y = 3$ or $y = -1$. Therefore $y = 3$ and $y = -1$ are the horizontal tangents to the curve.
Vertical tangents:
The curve has vertical tangents when the denominator in y' is 0 which gives $y = 1$.
At $y = 1$, $x = \pm 2$
Therefore, $x = \pm 2$ are the vertical tangents to curve.



41. **(a)** $x^2y^2 + 3y = 4x$
To find the derivative of y, we use Implicit differentiation.

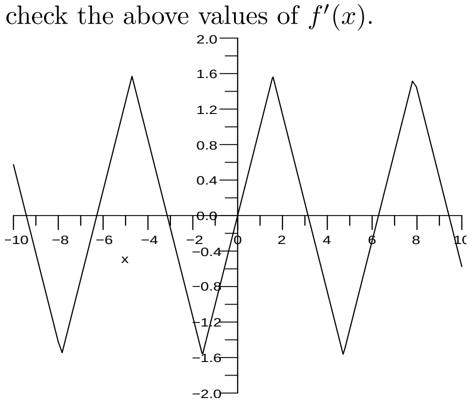
(b) $x^2y + 3y = 4x$
The derivative of y can be found directly and implicitly.

(c) $3xy + 6x^2\cos x = y\sin x$
The derivative of y can be found directly and implicitly.

(d) $3xy + 6x^2\cos y = y\sin x$

By using Implicit differentiation we can find the derivative of y.

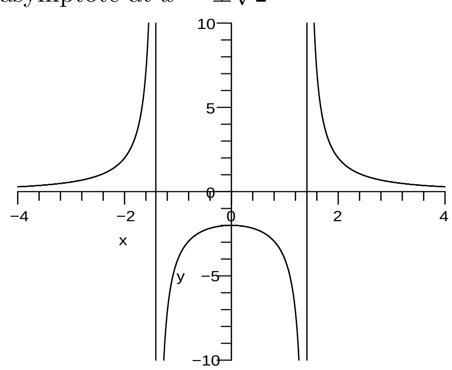
42. $f(x) = \sin^{-1}(\sin x)$
$= \sin^{-1}(\sin[2n\pi - (2n\pi - x)])$
$= \sin^{-1}(-\sin(2n\pi - x))$
$= -\sin^{-1}[\sin(2n\pi - x)]$
In the interval $\left(2n\pi - \dfrac{\pi}{2},\ 2n\pi + \dfrac{\pi}{2}\right)$,
$-\dfrac{\pi}{2} \le 2n\pi - x \le \dfrac{\pi}{2}$.
So, $f(x) = -(2n\pi - x) = x - 2n\pi$.
Again,
$f(x) = \sin^{-1}(\sin x)$
$= \sin^{-1}(\sin[(2n\pi + \pi) - (2n\pi + \pi - x)])$
$= \sin^{-1}(\sin(2n\pi + \pi - x))$
$= \sin^{-1}[\sin(2n\pi + \pi - x)]$
In the interval $\left[2n\pi + \pi - \dfrac{\pi}{2},\ 2n\pi + \pi + \dfrac{\pi}{2}\right]$,
$-\frac{\pi}{2} \le 2n\pi + \pi - x \le \frac{\pi}{2}$.
So, $f(x) = (2n+1)\pi - x$.
Therefore $f'(x) = 1$ for all $x \in \left(2n\pi - \frac{\pi}{2},\ 2n\pi + \frac{\pi}{2}\right)$ and $f'(x) = -1$ for all $x \in \left(2n\pi + \pi - \frac{\pi}{2},\ 2n\pi + \pi + \frac{\pi}{2}\right)$. At the points $x = n\pi \pm \frac{\pi}{2}$, $f'(x)$ is not defined. Here n is any integer.
From the graph of $f(x)$ in below, we can check the above values of $f'(x)$.



43. Let $y = \sin^{-1}x + \cos^{-1}x$
$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}} = 0$$
Therefore, $y = c$, where c is a constant. To determine c, substitute any convenient value of x, such as $x = 0$
$\sin^{-1}x + \cos^{-1}x = c$
$\sin^{-1}0 + \cos^{-1}0 = c$, so $c = \dfrac{\pi}{2}$
Thus $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$

44. Let $y = \sin^{-1}\left(\dfrac{x}{\sqrt{x^2+1}}\right)$
$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \left(\frac{x}{\sqrt{x^2+1}}\right)^2}} \cdot \frac{d}{dx}\left(\frac{x}{\sqrt{x^2+1}}\right)$$
$$= \left(\frac{1}{\sqrt{1 - \frac{x^2}{x^2+1}}}\right) \cdot \left(\frac{\sqrt{x^2+1} - x(1/2)(x^2+1)^{-\frac{1}{2}}(2x)}{x^2+1}\right)$$
$$= \frac{1 - \frac{x^2}{x^2+1}}{\sqrt{1 - \frac{x^2}{x^2+1}}} \cdot \frac{\sqrt{x^2+1}}{x^2+1}$$
$$= \frac{\sqrt{1 - \frac{x^2}{x^2+1}}}{\sqrt{x^2+1}} \cdot \left(\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}}\right)$$
$$= \frac{1}{1+x^2}$$
Thus $\sin^{-1}\left(\dfrac{x}{\sqrt{x^2+1}}\right) = y = \displaystyle\int \frac{1}{1+x^2}dx$
$= \tan^{-1}(x) + c$ for some constant c.
Substitute $x = 0$ in to the above expression to find $c = 0$ and so
$\sin^{-1}\left(\dfrac{x}{\sqrt{x^2+1}}\right) = \tan^{-1}x$

45. $\dfrac{d}{dx}(x^2y - 2y) = \dfrac{d}{dx}(4)$
$2xy + x^2y' - 2y' = 0$
$y'(x^2 - 2) = -2xy$
$y' = \dfrac{2xy}{(2 - x^2)}$
The derivative is undefined at $x = \pm\sqrt{2}$, suggesting that there might be vertical tangent lines at these points. Similarly, $y' = 0$ at $y = 0$ suggesting that there might be a horizontal tangent line at this point. However, plugging $x = \pm\sqrt{2}$ into the original equation gives $0 = 4$, a contradiction which shows that there are no points on the curve with x value $\pm\sqrt{2}$. Likewise, plugging $y = 0$ in the original equation gives $0 = 4$. Again,this is a contradiction which shows that there are no points on the graph with y value of 4. Sketching the graph, we see that there is a horizontal asymptote at $y = 0$ and vertical asymptote at $x = \pm\sqrt{2}$



46. For the first type of curve, $y + xy' = 0$ and $y' = \frac{-y}{x}$.
For the second type of curve, $2x - 2yy' = 0$ and $y' = \frac{x}{y}$.
At any point of intersection, the tangent line to the first curve is perpendicular to the tangent line to the second curve.

47. If $y_1 = \frac{c}{x}$ then $y_1' = -\frac{c}{x^2} = -\frac{y_1}{x}$.
If $y_2{}^2 = x^2 + k$ then $2y_2y_2{}' = 2x$ and $y_2{}' = \frac{x}{y_2}$. If we are at a particular point (x_0, y_0) on both graphs, this means $y_1(x_0) = y_0 = y_2(x_0)$ and
$y_1{}'.y_2{}' = \left(\frac{-y_0}{x_0}\right).\left(\frac{x_0}{y_0}\right) = -1.$
This means that the slopes are negative reciprocals and the curves are orthogonal.

48. For the first type of curve, $2x + 2yy' = c$ and $y' = \frac{c - 2x}{2y}$.
For the second type of curve, $2x + 2yy' = ky'$ and $y' = \frac{2x}{k - 2y}$.
Multiply the first x/x and the second by y/y.
This gives $y' = \frac{cx - 2x^2}{2xy} = \frac{y^2 - x^2}{2xy}$, and $y' = \frac{2xy}{ky - y^2} = \frac{2xy}{x^2 - y^2}$.
These are negative reciprocals of each other, so the families of the curve are orthogonal.

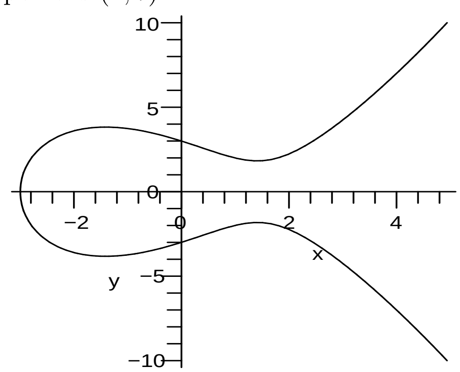
49. For the first type of curve, $y' = 3cx^2$.
For the second type of curve, $2x + 6yy' = 0$,
$y' = -\frac{2x}{6y} = -\frac{x}{3y} = -\frac{x}{3cx^3} = -\frac{1}{3cx^2}$.
These are negative reciprocals of each other, so the families of the curve are orthogonal.

50. For the first type of curve, $y' = 4cx^3$.
For the second type of curve, $2x + 8yy' = 0$.
$y' = \frac{-2x}{8y} = \frac{-x}{4y} = \frac{-x}{4cx^4} = \frac{-1}{4cx^3}$.
These are negative reciprocals of each other, so the families of the curve orthogonal.

51. Conjecture: The family of functions $\{y_1 = cx^n\}$ is orthogonal to the family of functions $\{x^2 + ny^2 = k\}$ wherever $n \neq 0$.
If $y_1 = cx^n$, then $y_1{}' = ncx^{n-1} = \frac{ny_1}{x}$.
If $ny_2{}^2 = -x^2 + k$, then $2ny_2.(y_2{}') = -2x$ and $y_2{}' = \frac{-x}{ny_2}$.
If we are at a particular point (x_0, y_0) on both graphs, this means $y_1(x_0) = y_0 = y_2(x_0)$ and
$y_1{}'.y_2{}' = \left(\frac{ny_0}{x_0}\right).\left(-\frac{x_0}{ny_0}\right) = -1.$
This means that the slopes are negative reciprocals and the curves are orthogonal.

52. The domain of the function $sin^{-1}x$ is $[-1, 1]$ and the domain of the function $\sec^{-1}x$ is $(-\infty, -1) \cup (1, \infty)$. Therefore, the function $sin^{-1}x + \sec^{-1}x$ is not defined.

53. **(a)** Both of the points$(-3, 0)$ and $(0, 3)$ are on the curve:
$0^2 = (-3)^3 - 6(-3) + 9$
$3^2 = 0^3 - 6(0) + 9$
The equation of the line through these points has slope$= \frac{0 - 3}{-3 - 0} = 1$ and y-intercept 3, so $y = x + 3$.
This line intersects the curve at:
$y^2 = x^3 - 6x + 9$
$(x + 3)^2 = x^3 - 6x + 9$
$x^2 + 6x + 9 = x^3 - 6x + 9$
$x^3 - 12x - x^2 = 0$
$x\left(x^2 - x - 12\right) = 0$
Therefore $x = 0, -3$ or 4 and so third point is $(4, 7)$.



