Chapter 2

Applications of Differentiation

Exercise Set 2.1

1. $f(x) = x^2 + 6x - 3$

First, find the critical points.

$f'(x) = 2x + 6$

$f'(x)$ exists for all real numbers. We solve

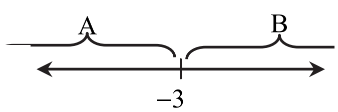
$f'(x) = 0$

$2x + 6 = 0$

$2x = -6$

$x = -3$

The only critical value is -3. We use -3 to divide the real number line into two intervals, A: $(-\infty, -3)$ and B:$(-3, \infty)$.



We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test -4, $f'(-4) = 2(-4) + 6 = -2 < 0$

B: Test 0, $f'(0) = 2(0) + 6 = 6 > 0$

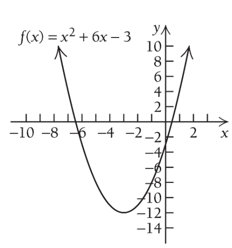
We see that $f(x)$ is decreasing on $(-\infty, -3)$ and increasing on $(-3, \infty)$, and the change from decreasing to increasing indicates that a relative minimum occurs at $x = -3$. We substitute into the original equation to find $f(-3)$:

$f(-3) = (-3)^2 + 6(-3) - 3 = -12$

Thus, there is a relative minimum at $(-3, -12)$.

We use the information obtained to sketch the graph. Other function values are listed below.

x	$f(x)$
-6	-3
-5	-8
-4	-11
-3	-12
-2	-11
-1	-8
0	-3



2. $f(x) = x^2 + 4x + 5$

$f'(x) = 2x + 4$

$f'(x)$ exists for all real numbers. Solve

$f'(x) = 0$

$2x + 4 = 0$

$2x = -4$

$x = -2$

The only critical value is -2. We use -2 to divide the real number line into two intervals, A: $(-\infty, -2)$ and B: $(-2, \infty)$.

A: Test -3, $f'(-3) = 2(-3) + 4 = -2 < 0$

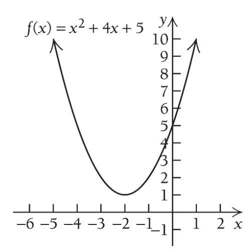
B: Test 0, $f'(0) = 2(0) + 4 = 4 > 0$

We see that $f(x)$ is decreasing on $(-\infty, -2)$ and increasing on $(-2, \infty)$, there is a relative minimum at $x = -2$.

$f(-2) = (-2)^2 + 4(-2) + 5 = 1$

Thus, there is a relative minimum at $(-2, 1)$. We sketch the graph.

x	$f(x)$
-5	10
-4	5
-3	2
-2	1
-1	2
0	5
1	10



3. $f(x) = 2 - 3x - 2x^2$

First, find the critical points.

$f'(x) = -3 - 4x$

$f'(x)$ exists for all real numbers. We solve

$f'(x) = 0$

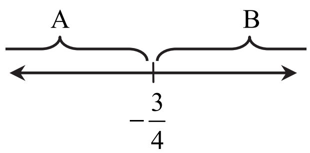
$-3 - 4x = 0$

$x = -\frac{3}{4}$

The solution is continued on the next page.

The only critical value is $-\frac{3}{4}$. We use $-\frac{3}{4}$ to divide the real number line into two intervals,

A: $\left(-\infty,-\frac{3}{4}\right)$ and B: $\left(-\frac{3}{4},\infty\right)$.



We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test -1, $f'(-1)=-3-4(-1)=1>0$

B: Test 0, $f'(0)=-3-4(0)=-3<0$

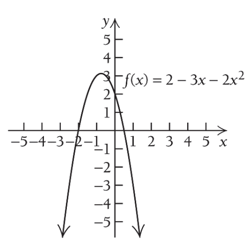
We see that $f(x)$ is increasing on $\left(-\infty,-\frac{3}{4}\right)$ and decreasing on $\left(-\frac{3}{4},\infty\right)$, and the change from increasing to decreasing indicates that a relative maximum occurs at $x=-\frac{3}{4}$. We substitute into the original equation to find $f\left(-\frac{3}{4}\right)$:

$$f\left(-\frac{3}{4}\right)=2-3\left(-\frac{3}{4}\right)-2\left(-\frac{3}{4}\right)^2=\frac{25}{8}$$

Thus, there is a relative maximum at $\left(-\frac{3}{4},\frac{25}{8}\right)$.

We use the information obtained to sketch the graph. Other function values are listed below.

x	$f(x)$
-3	-7
-2	0
-1	3
$-\frac{3}{4}$	$\frac{25}{8}$
0	2
1	-3
2	-12



4. $f(x)=5-x-x^2$

$f'(x)=-1-2x$

$f'(x)$ exists for all real numbers. Solve

$$f'(x)=0$$
$$-1-2x=0$$
$$-2x=1$$
$$x=-\frac{1}{2}$$

The only critical value is $-\frac{1}{2}$. We use $-\frac{1}{2}$ to divide the real number line into two intervals,

A: $\left(-\infty,-\frac{1}{2}\right)$ and B: $\left(-\frac{1}{2},\infty\right)$:

A: Test -1, $f'(-1)=-1-2(-1)=1>0$

B: Test 0, $f'(0)=-1-2(0)=-1<0$

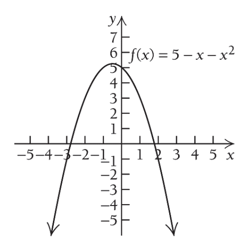
We see that $f(x)$ is increasing on $\left(-\infty,-\frac{1}{2}\right)$ and decreasing on $\left(-\frac{1}{2},\infty\right)$, which indicates there is a relative maximum at $x=-\frac{1}{2}$.

$$f\left(-\frac{1}{2}\right)=5-\left(-\frac{1}{2}\right)-\left(-\frac{1}{2}\right)^2=\frac{21}{4}$$

Thus, there is a relative maximum at $\left(-\frac{1}{2},\frac{21}{4}\right)$.

We sketch the graph.

x	$f(x)$
-3	-1
-2	3
-1	5
$-\frac{1}{2}$	$\frac{21}{4}$
0	5
1	3
2	-1



5. $F(x)=0.5x^2+2x-11$

First, find the critical points.

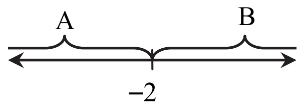
$F'(x)=x+2$

$F'(x)$ exists for all real numbers. We solve:

$$F'(x)=0$$
$$x+2=0$$
$$x=-2$$

The solution is continued on the next page

The only critical value is -2. We use -2 to divide the real number line into two intervals, A: $(-\infty,-2)$ and B:$(-2,\infty)$.



We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test $-3, F'(-3)=(-3)+2=-1<0$

B: Test 0, $\quad F'(0)=(0)+2=2>0$

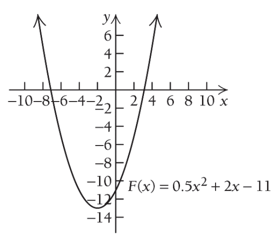
We see that $F(x)$ is decreasing on $(-\infty,-2)$ and increasing on $(-2,\infty)$, and the change from decreasing to increasing indicates that a relative minimum occurs at $x=-2$. We substitute into the original equation to find $F(-2)$:

$F(-2)=0.5(-2)^2+2(-2)-11=-13$

Thus, there is a relative minimum at $(-2,-13)$.

We use the information obtained to sketch the graph. Other function values are listed below.

x	$F(x)$
-5	$-\frac{17}{2}$
-4	-11
-3	$-\frac{25}{2}$
-2	-13
-1	$-\frac{25}{2}$
0	-11
1	$-\frac{17}{2}$



6. $g(x)=1+6x+3x^2$

$g'(x)=6+6x$

$g'(x)$ exists for all real numbers. Solve

$g'(x)=0$

$6+6x=0$

$x=-1$

The only critical value is -1. We use -1 to divide the real number line into two intervals, A: $(-\infty,-1)$ and B: $(-1,\infty)$:

A: Test $-2,\ g'(-2)=6+6(-2)=-6<0$

B: Test 0, $\quad g'(0)=6+6(0)=6>0$

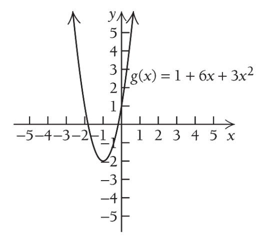
We see that $g'(x)$ is decreasing on $(-\infty,-1)$ and increasing on $(-1,\infty)$, which indicates there is a relative minimum at $x=-1$.

$g(-1)=1+6(-1)+3(-1)^2=-2$

Thus, there is a relative minimum at $(-1,-2)$.

We sketch the graph.

x	$g(x)$
-4	25
-3	10
-2	1
-1	-2
0	1
1	10
2	25



7. $g(x)=x^3+\frac{1}{2}x^2-2x+5$

First, find the critical points.

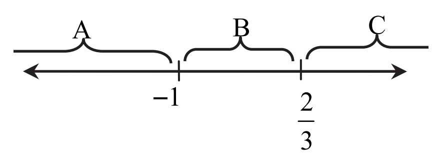
$g'(x)=3x^2+x-2$

$g'(x)$ exists for all real numbers. We solve

$$g'(x)=0$$
$$3x^2+x-2=0$$
$$(3x-2)(x+1)=0$$
$$3x-2=0 \quad \text{or} \quad x+1=0$$
$$x=\frac{2}{3} \quad \text{or} \quad x=-1$$

The critical values are -1 and $\frac{2}{3}$. We use them to divide the real number line into three intervals,

A: $(-\infty,-1)$, B: $\left(-1,\frac{2}{3}\right)$, and C:$\left(\frac{2}{3},\infty\right)$.



We use a test value in each interval to determine the sign of the derivative in each interval.

The solution is continued on the next page.

A: Test -2,

$g'(-2) = 3(-2)^2 + (-2) - 2 = 8 > 0$

B: Test 0,

$g'(0) = 3(0)^2 + (0) - 2 = -2 < 0$

C: Test 1,

$g'(1) = 3(1)^2 + (1) - 2 = 2 > 0$

We see that $g(x)$ is increasing on $(-\infty, -1)$, decreasing on $\left(-1, \frac{2}{3}\right)$, and increasing on $\left(\frac{2}{3}, \infty\right)$. So there is a relative maximum at $x = -1$ and a relative minimum at $x = \frac{2}{3}$.

We find $g(-1)$:

$$g(-1) = (-1)^3 + \frac{1}{2}(-1)^2 - 2(-1) + 5$$
$$= -1 + \frac{1}{2} + 2 + 5 = \frac{13}{2}$$

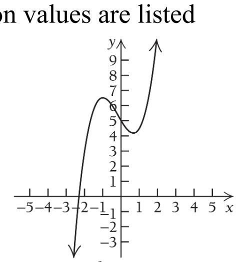
Then we find $g\left(\frac{2}{3}\right)$:

$$g\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^3 + \frac{1}{2}\left(\frac{2}{3}\right)^2 - 2\left(\frac{2}{3}\right) + 5$$
$$= \frac{8}{27} + \frac{2}{9} - \frac{4}{3} + 5 = \frac{113}{27}$$

There is a relative maximum at $\left(-1, \frac{13}{2}\right)$, and there is a relative minimum at $\left(\frac{2}{3}, \frac{113}{27}\right)$.

We use the information obtained to sketch the graph. Other function values are listed

x	$g(x)$
-2	3
0	5
1	$\frac{9}{2}$
2	11



$g(x) = x^3 + \frac{1}{2}x^2 - 2x + 5$

8. $G(x) = x^3 - x^2 - x + 2$

$G'(x) = 3x^2 - 2x - 1$

$G'(x)$ exists for all real numbers. We solve

$$G'(x) = 0$$
$$3x^2 - 2x - 1 = 0$$
$$(3x + 1)(x - 1) = 0$$
$$3x + 1 = 0 \quad \text{or} \quad x - 1 = 0$$
$$3x = -1 \quad \text{or} \quad x = 1$$
$$x = -\frac{1}{3} \quad \text{or} \quad x = 1$$

The critical values are $-\frac{1}{3}$ and 1. We use them to divide the real number line into three intervals,

A: $\left(-\infty, -\frac{1}{3}\right)$, B: $\left(-\frac{1}{3}, 1\right)$, and C: $(1, \infty)$.

A: Test -1,

$G'(-1) = 3(-1)^2 - 2(-1) - 1 = 4 > 0$

B: Test 0,

$G'(0) = 3(0)^2 - 2(0) - 1 = -1 < 0$

C: Test 2,

$G'(2) = 3(2)^2 - 2(2) - 1 = 7 > 0$

We see that $G(x)$ is increasing on $\left(-\infty, -\frac{1}{3}\right)$, decreasing on $\left(-\frac{1}{3}, 1\right)$, and increasing on $(1, \infty)$. So there is a relative maximum at $x = -\frac{1}{3}$ and a relative minimum at $x = 1$.

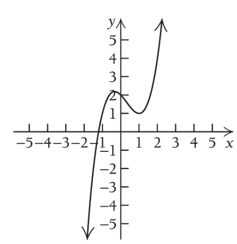
$$G\left(-\frac{1}{3}\right) = \left(-\frac{1}{3}\right)^3 - \left(-\frac{1}{3}\right)^2 - \left(-\frac{1}{3}\right) + 2 = \frac{59}{27}$$

$$G(1) = (1)^3 - (1)^2 - (1) + 2 = 1$$

There is a relative maximum at $\left(-\frac{1}{3}, \frac{59}{27}\right)$, and there is a relative minimum at $(1, 1)$.

We use the information obtained to sketch the graph. Other function values are listed below.

x	$G(x)$
-2	-8
-1	1
0	2
2	4
3	17



$G(x) = x^3 - x^2 - x + 2$

9. $f(x)=x^3-3x^2$

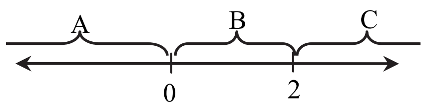
First, find the critical points.

$f'(x)=3x^2-6x$

$f'(x)$ exists for all real numbers. We solve

$$f'(x)=0$$
$$3x^2-6x=0$$
$$3x(x-2)=0$$
$$x=0 \quad \text{or} \quad x=2$$

The critical values are 0 and 2. We use them to divide the real number line into three intervals, A: $(-\infty,0)$, B: $(0,2)$, and C: $(2,\infty)$.



We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test -1, $f'(-1)=3(-1)^2-6(-1)=9>0$

B: Test 1, $f'(1)=3(1)^2-6(1)=-3<0$

C: Test 3, $f'(3)=3(3)^2-6(3)=9>0$

We see that $f(x)$ is increasing on $(-\infty,0)$, decreasing on $(0,2)$, and increasing on $(2,\infty)$. So there is a relative maximum at $x=0$ and a relative minimum at $x=2$.

We find $f(0)$:

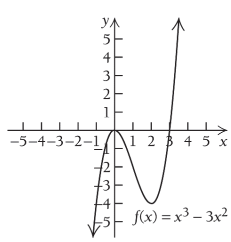
$$f(0)=(0)^3-3(0)^2=0.$$

Then we find $f(2)$:

$$f(2)=(2)^3-3(2)^2=-4.$$

There is a relative maximum at $(0,0)$, and there is a relative minimum at $(2,-4)$. We use the information obtained to sketch the graph. Other function values are listed below.

x	$f(x)$
-2	-20
-1	-4
1	-2
3	0
4	16



10. $f(x)=x^3-3x+6$

$f'(x)=3x^2-3$

$f'(x)$ exists for all real numbers. We solve

$$f'(x)=0$$
$$3x^2-3=0$$
$$3x^2=3$$
$$x^2=1$$
$$x=\pm 1$$

The critical values are -1 and 1. We use them to divide the real number line into three intervals, A: $(-\infty,-1)$, B: $(-1,1)$, and C: $(1,\infty)$.

A: Test -3, $f'(-3)=3(-3)^2-3=24>0$

B: Test 0, $f'(0)=3(0)^2-3=-3<0$

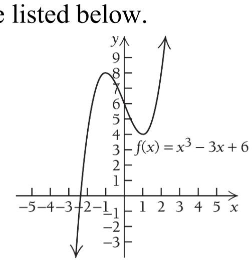
C: Test 2, $f'(2)=3(2)^2-3=9>0$

We see that $f(x)$ is increasing on $(-\infty,-1)$, decreasing on $(-1,1)$, and increasing on $(1,\infty)$. So there is a relative maximum at $x=-1$ and a relative minimum at $x=1$.

$$f(-1)=(-1)^3-3(-1)+6=-1+3+6=8$$
$$f(1)=(1)^3-3(1)+6=1-3+6=4$$

There is a relative maximum at $(-1,8)$, and there is a relative minimum at $(1,4)$. We use the information obtained to sketch the graph. Other function values are listed below.

x	$f(x)$
-3	-12
-2	4
0	6
2	8
3	24



11. $f(x) = x^3 + 3x$

First, find the critical points.

$f'(x) = 3x^2 + 3$

$f'(x)$ exists for all real numbers. We solve

$f'(x) = 0$

$3x^2 + 3 = 0$

$x^2 = -1$

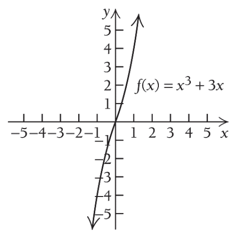
There are no real solutions to this equation. Therefore, the function does not have any critical values.

We test a point

$f'(0) = 3(0)^2 + 3 = 3 > 0$

We see that $f(x)$ is increasing on $(-\infty, \infty)$, and that there are no relative extrema. We use the information obtained to sketch the graph. Other function values are listed below.

x	$f(x)$
−2	−14
−1	−4
0	0
1	4
2	14



12. $f(x) = 3x^2 + 2x^3$

$f'(x) = 6x + 6x^2$

$f'(x)$ exists for all real numbers. We solve

$f'(x) = 0$

$6x + 6x^2 = 0$

$6x(1 + x) = 0$

$6x = 0 \quad \text{or} \quad x + 1 = 0$

$x = 0 \quad \text{or} \quad x = -1$

We know the critical values are -1 and 0. We use them to divide the real number line into three intervals,

A: $(-\infty, -1)$, B: $(-1, 0)$, and C: $(0, \infty)$.

A: Test -2,

$f'(-2) = 6(-2) + 6(-2)^2 = 12 > 0$

B: Test $-\frac{1}{2}$,

$f'\left(-\frac{1}{2}\right) = 6\left(-\frac{1}{2}\right) + 6\left(-\frac{1}{2}\right)^2 = -\frac{3}{2} < 0$

C: Test 1,

$f'(1) = 6(1) + 6(1)^2 = 12 > 0$

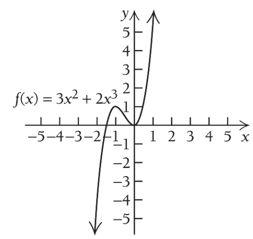
We see that $f(x)$ is increasing on $(-\infty, -1)$, decreasing on $(-1, 0)$, and increasing on $(0, \infty)$. So there is a relative maximum at $x = -1$ and a relative minimum at $x = 0$.

$f(-1) = 3(-1)^2 + 2(-1)^3 = 1$

$f(0) = 3(0)^2 + 2(0)^3 = 0$

There is a relative maximum at $(-1, 1)$, and there is a relative minimum at $(0, 0)$. We use the information obtained to sketch the graph. Other function values are listed below.

x	$f(x)$
−3	−27
−2	−4
$\frac{1}{2}$	1
2	28



13. $F(x) = 1 - x^3$

First, find the critical points.

$F'(x) = -3x^2$

$F'(x)$ exists for all real numbers. We solve

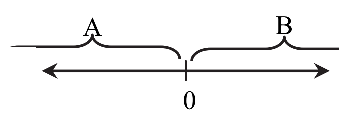
$F'(x) = 0$

$-3x^2 = 0$

$x = 0$

The only critical value is 0. We use 0 to divide the real number line into two intervals,

A: $(-\infty, 0)$, and B: $(0, \infty)$.



We use a test value in each interval to determine the sign of the derivative in each interval.

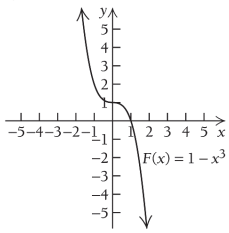
A: Test -1, $F'(-1) = -3(-1)^2 = -3 < 0$

B: Test 1, $F'(1) = -3(1)^2 = -3 < 0$

We see that $F(x)$ is decreasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$, so the function has no relative extema. We use the information obtained to sketch the graph on the next page.

Using the information from the previous page and deteriming other function values are listed below, we sketch the graph.

x	$F(x)$
-2	9
-1	2
0	1
1	0
2	-7



14. $g(x) = 2x^3 - 16$

First, find the critical points.

$g'(x) = 6x^2$

$g'(x)$ exists for all real numbers. We solve

$g'(x) = 0$

$6x^2 = 0$

$x = 0$

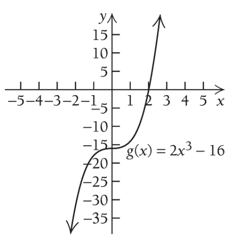
The only critical value is 0. We use 0 to divide the real number line into two intervals, A: $(-\infty, 0)$, and B: $(0, \infty)$.

A: Test -1, $g'(-1) = 6(-1)^2 = 6 > 0$

B: Test 1, $g'(1) = 6(1)^2 = 6 > 0$

We see that $g(x)$ is increasing on $(-\infty, 0)$ and increasing on $(0, \infty)$, so the function has no relative extema. We use the information obtained to sketch the graph. Other function values are listed below.

x	$g(x)$
-2	-32
-1	-18
0	-16
1	-14
2	0
3	38



15. $G(x) = x^3 - 6x^2 + 10$

First, find the critical points.