(b) $3^2 = (-1)^3 - 6(-1) + 4$ is true.
$2yy' = 3x^2 - 6$, so $y' = \frac{3x^2 - 6}{2y}$ and at $(-1, 3)$ the slope is -$\frac{1}{2}$. The line is $y = -\frac{1}{2}(x + 1) + 3$.
To find the other point of intersection, substitute the equation of the line in to the equation for the elliptic curve and simplify:
$\left(\frac{-1}{2}x + \frac{5}{2}\right)^2 = x^3 - 6x + 4$
$x^2 - 10x + 25 = 4x^3 - 24x + 16$
$4x^3 - x^2 - 14x - 9 = 0.$
We know already that $x = -1$ is a solution(actually a double solution) so we can factor out $(x + 1)$. Long division

yields $(x+1)^2(4x^2-9)$.
The second point has x-coordinate $\frac{9}{4}$, which can be substituted into the equation for the line to get $y = \dfrac{11}{8}$.

54. The equation of the circle is $x^2 + (y-c)^2 = r^2$. Differentiating implicitly gives
$2x + 2(y-c).y' = 0$ so $y' = \dfrac{x}{(c-y)}$.

At the point of tangency, the derivatives must be the same. Since the derivative of $y = x^2$ is $2x$, we must solve the equation $2x = \dfrac{x}{(c-y)}$. This gives $y = c - \dfrac{1}{2}$, as desired. Since $y = x^2$, plugging, $y = c - \dfrac{1}{2}$ into the equation of the circle gives
$$\left(c-\frac{1}{2}\right)^2 + \left(c-\frac{1}{2}-c\right)^2 = r^2$$
$$c - \frac{1}{2} + \frac{1}{4} = r^2$$
$$c = r^2 + \frac{1}{4}$$

55. The viewing angle is given by the formula $\theta(x) = \tan^{-1}\left(\dfrac{3}{x}\right) - \tan^{-1}\left(\dfrac{1}{x}\right)$.

This will be maximum where the derivative is zero.
$$\theta'(x) = \frac{1}{1+\left(\frac{3}{x}\right)^2}.\frac{-3}{x^2} - \frac{1}{1+x^2}.\frac{-1}{x^2}$$
$$= \frac{1}{1+x^2} - \frac{3}{9+x^2}.$$
This is zero when $\dfrac{1}{1+x^2} = \dfrac{3}{9+x^2} \Rightarrow x^2 = 3 \Rightarrow x = \sqrt{3}$

56. If A is the viewing angle formed between the rays from the person's eye to the top of the frame and to the bottom of the frame, and if x is the distance between the person and the wall, then since the frame extends from 6 to 8 feet, we have $\tan A = \frac{2}{x}$, or $A = \arctan\left(\dfrac{2}{x}\right)$.

Then
$$\frac{dA}{dx} = \frac{1}{1+\left(\frac{2}{x}\right)^2}.\left(\frac{-2}{x^2}\right) = \frac{-2}{x^2+4}$$
Since the derivative is negative, the angle is decreasing function of x. Strictly speaking $\arctan\left(\dfrac{2}{x}\right)$ is undefined at $x = 0$ but $\arctan\left(\dfrac{2}{x}\right) \to \dfrac{\pi}{2}$ as $x \to 0$. The angle a continues to enlarge(upto a right angle) as x decreases to zero. In this case, the maximal viewing angle is not a feasible one.

57. $x^2 + y^2 = 9$
Differentiating the above equation implicitly, we get $2x + 2yy' = 0$
$x + yy' = 0 \Rightarrow y' = -\dfrac{x}{y}$

At $(2.9, 0.77)$, y' gives slope of the tangent.
$y'|_{(2.9,0.77)} = \dfrac{-2.9}{0.77} = -3.77$

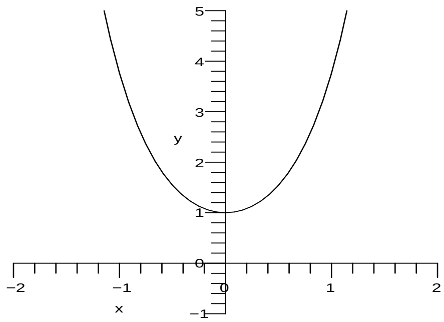
Therefore the equation of the tangent line is $y - 0.77 = -3.77(x-2.9) \Rightarrow y = -3.77x + 11.7$
Let (x_1, y_1) be any point on the line such that the distance is 300 feet. Therefore $(x_1-2.9)^2 + (y_1-0.77)^2 = 300^2$. Substitute the value of x as x_1 and y_1, as $y_1 = -3.77x_1 + 11.7$ into the above equation we get,
$(x_1-2.9)^2 + (-3.77x_1 + 11.7 - 0.77)^2 = 90000$
$(x_1-2.9)^2 + (-3.77x_1 + 10.93)^2 = 90000$
$15.21{x_1}^2 - 88.41x_1 - 89872.13 = 0$
Solving the above quadratic equation, we get $x_1 = 79.83$, $x_1 = -74.02$

Since the sling shot is rotating in the counter clockwise direction, we have to consider the negative value of x_1. Therefore substituting the negative value of x_1 into the equation, $y_{1=} - 3.77x + 11.70$

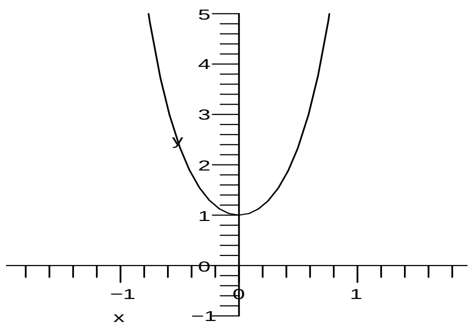
we get $y_1 = -3.77(-74.02) + 11.7 = 290.75$
Therefore $(-74.02, 290.75)$ is the required point.

2.9 The Hyperbolic Functions

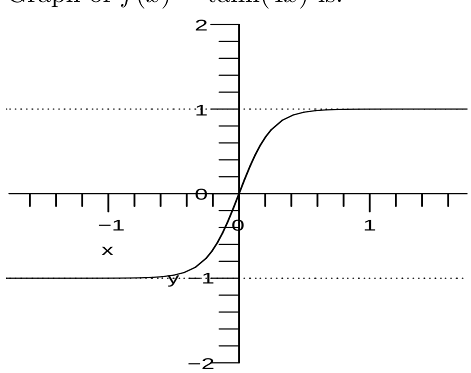
1. Graph of $f(x) = \cosh(2x)$ is:



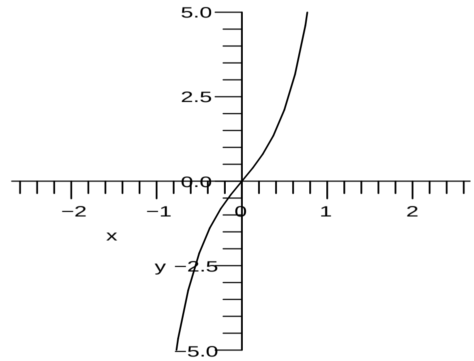
2. Graph of $f(x) = \cosh(3x)$ is:



3. Graph of $f(x) = \tanh(4x)$ is:



4. Graph of $f(x) = \sinh(3x)$ is:



5. **(a)**
$$\begin{aligned} f'(x) &= \frac{d}{dx}(\cosh 4x) \\ &= \sinh 4x \frac{d}{dx}(4x) \\ &= 4 \sinh 4x \end{aligned}$$

(b)
$$\begin{aligned} f'(x) &= \frac{d}{dx} \cosh^4 x \\ &= \frac{d}{dx} (\cosh x)^4 \\ &= 4(\cosh x)^3 (\sinh x) \\ &= 4 \sinh x \cdot \cosh^3 x \end{aligned}$$

6. **(a)**
$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\sinh\left(\sqrt{x}\right)\right) \\ &= \cosh\left(\sqrt{x}\right) \frac{d}{dx}\left(\sqrt{x}\right) \\ &= \cosh\left(\sqrt{x}\right)\left(\frac{1}{2\sqrt{x}}\right) \\ &= \frac{\cosh\left(\sqrt{x}\right)}{2\sqrt{x}} \end{aligned}$$

(b)
$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\sqrt{\sinh x}\right) \\ &= \frac{1}{2\sqrt{\sinh x}} \frac{d}{dx}(\sinh x) \\ &= \frac{1}{2\sqrt{\sinh x}}(\cosh x) \\ &= \frac{\cosh x}{2\sqrt{\sinh x}} \end{aligned}$$

7. **(a)**
$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\tanh x^2\right) \\ &= \operatorname{sech}^2 x^2 \cdot \frac{d}{dx}\left(x^2\right) \\ &= \left(\operatorname{sech}^2 x^2\right) \cdot (2x) \\ &= 2x \operatorname{sech}^2 x^2 \end{aligned}$$

(b)
$$\begin{aligned} f'(x) &= \frac{d}{dx}(\tanh x)^2 \\ &= 2 \tanh x \operatorname{sech}^2 x \end{aligned}$$

8. **(a)** $f'(x) = \frac{d}{dx}(\text{sech}\,3x)$
$= -\text{sech}\,3x\tanh 3x\frac{d}{dx}(3x)$
$= -3\ \text{sech}\,3x\tanh 3x$

(b) $f'(x) = \frac{d}{dx}(\text{csch}\,x)^3$
$= 3\left(\text{csch}^2 x\right)\frac{d}{dx}(\text{csch}\ x)$
$= 3\left(\text{csch}^2 x\right)(-\text{csch}\ x\coth x)$
$= -3\ \text{csch}^3 x\coth x$

9. **(a)** $f'(x) = \frac{d}{dx}\left(x^2\sinh 5x\right)$
$= x^2\frac{d}{dx}(\sinh 5x) + \sinh 5x\frac{d}{dx}\left(x^2\right)$
$= x^2\cosh 5x\frac{d}{dx}(5x) + \sinh 5x\,(2x)$
$= 5x^2\cosh 5x + 2x\sinh 5x$

(b) $f(x) = \frac{x^2+1}{\text{csch}^3 x} = (x^2+1)\sinh^3 x$
$f'(x) = 2x\sinh^3 x + (x^2+1)\frac{d}{dx}(\sinh^3 x)$
$= 2x\sinh^3 x + (x^2+1)3\sinh^2 x\cosh x$
$= 2x\sinh^3 x + 3(x^2+1)\sinh^2 x\cosh x$

10. **(a)** $f'(x) =$
$\frac{d}{dx}\left(\frac{\cosh 4x}{x+2}\right)$
$= \frac{(x+2)\frac{d}{dx}\cosh 4x - \cosh 4x\frac{d}{dx}(x+2)}{(x+2)^2}$
$= \frac{(x+2)\sinh 4x\,(4) - \cosh 4x\,(1)}{(x+2)^2}$
$= \frac{4\,(x+2)\sinh 4x - \cosh 4x}{(x+2)^2}$

(b) $f'(x) =$
$\frac{d}{dx}\left(x^2\tanh\left(x^3+4\right)\right)$
$= x^2\frac{d}{dx}\tanh\left(x^3+4\right)$
$+ \tanh\left(x^3+4\right)\frac{d}{dx}\left(x^2\right)$
$= x^2\text{sech}^2\left(x^3+4\right)\left(3x^2\right)$
$+ \tanh\left(x^3+4\right)(2x)$
$= 3x^4\text{sech}^2\left(x^3+4\right) + 2x\tanh\left(x^3+4\right)$

11. **(a)** $f'(x) = \frac{d}{dx}\left(\cosh^{-1}2x\right)$
$= \frac{1}{\sqrt{(2x)^2-1}}\frac{d}{dx}(2x)$
$= \frac{2}{\sqrt{4x^2-1}}$

(b) $f'(x) = \frac{d}{dx}\left(\sinh^{-1}x^2\right)$
$= \frac{1}{\sqrt{1+x^4}}\frac{d}{dx}\left(x^2\right)$
$= \frac{2x}{\sqrt{1+x^4}}$

12. **(a)** $f'(x) = \frac{d}{dx}\left(\tanh^{-1}3x\right)$
$= \frac{1}{1-(3x)^2}\frac{d}{dx}(3x)$
$= \frac{3}{1-9x^2}$

(b) $f'(x) =$
$\frac{d}{dx}\left(x^2\cosh^{-1}4x\right)$
$= x^2\frac{d}{dx}\left(\cosh^{-1}4x\right) + \cosh^{-1}4x\frac{d}{dx}\left(x^2\right)$
$= x^2\frac{1}{\sqrt{(4x)^2-1}}(4) + \cosh^{-1}4x\,(2x)$
$= \frac{4x^2}{\sqrt{16x^2-1}} + 2x\cosh^{-1}4x$

13. $\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x+e^{-x}}{2}\right)$
$= \frac{e^x-e^x}{2} = \sinh x$

$\frac{d}{dx}(\tanh x)$
$= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)$
$= \frac{\cosh x\frac{d}{dx}(\sinh x) - \sinh x\frac{d}{dx}(\cosh x)}{\cosh^2 x}$
$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \text{sech}^2 x$

14. $\frac{d}{dx}[\coth x] = \frac{d}{dx}\left[\frac{\cosh x}{\sinh x}\right]$
$= \frac{\sinh x\cdot\sinh x - \cosh x\cdot\cosh x}{(\sinh x)^2}$
$= \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x}$
$= \frac{-1}{\sinh^2 x}$
$= \text{csch}^2 x$

$$\frac{d}{dx}[\operatorname{sech} x] = \frac{d}{dx}\left[\frac{1}{\cosh x}\right] = -\frac{1}{\cosh^2 x}\sinh x = -\frac{1}{\cosh x}\cdot\frac{\sinh x}{\cosh x} = -\operatorname{sech} x\tanh x$$

$$\frac{d}{dx}[\operatorname{csch} x] = \frac{d}{dx}\left[\frac{1}{\sinh x}\right] = -\frac{1}{\sinh^2 x}\cosh x = -\frac{1}{\sinh x}\cdot\frac{\cosh x}{\sinh x} = -\operatorname{csch} x\coth x$$

15. First, $e^x > e^{-x}$ if $x > 0$ and $e^x < e^{-x}$ if $x < 0$. Since $\sinh x = \frac{e^x - e^{-x}}{2}$, we have that $e^x - e^{-x} > 0$ if $x > 0$ and $e^x - e^{-x} < 0$ if $x < 0$. Therefore $\sinh x > 0$ if $x > 0$ and $\sinh x < 0$ if $x < 0$.

16. $\cosh^2 x - \sinh^2 x$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$
$$= \frac{1}{4}\left[\left(e^{2x} + 2 + e^{-2x}\right) - \left(e^{2x} - 2 + e^{-2x}\right)\right]$$
$$= \frac{1}{4}(4) = 1$$

17. If $y = \cosh^{-1}x$ then $x = \cosh y$ and $x = \frac{e^y + e^{-y}}{2}$.

Also $\sinh y = \frac{e^y - e^{-y}}{2}$. Then

$$e^y = \cosh y + \sinh y = \cosh y + \sqrt{\sinh^2 y} = \cosh y + \sqrt{\cosh^2 y - 1} = x + \sqrt{x^2 - 1}$$

So, $y = \cosh^{-1}x = \ln\left(x + \sqrt{x^2 - 1}\right)$

18. If $y = \tanh^{-1}x$ then $x = \tanh y$ and $x = \frac{e^y - e^{-y}}{e^y + e^{-y}}$

Applying Componendo and Dividendo Rule,

$$\frac{1+x}{1-x} = \frac{2e^y}{2e^{-y}}$$
$$\frac{1+x}{1-x} = e^{2y}$$
$$e^{2y} = \frac{1+x}{1-x}$$
$$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

19. $\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = e^x$

20. $\cosh(-x) = \frac{e^{-x} + e^x}{2} = \cosh x$

$\sinh(-x) = \frac{e^{-x} - e^x}{2} = -\sinh x$

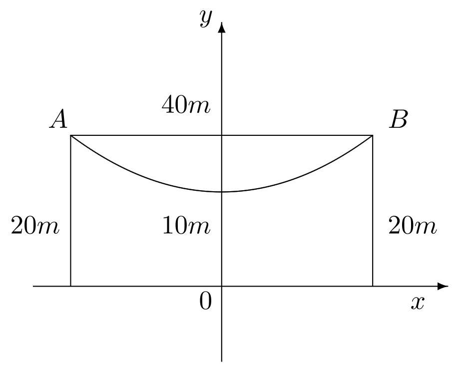
21. Since e^{-x} term tend to 0 as x tend to ∞.

$$\lim_{x\to\infty}\frac{e^x - e^{-x}}{e^x + e^{-x}} = 1,$$
$$\lim_{x\to-\infty}\frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to\infty}\frac{e^{-x} - e^x}{e^{-x} + e^x} = -1$$

22. $\tanh x = \frac{\frac{e^x+e^{-x}}{2}}{\frac{e^x+e^{-x}}{2}}\cdot\frac{2e^x}{2e^x} = \frac{e^{2x} - 1}{e^{2x} + 1}$

23. Given, $y = a\cosh\left(\frac{x}{b}\right)$. The hanging cable is as shown in the figure: From figure, $a = 10$ and $y = 10\cosh\left(\frac{x}{b}\right)$. The point $B\,(20, 20)$ is on the catenary.

$$\Rightarrow 20 = 10\cosh\left(\frac{20}{b}\right)$$
$$\Rightarrow 2 = \cosh\left(\frac{20}{b}\right)$$
$$\Rightarrow \frac{20}{b} = \cosh^{-1}(2) = \ln(2 + \sqrt{3})$$
$$\left[\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)\right]$$
$$\Rightarrow b = \frac{20}{\ln(2 + \sqrt{3})}$$



24. Given, $y = a\cosh\left(\frac{x}{b}\right)$. The hanging cable is as shown in the figure: From figure, $a = 10$ and $y = 10\cosh\left(\frac{x}{b}\right)$. Let

$A \equiv (-x_1, 30)$ and $B \equiv (x_2, 20)$ such that $AB = 40$.

$$d(A,B) = \sqrt{(x_1 + x_2)^2 + 100}$$
$$1600 = (x_1 + x_2)^2 + 100$$
$$x_1 + x_2 = \sqrt{1500} \quad ...(1)$$

The point $A\,(-x_1, 10)$ is on the catenary.

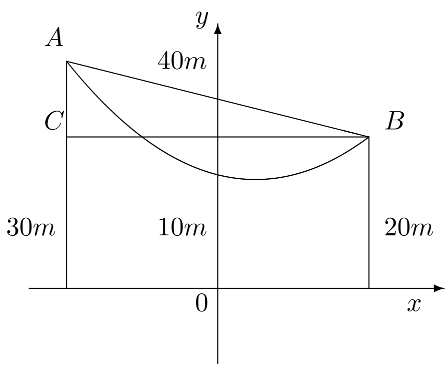
$$30 = 10\cosh\left(\frac{-x_1}{b}\right)$$
$$3 = \cosh\left(\frac{x_1}{b}\right)$$
$$\Rightarrow x_1 = b\cosh^{-1}(3)$$
$$\Rightarrow x_1 = b\ln(3 + \sqrt{8})$$

The point $A\,(x_2, 20)$ is on the catenary.

$$20 = 10\cosh\left(\frac{x_2}{b}\right)$$
$$2 = \cosh\left(\frac{x_2}{b}\right)$$
$$\Rightarrow x_2 = b\cosh^{-1}(2)$$
$$\Rightarrow x_2 = b\ln(2 + \sqrt{3})$$

By using (1),

$$b\ln[(3+\sqrt{8})(2+\sqrt{3})] = \sqrt{1500}$$
$$b = \frac{\sqrt{1500}}{\ln[(3+\sqrt{8})(2+\sqrt{3})]}$$



25. **(a)** Given that

$$v(t) = -\sqrt{\frac{mg}{k}}\tanh\left\{\sqrt{\frac{kg}{m}}\,t\right\}$$

Now, find terminal velocity(V)

$$V = \lim_{t\to\infty} v(t)$$
$$= -\sqrt{\frac{mg}{k}}\lim_{t\to\infty}\ \tanh\left\{\sqrt{\frac{kg}{m}}\ t\right\}$$
$$= -\sqrt{\frac{mg}{k}}\lim_{t\to\infty}\ \tanh\{ct\}$$

By putting $\sqrt{\frac{kg}{m}} = c$,

$$V = -\sqrt{\frac{mg}{k}}\lim_{t\to\infty}\frac{\sinh ct}{\cosh ct}$$
$$= -\sqrt{\frac{mg}{k}}\lim_{t\to\infty}\frac{\left(\frac{e^{ct}-e^{-ct}}{2}\right)}{\left(\frac{e^{ct}+e^{-ct}}{2}\right)}$$
$$= -\sqrt{\frac{mg}{k}}\lim_{t\to\infty}\frac{e^{2ct}-1}{e^{2ct}+1}$$
$$= -\sqrt{\frac{mg}{k}}\lim_{t\to\infty}\frac{\frac{d}{dt}\{e^{2ct}-1\}}{\frac{d}{dt}\{e^{2ct}+1\}}$$

By L'Hospital's rule,

$$= -\sqrt{\frac{mg}{k}}\lim_{t\to\infty}\frac{2ce^{2ct}}{2ce^{2ct}}$$
$$= -\sqrt{\frac{mg}{k}}\,(1)$$
$$\lim_{t\to\infty} v(t) = -\sqrt{\frac{mg}{k}}$$

(b) From (a), we get

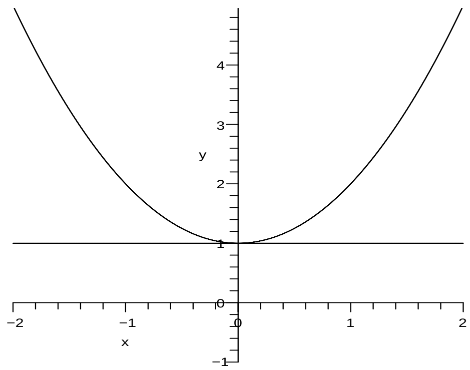
$$V = -\sqrt{\frac{mg}{k}}$$
$$V^2 = \frac{mg}{k}$$
$$mg = kV^2$$

26. For the first skydiver:
Terminal velocity is -80m/s.
Distance in 2 seconds is 19.41m.
Distance in 4 second is 75.45m.
For the second skydiver:
Terminal velocity is -40m/s.
Distance in 2 seconds is 18.86m.
Distance in 4 seconds is 68.35m.