$G'(x) = 3x^2 - 12x$

$G'(x)$ exists for all real numbers. We solve

$G'(x) = 0$

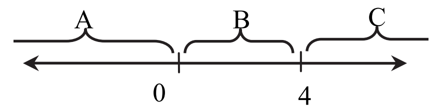
$x^2 - 4x = 0$ Dividing by 3

$x(x-4) = 0$

$x = 0$ or $x - 4 = 0$

$x = 0$ or $x = 4$

The critical values are 0 and 4. We use them to divide the real number line into three intervals, A: $(-\infty, 0)$, B: $(0, 4)$, and C: $(4, \infty)$.



We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test -1, $G'(-1) = 3(-1)^2 - 12(-1) = 15 > 0$

B: Test 1, $G'(1) = 3(1)^2 - 12(1) = -9 < 0$

C: Test 5, $G'(5) = 3(5)^2 - 12(5) = 15 > 0$

We see that $G(x)$ is increasing on $(-\infty, 0)$, decreasing on $(0, 4)$, and increasing on $(4, \infty)$. So there is a relative maximum at $x = 0$ and a relative minimum at $x = 4$.

We find $G(0)$:

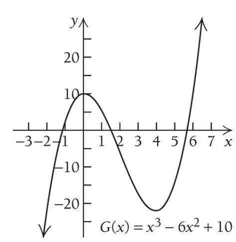
$$G(0) = (0)^3 - 6(0)^2 + 10 = 10$$

Then we find $G(4)$:

$$G(4) = (4)^3 - 6(4)^2 + 10 = 64 - 96 + 10 = -22$$

There is a relative maximum at $(0, 10)$, and there is a relative minimum at $(4, -22)$. We use the information obtained to sketch the graph. Other function values are listed below.

x	$G(x)$
-2	-22
-1	3
1	5
2	-6
3	-17



16. $f(x) = 12 + 9x - 3x^2 - x^3$

$f'(x) = 9 - 6x - 3x^2$

$f'(x)$ exists for all real numbers. Solve

$$f'(x) = 0$$
$$9 - 6x - 3x^2 = 0$$
$$x^2 + 2x - 3 = 0 \quad \text{Dividing by } -3$$
$$(x+3)(x-1) = 0$$
$$x + 3 = 0 \quad \text{or} \quad x - 1 = 0$$
$$x = -3 \quad \text{or} \quad x = 1$$

The critical values are -3 and 1. We use them to divide the real number line into three intervals, A: $(-\infty, -3)$, B: $(-3, 1)$, and C: $(1, \infty)$.

We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test -4,

$$f'(-4) = 9 - 6(-4) - 3(-4)^2 = -15 < 0$$

B: Test 0,

$$f'(0) = 9 - 6(0) - 3(0)^2 = 9 > 0$$

C: Test 2,

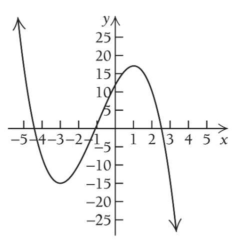
$$f'(2) = 9 - 6(2) - 3(2)^2 = -15 < 0$$

We see that $f(x)$ is decreasing on $(-\infty, -3)$, increasing on $(-3, 1)$, and decreasing on $(1, \infty)$. So there is a relative minimum at $x = -3$ and a relative maximum at $x = 1$.

$$f(-3) = 12 + 9(-3) - 3(-3)^2 - (-3)^3 = -15$$
$$f(1) = 12 + 9(1) - 3(1)^2 - (1)^3 = 17$$

There is a relative minimum at $(-3, -15)$, and there is a relative maximum at $(1, 17)$. We use the information obtained to sketch the graph. Other function values are listed below.

x	$f(x)$
-5	17
-4	-8
-2	-10
-1	1
0	12
2	10
3	-15



$f(x) = 12 + 9x - 3x^2 - x^3$

17. $g(x) = x^3 - x^4$

First, find the critical points.

$g'(x) = 3x^2 - 4x^3$

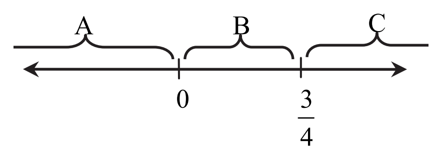
$g'(x)$ exists for all real numbers. We solve

$$g'(x) = 0$$
$$3x^2 - 4x^3 = 0$$
$$x^2(3 - 4x) = 0$$
$$x^2 = 0 \quad \text{or} \quad 3 - 4x = 0$$
$$x = 0 \quad \text{or} \quad -4x = -3$$
$$x = 0 \quad \text{or} \quad x = \frac{3}{4}$$

The critical values are 0 and $\frac{3}{4}$.

We use the critical values to divide the real number line into three intervals,

A: $(-\infty, 0)$, B: $\left(0, \frac{3}{4}\right)$, and C: $\left(\frac{3}{4}, \infty\right)$.



We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test -1, $g'(-1) = 3(-1)^2 - 4(-1)^3 = 7 > 0$

B: Test $\frac{1}{2}$, $g'\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right)^3$

$$= 3\left(\frac{1}{4}\right) - 4\left(\frac{1}{8}\right) = \frac{1}{4} > 0$$

C: Test 1, $g'(1) = 3(1)^2 - 4(1)^3 = -1 < 0$

We see that $g(x)$ is increasing on $(-\infty, 0)$ and $\left(0, \frac{3}{4}\right)$, and is decreasing on $\left(\frac{3}{4}, \infty\right)$. So there is no relative extrema at $x = 0$ but there is a relative maximum at $x = \frac{3}{4}$.

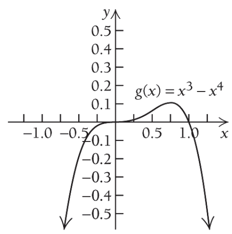
We find $g\left(\frac{3}{4}\right)$:

$$g\left(\frac{3}{4}\right) = \left(\frac{3}{4}\right)^3 - \left(\frac{3}{4}\right)^4 = \frac{27}{64} - \frac{81}{256} = \frac{27}{256}$$

The solution is continued on the next page.

From the previous page, we determine there is a relative maximum at $\left(\frac{3}{4},\frac{27}{256}\right)$. We use the information obtained to sketch the graph. Other function values are listed below.

x	$g(x)$
-2	-24
-1	-2
0	0
$\frac{1}{2}$	$\frac{1}{16}$
1	0
2	-8



18. $f(x)=x^4-2x^3$

$f'(x)=4x^3-6x^2$

$f'(x)$ exists for all real numbers. Solve

$$f'(x)=0$$
$$4x^3-6x^2=0$$
$$2x^2(2x-3)=0$$
$$x^2=0 \quad \text{or} \quad 2x-3=0$$
$$x=0 \quad \text{or} \quad x=\frac{3}{2}$$

The critical values are 0 and $\frac{3}{2}$. We use them to divide the real number line into three intervals,

A: $(-\infty,0)$, B: $\left(0,\frac{3}{2}\right)$, and C: $\left(\frac{3}{2},\infty\right)$.

A: Test -1, $f'(-1)=4(-1)^3-6(-1)^2=-10<0$

B: Test 1, $f'(1)=4(1)^3-6(1)^2=-2<0$

C: Test 2, $f'(2)=4(2)^3-6(2)^2=8>0$

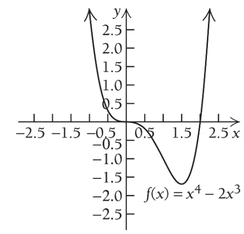
Since $f(x)$ is decreasing on both $(-\infty,0)$ and $\left(0,\frac{3}{2}\right)$, and increasing on $\left(\frac{3}{2},\infty\right)$, there is no relative extrema at $x=0$ but there is a relative minimum at $x=\frac{3}{2}$.

$$f\left(\frac{3}{2}\right)=\left(\frac{3}{2}\right)^4-2\left(\frac{3}{2}\right)^3=-\frac{27}{16}$$

There is a relative minimum at $\left(\frac{3}{2},-\frac{27}{16}\right)$.

We use the information obtained to sketch the graph. Other function values are listed below.

x	$f(x)$
-2	32
-1	3
0	0
1	-1
2	0
3	27



19. $f(x)=\frac{1}{3}x^3-2x^2+4x-1$

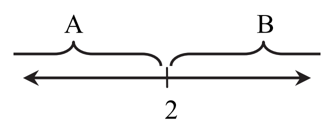
First, find the critical points.

$f'(x)=x^2-4x+4$

$f'(x)$ exists for all real numbers. We solve

$$f'(x)=0$$
$$x^2-4x+4=0$$
$$(x-2)^2=0$$
$$x=2$$

The only critical value is 2.
We divide the real number line into two intervals,
A: $(-\infty,2)$ and B: $(2,\infty)$.



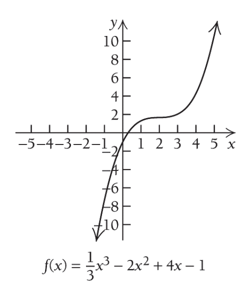
We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test 0, $f'(0)=(0)^2-4(0)+4=4>0$

B: Test 3, $f'(3)=(3)^2-4(3)+4=1>0$

We see that $f(x)$ is increasing on both $(-\infty,2)$ and $(2,\infty)$. Therefore, there are no relative extrema. We use the information obtained to sketch the graph. Other function values are listed below.

x	$f(x)$
-3	-40
-2	$-\frac{59}{3}$
-1	$-\frac{22}{3}$
0	-1
1	$\frac{4}{3}$
2	$\frac{5}{3}$
3	2



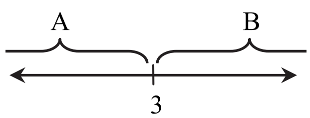
20. $F(x) = -\frac{1}{3}x^3 + 3x^2 - 9x + 2$

$F'(x) = -x^2 + 6x - 9$

$F'(x)$ exists for all real numbers. Solve

$$F'(x) = 0$$
$$-x^2 + 6x - 9 = 0$$
$$x^2 - 6x + 9 = 0$$
$$(x-3)^2 = 0$$
$$x - 3 = 0$$
$$x = 3$$

The only critical value is 3. We divide the real number line into two intervals,
A: $(-\infty, 3)$ and B: $(3, \infty)$.



We use a test value in each interval to determine the sign of the derivative in each interval.

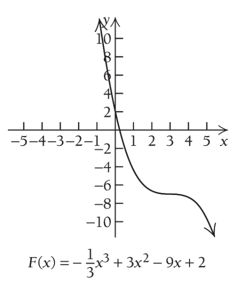
A: Test 0, $F'(0) = -(0)^2 + 6(0) - 9 = -9 < 0$

B: Test 4, $F'(4) = -(4)^2 + 6(4) - 9 = -1 < 0$

We see that $F(x)$ is decreasing on both $(-\infty, 3)$ and $(3, \infty)$. Therefore, there are no relative extrmea.

We use the information obtained to sketch the graph. Other function values are listed below.

x	$F(x)$
-3	65
-2	$\frac{104}{3}$
-1	$\frac{43}{3}$
0	2
1	$-\frac{13}{3}$
2	$-\frac{20}{3}$
3	-7



21. $f(x) = 3x^4 - 15x^2 + 12$

First, find the critical points.

$f'(x) = 12x^3 - 30x$

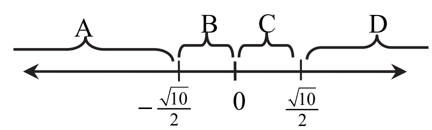
$f'(x)$ exists for all real numbers. We solve

$$f'(x) = 0$$
$$12x^3 - 30x = 0$$
$$6x(2x^2 - 5) = 0$$
$$6x = 0 \quad \text{or} \quad 2x^2 - 5 = 0$$
$$x = 0 \quad \text{or} \quad x^2 = \frac{5}{2}$$
$$x = 0 \quad \text{or} \quad x = \pm\frac{\sqrt{10}}{2}$$

The critical values are 0, $\frac{\sqrt{10}}{2}$ and $-\frac{\sqrt{10}}{2}$. We use them to divide the real number line into four intervals,

A: $\left(-\infty, -\frac{\sqrt{10}}{2}\right)$, B: $\left(-\frac{\sqrt{10}}{2}, 0\right)$,

C: $\left(0, \frac{\sqrt{10}}{2}\right)$, and D: $\left(\frac{\sqrt{10}}{2}, \infty\right)$.



We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test -2,

$f'(-2) = 12(-2)^3 - 30(-2) = -36 < 0$

B: Test -1,

$f'(-1) = 12(-1)^3 - 30(-1) = 18 > 0$

C: Test 1,

$f'(1) = 12(1)^3 - 30(1) = -18 < 0$

D: Test 2,

$f'(2) = 12(2)^3 - 30(2) = 36 > 0$

The solution is continued on the next page.

From the previous page, we see that $f(x)$ is decreasing on $\left(-\infty,-\frac{\sqrt{10}}{2}\right)$, increasing on $\left(-\frac{\sqrt{10}}{2},0\right)$, decreasing again on $\left(0,\frac{\sqrt{10}}{2}\right)$, and increasing again on $\left(\frac{\sqrt{10}}{2},\infty\right)$. Thus, there is a relative minimum at $x=-\frac{\sqrt{10}}{2}$, a relative maximum at $x=0$, and another relative minimum at $x=\frac{\sqrt{10}}{2}$.

We find $f\left(-\frac{\sqrt{10}}{2}\right)$:

$$f\left(-\frac{\sqrt{10}}{2}\right)=3\left(-\frac{\sqrt{10}}{2}\right)^4-15\left(-\frac{\sqrt{10}}{2}\right)^2+12$$
$$=-\frac{27}{4}$$

Then we find $f(0)$:

$$f(0)=3(0)^4-15(0)^2+12=12$$

Then we find $f\left(\frac{\sqrt{10}}{2}\right)$:

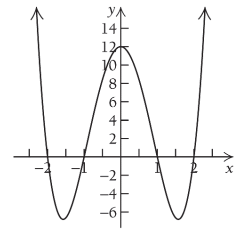
$$f\left(\frac{\sqrt{10}}{2}\right)=3\left(\frac{\sqrt{10}}{2}\right)^4-15\left(\frac{\sqrt{10}}{2}\right)^2+12$$
$$=-\frac{27}{4}$$

There are relative minima at $\left(-\frac{\sqrt{10}}{2},-\frac{27}{4}\right)$ and $\left(\frac{\sqrt{10}}{2},-\frac{27}{4}\right)$.

There is a relative maximum at $(0,12)$.

We use the information obtained above to sketch the graph. Other function values are listed at the top of the next column.

x	$f(x)$
−3	120
−2	0
−1	0
1	0
2	0
3	120



$f(x)=3x^4-15x^2+12$

22. $g(x)=2x^4-20x^2+18$

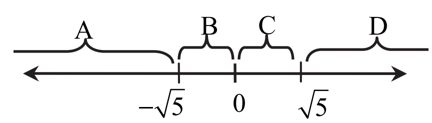
$g'(x)=8x^3-40x$

$g'(x)$ exists for all real numbers. We solve

$$g'(x)=0$$
$$8x^3-40x=0$$
$$8x\left(x^2-5\right)=0$$
$$8x=0 \quad \text{or} \quad x^2-5=0$$
$$x=0 \quad \text{or} \quad x^2=5$$
$$x=0 \quad \text{or} \quad x=\pm\sqrt{5}$$

The critical values are 0, $\sqrt{5}$ and $-\sqrt{5}$. We use them to divide the real number line into four intervals,

A: $\left(-\infty,-\sqrt{5}\right)$, B: $\left(-\sqrt{5},0\right)$,
C: $\left(0,\sqrt{5}\right)$, and D: $\left(\sqrt{5},\infty\right)$.



We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test −3,
$$g'(-3)=8(-3)^3-40(-3)=-96<0$$
B: Test −1,
$$g'(-1)=8(-1)^3-40(-1)=32>0$$
C: Test 1,
$$g'(1)=8(1)^3-40(1)=-32<0$$
D: Test 3,
$$g'(3)=8(3)^3-40(3)=96>0$$

The solution is continued on the next page.

From the previoius page, we see that $g(x)$ is decreasing on $(-\infty,-\sqrt{5})$, increasing on $(-\sqrt{5},0)$, decreasing again on $(0,\sqrt{5})$, and increasing again on $(\sqrt{5},\infty)$. Thus, there is a relative minimum at $x=-\sqrt{5}$, a relative maximum at $x=0$, and another relative minimum at $x=\sqrt{5}$.

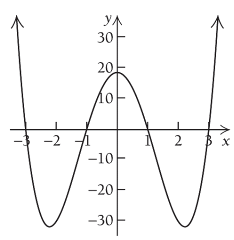
$g(-\sqrt{5})=2(-\sqrt{5})^4-20(-\sqrt{5})^2+18=-32$

$g(0)=2(0)^4-20(0)^2+18=18$

$g(\sqrt{5})=2(\sqrt{5})^4-20(\sqrt{5})^2+18=-32$

There are relative minima at $(-\sqrt{5},-32)$ and $(\sqrt{5},-32)$. There is a relative maximum at $(0,18)$ We use the information obtained to sketch the graph. Other function values are listed below.

x	$g(x)$
-4	210
-3	0
-1	0
1	0
3	0
4	210



$g(x)=2x^4-20x^2+18$

23. $G(x)=\sqrt[3]{x+2}=(x+2)^{1/3}$

First, find the critical points.

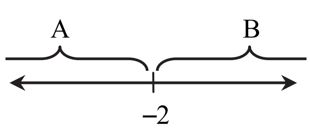
$G'(x)=\frac{1}{3}(x+2)^{-2/3}(1)$

$=\dfrac{1}{3(x+2)^{2/3}}$

$G'(x)$ does not exist when $x=-2$. The equation $G'(x)=0$ has no solution, therefore, the only critical value is $x=-2$.

We use -2 to divide the real number line into two intervals,

A: $(-\infty,-2)$ and B: $(-2,\infty)$:



We use a test value in each interval to determine the sign of the derivative in each interval.

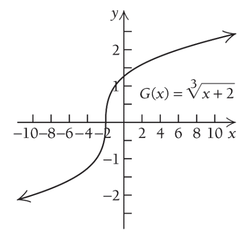
A: Test -3, $G'(-3)=\dfrac{1}{3(-3+2)^{2/3}}=\dfrac{1}{3}>0$

B: Test -1, $G'(-1)=\dfrac{1}{3(-1+2)^{2/3}}=\dfrac{1}{3}>0$

We see that $G(x)$ is increasing on both $(-\infty,-2)$ and $(-2,\infty)$. Thus, there are no relative extrema for $G(x)$.

We use the information obtained to sketch the graph. Other function values are listed below.

x	$G(x)$
-10	-2
-3	-1
-2	0
-1	1
6	2



24. $F(x)=\sqrt[3]{x-1}=(x-1)^{1/3}$

$F'(x)=\frac{1}{3}(x-1)^{-2/3}(1)$

$=\dfrac{1}{3(x-1)^{2/3}}$

$F'(x)$ does not exist when $3(x-1)^{2/3}=0$, which means that $F'(x)$ does not exist when $x=1$. The equation $F'(x)=0$ has no solution, therefore, the only critical value is $x=1$.

We use 1 to divide the real number line into two intervals,

A: $(-\infty,1)$ and B: $(1,\infty)$.

We use a test value in each interval to determine the sign of the derivative in each interval.

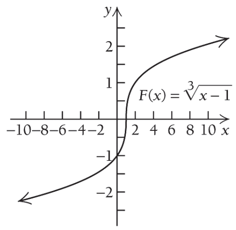
A: Test 0, $F'(0)=\dfrac{1}{3(0-1)^{2/3}}=\dfrac{1}{3}>0$

B: Test 2, $F'(2)=\dfrac{1}{3(2-1)^{2/3}}=\dfrac{1}{3}>0$

We see that $F(x)$ is increasing on both $(-\infty,1)$ and $(1,\infty)$. Thus, there are no relative extrema for $F(x)$. We use the information obtained to sketch the graph at the top of the next page.

Using the information from the previous page, we sketch the graph. Other function values are listed.

x	$F(x)$
-7	-2
0	-1
1	0
2	1
9	2



25. $f(x) = 1 - x^{2/3}$

First, find the critical points.

$f'(x) = \frac{-2}{3}x^{-1/3}$

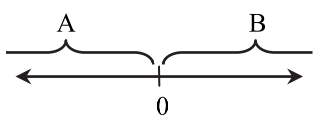
$= \frac{-2}{3\sqrt[3]{x}}$

$f'(x)$ does not exist when

$3\sqrt[3]{x} = 0$, which means that $f'(x)$ does not exist when $x = 0$. The equation $f'(x) = 0$ has no solution, therefore, the only critical value is $x = 0$.

We use 0 to divide the real number line into two intervals,

A: $(-\infty, 0)$ and B: $(0, \infty)$:



We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test -1, $f'(-1) = -\frac{2}{3\sqrt[3]{-1}} = \frac{2}{3} > 0$

B: Test 1, $f'(1) = -\frac{2}{3\sqrt[3]{1}} = -\frac{2}{3} < 0$

We see that $f(x)$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. Thus, there is a relative maximum at $x = 0$.

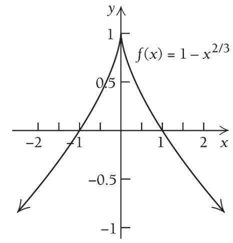
We find $f(0)$:

$f(0) = 1 - (0)^{2/3} = 1$.

Therefore, there is a relative maximum at $(0,1)$.

We use the information obtained to sketch the graph. Other function values are listed at the top of the next column.

x	$f(x)$
-8	-3
-1	0
1	0
8	-3



26. $f(x) = (x+3)^{2/3} - 5$

$f'(x) = \frac{2}{3}(x+3)^{-1/3}$

$= \frac{2}{3(x+3)^{1/3}}$

$f'(x)$ does not exist when $x = -3$. The equation $f'(x) = 0$ has no solution, therefore, the only critical value is $x = -3$.