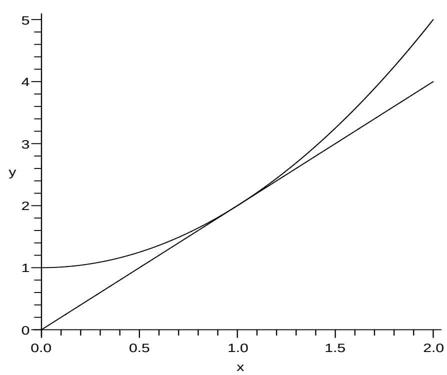
27. For an initial velocity $v_0 = 2000$, we set the derivative of the velocity equal to 0 and solve the resulting equation in a CAS. The maximum acceleration of -9.797 occurs at about 206 seconds.

2.10 The Mean Value Theorem

1. $f(x) = x^2 + 1\,, [-2,\ 2]$
$f(-2) = 5 = f(2)$.
As a polynomial $f(x)$ is continious on $[-2,\ 2]$, differentiable on $(-2,\ 2)$, and the condition's of Roll's Theorem hold. There exists $c \in (-2,\ 2)$ such that $f'(c) = 0$. But $f'(c) = 2c \Rightarrow c = 0$



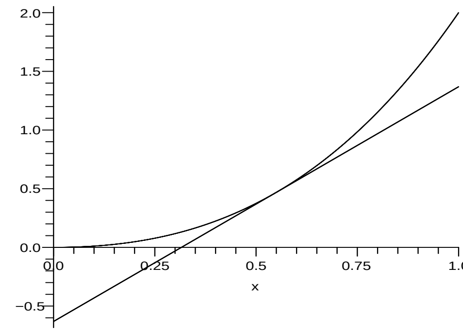
2. $f(x) = x^2 + 1, [0, 2]$
$f(x)$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$, so the conditions of the Mean Value Theorem hold. We need to find c so that
$f'(c) = \dfrac{f(2) - f(0)}{2 - 0} = \dfrac{5 - 1}{2 - 0} = 2.$
$f'(x) = 2x = 2$ when $x = 1$, so $c = 1$.



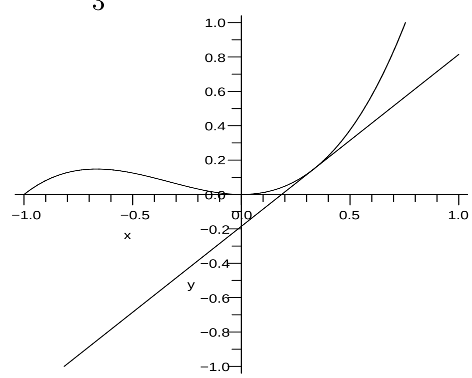
3. $f(x) = x^3 + x^2$ on $[0, 1]$ with $f(0) = 0$, $f(1) = 2$. As a polynomial, $f(x)$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Since the conditions of the Mean Value Theorem hold, there exists a number $c \in (0, 1)$ such that
$f'(c) = \dfrac{f(1) - f(0)}{1 - 0} = \dfrac{2 - 0}{1 - 0} = 2.$
But $f'(c) = 3c^2 + 2c$.
$\Rightarrow 3c^2 + 2c = 2 \Rightarrow 3c^2 + 2c - 2 = 0.$
By the quadratic formula,

$$c = \frac{-2 \pm \sqrt{2^2 - 4(3)(-2)}}{2(3)} = \frac{-2 \pm \sqrt{28}}{6} = \frac{-2 \pm 2\sqrt{7}}{6} = \frac{-1 \pm \sqrt{7}}{3}$$

$\Rightarrow c \approx -1.22$ or $c \approx 0.55$
But since $-1.22 \notin (0, 1)$, we accept only the other alternatives: $c = \dfrac{-1 \pm \sqrt{7}}{3} \approx 0.55$



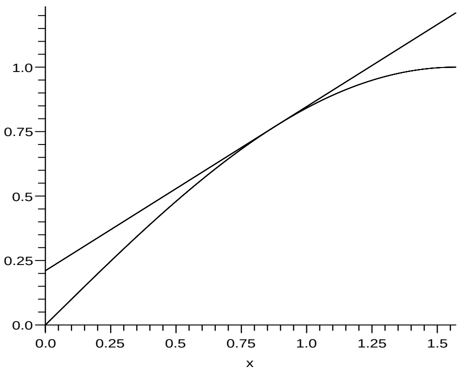
4. $f(x) = x^3 + x^2$ on $[-1, 1]$
$f(x)$ is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. So the conditions of the Mean Value Theorem hold. We need to find c so that
$f'(c) = \dfrac{f(1) - f(-1)}{1 - (-1)} = \dfrac{2 - 0}{2} = 1.$
$f'(x) = 3x^2 + 2x = 1$ when $x = -1$ or $x = \dfrac{1}{3}$, so $c = \dfrac{1}{3}$



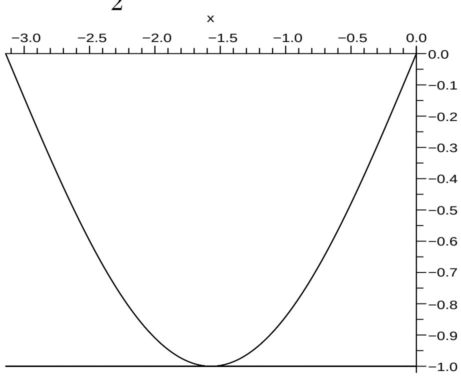
5. $f(x) = \sin x, \ [0, \pi/2]$,
$f(0) = 0, \ f(\pi/2) = 1.$
As a trig function, $f(x)$ is continuous on $[0, 2]$ and differintiable 0n $(0, \pi/2)$. The conditions of the Mean Value Theorem hold, and there exists $c \in (0, \pi/2)$ such that

$$f'(c) = \frac{f(\frac{\pi}{2}) - f(0)}{\frac{\pi}{2} - 0} = \frac{1 - 0}{\frac{\pi}{2} - 0} = \frac{2}{\pi}$$

But $f'(c) = \cos(c)$, and c is to be in the first quadrant, therefore $c = \cos^{-1}\left(\frac{2}{\pi}\right) \approx .88$



6. $f(x) = \sin x$, $[-\pi, 0]$
$f(x)$ is continuous on $[-\pi, 0]$, and differentiable on $(-\pi, 0)$. Also, $\sin(-\pi) = 0 = \sin(0)$. So the conditions of Roll's Theorem hold. We need to find c so that $f'(c) = 0$.
$f'(x) = \cos x = 0$, on $(-\pi, 0)$ when $x = -\dfrac{\pi}{2}$, so $c = -\dfrac{\pi}{2}$.



7. Let $f(x) = x^3 + 5x + 1$. As a polynomial, $f(x)$ is continuous and differentiable, for all x, with $f'(x) = 3x^2 + 5$, which is positive for all x. So $f(x)$ is strictly increasing for all x. Therefore the equation can have at most one solution. Since $f(x)$ is negative at $x = -1$ and positive at $x = 1$ and $f(x)$ is continuous, there must be a solution to $f(x) = 0$.

8. The derivative is $3x^2 + 4 > 0$ for all x. Therefore the function is strictly increasing, and so the equation can have at most one solution. Because the function is negative at $x = 0$ and positive at $x = 1$, and continuous, we know the equation has exactly one solution.

9. Let $f(x) = x^4 + 3x^2 - 2$. The derivative is $f'(x) = 4x^3 + 6x$. This is nagative for negative x, and positive for positive x so $f(x)$ strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, \infty)$. Since $f(0) = -2 \neq 0$, $f(x)$ can have at most one zero for $x < 0$ and one zero for $x > 0$. The function is continuous everywhere and $f(-1) = 2 = f(1)$, $f(0) < 0$. Therefore $f(x) = 0$ has exactly one solution between $x = -1$ and $x = 0$, and $f(x) = 0$ has exactly one solution between $x = 0$ and $x = 1$, and no other solutions.

10. Let $f(x) = x^4 + 6x^2 - 1$. The derivative is $f'(x) = 4x^3 + 12x$. This is nagative for negative x, and positive for positive x so $f(x)$ is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, \infty)$. Since $f(0) = -1 \neq 0$, $f(x)$ can have at most one zero for $x < 0$ and one zero for $x > 0$. The function is continuous everywhere and $f(-1) = 6 = f(1)$, $f(0) < 0$. Therefore $f(x) = 0$ has exactly one solution between $x = -1$ and $x = 0$, exactly one solution between $x = 0$ and $x = 1$, and no other solutions.

11. $f(x) = x^3 + ax + b$, $a > 0$. Any cubic(actyally any odd degree) polynomial heads in opposite directions $(\pm\infty)$ as x goes to the oppositely signed infinities, and therefore by the Intermediate Value Theorem $f(x)$ has atleast one root. For the uniqueness, we look at the derivative, in this case $3x^2 + a$. Because $a > 0$ by assumption, this expression is strictly positive. The function is strictly increasing and can have at most one root. Hence $f(x)$ has exactly one root.

12. The derivative is $f'(x) = 4x^3 + 2ax$. This is nagative for negative x, and positive for positive x so $f(x)$ is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, \infty)$, So can have at most one zero for $x < 0$ and one zero for $x > 0$. The function is continuous everywhere and $f(0) = -b < 0$ and $\lim\limits_{x\to\pm\infty} f(x) = \infty$, therefore $f(x)$ has exactly one solution for $x < 0$, and similarly exactly one solution for $x > 0$, and no other solutions.