We use -3 to divide the real number line into two intervals, A: $(-\infty, -3)$ and B: $(-3, \infty)$:

A: Test -4, $f'(-4) = \frac{2}{3(-4+3)^{1/3}} = -\frac{2}{3} < 0$

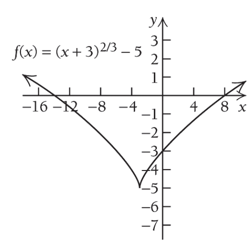
B: Test -2, $f'(-2) = \frac{2}{3(-2+3)^{1/3}} = \frac{2}{3} > 0$

We see that $f(x)$ is decreasing on $(-\infty, -3)$ and increasing on $(-3, \infty)$. Thus, there is a relative minimum at $x = -3$.

$f(-3) = (-3+3)^{2/3} - 5 = -5$:

Therefore, there is a relative minimum at $(-3, -5)$. We use the information obtained to sketch the graph. Other function values are listed below.

x	$f(x)$
-11	-1
-4	-4
-2	-4
5	-1



27. $G(x)=\dfrac{-8}{x^2+1}=-8\left(x^2+1\right)^{-1}$

First, find the critical points.

$$G'(x)=-8(-1)\left(x^2+1\right)^{-2}(2x)$$
$$=\frac{16x}{\left(x^2+1\right)^2}$$

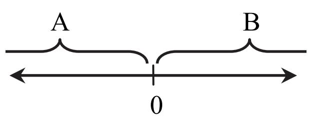
$G'(x)$ exists for all real numbers. Setting the derivative equal to zero, we have:

$$G'(x)=0$$
$$\frac{16x}{\left(x^2+1\right)^2}=0$$
$$16x=0$$
$$x=0$$

The only critical value is 0.

We use 0 to divide the real number line into two intervals,

A: $(-\infty,0)$ and B: $(0,\infty)$:



We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test -1, $G'(-1)=\dfrac{16(-1)}{\left((-1)^2+1\right)^2}=\dfrac{-16}{4}=-4<0$

B: Test 1, $G'(1)=\dfrac{16(1)}{\left((1)^2+1\right)^2}=\dfrac{16}{4}=4>0$

We see that $G(x)$ is decreasing on $(-\infty,0)$ and increasing on $(0,\infty)$. Thus, a relative minimum occurs at $x=0$.

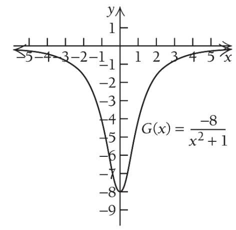
We find $G(0)$:

$$G(0)=\frac{-8}{(0)^2+1}=-8$$

Thus, there is a relative minimum at $(0,-8)$.

We use the information obtained to sketch the graph. Other function values are listed at the top of the next column.

x	$G(x)$
-3	$-\frac{4}{5}$
-2	$-\frac{8}{5}$
-1	-4
1	-4
2	$-\frac{8}{5}$
3	$-\frac{4}{5}$



28. $F(x)=\dfrac{5}{x^2+1}=5\left(x^2+1\right)^{-1}$

$$F'(x)=5(-1)\left(x^2+1\right)^{-2}(2x)$$
$$=\frac{-10x}{\left(x^2+1\right)^2}$$

$F'(x)$ exists for all real numbers. We solve

$$F'(x)=0$$
$$\frac{-10x}{\left(x^2+1\right)^2}=0$$
$$x=0$$

The only critical values is 0.

We use 0 to divide the real number line into two intervals,

A: $(-\infty,0)$ and B: $(0,\infty)$:

A: Test -1,

$$F'(-1)=\frac{-10(-1)}{\left((-1)^2+1\right)^2}=\frac{10}{4}=\frac{5}{2}>0$$

B: Test 1,

$$F'(1)=\frac{-10(1)}{\left((1)^2+1\right)^2}=\frac{-10}{4}=-\frac{5}{2}<0$$

We see that $F(x)$ is increasing on $(-\infty,0)$ and decreasing on $(0,\infty)$. Thus, a relative maximum occurs at $x=0$.

We find $F(0)$:

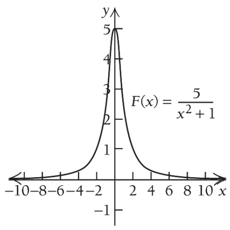
$$F(0)=\frac{5}{(0)^2+1}=5$$

Thus, there is a relative maximum at $(0,5)$.

The solution is continued on the next page.

We use the information obtained on the previous page to sketch the graph. Other function values are listed below.

x	$F(x)$
-3	$\frac{1}{2}$
-2	1
-1	$\frac{5}{2}$
1	$\frac{5}{2}$
2	1
3	$\frac{1}{2}$



29. $g(x)=\dfrac{4x}{x^2+1}$

First, find the critical points.

$$g'(x)=\frac{(x^2+1)(4)-4x(2x)}{(x^2+1)^2} \qquad \text{Quotient Rule}$$

$$=\frac{4x^2+4-8x^2}{(x^2+1)^2}$$

$$=\frac{4-4x^2}{(x^2+1)^2}$$

$g'(x)$ exists for all real numbers. We solve

$$g'(x)=0$$

$$\frac{4-4x^2}{(x^2+1)^2}=0$$

$$4-4x^2=0 \qquad \text{Multiplying by } (x^2+1)^2$$

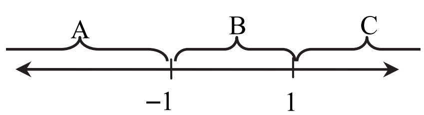
$$x^2-1=0 \qquad \text{Dividing by } -4$$

$$x^2=1$$

$$x=\pm\sqrt{1}$$

$$x=\pm 1$$

The critical values are -1 and 1. We use them to divide the real number line into three intervals, A: $(-\infty,-1)$, B: $(-1,1)$, and C:$(1,\infty)$.



We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test -2, $g'(-2)=\dfrac{4-4(-2)^2}{\left((-2)^2+1\right)^2}=-\dfrac{12}{25}<0$

B: Test 0, $g'(0)=\dfrac{4-4(0)^2}{\left((0)^2+1\right)^2}=4>0$

C: Test 2, $g'(2)=\dfrac{4-4(2)^2}{\left((2)^2+1\right)^2}=-\dfrac{12}{25}<0$

We see that $g(x)$ is decreasing on $(-\infty,-1)$, increasing on $(-1,1)$, and decreasing again on $(1,\infty)$. So there is a relative minimum at $x=-1$ and a relative maximum at $x=1$.

We find $g(-1)$:

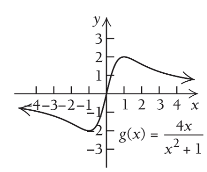
$$g(-1)=\frac{4(-1)}{(-1)^2+1}=\frac{-4}{2}=-2$$

Then we find $g(1)$:

$$g(1)=\frac{4(1)}{(1)^2+1}=\frac{4}{2}=2$$

There is a relative minimum at $(-1,-2)$, and there is a relative maximum at $(1,2)$. We use the information obtained to sketch the graph. Other function values are listed below.

x	$g(x)$
-3	$-\frac{6}{5}$
-2	$-\frac{8}{5}$
0	0
2	$\frac{8}{5}$
3	$\frac{6}{5}$



30. $g(x)=\dfrac{x^2}{x^2+1}$

$$g'(x)=\frac{(x^2+1)(2x)-x^2(2x)}{(x^2+1)^2}$$

$$g'(x)=\frac{2x}{(x^2+1)^2}$$

The solution is continued on the next page.

From the previous page, $g'(x)$ exists for all real numbers. We solve

$$g'(x) = 0$$
$$\frac{2x}{\left(x^2+1\right)^2} = 0$$
$$x = 0$$

The only critical values is 0.
We use 0 to divide the real number line into two intervals,
A: $(-\infty, 0)$ and B: $(0, \infty)$:
A: Test -1,

$$g'(-1) = \frac{2(-1)}{\left((-1)^2+1\right)^2} = \frac{-2}{4} = -\frac{1}{2} < 0$$

B: Test 1,

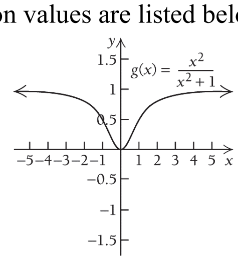
$$g'(1) = \frac{2(1)}{\left((1)^2+1\right)^2} = \frac{2}{4} = \frac{1}{2} > 0$$

We see that $g(x)$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. Thus, a relative minimum occurs at $x = 0$.
We find $g(0)$:

$$g(0) = \frac{(0)^2}{(0)^2+1} = 0$$

Thus, there is a relative minimum at $(0, 0)$. We use the information obtained to sketch the graph. Other function values are listed below.

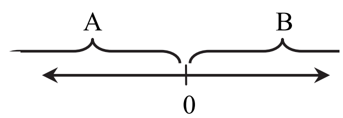
x	$g(x)$
-3	$\frac{9}{10}$
-2	$\frac{4}{5}$
-1	$\frac{1}{2}$
1	$\frac{1}{2}$
2	$\frac{4}{5}$
3	$\frac{9}{10}$



31. $f(x) = \sqrt[3]{x} = (x)^{1/3}$
First, find the critical points.

$$f'(x) = \frac{1}{3}(x)^{-2/3} = \frac{1}{3(x)^{2/3}} = \frac{1}{3 \cdot \sqrt[3]{x^2}}$$

$f'(x)$ does not exist when $x = 0$. The equation $f'(x) = 0$ has no solution, therefore, the only critical value is $x = 0$.
We use 0 to divide the real number line into two intervals,
A: $(-\infty, 0)$ and B: $(0, \infty)$:



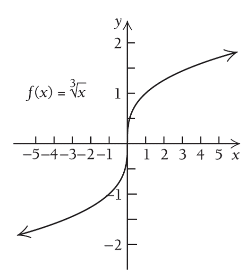
We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test -1, $f'(-1) = \frac{1}{3\sqrt[3]{(-1)^2}} = \frac{1}{3} > 0$

B: Test 1, $f'(1) = \frac{1}{3\left(\sqrt[3]{(1)^2}\right)} = \frac{1}{3} > 0$

We see that $f(x)$ is increasing on both $(-\infty, 0)$ and $(0, \infty)$. Thus, there are no relative extrema for $f(x)$. We use the information obtained to sketch the graph. Other function values are listed below.

x	$f(x)$
-8	-2
-1	-1
0	0
1	1
8	2



32. $f(x) = (x+1)^{1/3}$

$$f'(x) = \frac{1}{3}(x+1)^{-2/3}(1) = \frac{1}{3(x+1)^{2/3}}$$

$f'(x)$ does not exist when $x = -1$. The equation $f'(x) = 0$ has no solution, therefore, the only critical value is $x = -1$.

Using the information from the previous page, we use -1 to divide the real number line into two intervals,

A: $(-\infty,-1)$ and B: $(-1,\infty)$:

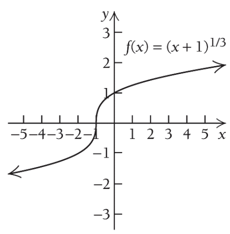
A: Test -2, $f'(-2)=\dfrac{1}{3(-2+1)^{2/3}}=\dfrac{1}{3}>0$

B: Test 0, $f'(0)=\dfrac{1}{3(0+1)^{2/3}}=\dfrac{1}{3}>0$

We see that $f(x)$ is increasing on both intervals, Thus, there are no relative extrema for $f(x)$.

We use the information obtained to sketch the graph. Other function values are listed below.

x	$f(x)$
−9	−2
−2	−1
−1	0
0	1
7	2



33. $g(x)=\sqrt{x^2+2x+5}=\left(x^2+2x+5\right)^{1/2}$

First, find the critical points.

$$g'(x)=\frac{1}{2}\left(x^2+2x+5\right)^{-1/2}(2x+2)$$
$$=\frac{2(x+1)}{2\left(x^2+2x+5\right)^{1/2}}$$
$$=\frac{x+1}{\sqrt{x^2+2x+5}}$$

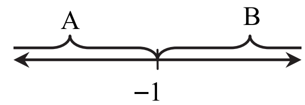
The equation $x^2+2x+5=0$ has no real-number solution, so $g'(x)$ exists for all real numbers. Next we find out where the derivative is zero. We solve

$$g'(x)=0$$
$$\frac{x+1}{\sqrt{x^2+2x+5}}=0$$
$$x+1=0$$
$$x=-1$$

The only critical value is -1.

We use -1 to divide the real number line into two intervals,

A: $(-\infty,-1)$ and B: $(-1,\infty)$:



We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test -2,

$$g'(-2)=\frac{(-2)+1}{\sqrt{(-2)^2+2(-2)+5}}=\frac{-1}{\sqrt{5}}<0$$

B: Test 0,

$$g'(0)=\frac{(0)+1}{\sqrt{(0)^2+2(0)+5}}=\frac{1}{\sqrt{5}}>0$$

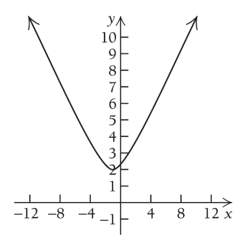
We see that $g(x)$ is decreasing on $(-\infty,-1)$ and increasing on $(-1,\infty)$, and the change from decreasing to increasing indicates that a relative minimum occurs at $x=-1$. We substitute into the original equation to find $g(-1)$:

$$g(-1)=\sqrt{(-1)^2+2(-1)+5}=\sqrt{4}=2$$

Thus, there is a relative minimum at $(-1,2)$.

We use the information obtained to sketch the graph. Other function values are listed below.

x	$g(x)$
−4	3.61
−2	2.24
0	2.24
1	2.83
3	4.47



$g(x)=\sqrt{x^2+2x+5}$

34. $F(x)=\dfrac{1}{\sqrt{x^2+1}}=\left(x^2+1\right)^{-1/2}$

$$F'(x)=\left(-\frac{1}{2}\right)\left(x^2+1\right)^{-3/2}(2x)$$
$$=\frac{-x}{\left(x^2+1\right)^{3/2}}$$

$F'(x)$ exists for all real numbers.

The solution is continued on the next page.

Using the derivative from the previous page,
We solve

$$F'(x) = 0$$

$$\frac{-x}{\left(x^2+1\right)^{3/2}} = 0$$

$$x = 0$$

The only critical value is 0.
We use 0 to divide the real number line into two intervals,
A: $(-\infty, 0)$ and B: $(0, \infty)$:
A: Test -1,

$$F'(-1) = \frac{-(-1)}{\left((-1)^2+1\right)^{3/2}} = \frac{1}{\sqrt{8}} > 0$$

B: Test 1,

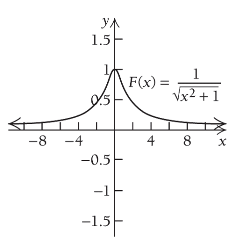
$$F'(1) = \frac{-1}{\left((1)^2+1\right)^{3/2}} = \frac{-1}{\sqrt{8}} < 0$$

We see that $F(x)$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. Thus, a relative maximum occurs at $x = 0$.

$$F(0) = \frac{1}{\sqrt{(0)^2+1}} = 1$$

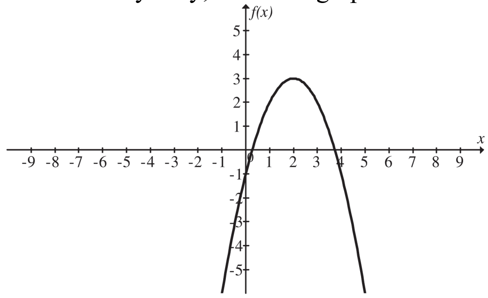
Thus, there is a relative maximum at $(0,1)$. We use the information obtained to sketch the graph. Other function values are listed below.

x	$F(x)$
-3	0.32
-2	0.45
-1	0.71
1	0.71
2	0.45
3	0.32

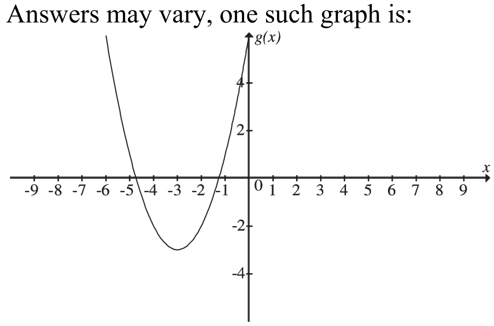


35. – 68. Left to the student.

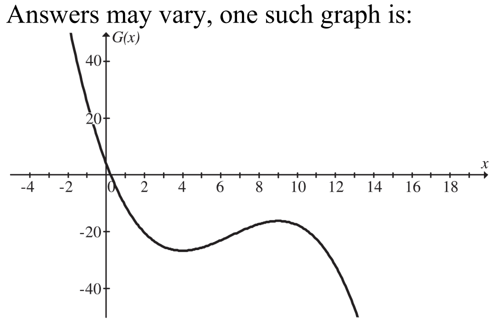
69. Answers may vary, one such graph is:



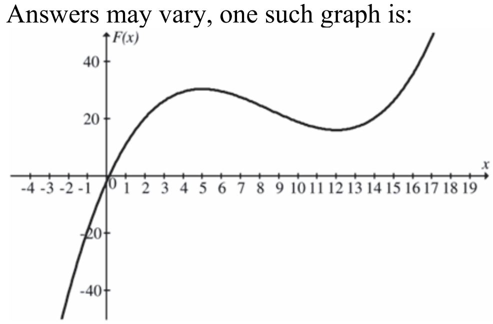
70. Answers may vary, one such graph is:



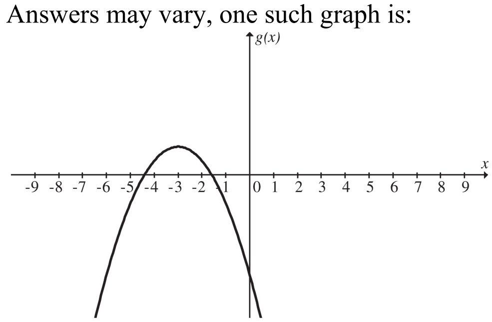
71. Answers may vary, one such graph is:



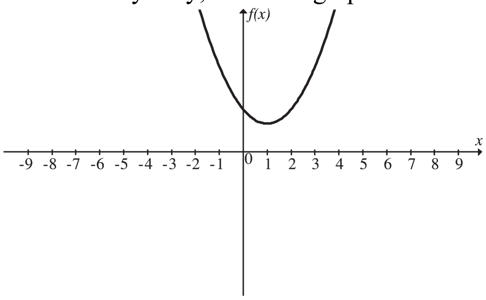
72. Answers may vary, one such graph is:



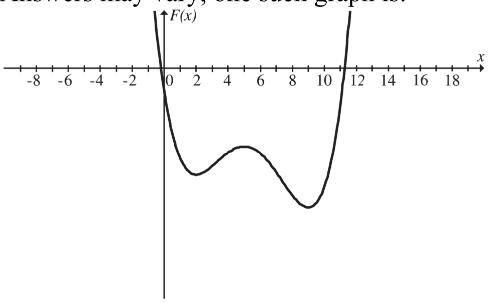
73. Answers may vary, one such graph is:



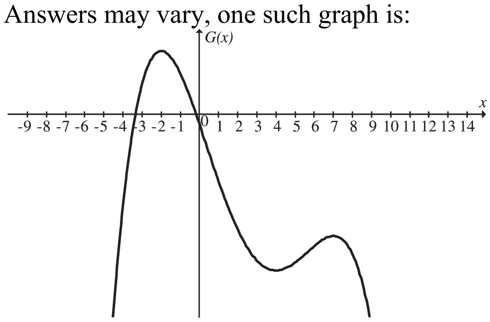
74. Answers may vary, one such graph is:



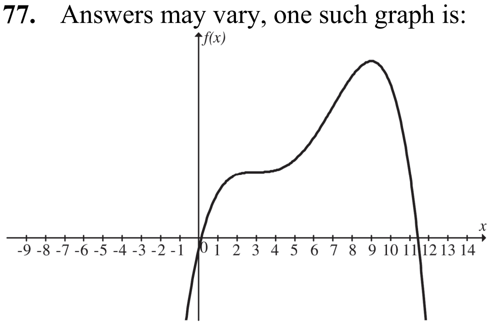
75. Answers may vary, one such graph is:



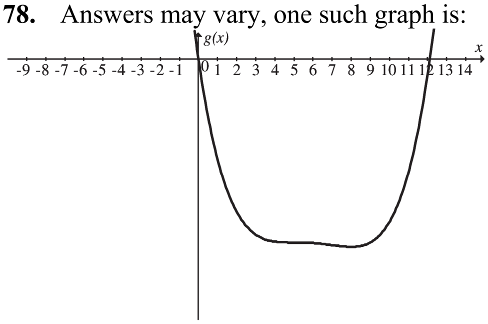
76. Answers may vary, one such graph is:



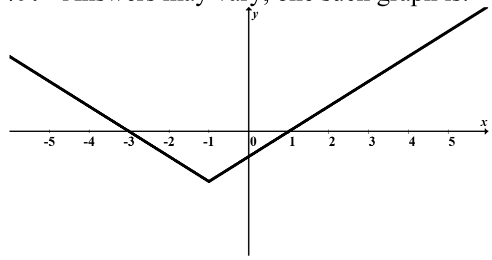
77. Answers may vary, one such graph is:



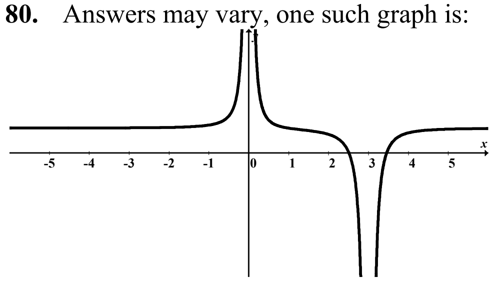
78. Answers may vary, one such graph is:



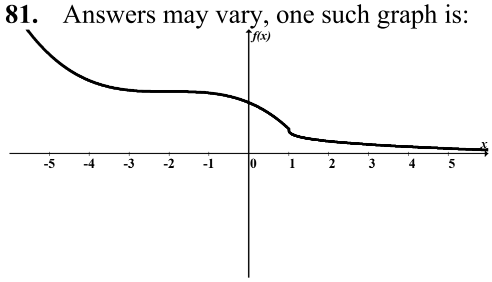
79. Answers may vary, one such graph is:



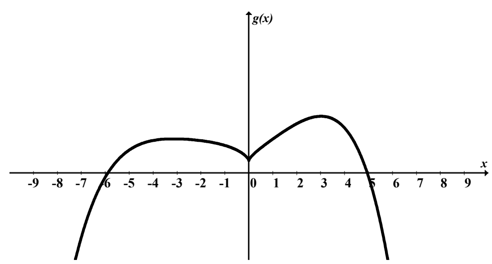
80. Answers may vary, one such graph is:



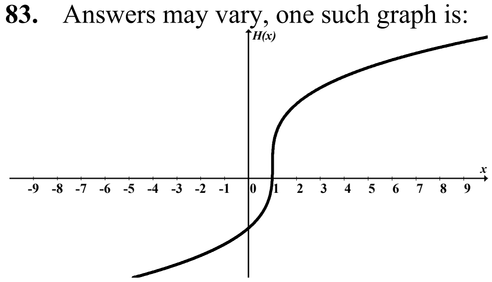
81. Answers may vary, one such graph is:



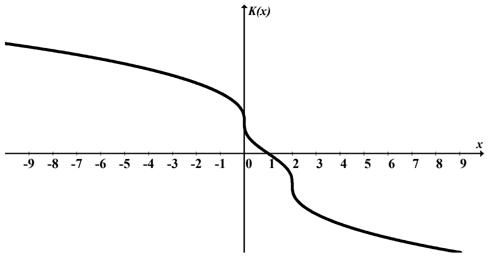
82. Answers may vary, one such graph is:



83. Answers may vary, one such graph is:



84. Answers may vary, one such graph is:



85. The critical value of a function f is an interior value c of its domain at which the tangent to the graph is horizontal $\left(f'(c)=0\right)$ or the tangent is vertical $\left(f'(c) \text{ does not exist}\right)$.
The critical values for this graph are $x_1, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}$.

86. The function is increasing on intervals (a,b) and (c,d). A line tangent to the curve at any point on either of these intervals has a positive slope. Thus, the function is increasing on the intervals for which the first derivative is positive. Similarly, we see that on the intervals (b,c) and (d,e) the function is decreasing. A line tangent to the curve at any point on either of these intervals has a negative slope. Thus, the function is decreasing on the intervals for which first derivative is negative.

87. Letting t be years since 2006 and E be thousand of employees, we have the function:

$$E(t)=107.833t^3-971.369t^2+2657.917t+50347.83$$

First, we find the critical points.

$$E'(t)=323.499t^2-1942.738t+2657.917$$

$E'(t)$ exists for all real numbers. Solve

$$E'(t)=0$$

$$323.499t^2-1942.738t+2657.917=0$$

Using the quadratic formula, we have:

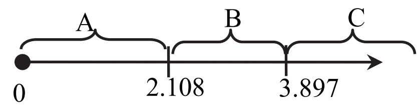
$$t=\frac{1942.738\pm\sqrt{(-1942.738)^2-4(323.499)(2657.917)}}{2(323.499)}$$

$$=\frac{1942.738\pm\sqrt{334,896.970312}}{646.998}$$

$t\approx 2.108$ or $t\approx 3.897$

There are two critical values.

We use them to divide the interval $[0,\infty)$ into three intervals:
A: $[0,2.108)$ B: $(2.08,3.897)$, and C:(3.897∞)



Next, we test a point in each interval to determine the sign of the derivative.

A: Test 1,

$$E'(1)=323.499(1)^2-1942.738(1)+2657.917$$
$$=1038.678>0$$

B: Test 3,

$$E'(3)=323.499(3)^2-1942.738(3)+2657.917$$
$$=-258.806<0$$

C: Test 4,

$$E'(4)=323.499(4)^2-1942.738(4)+2657.917$$
$$=62.949>0$$

Since, $E(t)$ is increasing on $[0,2.108)$ and decreasing on $(2.108,3.897)$ and there is a relative maximum at $t=2.108$.

$$E(2.108)=107.833(2.108)^3-971.369(2.108)^2+2657.917(2.108)+50347.83$$

$\approx 52,644.383$

There is a relative minimum at $(2.108,\ 52,644.383)$.

Since, $E(t)$ is decreasing on $(2.108,3.897)$ and increasing $[3.897,\infty)$ on and there is a relative minimum at $t=3.897$.

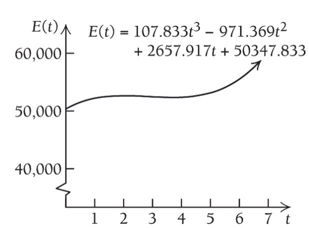
$$E(3.897)=107.833(3.897)^3-971.369(3.897)^2+2657.917(3.897)+50347.83$$

$\approx 52,335.73$

There is a relative maxnimum at $(3.897,\ 52,335.73)$.

We sketch the graph.

t	$T(t)$
0	50,347
1	52,142
3	52,491
5	52,832
8	64,654



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88. $N(a) = -a^2 + 300a + 6, \qquad 0 \le a \le 300$

$N'(a) = -2a + 300$

$N'(a)$ exists for all real numbers. Solve,

$$N'(a) = 0$$
$$-2a + 300 = 0$$
$$-2a = -300$$
$$a = 150$$

The only critical value is 150. We divide the interval $[0,300]$ into two intervals,

A: $[0,150)$ and B: $(150,300]$.

A: Test 100,

$N'(100) = -2(100) + 300 = 100 > 0$

B: Test 200,

$N'(200) = -2(200) + 300 = -100 < 0$

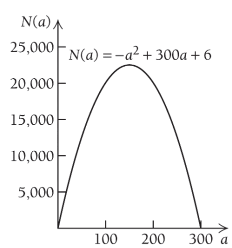
Since, $N(a)$ is increasing on $[0,150)$ and decreasing on $(150,300]$, there is a relative maximum at $x = 150$.

$N(150) = -(150)^2 + 300(150) + 6 = 22{,}506$

There is a relative maximum at $(150,\ 22{,}506)$.

We sketch the graph.

a	$N(a)$
0	6
100	20,006
200	20,006
300	6



89. $f(t) = 0.00259t^2 - 0.457t + 36.237$

First, we find the critical points.

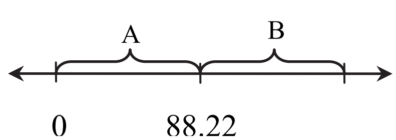
$f'(t) = 0.00518t - 0.457$

$f'(x)$ exists everywhere, so we solve

$$f'(t) = 0$$
$$0.00518t - 0.457 = 0$$
$$t = 88.22$$

The only critical value is about 88.22 we use it to break up the interval $(0,\infty)$ into two intervals

A: $(0,88.28)$ and B:$(88.22,\infty)$.



A: Test 20,

$f'(20) = 0.00518(20) - 0.457 = -0.3534 < 0$

B: Test 100,

$f'(100) = 0.00518(100) - 0.457 = 0.061 > 0$

We see that $f(t)$ is decreasing on $(0,88.22)$ and increasing on $(88.22,\infty)$, so there is a relative minimum at $t = 88.22$.

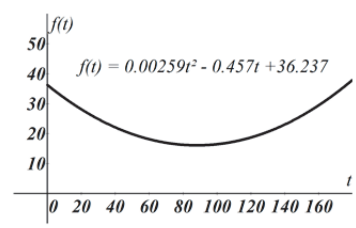
$f(88.22)$

$= 0.00259(88.22)^2 - 0.457(88.22) + 36.237$

≈ 16.09

There is a relative minimum at about $(88.22,16.08)$. Thus, the latitude that is closest to the equator at which the full eclipse could be view is 16.08 degrees south and will occur 88.22 minutes after the start of the eclipse. We use the information obtained above to sketch the graph. Other function values are listed below.

x	$f(x)$
20	28.133
30	24.858
40	22.101
60	18.141
80	16.253



90. $T(t) = -0.1t^2 + 1.2t + 98.6, \quad 0 \le t \le 12$

First, we find the critical points.

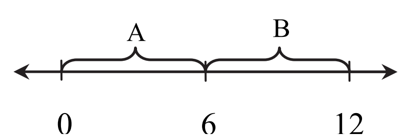
$T'(t) = -0.2t + 1.2$

$T'(t)$ exists for all real numbers. Solve

$$T'(t) = 0$$
$$-0.2t + 1.2 = 0$$
$$t = 6$$

The only critical value is 6. We use it to divide the interval $[0,12]$ into two intervals:

A: $[0,6)$ and B: $(6,12]$



The solution is continued on the next page.

Next, we test a point in each interval found on the previous page to determine the sign of the derivative.

A: Test 1, $T'(1) = -0.2(1) + 1.2 = 1.0 > 0$

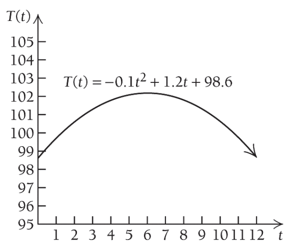
B: Test 7, $T'(7) = -0.2(7) + 1.2 = -0.2 < 0$

Since, $T(t)$ is increasing on $[0,6)$ and decreasing on $(6,12]$, there is a relative maximum at $t = 6$.

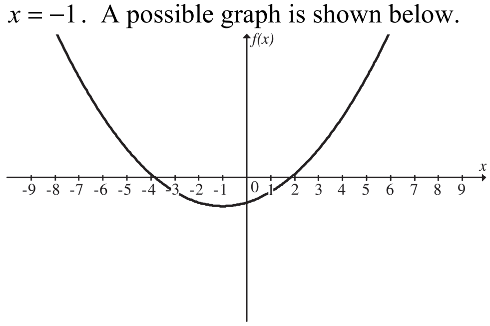
$T(6) = -0.1(6)^2 + 1.2(6) + 98.6 = 102.2$

There is a relative maximum at $(6, 102.2)$. We sketch the graph.

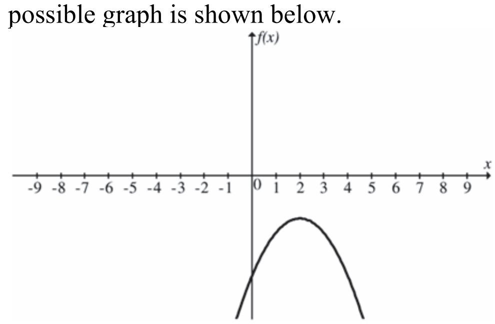
t	$T(t)$
0	98.6
3	101.3
5	102.1
7	102.1
8	101.8
12	98.6



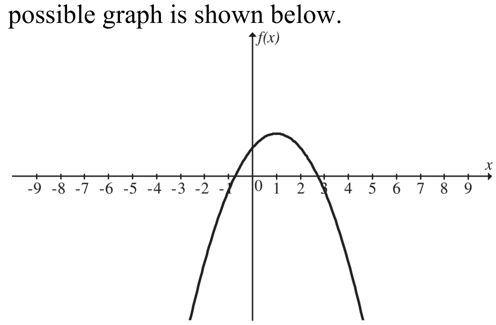
91. The derivative is negative over the interval $(-\infty, -1)$ and positive over the interval $(-1, \infty)$. Furthermore, it is equal to zero when $x = -1$. This means that the function is decreasing over the interval $(-\infty, -1)$, increasing over the interval $(-1, \infty)$ and has a horizontal tangent at $x = -1$. A possible graph is shown below.



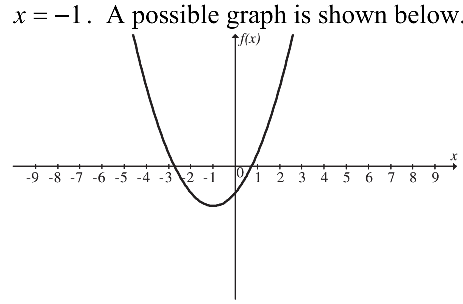
92. The derivative is positive over the interval $(-\infty, 2)$ and negative over the interval $(2, \infty)$. Furthermore, it is equal to zero when $x = 2$. This means that the function is increasing over the interval $(-\infty, 2)$, decreasing over the interval $(2, \infty)$ and has a horizontal tangent at $x = 2$. A possible graph is shown below.



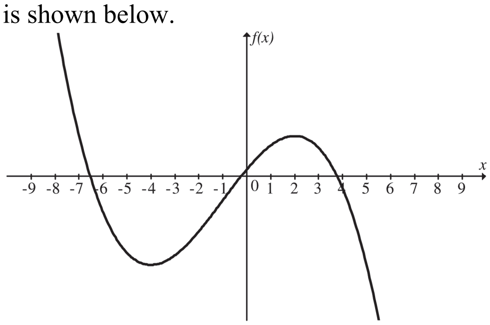
93. The derivative is positive over the interval $(-\infty, 1)$ and negative over the interval $(1, \infty)$. Furthermore, it is equal to zero when $x = 1$. This means that the function is increasing over the interval $(-\infty, 1)$, decreasing over the interval $(1, \infty)$ and has a horizontal tangent at $x = 1$. A possible graph is shown below.



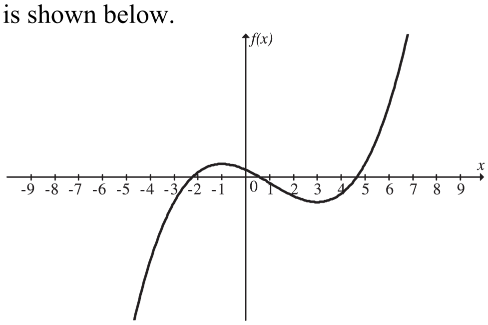
94. The derivative is negative over the interval $(-\infty, -1)$ and positive over the interval $(-1, \infty)$. Furthermore, it is equal to zero when $x = -1$. This means that the function is decreasing over the interval $(-\infty, -1)$, increasing over the interval $(-1, \infty)$ and has a horizontal tangent at $x = -1$. A possible graph is shown below.



95. The derivative is positive over the interval $(-4,2)$ and negative over the intervals $(-\infty,-4)$ and $(2,\infty)$. Furthermore, it is equal to zero when $x=-4$ and $x=2$. This means that the function is decreasing over the interval $(-\infty,-4)$, then increasing over the interval $(-4,2)$, and then decreasing again over the interval $(2,\infty)$. The function has horizontal tangents at $x=-4$ and $x=2$. A possible graph is shown below.

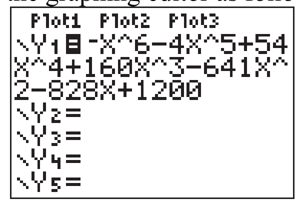


96. The derivative is negative over the interval $(-1,3)$ and intervals and positive over the intervals $(-\infty,-1)$ and $(3,\infty)$. Furthermore, it is equal to zero when $x=-1$ and $x=3$. This means that the function is increasing over the interval $(-\infty,-1)$, then decreasing over the interval $(-1,3)$, and then increasing again over the interval $(3,\infty)$. The function has horizontal tangents at $x=-1$ and $x=3$. A possible graph is shown below.

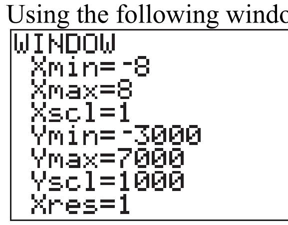


97. $f(x)=-x^6-4x^5+54x^4+160x^3-641x^2$
$-828x+1200$

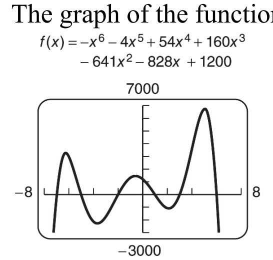
Using the calculator we enter the function into the graphing editor as follows:



Using the following window:



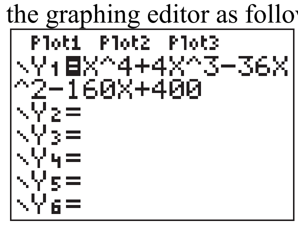
The graph of the function is:



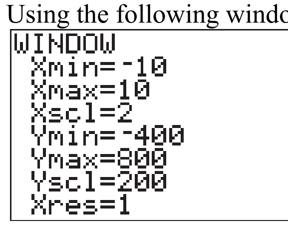
We find the relative extrema using the minimum/maximum feature on the calculator. There are relative minima at $(-3.683,-2288.03)$ and $(2.116,-1083.08)$.
There are relative maxima at $(-6.262,3213.8)$, $(-0.559,1440.06)$, and $(5.054,6674.12)$.

98. $f(x)=x^4+4x^3-36x^2-160x+400$

Using the calculator we enter the function into the graphing editor as follows:

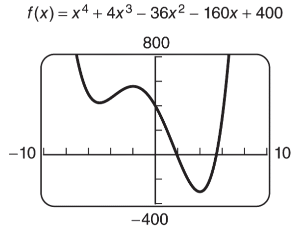


Using the following window:



The solution is continued on the next page.

The graph of the function is:

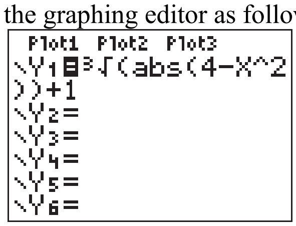


We find the relative extrema using the minimum/maximum feature on the calculator. There are relative minima at $(-5,425)$ and $(4,-304)$.

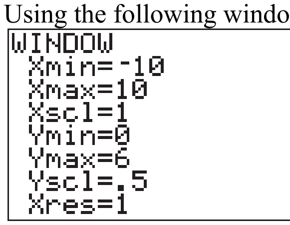
There is a relative maximum at $(-2,560)$.

99. $f(x)=\sqrt[3]{\left|4-x^2\right|}+1$

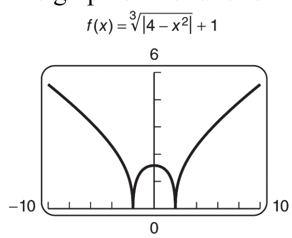
Using the calculator we enter the function into the graphing editor as follows:



Using the following window:



The graph of the function is:

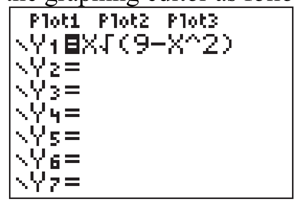


We find the relative extrema using the minimum/maximum feature on the calculator. There are relative minima at $(-2,1)$ and $(2,1)$.

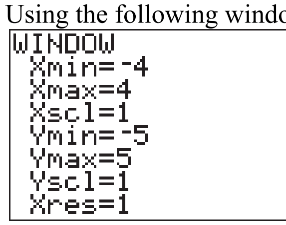
There is a relative maximum at $(0,2.587)$.

100. $f(x)=x\sqrt{9-x^2}$

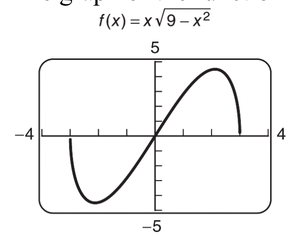
Using the calculator we enter the function into the graphing editor as follows:



Using the following window:



The graph of the function is:

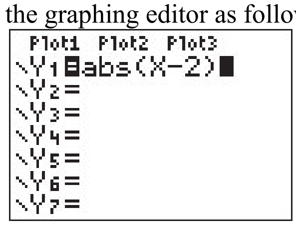


Notice, the calculator has trouble drawing the graph. The graph should continue to the *x*-intercepts at $(-3,0)$ and $(3,0)$. Fortunately, this does not hinder our efforts to find the extrema. We find the relative extrema using the minimum/maximum feature on the calculator. There is a relative minimum at $(-2.12,-4.5)$.

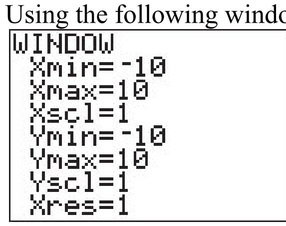
There is a relative maximum at $(2.12,4.5)$.

101. $f(x)=|x-2|$

Using the calculator we enter the function into the graphing editor as follows:

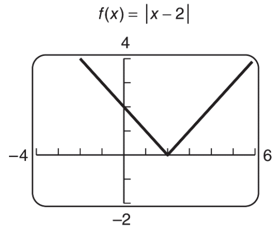


Using the following window:



The solution is continued on the next page.

The graph of the function is:



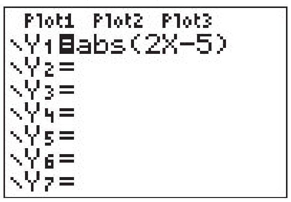
We find the relative extrema using the minimum/maximum feature on the calculator.
The graph is decreasing over the interval $(-\infty, 2)$.

The graph is increasing over the interval $(2, \infty)$.

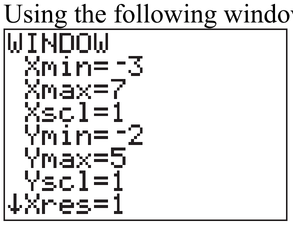
There is a relative minimum at $(2, 0)$.

The derivative does not exist at $x = 2$

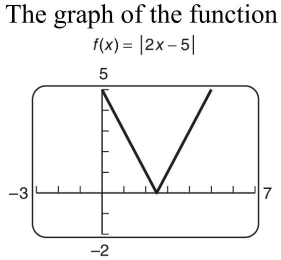
102. $f(x) = |2x - 5|$



Using the following window:



The graph of the function is:



We find the relative extrema using the minimum/maximum feature on the calculator.
The graph is decreasing over the interval $\left(-\infty, \frac{5}{2}\right)$.

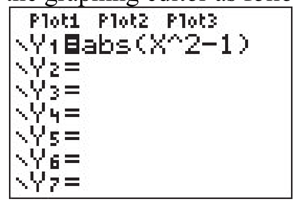
The graph is increasing over the interval $\left(\frac{5}{2}, \infty\right)$.