13. $f(x) = x^5 + ax^3 + bx + c$, $a > 0$, $b > 0$. Here is another odd degree polynomial(see #11) with atleast one root. $f'(x) = 5x^4 + 3ax^2 + b$ is evidently strictly positive because of our assumption about a, b. Exactly as in #11, $f(x)$ has exactly one root.

14. A third degree polynomial $p(x)$ has atleast one zero because

$$\lim_{x\to\pm\infty} p(x) = \pm \lim_{x\to\infty} p(x) = \pm\infty,$$

and it is continuous. Say this zero is at $x = c$. Then we know $p(x)$ factors into

$p(x) = (x-c)q(x)$, where $q(x)$ is a quadratic polynomial. Quadratic polynomial have atmost two zeros so $p(x)$ has atmost three zeros.

15. $f(x) = x^2$.
One candidate: $g_0(x) = kx^3$.
Because we require $x^2 = g_0'(x) = 3kx^2$, we must have $3k = 1$, $k = 1/3$.
Most general solution: $g(x) = g_0(x) + c = x^3/3 + c$, where c is an arbitrary constant.

16. If $g'(x) = 9x^4$, then $g(x) = \frac{9}{5}x^5 + c$ for any constant c.

17. Although the obvious first candidate is $g_0(x) = -1/x$, due to disconnection of the domain by the discontinuity at $x = 0$, we could add different constants, one for negative x, another for positive x. Thus the most general solution is:
$$g(x) = \begin{cases} -1/x + a & \text{when } x > 0 \\ -1/x + b & \text{when } x < 0 \end{cases}$$

18. If $g'(x) = \sqrt{x}$, then $g(x) = \frac{2}{3}x^{3/2} + c$ for any constant c.

19. If $g'(x) = \sin x$, then $g(x) = -\cos x + c$ for any constant c.

20. If $g'(x) = \cos x$, then $g(x) = \sin x + c$ for any constant c.

21. If $g'(x) = \dfrac{4}{1+x^2}$ then $g(x) = 4\tan^{-1}(x) + c$.

22. If $g'(x) = \dfrac{2}{\sqrt{1-x^2}}$ then $g(x) = 2\sin^{-1}(x) + c$.

23. If derivative $g'(x)$ is positive at a single point $x = b$, then $g(x)$ is an increasing function for x sufficiently near b, i .e., $g(x) > g(b)$ for $x > b$ but sufficiently near b. In this problem ,we will apply that remark to f' at $x = 0$, and conclude from $f''(0) > 0$ that $f'(x) > f'(0) = 0$ for $x > 0$ but sufficiently small. This being true about the derivative f', it tells us that f itself is increasing on some interval $(0, a)$ and in particular that $f(x) > f(0) = 0$ for $0 < x < a$. On the other side(the nagative side) f' is negative, f is decreasing(to zero) and therefore likewise positive. In summary, $x = 0$ is a genuine relative minimum.

24. The function $\cos x$ is continuous and differentiable everywhere, so for any u and v we can apply the Mean Value Theorem to get $\dfrac{\cos u - \cos v}{u - v} = \sin c$ for some c between u and v. We know $-1 \le \sin x \le 1$, so taking absolute values, we get $\left|\dfrac{\cos u - \cos v}{u - v}\right| \le 1$, or $|\cos u - \cos v| \le |u - v|$.

25. Consider the function $g(x) = x - \sin x$, obviously with $g(0) = 0$ and $g'(x) = 1 - \cos x$. If there was ever point $a > 0$ with $\sin(a) \ge a$, $(g(a) \le 0)$, then by the MVT applied to go g on the interval $[0, a]$, there would be a point c $(0 < c < a)$ with
$$g'(c) = \frac{g(a) - g(0)}{a - 0} = \frac{g(a)}{a} \le 0.$$
This would read $1 - \cos c = g'(c) \le 0$ or $\cos c \ge 1$. The latter condition is possible only if $\cos(c) = 1$ and $\sin(c) = 0$, in which case c(being positive) would be at minimum π. But even this unlikelycase we still would have $\sin(a) \le 1 < \pi \le c < a$.
Since $\sin a < a$ for all $a > 0$, we have $-\sin a > -a$ for all $a > 0$, but $-\sin a = \sin(-a)$ so we have $\sin(-a) > -a$ for all $a > 0$. This is the same as saying $\sin a > a$ for all $a < 0$ so in absolute value we have $|\sin a| < |a|$ for all $a \ne 0$.
Thus the only possible solution to the equation $\sin x = x$ is $x = 0$, which we know to be true.

26. The function $\tan^{-1} x$ is continuous and differentiable everywhere, so for any $a \ne 0$ we can apply the Mean Value Theorem to get $\dfrac{\tan^{-1}a - \tan^{-1}0}{a - 0} = \dfrac{1}{1+c^2}$ for some c between 0 and a. Taking absolute values, we get $\left|\dfrac{\tan^{-1}a}{a}\right| = \dfrac{1}{1+c^2} < 1$, so $\left|\tan^{-1}a\right| < |a|$ for $a \ne 0$. This means that the only solution to $\tan^{-1}x = x$ is $x = 0$.

27. Since the inverse sine function is increasing on the interval [0, 1) (it has positive derivative) we start from the previously proven inequality $\sin x < x$ for $0 < x$. If indeed $0 < x < 1$, we can apply the inverse sine and conclude $x = \sin^{-1}(\sin x) < \sin^{-1}(x)$.

28. The function $\tan x$ is continuous and differentiable for $|x| < \pi/2$, so for any $a \ne 0$ in $(-\pi/2, \pi/2)$, we can apply the Mean Value Theorem to get $\dfrac{\tan a - \tan 0}{a - 0} = \sec^2 c$ for some c between 0 and a. Taking absolute values, we get $\left|\dfrac{\tan a}{a}\right| = \left|\sec^2 c\right| > 1$, so

$|\tan a| > |a|$ for $a \neq 0$. Of course $\tan 0 = 0$, so $|\tan a| \geq |a|$ for all $|a| < \pi/2$.

29. If $f'(x) > 0$ for all x then for each $(a,\, b)$ with $a < b$ we know there exists $c \in (a,\, b)$ such that
$$\frac{f(b) - f(a)}{b - a} = f'(c) > 0.$$
$a < b$ makes the denominator positive, and so we must have the numerator also positive, which implies $f(a) < f(b)$.

30. Let $a < b$. f is differentiable on $(a,\, b)$ and continuous on $[a,\, b]$, since it is differentiable for all x. This means that
$$\frac{f(b) - f(a)}{b - a} = f'(c)$$
for some $c \in (a,\, b)$. Therefore $f(b) - f(a) = f'(c)\,(b - a)$ is negative, and $f(a) > f(b)$.

31. $f'(x) = 3x^2 + 5$. This is positive for all x, so $f(x)$ is increasing.

32. $f'(x) = 5x^4 + 9x^2 \geq 0$ for all x. $f' = 0$ only at $x = 0$, so $f(x)$ is increasing.

33. $f'(x) = -3x^2 - 3$. This is nagative for all x, so $f(x)$ is decreasing.

34. $f'(x) = 4x^3 + 4x$ is negative for negative x, and positive for positive x, so $f(x)$ is neither an increasing function nor a decreasing function.

35. $f'(x) = e^x$. This is positive for all x, so $f(x)$ is increasing.

36. $f'(x) = -e^{-x} < 0$ for all x, so $f(x)$ is a decreasing function.

37. $f'(x) = \dfrac{1}{x}$
$f'(x) > 0$ for $x > 0$, that is, for all x in the domain of f. So $f(x)$ is increasing.

38. $f'(x) = \dfrac{1}{x^2}.2x = \dfrac{2}{x}$ is negative for negative x, and positive for positive x, so $f(x)$ is neither an increasing function nor a decreasing function.

39. The average velocity on $[a,\, b]$ is $\dfrac{s(b) - s(a)}{b - a}$.
By the Mean Value Theorem, there exists a $c \in (a,\, b)$ such that $s'(c) = \dfrac{s(b) - s(a)}{b - a}$
Thus, the instantaneous velocity at $t = c$ is equal to the average velocity between times $t = a$ and $t = b$.

40. Let $f(t)$ be the distance the first runner has gone after time t and let $g(t)$ be the distance the second runner has gone after time t. The functions $f(t)$ and $g(t)$ will be continuous and differentiable. Let $h(t) = f(t) - g(t)$. At $t = 0$, $f(0) = 0$ and $g(0) = 0$ so $h(0) = 0$. At $t = a$, $f(a) > g(a)$ so $h(a) > 0$. Similarly, at $t = b$, $f(b) < g(b)$ so $h(b) < 0$. Thus, by the Intermideate Value Theorem, there is time $t = t_0$ for $t_0 \in (a, b)$ where $h(t_0) = 0$. Rolle's Theorem then says that there is time $t = c$ where $c \in (0,\, t_0)$ such that $h'(c) = 0$. But $h'(t) = f'(t) - g'(t)$, so $h'(c) = f'(c) - g'(c) = 0$ implies that $f'(c) = g'(c)$, i.e., at time $t = c$ the runners are going exactly the same speed.

41. Define $h(x) = f(x) - g(x)$. Then h is differentiable because f and g are, and $h(a) = h(b) = 0$. Apply Rolle's Theorem to h on $[a,\, b]$ to conclude thet there exists $c \in (a, b)$ such that $h'(c) = 0$. Thus, $f'(c) = g'(c)$, and so f and g have parallel tangent lines at $x = c$.

42. As in #41, let $h(x) = f(x) - g(x)$. Again, h is continuous and differentiable on the appropriate intervals because f and g are. Since $f(a) - f(b) = g(a) - g(b)$ (by assumption), we have $f(a) = g(a) - g(b) + f(b)$. Then,
$$\begin{aligned} h(a) &= f(a) - g(a) \\ &= g(a) - g(b) + f(b) - g(a) \\ &= f(b) - g(b) = h(b). \end{aligned}$$
Rolle's Theorem then tells us that there exists $c \in (a,\, b)$ such that $h'(c) = 0$ or $f'(c) = g'(c)$ so that f and g have parallel tangent lines at $x = c$.

43. $f(x) = 1/x$ on $[-1,\, 1]$. We easily see that $f(1) = 1$, $f(-1) = -1$, and $f'(x) = -1/x^2$. If we try to find the c in the interval $(-1,\, 1)$ for which
$$f'(c) = \frac{f(1) - f(1)}{1 - (-1)} = \frac{1 - (-1)}{1 - (-1)} = 1,$$
the equation would be $-1/c^2 = 1$ or $c^2 = -1$. There is of course no such c, and the explanation is that the function is not defined for $x = 0 \in (-1,\, 1)$ and so the function is not continuous.
The hypotheses for the Mean Value Theorem are not fulfilled.