There is a relative minimum at $\left(\frac{5}{2}, 0\right)$.

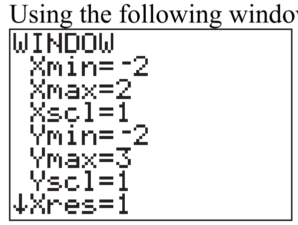
The derivative does not exist at $x = \frac{5}{2}$

103. $f(x) = |x^2 - 1|$

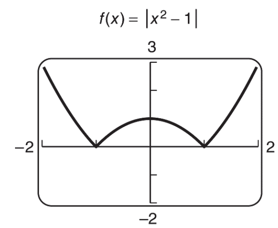
Using the calculator we enter the function into the graphing editor as follows:



Using the following window:



The graph of the function is:



We find the relative extrema using the minimum/maximum feature on the calculator.
The graph is decreasing over the interval $(-\infty, -1)$ and $(0, 1)$.

The graph is increasing over the interval $(-1, 0)$ and $(2, \infty)$.

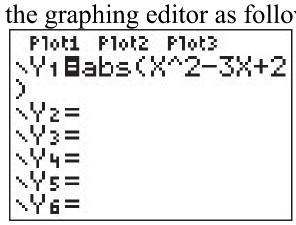
There are relative mimima at $(-1, 0)$ and $(1, 0)$.

There is a relative maximum at $(0, 1)$.

The derivative does not exist at $x = -1$ and $x = 1$.

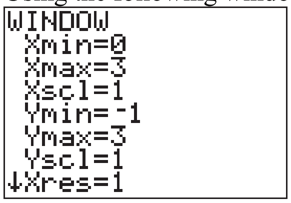
104. $f(x) = |x^2 - 3x + 2|$

Using the calculator we enter the function into the graphing editor as follows:

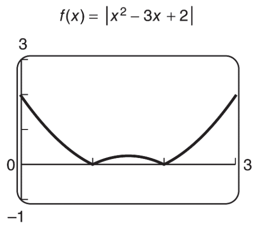


The solution is continued on the next page.

Using the following window:



The graph of the function is:



We find the relative extrema using the minimum/maximum feature on the calculator. The graph is decreasing over the interval $(-\infty, 1)$ and $\left(\frac{3}{2}, 2\right)$.

The graph is increasing over the interval $\left(1, \frac{3}{2}\right)$ and $(2, \infty)$.

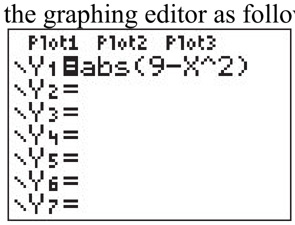
There are relative mimima at $(1, 0)$ and $(2, 0)$.

There is a relative maximum at $\left(\frac{3}{2}, \frac{1}{4}\right)$.

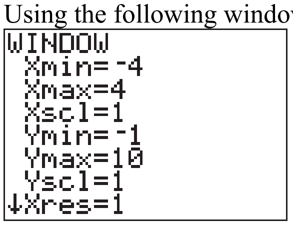
The derivative does not exist at $x = 1$ and $x = 2$.

105. $f(x) = |9 - x^2|$

Using the calculator we enter the function into the graphing editor as follows:

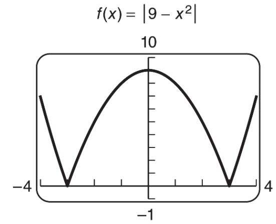


Using the following window:



The solution is continued on the next column.

The graph of the function is:



We find the relative extrema using the minimum/maximum feature on the calculator. The graph is decreasing over the interval $(-\infty, -3)$ and $(0, 3)$.

The graph is increasing over the interval $(-3, 0)$ and $(3, \infty)$.

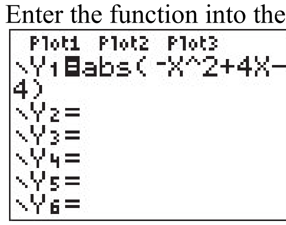
There are relative mimima at $(-3, 0)$ and $(3, 0)$.

There is a relative maximum at $(0, 9)$.

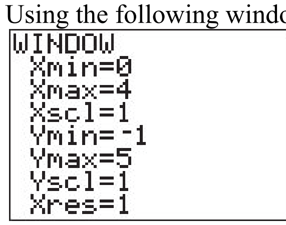
The derivative does not exist at $x = -3$ and $x = 3$.

106. $f(x) = |-x^2 + 4x - 4|$

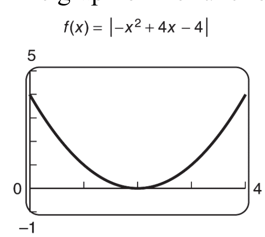
Enter the function into the graphing editor:



Using the following window:



The graph of the function is:



We find the relative extrema using the minimum/maximum feature on the calculator. The graph is decreasing over the interval $(-\infty, 2)$.

The graph is increasing over the interval $(2, \infty)$.

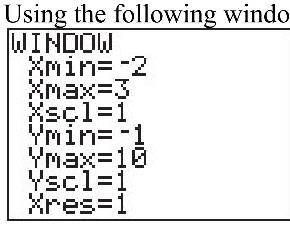
There is a relative minimum at $(2, 0)$.

The derivative exists for all values of x.

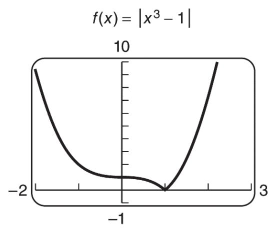
107. $f(x)=\left|x^3-1\right|$

Using the calculator we enter the function into the graphing editor as follows:

Using the following window:



The graph of the function is:



We find the relative extrema using the minimum/maximum feature on the calculator. The graph is decreasing over the interval $(-\infty,-1)$.

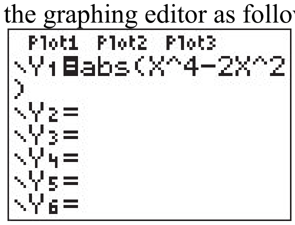
The graph is increasing over the interval $(1,\infty)$.

There is a relative minimum at $(1,0)$.

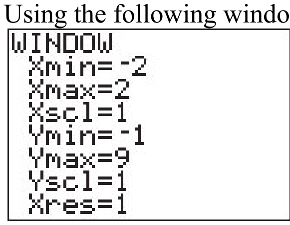
The derivative does not exist at $x=1$.

108. $f(x)=\left|x^4-2x^2\right|$

Using the calculator we enter the function into the graphing editor as follows:

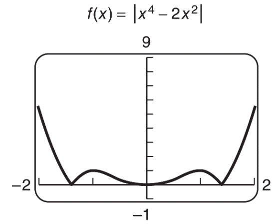


Using the following window:



The solution is continued at the top of the next column.

The graph of the function is:



We find the relative extrema using the minimum/maximum feature on the calculator. The graph is decreasing over the interval $(-\infty,-1.41),(-1,0)$ and $(0,1.41)$.

The graph is increasing over the interval $(-1.41,0),(0,1)$ and $(1.41,\infty)$.

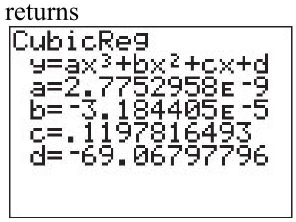
There are relative mimima at $(-1.41,0),(0,0)$ and $(1.41,0)$.

There are relative maxima at $(-1,0)$ and $(1,0)$.

The derivative does not exist at $x=-1.41$ and $x=1.41$.

109.

a) We enter the data into the calculator and run a cubic regression. The calculator returns



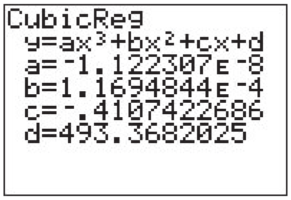
When we try to run a quartic regression, the calculator returns a domain error. Therefore, the cubic regression fits best.

b) The domain of the function is the set of nonnegative real numbers. Realistically, there would be some upper limit upon daily caloric intake.

c) The cubic regression model appears to have a relative minimum at $(4316,77.85)$ and it appears to have a relative maximum at $(3333,79.14)$. This leads us to believe that eating too many calories might shorten life expectancy.

110.

a) The cubic function fits best. In fact some calculators will return an error message when an attempt is made to fit a quartic function to the data.



b) The domain of the function is the set of nonnegative real numbers. Realistically, there would be some upper limit upon daily caloric intake.

c) The cubic regression model does not appears to have a relative extrema. The greater the daily caloric intake, the lower the infant mortality.

111.

a) Answers will vary. In Exercises 1-16 the function is given in equation form. The most accurate way to select an appropriate viewing window, one should first determine the domain, because that will help determine the x-range. For polynomials the domain is all real numbers, so we will typically select a x-range that is symmetric about 0. Next, you should find the critical values and make sure that your x-range contains them. Finally, you should determine the x-intercepts and make sure the x-range includes them. To find the y-range, you should find the y-values of the critical points and make sure the y-range includes those values. You should also make sure that the y-range includes the y-intercept.
To avoid the calculations required to find the relative extrema and the zeros as described above, we can determine a good window by using the table screen on the calculator and observing the appropriate y-values for selected x-values.

b) Answers will vary. When the equations are somewhat complex, the best way to determine a viewing window is to use the table screen on the calculator and observing appropriate y-values for selected x-values. You will need to set your table to accept selected x-values. Enter the table set up feature on your calculator and turn on the ask feature for your independent variable. This will allow you to enter an x-value and the calculator will return the y-value. You should make your ranges large enough so that all the data points will be easily viewed in the window.

Exercise Set 2.2

1. $f(x)=4-x^2$

First, find $f'(x)$ and $f''(x)$.

$f'(x)=-2x$

$f''(x)=-2$

Next, find the critical points of $f(x)$. Since $f'(x)$ exists for all real numbers x, the only critical points occur when $f'(x)=0$.

$f'(x)=0$

$-2x=0$

$x=0$

We find the function value at $x=0$.

$f(0)=4-(0)^2=4$.

The critical point is $(0,4)$.

Next, we apply the Second Derivative test.

$f''(x)=-2$

$f''(0)=-2<0$

Therefore, $f(0)=4$ is a relative maximum.

2. $f(x)=5-x^2$

$f'(x)=-2x$

$f''(x)=-2$

$f'(x)$ exists for all real numbers. The only critical points occur when $f'(x)=0$.

$f'(x)=0$

$-2x=0$

$x=0$

We find the function value at $x=0$.

$f(0)=5-(0)^2=5$.

The critical point is $(0,5)$.

Next, we apply the Second Derivative test.

$f''(x)=-2$

$f''(0)=-2<0$

Therefore, $f(0)=5$ is a relative maximum.

3. $f(x)=x^2+x-1$

First, find $f'(x)$ and $f''(x)$.

$f'(x)=2x+1$

$f''(x)=2$

Next, find the critical points of $f(x)$. Since $f'(x)$ exists for all real numbers x, the only critical points occur when $f'(x)=0$.

$2x+1=0$

$2x=-1$

$x=-\frac{1}{2}$

We find the function value at $x=-\frac{1}{2}$.

$f\left(-\frac{1}{2}\right)=\left(-\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)-1=\frac{1}{4}-\frac{1}{2}-1=-\frac{5}{4}$.

The critical point is $\left(-\frac{1}{2},-\frac{5}{4}\right)$.

Next, we apply the Second Derivative test.

$f''(x)=2$

$f''\left(-\frac{1}{2}\right)=2>0$

Therefore, $f\left(-\frac{1}{2}\right)=-\frac{5}{4}$ is a relative minimum.

4. $f(x)=x^2-x$

$f'(x)=2x-1$

$f''(x)=2$

$f'(x)$ exists for all real numbers. The only critical points occur when $f'(x)=0$.

$f'(x)=0$

$2x-1=0$

$x=\frac{1}{2}$

We find the function value at $x=\frac{1}{2}$.

$f\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)=\frac{1}{4}-\frac{1}{2}=-\frac{1}{4}$.

The critical point is $\left(\frac{1}{2},-\frac{1}{4}\right)$.

Next, we apply the Second Derivative test.

$f''(x)=2$

$f''\left(\frac{1}{2}\right)=2>0$

Therefore, $f\left(\frac{1}{2}\right)=-\frac{1}{4}$ is a relative minimum.

5. $f(x) = -4x^2 + 3x - 1$

First, find $f'(x)$ and $f''(x)$.

$f'(x) = -8x + 3$

$f''(x) = -8$

Next, find the critical points of $f(x)$. Since $f'(x)$ exists for all real numbers x, the only critical points occur when $f'(x) = 0$.

$$f'(x) = 0$$
$$-8x + 3 = 0$$
$$-8x = -3$$
$$x = \frac{3}{8}$$

We find the function value at $x = \frac{3}{8}$.

$$f\left(\frac{3}{8}\right) = -4\left(\frac{3}{8}\right)^2 + 3\left(\frac{3}{8}\right) - 1 = -4\left(\frac{9}{64}\right) + \frac{9}{8} - 1 = -\frac{9}{16} + \frac{18}{16} - \frac{16}{16} = -\frac{7}{16}.$$

The critical point is $\left(\frac{3}{8}, -\frac{7}{16}\right)$.

Next, we apply the Second Derivative test.

$f''(x) = -8$

$f''\left(\frac{3}{8}\right) = -8 < 0$

Therefore, $f\left(\frac{3}{8}\right) = -\frac{7}{16}$ is a relative maximum.

6. $f(x) = -5x^2 + 8x - 7$

$f'(x) = -10x + 8$

$f''(x) = -10$

$f'(x)$ exists for all real numbers. The only critical points occur when $f'(x) = 0$.

$$f'(x) = 0$$
$$-10x + 8 = 0$$
$$x = \frac{4}{5}$$

We find the function value at $x = \frac{4}{5}$.

$$f\left(\frac{4}{5}\right) = -5\left(\frac{4}{5}\right)^2 + 8\left(\frac{4}{5}\right) - 7 = -\frac{16}{5} + \frac{32}{5} - \frac{35}{5} = -\frac{19}{5}.$$

The critical point is $\left(\frac{4}{5}, -\frac{19}{5}\right)$.

Next, we apply the Second Derivative test.

$f''(x) = -10$

$f''\left(\frac{4}{5}\right) = -10 < 0$

Therefore, $f\left(\frac{4}{5}\right) = -\frac{19}{5}$ is a relative maximum.

7. $f(x) = x^3 - 12x - 1$

First, find $f'(x)$ and $f''(x)$.

$f'(x) = 3x^2 - 12$

$f''(x) = 6x$

Next, find the critical points of $f(x)$. Since $f'(x)$ exists for all real numbers x, the only critical points occur when $f'(x) = 0$.

$$f'(x) = 0$$
$$3x^2 - 12 = 0$$
$$3x^2 = 12$$
$$x^2 = 4$$
$$x = \pm\sqrt{4}$$
$$x = \pm 2$$

There are two critical values.

First, we find the function value at $x = -2$.

$$f(-2) = (-2)^3 - 12(-2) - 1 = -8 + 24 - 1 = 15.$$

The critical point is $(-2, 15)$.

Next, we apply the Second Derivative test.

$f''(x) = 6x$

$f''(-2) = 6(-2) = -12 < 0$

Therefore, $f(-2) = 15$ is a relative maximum.

The solution is continued on the next page.

Next, we find the function value at $x = 2$.

$$f(2) = (2)^3 - 12(2) - 1$$
$$= 8 - 24 - 1$$
$$= -17$$

The critical point is $(2, -17)$.

Next, we apply the Second Derivative test.

$f''(x) = 6x$

$f''(2) = 6(2) = 12 > 0$

Therefore, $f(2) = -17$ is a relative minimum.

8. $f(x) = 8x^3 - 6x + 1$

$f'(x) = 24x^2 - 6$

$f''(x) = 48x$

$f'(x)$ exists for all real numbers. The only critical points occur when $f'(x) = 0$.

$$f'(x) = 0$$
$$24x^2 - 6 = 0$$
$$4x^2 - 1 = 0$$
$$x^2 = \frac{1}{4}$$
$$x = \pm\frac{1}{2}$$

There are two critical values $x = -\frac{1}{2}$ and $x = \frac{1}{2}$.

We find the function value at $x = -\frac{1}{2}$

$$f\left(-\frac{1}{2}\right) = 8\left(-\frac{1}{2}\right)^3 - 6\left(-\frac{1}{2}\right) + 1$$
$$= 3$$

The critical point is $\left(-\frac{1}{2}, 3\right)$.

Next, we apply the Second Derivative test.

$f''(x) = 48x$

$f''\left(-\frac{1}{2}\right) = 48\left(-\frac{1}{2}\right) = -24 < 0$

Therefore, $f\left(-\frac{1}{2}\right) = 3$ is a relative maximum.

Now, we find the function value at $x = \frac{1}{2}$.

$$f\left(\frac{1}{2}\right) = 8\left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right) + 1$$
$$= -1$$

The critical point is $\left(\frac{1}{2}, -1\right)$.

Next, we apply the Second Derivative test.

$f''(x) = 48x$

$f''\left(\frac{1}{2}\right) = 48\left(\frac{1}{2}\right) = 24 > 0$

Therefore, $f\left(\frac{1}{2}\right) = -1$ is a relative minimum.

9. $f(x) = x^3 - 27x$

a) Find $f'(x)$ and $f''(x)$.

$f'(x) = 3x^2 - 27$

$f''(x) = 6x$

The domain of f is $\mathbb{R}$.

b) Find the critical points of $f(x)$. Since $f'(x)$ exists for all real numbers x, the only critical points occur when $f'(x) = 0$.

$$3x^2 - 27 = 0$$
$$x^2 = 9$$
$$x = \pm 3$$

There are two critical values $x = -3$ and $x = 3$.

$f(-3) = (-3)^3 - 27(-3) = 54$.

The critical point on the graph is $(-3, 54)$.

$f(3) = (3)^3 - 27(3) = -54$.

The critical point on the graph is $(3, -54)$.

c) We apply the Second Derivative test to the critical points.

For $x = -3$

$f''(x) = 6x$

$f''(-3) = 6(-3) = -18 < 0$

The critical point $(-3, 54)$ is a relative maximum.

For $x = 3$

$f''(x) = 6x$

$f''(3) = 6(3) = 18 > 0$

The critical point $(3, -54)$ is a relative minimum.

The solution is continued on the next page.

If we use the critical values $x=-3$ and $x=3$ to divide the real line into three intervals, $(-\infty,-3),(-3,3)$, and $(3,\infty)$, we know from the extrema above, that $f(x)$ is increasing over the interval $(-\infty,-3)$, decreasing over the interval $(-3,3)$ and then increasing again over the interval $(3,\infty)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so we solve the equation

$$f''(x)=0$$
$$6x=0$$
$$x=0$$

Therefore, a possible inflection point occurs at $x=0$.

$f(0)=(0)^3-27(0)=0$.

A possible inflection point on the graph is the point $(0,0)$.

e) To determine concavity, we use the possible inflection point to divide the real number line into two intervals $A:(-\infty,0)$ and B:$(0,\infty)$. We test a point in each interval.

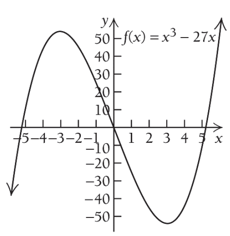
A: Test $-1: f''(-1)=6(-1)=-6<0$

B: Test $1: \quad f''(1)=6(1)=6>0$

Then, $f(x)$ is concave down on the interval $(-\infty,0)$ and concave up on the interval $(0,\infty)$, so $(0,0)$ is an inflection point.

f) Finally, we use the preceding information to sketch the graph of the function. Additional function values can also be calculated as needed.

x	$f(x)$
–4	44
–1	26
1	–26
4	–44



10. $f(x)=x^3-12x$

a) Find $f'(x)$ and $f''(x)$.

$$f'(x)=3x^2-12$$
$$f''(x)=6x$$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers. Solve:

$$f'(x)=0$$
$$f'(x)=0$$
$$3x^2-12=0$$
$$3x^2=12$$
$$x^2=4$$
$$x=\pm 2$$

There are two critical values $x=-2$ and $x=2$.

We find the function value at $x=-2$

$f(-2)=(-2)^3-12(-2)=16$.

The critical point on the graph is $(-2,16)$.

Next, we find the function value at $x=2$.

$f(2)=(2)^3-12(2)=-16$.

The critical point on the graph is $(2,-16)$.

c) Apply the Second Derivative test to the critical points.

For $x=-2$

$$f''(x)=6x$$
$$f''(-2)=6(-2)=-12<0$$

The critical point $(-2,16)$ is a relative maximum.

For $x=2$

$$f''(x)=6x$$
$$f''(2)=6(2)=12>0$$

We determine the critical point $(2,-16)$ is a relative minimum.

If we use the critical values $x=-2$ and $x=2$ to divide the real line into three intervals, $(-\infty,-2),(-2,2)$, and $(2,\infty)$, we know from the extrema above, that $f(x)$ is increasing over the interval $(-\infty,-2)$, decreasing over the interval $(-2,2)$ and then increasing again over the interval $(2,\infty)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so we solve the equation

$$f''(x)=0$$
$$6x=0$$
$$x=0$$

Therefore, a possible inflection point occurs at $x=0$.

The solution is continued on the next page.

We evaluate the function at $x=0$.

$f(0)=(0)^3-12(0)=0$.

This gives the point $(0,0)$ on the graph.

e) To determine concavity, we use the possible inflection point to divide the real number line into two intervals $A:(-\infty,0)$ and $B:(0,\infty)$. We test a point in each interval

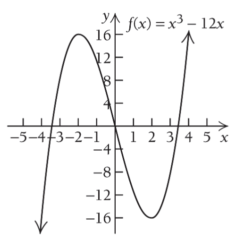
A: Test -1: $f''(-1)=6(-1)=-6<0$

B: Test 1: $f''(1)=6(1)=6>0$

Then, $f(x)$ is concave down on the interval $(-\infty,0)$ and concave up on the interval $(0,\infty)$, so $(0,0)$ is an inflection point.

f) Finally, we use the preceding information to sketch the graph of the function. Additional function values can also be calculated as needed.

x	$f(x)$
-3	9
-1	11
1	-11
3	-9



11. $f(x)=2x^3-3x^2-36x+28$

a) First, find $f'(x)$ and $f''(x)$.

$f'(x)=6x^2-6x-36$

$f''(x)=12x-6$

The domain of f is $\mathbb{R}$.

b) Find the critical points of $f(x)$. Since $f'(x)$ exists for all real numbers x, the only critical points occur when $f'(x)=0$.

$$f'(x)=0$$
$$6x^2-6x-36=0$$
$$x^2-x-6=0$$
$$(x-3)(x+2)=0$$
$$x-3=0 \text{ or } x+2=0$$
$$x=3 \text{ or } x=-2$$

There are two critical values $x=-2$ and $x=3$.

We find the function value at $x=-2$

$$f(-2)=2(-2)^3-3(-2)^2-36(-2)+28$$
$$=-16-12+72+28$$
$$=72$$

The critical point on the graph is $(-2,72)$.

Next, we find the function value at $x=3$.

$$f(3)=2(3)^3-3(3)^2-36(3)+28$$
$$=54-27-108+28$$
$$=-53$$

The critical point on the graph is $(3,-53)$.

c) Apply the Second Derivative test to the critical points.

For $x=-2$

$f''(x)=12x-6$

$f''(-2)=12(-2)-6=-30<0$

The critical point $(-2,72)$ is a relative maximum.

For $x=3$

$f''(x)=12x-6$

$f''(3)=12(3)-6=30>0$

The critical point $(3,-53)$ is a relative minimum.

We use the critical values $x=-2$ and $x=3$ to divide the real line into three intervals, A:$(-\infty,-2)$, B: $(-2,3)$, and C: $(3,\infty)$, we know from the extrema above, that $f(x)$ is increasing over the interval $(-\infty,-2)$, decreasing over the interval $(-2,3)$ and then increasing again over the interval $(3,\infty)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so we solve the equation

$$f''(x)=0$$
$$12x-6=0$$
$$x=\frac{1}{2}$$

Therefore, a possible inflection point occurs at $x=\frac{1}{2}$.

$$f\left(\frac{1}{2}\right)=2\left(\frac{1}{2}\right)^3-3\left(\frac{1}{2}\right)^2-36\left(\frac{1}{2}\right)+28$$
$$=\frac{19}{2}.$$

A possible inflection point on the graph is the point $\left(\frac{1}{2},\frac{19}{2}\right)$.

e) To determine concavity, we use the possible inflection point to divide the real number line into two intervals $A:\left(-\infty,\frac{1}{2}\right)$ and B: $\left(\frac{1}{2},\infty\right)$. We test a point in each interval

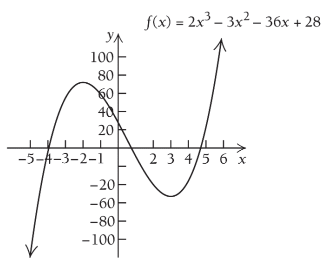
A: Test 0: $f''(0)=12(0)-6=-6<0$

B: Test 1: $f''(1)=12(1)-6=6>0$

Then, $f(x)$ is concave down on the interval $\left(-\infty,\frac{1}{2}\right)$ and concave up on the interval $\left(\frac{1}{2},\infty\right)$, so $\left(\frac{1}{2},\frac{19}{2}\right)$ is an inflection point.

f) We use the preceding information to sketch the graph of the function. Additional function values can also be calculated as needed.

x	$f(x)$
–3	55
–1	59
0	28
1	–9
4	–36



12. $f(x)=3x^3-36x-3$

a) First, find $f'(x)$ and $f''(x)$.

$f'(x)=9x^2-36$

$f''(x)=18x$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers. Solve:

$$f'(x)=0$$
$$9x^2-36=0$$
$$9x^2=36$$
$$x^2=4$$
$$x=\pm 2$$

There are two critical values $x=-2$ and $x=2$.

We find the function value at $x=-2$.

$f(-2)=3(-2)^3-36(-2)-3=45$.

The critical point on the graph is $(-2,45)$.

Next, we find the function value at $x=2$.

$f(2)=3(2)^3-36(2)-3=-51$.

The critical point on the graph is $(2,-51)$.

c) Apply the Second Derivative test to the critical points.

For $x=-2$

$f''(x)=18x$

$f''(-2)=18(-2)=-36<0$

The critical point $(-2,45)$ is a relative maximum.

For $x=2$

$f''(x)=18x$

$f''(2)=18(2)=36>0$

The critical point $(2,-51)$ is a relative minimum.

We use the critical values $x=-2$ and $x=2$ to divide the real line into three intervals, A: $(-\infty,-2)$, B: $(-2,2)$, and C: $(2,\infty)$, we know from the extrema above, that $f(x)$ is increasing over the interval $(-\infty,-2)$, decreasing over the interval $(-2,2)$ and then increasing again over the interval $(2,\infty)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so we solve the equation

$$f''(x)=0$$
$$18x=0$$
$$x=0$$

Therefore, a possible inflection point occurs at $x=0$.

$f(0)=3(0)^3-36(0)-3=-3$.

This gives the point $(0,-3)$ on the graph.

e) To determine concavity, we use the possible inflection point to divide the real number line into two intervals $A:(-\infty,0)$ and B: $(0,\infty)$. We test a point in each interval

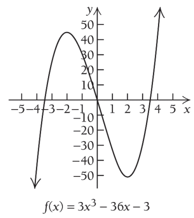
A: Test -1: $f''(-1)=18(-1)=-18<0$

B: Test 1: $f''(1)=18(1)=18>0$

Then, $f(x)$ is concave down on the interval $(-\infty,0)$ and concave up on the interval $(0,\infty)$, so $(0,-3)$ is an inflection point.

f) We use the preceding information to sketch the graph of the function. Additional function values can also be calculated as needed.

x	$f(x)$
–3	24
–1	30
1	–36
3	–30



13. $f(x) = 80 - 9x^2 - x^3$

a) First, find $f'(x)$ and $f''(x)$.

$f'(x) = -18x - 3x^2$

$f''(x) = -18 - 6x$

The domain of f is $\mathbb{R}$.

b) Next, find the critical points of $f(x)$.

Since $f'(x)$ exists for all real numbers x, the only critical points occur when $f'(x) = 0$.

$f'(x) = 0$

$-18x - 3x^2 = 0$

$-3x(x+6) = 0$

$-3x = 0$ or $x + 6 = 0$

$x = 0$ or $x = -6$

There are two critical values $x = -6$ and $x = 0$.

We find the function value at $x = -6$.

$f(-6) = 80 - 9(-6)^2 - (-6)^3$

$= 80 - 324 + 216$

$= -28$

The critical point on the graph is $(-6, -28)$.

Next, we find the function value at $x = 0$.

$f(0) = 80 - 9(0)^2 - (0)^3 = 80$.

The critical point on the graph is $(0, 80)$.

c) We Apply the Second Derivative test to the critical points.

For $x = -6$

$f''(x) = -18 - 6x$

$f''(-6) = -18 - 6(-6) = 18 > 0$

The critical point $(-6, -28)$ is a relative minimum.

For $x = 0$

$f''(x) = -18 - 6x$

$f''(0) = -18 - 6(0) = -18 < 0$

The critical point $(0, 80)$ is a relative maximum.

We use the critical values $x = -6$, and $x = 0$ to divide the real line into three intervals, A : $(-\infty, -6)$, B: $(-6, 0)$, and C: $(0, \infty)$, we know from the extrema above, that $f(x)$ is decreasing over the interval $(-\infty, -6)$, increasing over the interval $(-6, 0)$ and then decreasing again over the interval $(0, \infty)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so we solve the equation

$f''(x) = 0$

$-18 - 6x = 0$

$x = -3$

Therefore, a possible inflection point occurs at $x = -3$.

$f(-3) = 80 - 9(-3)^2 - (-3)^3 = 26$.

A possible inflection point on the graph is the point $(-3, 26)$.

e) To determine concavity, we use the possible inflection point to divide the real number line into two intervals $A: (-\infty, -3)$ and B: $(-3, \infty)$. We test a point in each interval

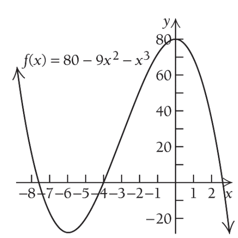
A: Test -4: $f''(-4) = -18 - 6(-4) = 6 > 0$

B: Test 0: $f''(0) = -18 - 6(0) = -18 < 0$

Then, $f(x)$ is concave up on the interval $(-\infty, -3)$ and concave down on the interval $(-3, \infty,)$, so $(-3, 26)$ is an inflection point.

f) We use the preceding information to sketch the graph of the function. Additional function values are also calculated.

x	$f(x)$
–9	80
–4	0
–2	52
0	80
2	36
3	–28



14. $f(x)=\frac{8}{3}x^3-2x+\frac{1}{3}$

a) Find $f'(x)$ and $f''(x)$.

$f'(x)=8x^2-2$

$f''(x)=16x$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers. Solve:

$$f'(x)=0$$
$$8x^2-2=0$$
$$8x^2=2$$
$$x^2=\frac{1}{4}$$
$$x=\pm\frac{1}{2}$$

There are two critical values $x=-\frac{1}{2}$ and $x=\frac{1}{2}$.

We find the function value at $x=-\frac{1}{2}$.

$$f\left(-\frac{1}{2}\right)=\frac{8}{3}\left(-\frac{1}{2}\right)^3-2\left(-\frac{1}{2}\right)+\frac{1}{3}$$
$$=1$$

The critical point on the graph is $\left(-\frac{1}{2},1\right)$.

Next, we find the function value at $x=\frac{1}{2}$.

$$f\left(\frac{1}{2}\right)=\frac{8}{3}\left(\frac{1}{2}\right)^3-2\left(\frac{1}{2}\right)+\frac{1}{3}.$$
$$=-\frac{1}{3}$$

The critical point on the graph is $\left(\frac{1}{2},-\frac{1}{3}\right)$.

c) Apply the Second Derivative test to the critical points.

For $x=-\frac{1}{2}$

$$f''(x)=16x$$
$$f''\left(-\frac{1}{2}\right)=16\left(-\frac{1}{2}\right)=-8<0$$

The critical point $\left(-\frac{1}{2},1\right)$ is a relative maximum.

For $x=\frac{1}{2}$

$$f''(x)=16x$$
$$f''\left(\frac{1}{2}\right)=16\left(\frac{1}{2}\right)=8>0$$

The critical point $\left(\frac{1}{2},-\frac{1}{3}\right)$ is a relative minimum.

Therefore, $f(x)$ is increasing over the interval $\left(-\infty,-\frac{1}{2}\right)$, decreasing over the interval $\left(-\frac{1}{2},\frac{1}{2}\right)$ and then increasing again over the interval $\left(\frac{1}{2},\infty\right)$.

d) We find the points of inflection. $f''(x)$ exists for all real numbers, so we solve the equation

$$f''(x)=0$$
$$16x=0$$
$$x=0$$

Therefore, a possible inflection point occurs at $x=0$.

$$f(0)=\frac{8}{3}(0)^3-2(0)+\frac{1}{3}=\frac{1}{3}.$$

This gives the point $\left(0,\frac{1}{3}\right)$ on the graph.

e) To determine concavity, we use the possible inflection point to divide the real number line into two intervals $A:(-\infty,0)$ and $B:(0,\infty)$. We test a point in each interval

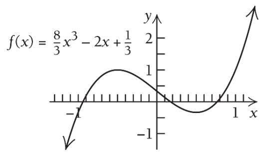
A: Test -1: $f''(-1)=16(-1)=-16<0$

B: Test 1: $f''(1)=16(1)=16>0$

Then, $f(x)$ is concave down on the interval $(-\infty,0)$ and concave up on the interval $(0,\infty)$, so $\left(0,\frac{1}{3}\right)$ is an inflection point.

f) We use the preceding information to sketch the graph of the function. Additional function values can also be calculated as needed.

x	$f(x)$
-2	-17
-1	$-\frac{1}{3}$
1	1
2	$\frac{53}{3}$



15. $f(x)=-x^3+3x-2$

a) First, find $f'(x)$ and $f''(x)$.

$f'(x)=-3x^2+3$

$f''(x)=-6x$

The domain of f is $\mathbb{R}$.

b) Find the critical points of $f(x)$. Since $f'(x)$ exists for all real numbers x, the only critical points occur when $f'(x)=0$.

$$f'(x)=0$$
$$-3x^2+3=0$$
$$-3(x^2-1)=0$$
$$x^2-1=0$$
$$x=\pm1$$

There are two critical values $x=-1$ and $x=1$.

We find the function value at $x=-1$.

$f(-1)=-(-1)^3+3(-1)-2=-4$.

The critical point on the graph is $(-1,-4)$.

We find the function value at $x=1$.

$f(1)=-(1)^3+3(1)-2=0$.

The critical point on the graph is $(1,0)$.

c) We apply the Second Derivative test to the critical points.

For $x=-1$

$f''(x)=-6x$

$f''(-1)=-6(-1)=6>0$

The critical point $(-1,-4)$ is a relative minimum.

For $x=1$

$f''(x)=-6x$

$f''(1)=-6(1)=-6<0$

The critical point $(1,0)$ is a relative maximum.

We use the critical values $x=-1$ and $x=1$ to divide the real line into three intervals, A : $(-\infty,-1)$, B: $(-1,1)$, and C: $(1,\infty)$, we know from the extrema above, that $f(x)$ is decreasing over the interval $(-\infty,-1)$, increasing over the interval $(-1,1)$ and then decreasing again over the interval $(1,\infty)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so we solve the equation

$$f''(x)=0$$
$$-6x=0$$
$$x=0$$

Therefore, a possible inflection point occurs at $x=0$.

$f(0)=-(0)^3+3(0)-2=-2$.

A possible inflection point on the graph is the point $(0,-2)$.

f) To determine concavity, we use the possible inflection point to divide the real number line into two intervals

$A:(-\infty,0)$ and B:$(0,\infty)$.

We test a point in each interval

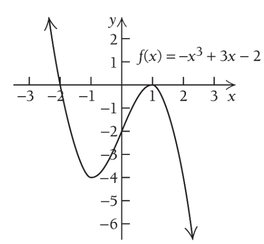
A: Test -1: $f''(-1)=-6(-1)=6>0$

B: Test 1: $f''(1)=-6(1)=-6<0$

Then, $f(x)$ is concave up on the interval $(-\infty,0)$ and concave down on the interval $(0,\infty)$, so $(0,-2)$ is an inflection point.

e) We use the preceding information to sketch the graph of the function. Additional function values can also be calculated as needed.

x	$f(x)$
-3	16
-2	0
2	-4
3	-20



16. $f(x)=-x^3+3x^2-4$

a) First, find $f'(x)$ and $f''(x)$.

$f'(x)=-3x^2+6x$

$f''(x)=-6x+6$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers. Solve:

$$f'(x)=0$$
$$-3x^2+6x=0$$
$$-3x(x-2)=0$$
$$-3x=0 \quad \text{or} \quad x-2=0$$
$$x=0 \quad \text{or} \quad x=2$$

There are two critical values $x=0$ and $x=2$.

We find the function value at $x=0$.

$$f(0)=-(0)^3+3(0)^2-4=-4$$

The critical point on the graph is $(0,-4)$.

Next, we find the function value at $x=2$.

$$f(2)=-(2)^3+3(2)^2-4=0.$$

The critical point on the graph is $(2,0)$.

c) Apply the Second Derivative test to the critical points.

For $x=0$

$$f''(x)=-6x+6$$
$$f''(0)=-6(0)+6=6>0$$

The critical point $(0,-4)$ is a relative minimum.

For $x=2$

$$f''(x)=-6x+6$$
$$f''(2)=-6(2)+6=-6<0$$

The critical point $(2,0)$ is a relative maximum.

Therefore, $f(x)$ is decreasing over the interval $(-\infty,0)$, increasing over the interval $(0,2)$ and then decreasing again over the interval $(2,\infty)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so we solve the equation

$$f''(x)=0$$
$$-6x+6=0$$
$$x=1$$

Therefore, a possible inflection point occurs at $x=1$.

$$f(1)=-(1)^3+3(1)^2-4$$
$$=-1+3-4$$
$$=-2$$

This gives the point $(1,-2)$ on the graph.

e) To determine concavity, we use the possible inflection point to divide the real number line into two intervals $A:(-\infty,1)$ and B:$(1,\infty)$. We test a point in each interval

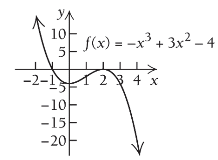
A: Test $0: f''(0)=-6(0)+6=6>0$

B: Test $2: f''(2)=-6(2)+6=-6<0$

Then, $f(x)$ is concave up on the interval $(-\infty,1)$ and concave down on the interval $(1,\infty)$, so $(1,-2)$ is an inflection point.

f) We use the preceding information to sketch the graph of the function. Additional function values can also be calculated as needed.

x	$f(x)$
-2	16
-1	0
3	-4
4	-20



17. $f(x)=3x^4-16x^3+18x^2$

a) First, find $f'(x)$ and $f''(x)$.

$$f'(x)=12x^3-48x^2+36x$$
$$f''(x)=36x^2-96x+36$$

The domain of f is $\mathbb{R}$.

b) Next, find the critical points of $f(x)$.

Since $f'(x)$ exists for all real numbers x, the only critical points occur when $f'(x)=0$.

$$f'(x)=0$$
$$12x^3-48x^2+36x=0$$
$$12x(x^2-4x+3)=0$$
$$12x(x-1)(x-3)=0$$
$$12x=0 \quad \text{or} \quad x-1=0 \quad \text{or} \quad x-3=0$$
$$x=0 \quad \text{or} \quad x=1 \quad \text{or} \quad x=3$$

There are three critical values $x=0$, $x=1$, and $x=3$.

Then

$$f(0)=3(0)^4-16(0)^3+18(0)^2=0$$
$$f(1)=3(1)^4-16(1)^3+18(1)^2=5$$
$$f(3)=3(3)^4-16(3)^3+18(3)^2=-27$$

Thus, the critical points $(0,0)$, $(1,5)$, and $(3,-27)$ are on the graph.

c) Apply the Second Derivative test to the critical points.

$f''(0)=36(0)^2-96(0)+36=36>0$

The critical point $(0,0)$ is a relative minimum.

$f''(1)=36(1)^2-96(1)+36=-24<0$

The critical point $(1,5)$ is a relative maximum.

$f''(3)=36(3)^2-96(3)+36=72>0$

The critical point $(3,-27)$ is a relative minimum.

We use the critical values $0, 1,$ and 3 to divide the real line into four intervals, $A:(-\infty,0)$, B:$(0,1)$, C:$(1,3)$ and D:$(3,\infty)$, we know from the extrema above, that $f(x)$ is decreasing over the intervals $(-\infty,0)$ and $(1,3)$ and $f(x)$ increasing over the intervals $(0,1)$ and $(3,\infty)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so we solve the equation $f''(x)=0$.

$$f''(x)=0$$
$$36x^2-96x+36=0$$
$$12\left(3x^2-8x+3\right)=0$$
$$3x^2-8x+3=0$$

Using the quadratic formula, we find that $x=\dfrac{4\pm\sqrt{7}}{3}$, so $x\approx 0.451$ or $x\approx 2.215$ are possible inflection points.

$f(0.451)\approx 2.321$

$f(2.215)\approx -13.358$

So, $(0.451, 2.321)$ and $(2.215, -13.358)$ are two more points on the graph.

e) To determine concavity, we use the possible inflection point to divide the real number line into three intervals $A:(-\infty,0.451)$, B:$(0.451, 2.215)$, and C:$(2.215,\infty)$.

We test a point in each interval to determine the sign of the second derivative.

A: Test 0:

$f''(0)=36(0)^2-96(0)+36=36>0$

B: Test 1:

$f''(1)=36(1)^2-96(1)+36=-24<0$

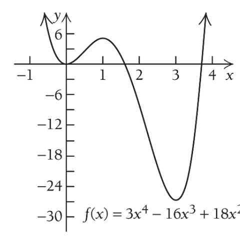
C: Test 3:

$f''(3)=36(3)^2-96(3)+36=72>0$

Then, $f(x)$ is concave up on the interval $(-\infty,0.451)$ and concave down on the interval $(0.451, 2.215)$ and concave up on the interval $(2.215,\infty)$, so $(0.451, 2.321)$ and $(2.215, -13.358)$ are inflection points.

f) We use the preceding information to sketch the graph of the function. Additional function values can also be calculated as needed.

x	$f(x)$
-1	37
2	-8
4	32



18. $f(x)=3x^4+4x^3-12x^2+5$

a) $f'(x)=12x^3+12x^2-24x$

$f''(x)=36x^2+24x-24$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers. Solve

$$f'(x)=0$$
$$12x^3+12x^2-24x=0$$
$$12x\left(x^2+x-2\right)=0$$
$$12x(x+2)(x-1)=0$$
$$12x=0 \quad \text{or} \quad x+2=0 \quad \text{or} \quad x-1=0$$
$$x=0 \quad \text{or} \quad x=-2 \quad \text{or} \quad x=1$$

There are three critical values $x=-2$, $x=0$, and $x=1$. Then

$$f(-2)=3(-2)^4+4(-2)^3-12(-2)^2+5$$
$$=-27$$
$$f(0)=3(0)^4+4(0)^3-12(0)^2+5=5$$
$$f(1)=3(1)^4+4(1)^3-12(1)^2+5=0$$

Thus, the critical points $(-2,-27)$, $(0,5)$, and $(1,0)$ are on the graph.

c) Apply the Second Derivative test to the critical points.

$f''(-2) = 36(-2)^2 + 24(-2) - 24 = 72 > 0$

The critical point $(-2, -27)$ is a relative minimum.