44. $f(x)$ is not continuous on $[-1,\, 2]$, and not differentiable on $(-1,\, 2)$. Can we find

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{\frac{1}{4} - 1}{3} = -\frac{1}{4}\,?$$

$f'(x) = -\frac{2}{x^3} = -\frac{1}{4}$ when $x = 2$. This is not in $(-1, 2)$, so no c makes the conclusion of Mean Value Theorem true.

45. $f(x) = \tan x$ on $[0, \pi]$, $f'(x) = \sec^2 x$. We know the tangent has a massive discontinuity at $x = \pi/2$, so as in #44, we should not be surprised if the Mean Value Theorem does not apply. As applied to the interval $[0, \pi]$ it would say

$$\sec^2 c = f'(c) = \frac{f(\pi) - f(0)}{\pi - 0} = \frac{\tan\pi - \tan 0}{\pi - 0} = 0.$$

But secant = 1/cosine is never 0 in the interval $(-1, 1)$, so no such c exists.

46. $f(x)$ is not differentiable on $(-1, 1)$. Can we find c with

$$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{1 - (-1)}{2} = 1\,?$$

$f'(x) = \frac{1}{3}x^{-2/3} = 1$ when $x = \pm\left(\frac{1}{3}\right)^{3/2}$. These are both in $(-1, 1)$, so we can use either of these as c to make the conclusion of Mean Value Theorem true.

47. $f(x) = \begin{cases} 2x & \text{when } x \le 0 \\ 2x - 4 & \text{when } x > 0 \end{cases}$

$f(x) = 2x - 4$ is continuous and differentiable on $(0, 2)$. Also, $f(0) = 0 = f(2)$. But $f'(x) = 2$ on $(0, 2)$, so there is no c such that $f'(c) = 0$. Rolle's Theorem requires that $f(x)$ be continuous on the closed interval, but we have a jump discontinuity at $x = 0$, which is enough to preclude the applicability of Rolle's.

48. $f(x) = x^2$ is counter-example. The flaw in the proof is that we do not have $f'(c) = 0$.

Ch. 2 Review Exercises

1. $\frac{3.4 - 2.6}{1.5 - 0.5} = \frac{0.8}{1} = 0.8$

2. C (large negative), B (small negative), A (small positive), and D (large positive)

3.
$$\begin{aligned} f'(2) &= \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h\to 0} \frac{(2+h)^2 - 2(2+h) - (0)}{h} \\ &= \lim_{h\to 0} \frac{4 + 4h + h^2 - 4 - 2h}{h} \\ &= \lim_{h\to 0} \frac{2h + h^2}{h} \\ &= \lim_{h\to 0} 2 + h = 2 \end{aligned}$$

4.
$$\begin{aligned} f'(1) &= \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x\to 1} \frac{1 + \frac{1}{x} - 2}{x - 1} \\ &= \lim_{x\to 1} \frac{\frac{-(x-1)}{x}}{x - 1} \\ &= \lim_{x\to 1} \frac{-1}{x} = -1 \end{aligned}$$

5.
$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{1+h} - 1}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{1+h} - 1}{h} \cdot \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1} \\ &= \lim_{h\to 0} \frac{1 + h - 1}{h(\sqrt{1+h} + 1)} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{1+h} + 1} = \frac{1}{2} \end{aligned}$$

6.
$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{x^3 - 2x}{x} \\ &= \lim_{x\to 0} x^2 - 2 = -2 \end{aligned}$$

7.
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^3 + (x+h) - (x^3 + x)}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 + h}{h} \\ &= \lim_{h\to 0} 3x^2 + 3xh + h^2 + 1 \\ &= 3x^2 + 1 \end{aligned}$$

8.
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{3}{x+h} - \frac{3}{x}}{h} \\ &= \lim_{h\to 0} \frac{\frac{3x - 3(x+h)}{x(x+h)}}{h} \\ &= \lim_{h\to 0} \frac{\frac{-3h}{x(x+h)}}{h} \\ &= \lim_{h\to 0} \frac{-3}{x(x+h)} = \frac{-3}{x^2} \end{aligned}$$

9. The point is $(1, 0)$. $y' = 4x^3 - 2$ so the slope at $x = 1$ is 2, and the equation of the tangent line is $y - 0 = 2(x - 1)$ or $y = 2x - 2$.

10. The point is $(0, 0)$. $y' = 2\cos 2x$, so the slope at $x = 0$ is 2, and the equation of the tangent line is $y = 2x$.

11. The point is $(0, 3)$. $y' = 6e^{2x}$, so the slope at $x = 0$ is 6, and the equation of the tangent line is $y - 3 = 6(x - 0)$ or $y = 6x + 3$.

12. The point is $(0, 1)$. $y' = \dfrac{2x}{2\sqrt{x^2+1}}$, so the slope at $x = 0$ is 0, and the equation of the tangent line is $y = 1$.

13. Find the slope to $y - x^2y^2 = x - 1$ at $(1, 1)$.
$\dfrac{d}{dx}(y - x^2y^2) = \dfrac{d}{dx}(x-1)$
$y' - 2xy^2 - x^2 2y \cdot y' = 1$
$y'(1 - x^2 2y) = 1 + 2xy^2$
$y' = \dfrac{1+2xy^2}{1-2x^2y}$
At $(1, 1)$:
$y' = \dfrac{1+2(1)(1)^2}{1-2(1)^2(1)} = \dfrac{3}{-1} = -3$
The equation of the tangent line is
$y - 1 = -3(x - 1)$ or $y = -3x + 4$.

14. Implicitly differentiating:
$2yy' + e^y + xe^y y' = -1$, and
$y' = \dfrac{-1-e^y}{2y+xe^y}$.
At $(2, 0)$ the slope is -1, and the equation of the tangent line is $y = -(x - 2)$.

15. $s(t) = -16t^2 + 40t + 10$
$v(t) = s'(t) = -32t + 40$
$a(t) = v'(t) = -32$

16. $s(t) = -9.8t^2 - 22t + 6$
$v(t) = s'(t) = -19.6t - 22$
$a(t) = s''(t) = -19.6$

17. $s(t) = 10e^{-2t}\sin 4t$
$$\begin{aligned}v(t) &= s'(t)\\ &= 10\left(-2e^{-2t}\sin 4t + 4e^{-2t}\cos 4t\right)\\ a(t) &= v'(t)\\ &= 10\cdot(-2)\left[-2e^{-2t}\sin 4t + e^{-2t}4\cos 4t\right]\\ &\quad + 10(4)\cdot\left[-2e^{-2t}\cos 4t - e^{-2t}4\sin 4t\right]\\ &= 160e^{-2t}\cos 4t - 120e^{-2t}\sin 4t\end{aligned}$$

18. $s(t) = \sqrt{4t+16} - 4$
$$\begin{aligned}v(t) &= s'(t) = \frac{4}{2\sqrt{4t+16}}\\ &= \frac{2}{\sqrt{4t+16}}\\ a(t) &= s''(t)\\ &= \frac{-2\cdot 4}{2(4t+16)^{3/2}} = \frac{-4}{(4t+16)^{3/2}}\end{aligned}$$

19. $v(t) = s'(t) = -32t + 40$
$v(1) = -32(1) + 40 = 8$
The ball is rising.
$v(2) = -32(2) + 40 = -24$
The ball is falling.

20. $v(t) = s'(t) = 20e^{-2t}(2\cos 4t - \sin 4t)$
$v(0) = 40$ and $v(\pi) = 40e^{-2\pi} \approx 0.075$. The mass attached to the spring is moving in the same direction, much faster at $t = 0$.

21. **(a)** $m_{\sec} = \dfrac{f(2)-f(1)}{2-1} = \dfrac{\sqrt{3}-\sqrt{2}}{1} \approx .318$

(b) $m_{\sec} = \dfrac{f(1.5)-f(1)}{1.5-1} = \dfrac{\sqrt{2.5}-\sqrt{2}}{.5} \approx .334$

(c) $m_{\sec} = \dfrac{f(1.1)-f(1)}{1.1-1} = \dfrac{\sqrt{2.1}-\sqrt{2}}{.1} \approx .349$

Best estimate for the slope of the tangent line: (c) (approximately .349).

22. Point at $x = 1$ is $(1, 7.3891)$.

(a) $m_{\sec} = \dfrac{f(2)-f(1)}{2-1} = \dfrac{e^4-e^2}{1} \approx 47.2091$

(b) $m_{\sec} = \dfrac{f(1.5)-f(1)}{1.5-1} = \dfrac{e^3-e^2}{.5} \approx 25.3928$

(c) $m_{\sec} = \dfrac{f(1.1)-f(1)}{1.1-1} = \dfrac{e^{2.2}-e^2}{.1} \approx 16.3590$

Best estimate for the slope of the tangent line: (c) (approximately 16.3590).

23. $f'(x) = 4x^3 - 9x^2 + 2$

24. $f'(x) = \dfrac{2}{3}x^{-1/3} - 8x$

25. $f'(x) = -\dfrac{3}{2}x^{-3/2} - 10x^{-3} = \dfrac{-3}{2x\sqrt{x}} - \dfrac{10}{x^3}$

26. $f'(x) = \dfrac{\sqrt{x}(-3+2x)}{x} - \dfrac{(2-3x+x^2)\frac{1}{2\sqrt{x}}}{x}$

27. $$\begin{aligned}f'(t) &= 2t(t+2)^3 + t^2\cdot 3(t+2)^2\cdot 1\\ &= 2t(t+2)^3 + 3t^2(t+2)^2\\ &= t(t+2)^2(5t+4)\end{aligned}$$

28. $f'(t) = 2t(t^3 - 3t + 2) + (t^2 + 1)(3t^2 - 3)$

29. $g'(x) = \dfrac{(3x^2-1)\cdot 1 - x(6x)}{(3x^2-1)^2}$
$= \dfrac{3x^2-1-6x^2}{(3x^2-1)^2}$
$= -\dfrac{3x^2+1}{(3x^2-1)^2}$

30. $g(x) = 3x - \dfrac{1}{x}$
$g'(x) = 3 + \frac{1}{x^2}$

31. $f'(x) = 2x\sin x + x^2\cos x$

32. $f'(x) = 2x\cos x^2$

33. $f'(x) = \sec^2\sqrt{x}\cdot\dfrac{1}{2\sqrt{x}}$

34. $f'(x) = \dfrac{1}{2\sqrt{\tan x}}\sec^2 x$

35. $f'(t) = \csc t\cdot 1 + t\cdot(-\csc t\cdot\cot t)$
$= \csc t - t\csc t\cot t$