$f''(0) = 36(0)^2 + 24(0) - 24 = -24 < 0$

The critical point $(0, 5)$ is a relative maximum.

$f''(1) = 36(1)^2 + 24(1) - 24 = 36 > 0$

The critical point $(1, 0)$ is a relative minimum.

Then $f(x)$ is decreasing over the intervals $(-\infty, -2)$ and $(0, 1)$ and $f(x)$ increasing over the intervals $(-2, 0)$ and $(1, \infty)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so we solve the equation

$$f''(x) = 0$$

$$36x^2 + 24x - 24 = 0$$

Using the quadratic formula, we find that $x = \dfrac{-1 \pm \sqrt{7}}{3}$, so $x \approx -1.215$ or $x \approx 0.549$ are possible inflection points.

$f(-1.215) \approx -13.358$

$f(0.549) \approx 2.321$

So, $(-1.215, -13.358)$ and $(0.549, 2.321)$ are two more points on the graph.

e) To determine concavity, we use the possible inflection points to divide the real number line into three intervals $A: (-\infty, -1.215)$, B: $(-1.215, 0.549)$, and C: $(0.549, \infty)$.

We test a point in each interval

A: Test $-2: f''(-2) = 72 > 0$

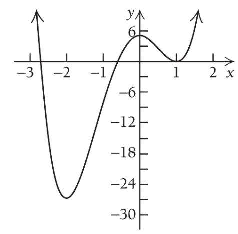
B: Test $0: f''(0) = -24 < 0$

C: Test 1: $f''(1) = 36 > 0$

Then, $f(x)$ is concave up on the interval $(-\infty, -1.215)$ and concave down on the interval $(-1.215, 0.549)$ and concave up on the interval $(0.549, \infty)$, so $(-1.215, -13.358)$ and $(0.549, 2.321)$ are inflection points.

f) We use the preceding information to sketch the graph of the function. Additional function values can also be calculated as needed.

x	$f(x)$
-3	32
-1	-8
2	37



$f(x) = 3x^4 + 4x^3 - 12x^2 + 5$

19. $f(x) = x^4 - 6x^2$

a) First, find $f'(x)$ and $f''(x)$.

$f'(x) = 4x^3 - 12x$

$f''(x) = 12x^2 - 12$

The domain of f is $\mathbb{R}$.

b) Find the critical points of $f(x)$. Since $f'(x)$ exists for all real numbers x, the only critical points occur when $f'(x) = 0$.

$$f'(x) = 0$$

$$4x^3 - 12x = 0$$

$$4x(x^2 - 3) = 0$$

$$4x = 0 \quad \text{or} \quad x^2 - 3 = 0$$

$$x = 0 \quad \text{or} \quad x = \pm\sqrt{3}$$

There are three critical values $-\sqrt{3}$, 0, and $\sqrt{3}$.

Then

$$f(-\sqrt{3}) = (-\sqrt{3})^4 - 6(-\sqrt{3})^2 = 9 - 6(3) = -9$$

$$f(0) = (0)^4 - 6(0)^2 = 0$$

$$f(\sqrt{3}) = (\sqrt{3})^4 - 6(\sqrt{3})^2 = 9 - 6(3) = -9$$

Thus, the critical points $(-\sqrt{3}, -9)$, $(0, 0)$, $(\sqrt{3}, -9)$ and are on the graph.

c) Apply the Second Derivative test to the critical points.

$f''\left(-\sqrt{3}\right)=12\left(-\sqrt{3}\right)^2-12$

$=12(3)-12=24>0$

The critical point $\left(-\sqrt{3},-9\right)$ is a relative minimum.

$f''(0)=12(0)^2-12=-12<0$

The critical point $(0,0)$ is a relative maximum.

$f''\left(\sqrt{3}\right)=12\left(\sqrt{3}\right)^2-12$

$=12(3)-12=24>0$

The critical point $\left(\sqrt{3},-9\right)$ is a relative minimum.

If we use the critical values $-\sqrt{3}$, 0, and $\sqrt{3}$ to divide the real line into four intervals, A: $\left(-\infty,-\sqrt{3}\right)$, B: $\left(-\sqrt{3},0\right)$, C: $\left(0,\sqrt{3}\right)$, and D: $\left(\sqrt{3},\infty\right)$

Then $f(x)$ is decreasing over the intervals $\left(-\infty,-\sqrt{3}\right)$ and $\left(0,\sqrt{3}\right)$, and $f(x)$ increasing over the intervals $\left(-\sqrt{3},0\right)$ and $\left(\sqrt{3},\infty\right)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so we solve the equation

$f''(x)=0$

$12x^2-12=0$

$x^2-1=0$

$x^2=1$

$x=\pm 1$

So $x=-1$ or $x=1$ are possible inflection points.

$f(-1)=(-1)^4-6(-1)^2=1-6=-5$

$f(1)=(1)^4-6(1)^2=1-6=-5$

So, $(-1,-5)$ and $(1,-5)$ are two more points on the graph.

e) To determine concavity, we use the possible inflection point to divide the real number line into three intervals $A:(-\infty,-1)$, B: $(-1,1)$, and C: $(1,\infty)$. We test a point in each interval

A: Test -2:

$f''(-2)=12(-2)^2-12=36>0$

B: Test 0:

$f''(0)=12(0)^2-12=-12<0$

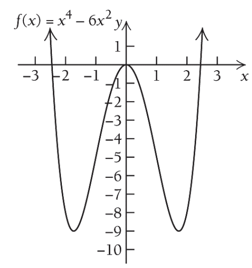
C: Test 2:

$f''(2)=12(2)^2-12=36>0$

Then, $f(x)$ is concave up on the intervals $(-\infty,-1)$ and $(1,\infty)$ and concave down on the interval $(-1,1)$, so $(-1,-5)$ and $(1,-5)$ are inflection points.

f) We use the preceding information to sketch the graph of the function. Additional function values can also be calculated as needed.

x	$f(x)$
–3	27
–2	–8
2	–8
3	27



20. $f(x)=2x^2-x^4$

a) $f'(x)=4x-4x^3$

$f''(x)=4-12x^2$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers. Solve

$f'(x)=0$

$4x-4x^3=0$

$4x\left(1-x^2\right)=0$

$4x(1-x)(1+x)=0$

$4x=0$ or $1-x=0$ or $1+x=0$

$x=0$ or $x=1$ or $x=-1$

There are three critical values $x=-1$, $x=0$, and $x=1$.

Then

$f(-1)=2(-1)^2-(-1)^4=1$

$f(0)=2(0)^2-(0)^4=0$

$f(1)=2(1)^2-(1)^4=1$

Thus, the critical points $(-1,1)$, $(0,0)$, and $(1,1)$ are on the graph.

c) Apply the Second Derivative test to the critical points.

$f''(-1) = 4 - 12(-1)^2 = -8 < 0$

The critical point $(-1, 1)$ is a relative maximum.

$f''(0) = 4 - 12(0)^2 = 4 > 0$

The critical point $(0, 0)$ is a relative minimum.

$f''(1) = 4 - 12(1)^2 = -8 < 0$

The critical point $(1, 1)$ is a relative maximum.

Then $f(x)$ is increasing over the intervals $(-\infty, -1)$ and $(0, 1)$, and $f(x)$ is decreasing over the intervals $(-1, 0)$ and $(1, \infty)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so we solve the equation

$$f''(x) = 0$$
$$4 - 12x^2 = 0$$
$$x^2 = \frac{1}{3}$$
$$x = \pm\sqrt{\frac{1}{3}} = \pm\frac{1}{\sqrt{3}}$$

So at $x = -\frac{1}{\sqrt{3}}$ and $x = \frac{1}{\sqrt{3}}$ there exists possible inflection points.

$$f\left(-\frac{1}{\sqrt{3}}\right) = \frac{5}{9}$$
$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{5}{9}.$$

So, $\left(-\frac{1}{\sqrt{3}}, \frac{5}{9}\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{5}{9}\right)$ are two more points on the graph.

e) To determine concavity, we use the possible inflection points to divide the real number line into three intervals $A: \left(-\infty, -\frac{1}{\sqrt{3}}\right)$, B: $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, and C: $\left(\frac{1}{\sqrt{3}}, \infty\right)$.

We test a point in each interval

A: Test -1: $f''(-1) = -8 < 0$

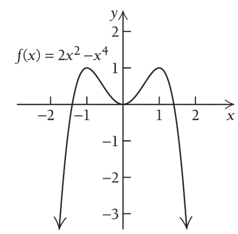
B: Test 0: $f''(0) = 4 > 0$

C: Test 1: $f''(1) = -8 < 0$

Therefore, $f(x)$ is concave down on the intervals $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$ and $f(x)$ is concave up on the interval $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, so $\left(-\frac{1}{\sqrt{3}}, \frac{5}{9}\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{5}{9}\right)$ are inflection points.

f) We use the preceding information to sketch the graph of the function. Additional function values can also be calculated as needed.

x	$f(x)$
-2	-8
-1	1
0	0
1	1
2	-8



21. $f(x) = x^3 - 6x^2 + 9x + 1$

a) First, find $f'(x)$ and $f''(x)$.

$f'(x) = 3x^2 - 12x + 9$

$f''(x) = 6x - 12$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all values of x, so the only critical points of f are where $f'(x) = 0$.

$$f'(x) = 0$$
$$3x^2 - 12x + 9 = 0$$
$$3(x^2 - 4x + 3) = 0$$
$$3(x-1)(x-3) = 0$$
$$x - 1 = 0 \quad \text{or} \quad x - 3 = 0$$
$$x = 1 \quad \text{or} \quad x = 3$$

The critical values are 1 and 3.

Then,

$f(1) = (1)^3 - 6(1)^2 + 9(1) + 1 = 5$

$f(3) = (3)^3 - 6(3)^2 + 9(3) + 1 = 1$

So, $(1, 5)$ and $(3, 1)$ are on the graph.

c) Applying the Second Derivative Test, we have:

$f''(1) = 6(1) - 12 = -6 < 0$

$f''(3) = 6(3) - 12 = 6 > 0$

Therefore, $(1, 5)$ is a relative maximum and $(3, 1)$ is a relative minimum.

The solution is continued on the next page.

If we use the points 1 and 3 to divide the real number line into three intervals, $(-\infty,1)$, $(1,3)$, and $(3,\infty)$. Using the extrema, we know $f(x)$ is increasing on the intervals $(-\infty,1)$ and $(3,\infty)$, and $f(x)$ is decreasing on the interval $(1,3)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers. Solve

$$f''(x)=0$$
$$6x-12=0$$
$$6x=12$$
$$x=2$$

There is a possible inflection point at $x=2$.

$$f(2)=(2)^3-6(2)^2+9(2)+1=3$$

Thus, the point $(2,3)$ on the graph is a possible inflection point.

e) To determine concavity, we use 2 to divide the real number line into two intervals, A: $(-\infty,2)$ and B: $(2,\infty)$. Then test a point in each interval.

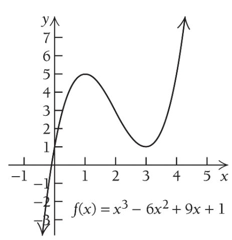
A: Test 0, $f''(0)=6(0)-12=-12<0$

B: Test 3, $f''(3)=6(3)-12=6>0$

Thus, $f(x)$ is concave down on the interval $(-\infty,2)$ and concave up on the interval $(2,\infty)$ and $(2,3)$ is an inflection point.

f) We sketch the graph. Additional points may be found as necessary.

x	$f(x)$
-2	-49
-1	-15
0	1
4	5
5	21



22. $f(x)=x^3-2x^2-4x+3$

a) $f'(x)=3x^2-4x-4$

$f''(x)=6x-4$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers. Solve

$$3x^2-4x-4=0$$
$$(3x+2)(x-2)=0$$
$$3x+2=0 \quad \text{or} \quad x-2=0$$
$$x=-\frac{2}{3} \quad \text{or} \quad x=2$$

The critical values are $-\frac{2}{3}$ and 2.

$$f\left(-\frac{2}{3}\right)=\left(-\frac{2}{3}\right)^3-2\left(-\frac{2}{3}\right)^2-4\left(-\frac{2}{3}\right)+3$$
$$=-\frac{8}{27}-\frac{8}{9}+\frac{8}{3}+3$$
$$=\frac{121}{27}$$

$$f(2)=(2)^3-2(2)^2-4(2)+3=-5$$

The critical points $\left(-\frac{2}{3},\frac{121}{27}\right)$ and $(2,-5)$ are on the graph.

c) Applying the Second Derivative Test, we have:

$$f''\left(-\frac{2}{3}\right)=6\left(-\frac{2}{3}\right)-4=-4-4=-8<0$$

$\left(-\frac{2}{3},\frac{121}{27}\right)$ is a relative maximum.

$$f''(2)=6(2)-4=12-4=8>0$$

$(2,-5)$ is a relative minimum.

Therefore $f(x)$ is increasing on $\left(-\infty,-\frac{2}{3}\right)$ and on $(2,\infty)$ and $f(x)$ is decreasing on $\left(-\frac{2}{3},2\right)$.

d) Find the points of inflection. $f''(x)$ exists for all values of $f(x)$, so the only possible inflection points occur when $f''(x)=0$.

$$6x-4=0$$
$$x=\frac{4}{6}=\frac{2}{3}$$

There is a possible inflection point at $x=\frac{2}{3}$.

$$f\left(\frac{2}{3}\right)=\left(\frac{2}{3}\right)^3-2\left(\frac{2}{3}\right)^2-4\left(\frac{2}{3}\right)+3$$
$$=\frac{8}{27}-\frac{8}{9}-\frac{8}{3}+3$$
$$=-\frac{7}{27}$$

Another point on the graph is $\left(\frac{2}{3},-\frac{7}{27}\right)$.

e) To determine concavity we use $\frac{2}{3}$ to divide the real number line into two intervals, A: $\left(-\infty, \frac{2}{3}\right)$ and B: $\left(\frac{2}{3}, \infty\right)$. Then test a point in each interval.

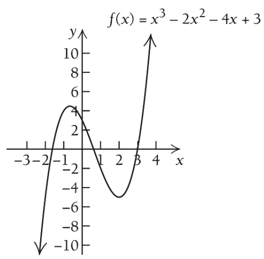
A: Test 0, $f''(0) = 6(0) - 4 = -4 < 0$

B: Test 1, $f''(1) = 6(1) - 4 = 2 > 0$

We see that f is concave down on $\left(-\infty, \frac{2}{3}\right)$ and concave up on $\left(\frac{2}{3}, \infty\right)$, so $\left(\frac{2}{3}, -\frac{7}{27}\right)$ is an inflection point.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
−2	−5
−1	4
0	3
1	−2
3	0
4	19



23. $f(x) = x^4 - 2x^3$

a) First, find $f'(x)$ and $f''(x)$.

$f'(x) = 4x^3 - 6x^2$

$f''(x) = 12x^2 - 12x$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all values of x, so the only critical points of f are where $f'(x) = 0$.

$$f'(x) = 0$$
$$4x^3 - 6x^2 = 0$$
$$2x^2(2x - 3) = 0$$
$$2x^2 = 0 \quad \text{or} \quad 2x - 3 = 0$$
$$x = 0 \quad \text{or} \quad x = \frac{3}{2}$$

The critical values are 0 and $\frac{3}{2}$.

$f(0) = (0)^4 - 2(0)^3 = 0$

$f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^4 - 2\left(\frac{3}{2}\right)^3 = -\frac{27}{16}$

So, $(0,0)$ and $\left(\frac{3}{2}, -\frac{27}{16}\right)$ are on the graph.

c) Applying the Second Derivative Test, we have:

$f''(0) = 12(0)^2 - 12(0) = 0$

The Second Derivative Test fails, we will have to use the First Derivative Test for $x = 0$. Divide $\left(-\infty, \frac{3}{2}\right)$ into two intervals, A: $(-\infty, 0)$ and B: $\left(0, \frac{3}{2}\right)$, and test a point in each interval.

A: Test -1,

$f'(-1) = 4(-1)^3 - 6(-1)^2 = -10 < 0$

B: Test 1,

$f'(1) = 4(1)^3 - 6(1)^2 = -2 < 0$

Since, f is decreasing on both intervals, $(0,0)$ is not a relative extremum.

We use the Second Derivative Test for $x = \frac{3}{2}$.

$f''\left(\frac{3}{2}\right) = 12\left(\frac{3}{2}\right)^2 - 12\left(\frac{3}{2}\right) = 9 > 0$

Therefore, $\left(\frac{3}{2}, -\frac{27}{16}\right)$ is a relative minimum.

Thus, $f(x)$ is increasing on the interval $\left(\frac{3}{2}, \infty\right)$, and $f(x)$ is decreasing on the intervals $(-\infty, 0)$ and $\left(0, \frac{3}{2}\right)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers. Solve

$$f''(x) = 0$$
$$12x^2 - 12x = 0$$
$$12x(x - 1) = 0$$
$$12x = 0 \quad \text{or} \quad x - 1 = 0$$
$$x = 0 \quad \text{or} \quad x = 1$$

There are a possible inflection points at $x = 0$ and $x = 1$.

$f(0) = 0$ found earlier

$f(1) = (1)^4 - 2(1)^3 = -1$

Thus, the points $(0,0)$ and $(1,-1)$ on the graph are possible inflection points.

e) Use 0 and 1 to divide the real number line into two intervals, A: $(-\infty,0)$, B: $(0,1)$, and C: $(1,\infty)$. Then test a point in each interval.

A: Test -1,

$$f''(-1)=12(-1)^2-12(-1)=24>0$$

B: Test $\frac{1}{2}$,

$$f''\left(\frac{1}{2}\right)=12\left(\frac{1}{2}\right)^2-12\left(\frac{1}{2}\right)=-3<0$$

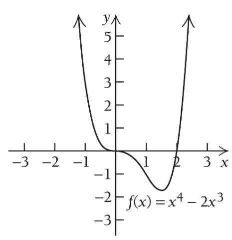
C: Test 2,

$$f''(2)=12(2)^2-12(2)=24>0$$

Thus, $f(x)$ is concave up on the intervals $(-\infty,0)$ and $(1,\infty)$ and concave down on the interval $(0,1)$. Also, the points $(0,0)$ and $(1,-1)$ are inflection points.

f) Using the preceding information, we sketch the graph. Additional points may be found as necessary.

x	$f(x)$
-2	32
-1	3
2	0
3	27



24. $f(x)=3x^4+4x^3$

a) $f'(x)=12x^3+12x^2$

$f''(x)=36x^2+24x$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers. Solve

$$12x^3+12x^2=0$$
$$12x^2(x+1)=0$$
$$12x^2=0 \quad \text{or} \quad x+1=0$$
$$x=0 \quad \text{or} \quad x=-1$$

The critical values are -1 and 0.

$$f(-1)=3(-1)^4+4(-1)^3=-1$$
$$f(0)=3(0)^4+4(0)^3=0$$

The critical points $(-1,-1)$ and $(0,0)$ are on the graph.

c) Applying the Second Derivative Test, we have:

$$f''(-1)=36(-1)^2+24(-1)=36-24$$
$$=12>0$$

So $(-1,-1)$ is a relative minimum.

$$f''(0)=36(0)^2+24(0)=0$$

The test fails. We will use the First Derivative Test. We use 0 to divide the interval $(-1,\infty)$ into two intervals,

A: $(-1,0)$ and B: $(0,\infty)$, and test a point in each interval.

A: Test $-\frac{1}{2}$,

$$f'\left(-\frac{1}{2}\right)=12\left(-\frac{1}{2}\right)^3+12\left(-\frac{1}{2}\right)^2$$
$$=\frac{3}{2}>0$$

B: Test 2,

$$f'(2)=12(2)^3+12(2)^2=144>0$$

f is increasing on both intervals $(-1,0)$ and $(0,\infty)$. Therefore, $(0,0)$ is not a relative extremum. Since $(-1,-1)$ is a relative minimum, we know that f is decreasing on $(-\infty,-1)$.

d) Find the points of inflection. $f''(x)$ exists for all values of x, so the only possible inflection points occur when $f''(x)=0$.

$$f''(x)=0$$
$$36x^2+24x=0$$
$$12x(3x+2)=0$$
$$12x=0 \quad \text{or} \quad 3x+2=0$$
$$x=0 \quad \text{or} \quad x=-\frac{2}{3}$$

There are a possible inflection points at $x=-\frac{2}{3}$ and $x=0$.

$$f\left(-\frac{2}{3}\right)=3\left(-\frac{2}{3}\right)^4+4\left(-\frac{2}{3}\right)^3=-\frac{16}{27}$$
$$f(0)=3(0)^4+4(0)^3=0$$

This gives one additional point $\left(-\frac{2}{3},-\frac{16}{27}\right)$ on the graph.

e) To determine concavity we use $-\frac{2}{3}$ and 0 to divide the real number line into three intervals, $A:\left(-\infty,-\frac{2}{3}\right)$, B: $\left(-\frac{2}{3},0\right)$, and C: $(0,\infty)$. Then test a point in each interval.

A: Test -1,

$f''(-1)=36(-1)^2+24(-1)=12>0$

B: Test $-\frac{1}{2}$,

$$f''\left(-\frac{1}{2}\right)=36\left(-\frac{1}{2}\right)^2+24\left(-\frac{1}{2}\right)$$
$$=-3<0$$

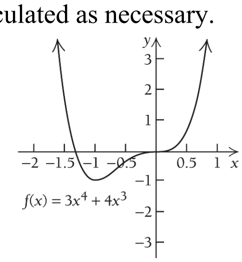
C: Test 1,

$f''(1)=36(1)^2+24(1)=60>0$

We see that f is concave up on the intervals $\left(-\infty,-\frac{2}{3}\right)$ and $(0,\infty)$, and concave down on the interval $\left(-\frac{2}{3},0\right)$, so both $\left(-\frac{2}{3},-\frac{16}{27}\right)$ and $(0,0)$ are inflection points.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-2	16
1	7
2	80



25. $f(x)=x^3-6x^2-135x$

a) $f'(x)=3x^2-12x-135$

$f''(x)=6x-12$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all values of x, so the only critical points of f are where $f'(x)=0$.

$$3x^2-12x-135=0$$
$$x^2-4x-45=0$$
$$(x-9)(x+5)=0$$
$$x-9=0 \quad \text{or} \quad x+5=0$$
$$x=9 \quad \text{or} \quad x=-5$$

The critical values are -5 and 9.