36. $f'(t) = 3\cos 3t\cos 4t - 4\sin 3t\sin 4t$

37. $u'(x) = 2e^{-x^2}(-2x) = -4xe^{-x^2}$

38. $u'(x) = 2(2e^{-x})(-2e^{-x}) = -8e^{-2x}$

39. $f'(x) = 1\cdot\ln x^2 + x\cdot\frac{1}{x^2}\cdot 2x$
$= \ln x^2 + 2$

40. $f'(x) = \dfrac{1}{2\sqrt{\ln x+1}}\cdot\dfrac{1}{x}$

41. $f'(x) = \dfrac{1}{2}\cdot\dfrac{1}{\sin 4x}\cdot\cos 4x\cdot 4 = 2\cot 4x$

42. $f'(x) = e^{\tan(x^2+1)}\cdot\sec^2(x^2+1)\cdot 2\cdot x$
$= 2xe^{\tan(x^2+1)}\sec^2(x^2+1)$

43. $f'(x) = 2\left(\dfrac{x+1}{x-1}\right)\dfrac{d}{dx}\left(\dfrac{x+1}{x-1}\right)$
$= 2\left(\dfrac{x+1}{x-1}\right)\dfrac{(x-1)-(x+1)}{(x-1)^2}$
$= 2\left(\dfrac{x+1}{x-1}\right)\dfrac{-2}{(x-1)^2}$
$= \dfrac{-4(x+1)}{(x-1)^3}$

44. $f'(x) = \dfrac{3}{2\sqrt{3x}}e^{\sqrt{3x}}$

45. $f'(t) = e^{4t}\cdot 1 + te^{4t}\cdot 4 = (1+4t)e^{4t}$

46. $f'(x) = \dfrac{(x-1)^2 6 - 6x\cdot 2(x-1)}{(x-1)^4}$

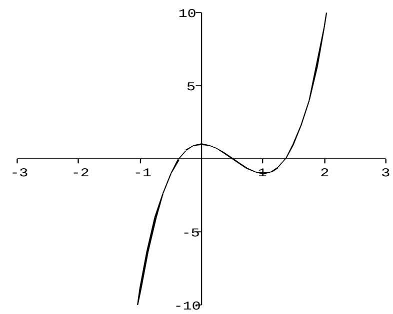
47. The given function is well defined only for $x = 0$. Hence it is not differentiable.

48. $f'(x) = \cos\left(\cos^{-1}\left(x^2\right)\right)\cdot\left(\dfrac{-2x}{\sqrt{1-\left(x^2\right)^2}}\right)$
$= \dfrac{-2x^3}{\sqrt{1-x^4}}$

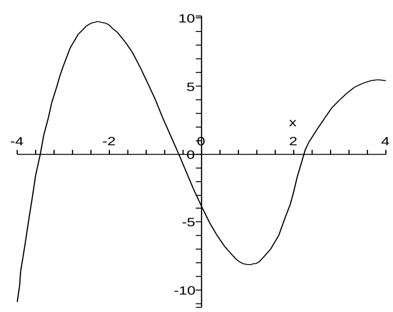
49. $\dfrac{1}{1+(\cos 2x)^2}\cdot(-2\sin 2x)$

50. $\dfrac{1}{3x^2\sqrt{(3x^2)^2-1}}\cdot 6x$

51. The derivative should look roughly like:



52. The derivative should look roughly like:



53. $f(x) = x^4 - 3x^3 + 2x^2 - x - 1$
$f'(x) = 4x^3 - 9x^2 + 4x - 1$
$f''(x) = 12x^2 - 18x + 4$

54. $f(x) = (x+1)^{1/2}$
$f'(x) = \dfrac{1}{2}(x+1)^{-1/2}$
$f''(x) = \dfrac{-1}{4}(x+1)^{-3/2}$
$f'''(x) = \dfrac{3}{8}(x+1)^{-5/2}$

55. $f(x) = xe^{2x}$
$f'(x) = 1\cdot e^{2x} + xe^{2x}\cdot 2 = e^{2x} + 2xe^{2x}$
$f''(x) = e^{2x}\cdot 2 + 2\cdot\left(e^{2x} + 2xe^{2x}\right)$
$= 4e^{2x} + 4xe^{2x}$
$f'''(x) = 4e^{2x}\cdot 2 + 4\left(e^{2x} + 2xe^{2x}\right)$
$= 12e^{2x} + 8xe^{2x}$

56. $f(x) = 4(x+1)^{-1}$
$f'(x) = -4(x+1)^{-2}$
$f''(x) = 8(x+1)^{-3}$

57. $f''(x) = 2 \cdot 2 \cdot \sec(2x) \cdot \sec(2x)\tan(2x) \cdot 2x$
$= 8\sec^2(2x)\tan(2x)$

58. Let $f(x) = [p(x)]^2$, where
$p(x) = x^6 - 3x^4 + 2x^3 - 7x + 1$
$p'(x) = 6x^5 - 12x^3 + 6x^2 - 7$
$p''(x) = 30x^4 - 36x^2 + 12x$
$p'''(x) = 120x^3 - 72x + 12$
$p^{(4)}(x) = 360x^2 - 72$
Then
$f^{(4)}(x) = 6[p''(x)]^2 + 8[p'(x)][p'''(x)] + 2[p(x)][p^{(4)}(x)]$

59. $f(x) = \sin 3x$
$f'(x) = \cos 3x \cdot 3 = 3\cos 3x$
$f''(x) = 3(-\sin 3x \cdot 3) = -9\sin 3x$
$f'''(x) = -9\cos 3x \cdot 3 = -27\cos 3x$
$f^{(26)}(x) = -3^{26}\sin 3x$

60. For $f(x) = e^{-2x}$, each derivative multiplies by a factor of -2, so
$f^{(31)}(x) = (-2)^{31}e^{-2x}$.

61. $R(t) = P(t)Q(t)$
$R'(t) = Q'(t) \cdot P(t) + Q(t) \cdot P'(t)$
$P(0) = 2.4(\$)$
$Q(0) = 12$ (thousands)
$Q'(t) = -1.5$ (thousands per year)
$P'(t) = 0.1$ ($ per year)
$R'(0) = (-1.5) \cdot (2.4) + 12 \cdot (0.1)$
$= -2.4$ (thousand $ per year)
Revenue is decreasing at a rate of $2400 per year.

62. The relative rate of change is $\frac{v'(t)}{v(t)}$. $v'(t) = 200(\frac{3}{2})^t \ln\frac{3}{2}$, so the relative rate of change is $\ln\frac{3}{2} \approx 0.4055$, giving an instantaneous percentage rate of change of 40.55%.

63. $f(t) = 4\cos 2t$
$v(t) = f'(t) = 4(-\sin 2t) \cdot 2$
$= -8\sin 2t$

(a) The velocity is zero when
$v(t) = -8\sin 2t = 0$, i.e., when
$2t = 0, \pi, 2\pi, \ldots$ so when
$t = 0, \pi/2, \pi, 3\pi/2, \ldots$
$f(t) = 4$ for $t = 0, \pi, 2\pi, \ldots$
$f(t) = 4\cos 2t = -4$ for
$t = \pi/2, 3\pi/2, \ldots$
The position of the spring when the velocity is zero is 4 or -4.

(b) The velocity is a maximum when
$v(t) = -8\sin 2t = 8$, i.e., when
$2t = 3\pi/2, 7\pi/2, \ldots$ so
$t = 3\pi/4, 7\pi/4, \ldots$
$f(t) = 4\cos 2t = 0$ for
$t = 3\pi/4, 7\pi/4, \ldots$
The position of the spring when the velocity is at a maximum is zero.

(c) Velocity is at a minimum when
$v(t) = -8\sin 2t = -8$, i.e., when
$2t = \pi/2, 5\pi/2, \ldots$ so
$t = \pi/4, 5\pi/4, \ldots$
$f(t) = 4\cos 2t = 0$ for
$t = \pi/4, 5\pi/4, \ldots$
The position of the spring when the velocity is at a minimum is also zero.

64. The velocity is given by
$f'(t) = -2e^{-2t}\sin 3t + 3e^{-2t}\cos 3t$.

65. $\frac{d}{dx}(x^2y - 3y^3) = \frac{d}{dx}(x^2+1)$
$2xy + x^2y' - 3 \cdot 3y^2 \cdot y' = 2x$
$y'(x^2 - 9y^2) = 2x - 2xy$
$y' = \dfrac{2x(1-y)}{x^2 - 9y^2}$

66. $\frac{d}{dx}(\sin(xy) + x^2) = \frac{d}{dx}(x - y)$
$\cos(xy)(y + xy') + 2x = 1 - y'$
$y' = \dfrac{1 - 2x - y\cos(xy)}{x\cos(xy) + 1}$.

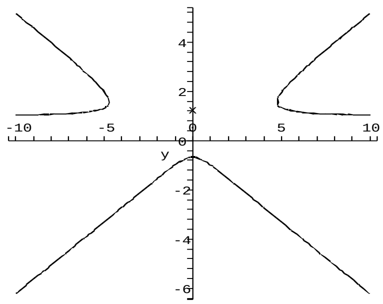
67. $\frac{d}{dx}\left(\frac{y}{x+1} - 3y\right) = \frac{d}{dx}\tan x$
$\dfrac{(x+1)y' - y \cdot (1)}{(x+1)^2} - 3y' = \sec^2 x$
$y'(x+1) - y = (x+1)^2(3y' + \sec^2 x)$
$y' = \dfrac{\sec^2 x(x+1)^2 + y}{(x+1)[1 - 3(x+1)]}$

68. $\frac{d}{dx}(x - 2y^2) = \frac{d}{dx}(3e^{x/y})$
$1 - 2yy' = 3e^{x/y} \cdot \dfrac{y - xy'}{y^2}$
$1 - 2yy' = \dfrac{3e^{x/y}}{y} - \dfrac{3e^{x/y}xy'}{y^2}$
$y' = \dfrac{\frac{3e^{x/y}}{y} - 1}{\frac{3xe^{x/y}}{y^2} - 2y}$

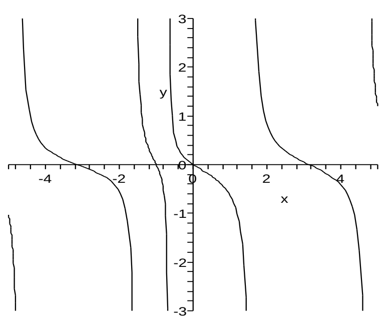
69. When $x = 0$, $-3y^3 = 1$, $y = \frac{-1}{\sqrt[3]{3}}$ (call this a).
From our formula (#65), we find $y' = 0$ at this point. To find y'', implicitly differentiate the first derivative (second line in #65):
$2(xy' + y) + (2xy' + x^2y'')$
$- 9\left[2y(y')^2 + y^2y''\right] = 2$
At $(0, a)$ with $y' = 0$, we find

$2a - 9a^2y'' = 2,$
$y'' = \frac{-2\sqrt[3]{3}}{9}\left(\sqrt[3]{3}+1\right)$

Below is a sketch of the graph of $x^2y - 3y^3 = x^2 + 1$.



70. Plugging in $x = 0$ gives $-2y = 0$ so $y = 0$. Plugging $(0, 0)$ into the formula for y' gives a slope of $-1/2$. Implicitly differentiating the third line of the solution to #37 gives
$y''(x+1) + y' - y'$
$= 2(x+1)(3y' + \sec^2 x)$
$+ (x+1)^2(3y'' + 2\sec x \cdot \sec x \tan x)$
Plugging in $x = 0$, $y = 0$ and $y' = -1/2$ gives
$y'' = 2(-3/2 + \sec^2(0))$
$+ (1)^2(3y'' + 2\sec^2(0)\tan(0))$
$y'' = 1 + 3y''.$
So at $x = 0$, $y'' = -1/2$.
The graph is:



71. $y' = 3x^2 - 12x = 3x(x-4)$

(a) $y' = 0$ for $x = 0$ $(y = 1)$, and $x = 4$ $(y = -31)$ so there are horizontal tangent lines at $(0, 1)$ and $(4, -31)$.

(b) y' is defined for all x, so there are no vertical tangent lines.

72. $y' = \frac{2}{3}x^{-1/3}$

(a) The derivative is never 0, so the tangent line is never horizontal.

(b) The derivative is undefined at $x = 0$ and the tangent is vertical there.

73. $\frac{d}{dx}(x^2y - 4y) = \frac{d}{dx}x^2$
$2xy + x^2y' - 4y' = 2x$
$y'(x^2 - 4) = 2x - 2xy$
$y' = \frac{2x - 2xy}{x^2 - 4} = \frac{2x(1-y)}{x^2 - 4}$

(a) $y' = 0$ when $x = 0$ or $y = 1$.
At $y = 1$, $x^2 \cdot 1 - 4 \cdot 1 = x^2$
$x^2 - 4 = x^2$
This is impossible, so there is no x for which $y = 1$.
At $x = 0$, $0^2 \cdot y - 4y = 0^2$, so $y = 0$.
Therefore, there is a horizontal tangent line at $(0, 0)$.

(b) y' is not defined when $x^2 - 4 = 0$, or $x = \pm 2$. At $x = \pm 2$, $4y - 4y = 4$ so the function is not defined at $x = \pm 2$. There are no vertical tangent lines.

74. $y' = 4x^3 - 2x = 2x(2x^2 - 1)$.

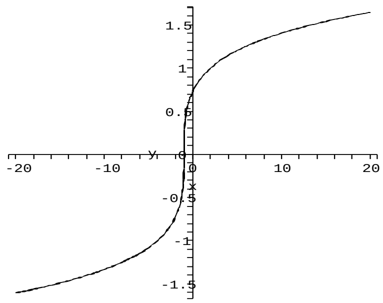
(a) The derivative is 0 at $x = 0$ and $x = \pm\sqrt{\frac{1}{2}}$, and the tangent line is horizontal at those points.

(b) The tangent line is never vertical.

75. $f(x)$ is continuous and differentiable for all x, and $f'(x) = 3x^2 + 7$, which is positive for all x. By Theorem 9.2, if the equation $f(x) = 0$ has two solutions, then $f'(x) = 0$ would have at least one solution, but it has none. We discussed at length (Section 2.9) why every odd degree polynomial has at least one root, so in this case there is exactly one root.

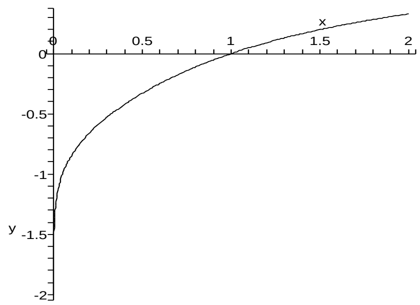
76. The derivative is $5x^4 + 9x^2$. This is non-negative for all x. $f(x)$ is increasing function so can have at most one zero. Since $f(0) = -2$, $f(1) = 2$, $f(x)$ has exactly one solution.

77. $f(x) = x^5 + 2x^3 - 1$ is a one-to-one function with $f(1) = 2$, $f'(1) = 11$. If g is the name of the inverse, then $g(2) = 1$ and
$g'(2) = \frac{1}{f'(g(2))} = \frac{1}{f'(1)} = \frac{1}{11}.$



78. Since $e^{0^3+2\cdot 0} = 1$, the derivative of the inverse at $x = 1$ will be one over the derivative of e^{x^3+2x} at $x = 0$. The derivative of e^{x^3+2x} is $(3x^2+2)e^{x^3+2x}$ and this is 2 when $x = 0$. Therefore the derivative of the inverse to e^{x^3+2x} at $x = 1$ is $1/2$.

The graph is the graph of e^{x^3+2x} reflected across the line $y = x$.



79. Let $a > 0$. We know that $f(x) = \cos x - 1$ is continuous and differentiable on the interval $(0, a)$. Also $f'(x) = \sin x \le 1$ for all x. The Mean Value Theorem implies that there exists some c in the interval $(0, a)$ such that $f'(c) = \sin c$. But

$$f'(c) = \frac{\cos a - 1 - (\cos 0 - 1)}{a - 0} = \frac{\cos a - 1}{a}.$$

Since this is equal to $\sin c$ and $\sin c \le 1$ for any c, we get that

$$\cos a - 1 \le a$$

as desired. This works for all positive a, but since $\cos x - 1$ is symmetric about the y axis, we get

$$|\cos x - 1| \le |x|.$$

They are actually equal at $x = 0$.

80. This is an example of a Taylor polynomial. Later, Taylor's theorem will be used to prove such inequalities. For now, one can use multiple derivatives and argue that the rate of the rate of the rate of change (etc.) increases as one moves left to right through the inequalities.

81. To show that $g(x)$ is continuous at $x = a$, we need to show that the limit as x approaches a of $g(x)$ exists and is equal to $g(a)$. But

$$\lim_{x\to a} g(x) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a},$$

which is the definition of the derivative of $f(x)$ at $x = a$. Since $f(x)$ is differentiable at $x = a$, we know this limit exists and is equal to $f'(a)$, which, in turn, is equal to $g(a)$. Thus $g(x)$ is continuous at $x = a$.

82. We have

$$\begin{aligned} &f(x) - T(x) \\ &\quad = f(x) - f(a) - f'(a)(x-a) \\ &\quad = \left(\frac{f(x) - f(a)}{x - a} - f'(a)\right)(x - a) \end{aligned}$$

Letting $e(x) = \dfrac{f(x) - f(a)}{x - a} - f'(a)$, we obtain the desired form. Since $f(x)$ is differentiable at $x = a$, we know that

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

so

$$\begin{aligned} \lim_{x\to a} e(x) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} - f'(a) \\ &= 0. \end{aligned}$$

83. $f(x) = x^2 - 2x$ on $[0, 2]$
$f(2) = 0 = f(0)$
If $f'(c) = \dfrac{f(2) - f(0)}{2 - 0} = \dfrac{0 - 0}{2} = 0$
then $2c - 2 = f'(c) = 0$ so $c = 1$.

84. $f(x)$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$, so the Mean Value Theorem applies. We need to find c so that
$f'(c) = \dfrac{f(2) - f(0)}{2 - 0} = \dfrac{6 - 0}{2 - 0} = 3.$
$f'(x) = 3x^2 - 1 = 3$ when $x = \sqrt{4/3}$, so $c = 2\sqrt{3}/3$.

85. $f(x) = 3x^2 - \cos x$
One trial: $g_o(x) = kx^3 - \sin x$

$g_o'(x) = 3kx^2 - \cos x$
Need $3k = 3$, $k = 1$, and the general solution is
$g(x) = g_o(x) + c = x^3 - \sin x + c$
for c an arbitrary constant.

86. If $g'(x) = x^3 - e^{2x}$, then $g(x)$ must be
$\frac{1}{4}x^4 - \frac{1}{2}e^{2x} + c$,
for any constant c.

87. $x = 1$ is to be double root of
$f(x) = (x^3 + 1) - [m(x-1) + 2]$
$= (x^3 + 1 - 2) - m(x-1)$
$= (x^3 - 1) - m(x-1)$
$= (x-1)\left[x^2 + x + 1 - m\right]$
Let $g(x) = x^2 + x + 1 - m$. Then $x = 1$ is a *double* root of f only if $(x-1)$ is a *factor* of g, in which case $g(1) = 0$. Therefore we require $0 = g(1) = 3 - m$ or $m = 3$. Now
$g(x) = x^2 + x - 2 = (x-1)(x+2)$,
$f(x) = (x-1)g(x) = (x-1)^2(x+2)$
and $x = 1$ is a double root.
The line tangent to the curve $y = x^3 + 1$ at the point $(1, 2)$ has slope $y' = 3x^2 = 3(1) = 3(= m)$. The equation of the tangent line is $y-2 = 3(x-1)$ or $y = 3x-1(= m(x-1)+2)$.

88. We are asked to find m so that
$x^3 + 2x - [m(x-2) + 12]$
$= x^3 + (2-m)x + (2m - 12)$
has a double root. A cubic with a double root factors as
$(x-a)^2(x-b)$
$= x^3 - (2a+b)x^2 + (2ab + a^2)x - a^2b$.
Equating like coefficients gives a system of equations
$2a + b = 0$,
$2ab + a^2 = 2 - m$, and
$-a^2b = 2m - 12$.
The first equation gives $b = -2a$. Substituting this into the second equation gives $m = 2 + 3a^2$. Substituting these results into the third equation gives a cubic polynomial in a with zeros $a = -1$ and $a = 2$. This gives two solutions: $m = 5$ and $m = 14$.
$f'(x) = 3x^2 + 2$, so $f'(2) = 14$. The tangent line at $(2, 12)$ is $y = 14(x-2) + 12$.
The second solution corresponds to the tangent line to $f(x)$ at $x = -1$, which happens to pass through the point $(2, 12)$.

89. Given, $f = \frac{1}{2L}\sqrt{\frac{T}{P}} \Rightarrow \frac{df}{dT} = \frac{1}{4L\sqrt{pT}}$ as T is an independent variable and p , L are constants. Tightening the string means increasing the tension, resulting in decrease in $\frac{df}{dT}$, which means there is a decrease in the rate of change of frequency with respect to the tension in the string. On the other end, loosening the string means decreasing the tension, resulting in increase in $\frac{df}{dT}$, which means there is a increase in the rate of change of frequency with respect to the tension in the string. Also, $f = \frac{1}{2L}\sqrt{\frac{T}{P}} \Rightarrow \frac{df}{dL} = -\frac{1}{2L^2}\sqrt{\frac{T}{p}}$.When the guitarist plays the notes by pressing the string against a fret; he is increasing the length and hence decreasing the rate of change of frequency of vibration with respect to the length of the string.