The function values are

$$f(-5)=(-5)^3-6(-5)^2-135(-5)$$
$$=-125-150+675$$
$$=400$$
$$f(9)=(9)^3-6(9)^2-135(9)$$
$$=729-486-1215$$
$$=-972$$

The critical points $(-5,400)$ and $(9,-972)$ are on the graph.

c) Applying the Second Derivative Test, we have:

$$f''(-5)=6(-5)-12=-30-12$$
$$=-42<0$$

The critical point $(-5,400)$ is a relative maximum.

$$f''(9)=6(9)-12=54-12$$
$$=42>0$$

The critical point $(9,-972)$ is a relative minimum.

If we use the points -5 and 9 to divide the real number line into three intervals $(-\infty,-5)$, $(-5,9)$, and $(9,\infty)$ we see that $f(x)$ is increasing on the intervals $(-\infty,-5)$ and $(9,\infty)$ and $f(x)$ is decreasing on the interval $(-5,9)$.

d) Find the points of inflection. $f''(x)$ exists for all values of x, so the only possible inflection points occur when $f''(x)=0$. We set the second derivative equal to zero and find possible inflection points.

$$6x-12=0$$
$$6x=12$$
$$x=2$$

The only possible inflection point is 2.

$$f(2)=(2)^3-6(2)^2-135(2)$$
$$=8-24-270$$
$$=-286$$

The point $(2,-286)$ is a possible inflection point on the graph.

e) To determine concavity we use 2 to divide the real number line into two intervals, $A:(-\infty,2)$ and B: $(2,\infty)$, Then test a point in each interval at the top of the next page.

Using the information from the previous page, we test a point in each interval.

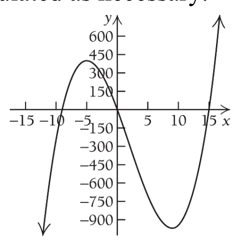
A: Test 0, $f''(0)=6(0)-12=-12<0$

B: Test 3, $f''(3)=6(3)-12=6>0$

We see that f is concave down on the interval $(-\infty,2)$ and concave up on the interval $(2,\infty)$, Therefore $(2,-286)$ is an inflection point.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-11	-572
-10	-250
-3	324
0	0
2	-286
5	-700
15	0
16	400



$f(x)=x^3-6x^2-135x$

26. $f'(x)=x^3-3x^2-144x-140$

a) $f'(x)=3x^2-6x-144$

$f''(x)=6x-6$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers. Solve:

$$f'(x)=0$$
$$3x^2-6x-144=0$$
$$x^2-2x-48=0$$
$$(x+6)(x-8)=0$$
$$x+6=0 \quad \text{or} \quad x-8=0$$
$$x=-6 \quad \text{or} \quad x=8 \quad \text{Critical Values}$$

$f(-6)=400$

$f(8)=-972$

So the critical points $(-6,400)$ and $(8,-972)$ are on the graph.

c) Using the Second Derivative Test, we have:

$f''(-6)=-42<0$

$f''(8)=42>0$

So, $(-6,400)$ is a relative maximum and $(8,-972)$ is a relative minimum.

Therefore, $f(x)$ is increasing on the intervals $(-\infty,-6)$ and $(8,\infty)$ and $f(x)$ is decreasing on the interval $(-6,8)$.

d) Find the inflection points. $f''(x)$ exists for all real numbers. Solve:

$$f''(x)=0$$
$$6x-6=0$$
$$x=1$$
$$f(1)=-286$$

The possible inflection point $(1,-286)$ is on the graph.

e) To determine concavity, use 1 to divide the real number line into two intervals, A: $(-\infty,1)$ and B: $(1,\infty)$ and test a point in each interval.

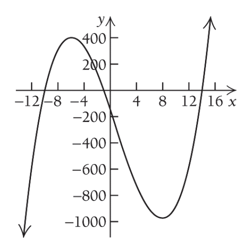
A: Test 0, $f''(0)=6(0)-6=-6<0$

B: Test 2, $f''(2)=6(2)-6=6>0$

Therefore, $f(x)$ is concave down on the interval $(-\infty,1)$ and concave up on the interval $(1,\infty)$. The point $(1,-286)$ is an inflection point.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-12	-572
-10	0
-3	238
0	-140
4	-700
14	0
15	400



$f(x)=x^3-3x^2-144x-140$

27. $f(x)=x^4-4x^3+10$

a) $f'(x)=4x^3-12x^2$

$f''(x)=12x^2-24x$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all values of x, so the only critical points of f are where $f'(x)=0$.

$4x^3-12x^2=0$

$4x^2(x-3)=0$

$4x^2=0$ or $x-3=0$

$x=0$ or $x=3$

The critical values are 0 and 3.

$f(0)=(0)^4-4(0)^3+10=10$

$f(3)=(3)^4-4(3)^3+10=-17$

The critical points $(0,10)$ and $(3,-17)$ are on the graph.

c) Applying the Second Derivative Test, we have:

$f''(0)=12(0)^2-24(0)=0$

The test fails, we will use the First Derivative Test.

Divide $(-\infty,3)$ into two intervals,

A: $(-\infty,0)$ and B: $(0,3)$, and test a point in each interval.

A: Test -1,

$f'(-1)=4(-1)^3-12(-1)^2=-16<0$

B: Test 1, $f'(1)=4(1)^3-12(1)^2=-8<0$

Since, f is decreasing on both intervals, $(0,10)$ is not a relative extremum.

We use the Second Derivative Test for $x=3$.

$f''(3)=12(3)^2-24(3)=36>0$

The critical point $(3,-17)$ is a relative minimum.

When we applied the First Derivative Test, we saw that $f(x)$ was decreasing on the intervals $(-\infty,0)$ and $(0,3)$. Since $(3,-17)$ is a relative minimum, we know that $f(x)$ is increasing on $(3,\infty)$.

d) Find the points of inflection. $f''(x)$ exists for all values of x, so the only possible inflection points occur when $f''(x)=0$.

$12x^2-24x=0$

$12x(x-2)=0$

$12x=0$ or $x-2=0$

$x=0$ or $x=2$

Possible points of inflection occur at $x=0$ and $x=2$.

$f(0)=(0)^4-4(0)^3+10=10$

$f(2)=(2)^4-4(2)^3+10=-6$

The points $(0,10)$ and $(2,-6)$ are possible inflection points on the graph.

e) To determine concavity we use 0 and 2 to divide the real number line into three intervals,

A: $(-\infty,0)$, B: $(0,2)$, and C: $(2,\infty)$, Then test a point in each interval.

A: Test -1,

$f''(-1)=12(-1)^2-24(-1)=36>0$

B: Test 1,

$f''(1)=12(1)^2-24(1)=-12<0$

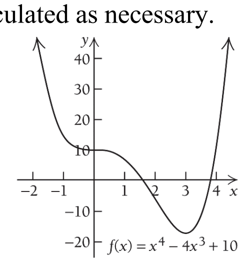
C: Test 3,

$f''(3)=12(3)^2-24(3)=36>0$

We see that f is concave up on the intervals $(-\infty,0)$ and $(2,\infty)$ and concave down on the interval $(0,2)$. Therefore both $(0,10)$ and $(2,-6)$ are inflection points.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-2	58
-1	15
1	7
4	10
5	135



28. $f(x)=\frac{4}{3}x^3-2x^2+x$

a) $f'(x)=4x^2-4x+1$

$f''(x)=8x-4$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers. Solve:

$f'(x)=0$

$4x^2-4x+1=0$

$(2x-1)^2=0$

$2x-1=0$

$x=\frac{1}{2}$ Critical Value

We evaluate the function at the critical value.

$f\left(\frac{1}{2}\right)=\frac{1}{6}$

So the critical point $\left(\frac{1}{2},\frac{1}{6}\right)$ is on the graph.

c) Using the Second Derivative Test, we have:

$f''\left(\frac{1}{2}\right)=0$ and the test fails. We will use the First Derivative Test.

Divide the real number line into two intervals, A: $\left(-\infty,\frac{1}{2}\right)$ and B: $\left(\frac{1}{2},\infty\right)$, and test a point in each interval.

A: Test 0, $f'(0)=4(0)^2-4(0)+1=1>0$

B: Test 1, $f'(1)=4(1)^2-4(1)+1=1>0$

Since, f is increasing on both intervals, $\left(\frac{1}{2},\frac{1}{6}\right)$ is not a relative extremum.

However, we now know that $f(x)$ is increasing over the intervals $\left(-\infty,\frac{1}{2}\right)$ and $\left(\frac{1}{2},\infty\right)$.

d) Find the inflection points. $f''(x)$ exists for all real numbers. Solve:

$$f''(x)=0$$
$$8x-4=0$$
$$x=\frac{1}{2}$$
$$f\left(\frac{1}{2}\right)=\frac{1}{6}$$

The possible inflection point is $\left(\frac{1}{2},\frac{1}{6}\right)$.

e) To determine concavity, use $\frac{1}{2}$ to divide the real number line into two intervals, A: $\left(-\infty,\frac{1}{2}\right)$ and B: $\left(\frac{1}{2},\infty\right)$ and test a point in each interval.

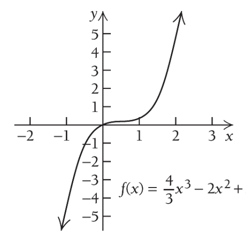
A: Test 0, $f''(0)=8(0)-4=-4<0$

B: Test 1, $f''(1)=8(1)-4=4>0$

Therefore, $f(x)$ is concave down on the interval $\left(-\infty,\frac{1}{2}\right)$ and concave up on the interval $\left(\frac{1}{2},\infty\right)$. The point $\left(\frac{1}{2},\frac{1}{6}\right)$ is an inflection point.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-6	-366
-3	-57
-1	$-\frac{13}{3}$
0	0
1	$\frac{1}{3}$
3	21
6	222



29. $f(x)=x^3-6x^2+12x-6$

a) $f'(x)=3x^2-12x+12$

$f''(x)=6x-12$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all values of x, so the only critical points of f are where $f'(x)=0$.

$$3x^2-12x+12=0$$
$$x^2-4x+4=0 \quad \text{Dividing by 3}$$
$$(x-2)^2=0$$
$$x-2=0$$
$$x=2$$

The critical value is 2.

$f(2)=(2)^3-6(2)^2+12(2)-6=2$

The critical point $(2,2)$ is on the graph.

c) Applying the Second Derivative Test, we have:

$f''(2)=6(2)-12=0$

The test fails, we will use the First Derivative Test.

Divide the real line into two intervals, A: $(-\infty,2)$ and B: $(2,\infty)$, and test a point in each interval on the next page.

Testing a point in each interval, we have:
A: Test 0,

$$f'(0)=3(0)^2-12(0)+12=12>0$$

B: Test 3,

$$f'(3)=3(3)^2-12(3)+12=3>0$$

Since, f is increasing on both intervals, $(2,2)$ is not a relative extremum.

When we applied the First Derivative Test, we saw that $f(x)$ was increasing on the intervals $(-\infty,2)$ and $(2,\infty)$.

d) Find the points of inflection. $f''(x)$ exists for all values of x, so the only possible inflection points occur when $f''(x)=0$.

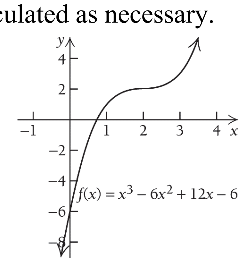
$$6x-12=0$$
$$6x=12$$
$$x=2$$

We have already seen that $f(2)=2$, so the point $(2,2)$ is a possible inflection point on the graph.

e) To determine concavity we use 2 to divide the real number line into two intervals,
A : $(-\infty,2)$ and B: $(2,\infty)$, Then test a point in each interval.
A: Test 0, $f''(0)=6(0)-12=-12<0$
B: Test 3, $f''(3)=6(3)-12=6>0$
We see that $f(x)$ is concave down on the interval $(-\infty,2)$ and concave up on the interval $(2,\infty)$. Therefore, the point $(2,2)$ is an inflection point.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-1	-25
0	-6
1	1
3	3
4	10



30. $f(x)=x^3+3x+1$

a) $f'(x)=3x^2+3$

$f''(x)=6x$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers, and the equation $f'(x)=0$ has no real solution. $\left[3x^2+3>0 \text{ for all } x\right]$. Thus, $f(x)$ has no critical points.

c) Since $f'(x)>0$ for all real numbers, $f(x)$ is increasing over the entire domain $(-\infty,\infty)$.

d) Find the points of inflection. Since $f''(x)$ exists for all real numbers, solve:

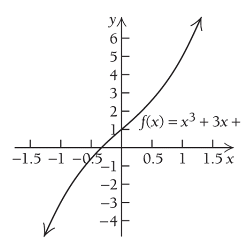
$$f''(x)=0$$
$$6x=0$$
$$x=0$$
$$f(0)=1$$

So the point $(0,1)$ is a possible inflection point on the graph.

e) To determine concavity, we use 0 to divide the real number line into two intervals.
A : $(-\infty,0)$ and B: $(0,\infty)$, Then test a point in each interval.
A: Test -1, $f''(-1)=6(-1)=-6<0$
B: Test 1, $f''(1)=6(1)=6>0$
We see that $f(x)$ is concave down on the interval $(-\infty,0)$ and concave up on the interval $(0,\infty)$. Therefore, the point $(0,1)$ is an inflection point.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-2	-13
-1	-3
1	5
2	15



31. $f(x)=5x^3-3x^5$

a) $f'(x)=15x^2-15x^4$

$f''(x)=30x-60x^3$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all values of x, so the only critical points of f are where $f'(x)=0$.

$15x^2-15x^4=0$

$15x^2\left(1-x^2\right)=0$

$15x^2=0 \quad \text{or} \quad 1-x^2=0$

$x=0 \quad \text{or} \quad x=\pm 1$

The critical values are $-1,\ 0,$ and 1.

$f(-1)=5(-1)^3-3(-1)^5=-2$

$f(0)=5(0)^3-3(0)^5=0$

$f(1)=5(1)^3-3(1)^5=2$

The critical points $(-1,-2)$, $(0,0)$ and $(1,2)$ are on the graph.

c) Applying the Second Derivative Test, we have:

$f''(-1)=30(-1)-60(-1)^3=30>0$

So, the critical point $(-1,-2)$ is a relative minimum.

$f''(0)=30(0)-60(0)^3=0$

The test fails, we will use the First Derivative Test.

Divide $(-1,1)$ into two intervals, A: $(-1,0)$ and B: $(0,1)$, and test a point in each interval.

A: Test $-\frac{1}{2}$,

$$f'\left(-\frac{1}{2}\right)=15\left(-\frac{1}{2}\right)^2-15\left(-\frac{1}{2}\right)^4=\frac{45}{16}>0$$

B: Test $\frac{1}{2}$,

$$f'\left(\frac{1}{2}\right)=15\left(\frac{1}{2}\right)^2-15\left(\frac{1}{2}\right)^4=\frac{45}{16}>0$$

Since, f is increasing on both intervals, $(0,0)$ is not a relative extremum.

We use the Second Derivative Test for $x=1$.

$f''(1)=30(1)-60(1)^3=-30<0$

The critical point $(1,2)$ is a relative maximum.

When we applied the First Derivative Test, we saw that $f(x)$ was increasing on the intervals $(-1,0)$ and $(0,1)$. Since $(-1,-2)$ is a relative minimum, we know that $f(x)$ is decreasing on $(-\infty,-1)$. Since $(1,2)$ is a relative maximum, we know that $f(x)$ is decreasing on $(1,\infty)$.

d) Find the points of inflection. $f''(x)$ exists for all values of x, so the only possible inflection points occur when $f''(x)=0$.

$30x-60x^3=0$

$30x\left(1-2x^2\right)=0$

$30x=0 \quad \text{or} \quad 1-2x^2=0$

$x=0 \quad \text{or} \quad x^2=\frac{1}{2}$

$x=0 \quad \text{or} \quad x=\pm\sqrt{\frac{1}{2}}=\pm\frac{1}{\sqrt{2}}$

$$f\left(-\frac{1}{\sqrt{2}}\right)=5\left(-\frac{1}{\sqrt{2}}\right)^3-3\left(-\frac{1}{\sqrt{2}}\right)^5=-1.237$$

$$f(0)=5(0)^3-3(0)^5=0$$

$$f\left(\frac{1}{\sqrt{2}}\right)=5\left(\frac{1}{\sqrt{2}}\right)^3-3\left(\frac{1}{\sqrt{2}}\right)^5=1.237$$

The points $\left(-\frac{1}{\sqrt{2}},-1.237\right)$, $(0,0)$ and $\left(\frac{1}{\sqrt{2}},1.237\right)$ are possible inflection points on the graph.

e) To determine concavity we use $-\frac{1}{\sqrt{2}},\ 0,$ and $\frac{1}{\sqrt{2}}$ to divide the real number line into four intervals, A: $\left(-\infty,-\frac{1}{\sqrt{2}}\right)$, B: $\left(-\frac{1}{\sqrt{2}},0\right)$, C: $\left(0,\frac{1}{\sqrt{2}}\right)$, and D: $\left(\frac{1}{\sqrt{2}},\infty\right)$.

Then test a point in each interval on the next page.

Testing the point in each interval, we have

A: Test -1, $f''(-1) = 30(-1) - 60(-1)^3$
$= 30 > 0$

B: Test $-\frac{1}{2}$,
$f''\left(-\frac{1}{2}\right) = 30\left(-\frac{1}{2}\right) - 60\left(-\frac{1}{2}\right)^3$
$= -\frac{15}{2} < 0$

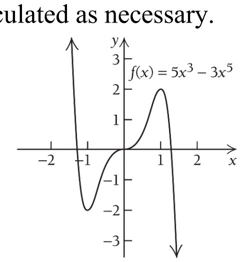
C: Test $\frac{1}{2}$,
$f''\left(\frac{1}{2}\right) = 30\left(\frac{1}{2}\right) - 60\left(\frac{1}{2}\right)^3$
$= \frac{15}{2} > 0$

D: Test 1 ,
$f''(1) = 30(1) - 60(1)^3$
$= -30 < 0$

We see that f is concave up on the intervals $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$ and $\left(0, \frac{1}{\sqrt{2}}\right)$ and concave down on the intervals $\left(-\frac{1}{\sqrt{2}}, 0\right)$ and $\left(\frac{1}{\sqrt{2}}, \infty\right)$. Therefore, the points $\left(-\frac{1}{\sqrt{2}}, -1.237\right)$, $(0,0)$ and $\left(\frac{1}{\sqrt{2}}, 1.237\right)$ are inflection points.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-2	56
$-\frac{1}{2}$	$-\frac{17}{32}$
$\frac{1}{2}$	$\frac{17}{32}$
2	-56



32. $f(x) = 20x^3 - 3x^5$

a) $f'(x) = 60x^2 - 15x^4$
$f''(x) = 120x - 60x^3$
The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers. Solve:
$f'(x) = 0$
$60x^2 - 15x^4 = 0$
$15x^2(4 - x^2) = 0$
$15x^2 = 0$ or $4 - x^2 = 0$
$x = 0$ or $x = \pm 2$
$f(-2) = -64$
$f(0) = 0$
$f(2) = 64$
So the critical points $(-2,-64)$, $(0,0)$, and $(2,64)$ are on the graph.

c) Applying the Second Derivative Test, we have:
$f''(-2) = 120(-2) - 60(-2)^3 = 240 > 0$
So, the critical point $(-2,-64)$ is a relative minimum.
Testing the remaining critical values we have:
$f''(0) = 120(0) - 60(0)^3 = 0$
The test fails, we will use the First Derivative Test.
Divide $(-2,2)$ into two intervals,
A: $(-2,0)$ and B: $(0,2)$, and test a point in each interval.
A: Test -1,
$f'(-1) = 60(-1)^2 - 15(-1)^4 = 45 > 0$
B: Test 1,
$f'(1) = 60(1)^2 - 15(1)^4 = 45 > 0$
Since, f is increasing on both intervals, $(0,0)$ is not a relative extremum.
We use the Second Derivative Test for $x = 2$.
$f''(2) = 120(2) - 60(2)^3 = -240 < 0$
The critical point $(2,64)$ is a relative maximum.

c) When we applied the First Derivative Test, we saw that $f(x)$ was increasing on the intervals $(-2,0)$ and $(0,2)$. Since $(-2,-64)$ is a relative minimum, we know that $f(x)$ is decreasing on $(-\infty,-2)$. Since $(2,64)$ is a relative maximum, we know that $f(x)$ is decreasing on $(2,\infty)$.

d) Find the inflection points. $f''(x)$ exists for all real numbers. Solve:

$120x - 60x^3 = 0$

$60x(2 - x^2) = 0$

$60x = 0$ or $2 - x^2 = 0$

$x = 0$ or $x^2 = 2$

$x = 0$ or $x = \pm\sqrt{2}$

$f(-\sqrt{2}) = -28\sqrt{2}$

$f(0) = 0$

$f(\sqrt{2}) = 28\sqrt{2}$

The points $(-\sqrt{2}, -28\sqrt{2})$, $(0,0)$ and $(\sqrt{2}, 28\sqrt{2})$ are possible inflection points on the graph.

e) To determine concavity we use $-\sqrt{2}$, 0, and $\sqrt{2}$ to divide the real number line into four intervals,

A: $(-\infty, -\sqrt{2})$, B: $(-\sqrt{2}, 0)$, C: $(0, \sqrt{2})$, and D: $(\sqrt{2}, \infty)$.

Then test a point in each interval.

A: Test -2, $f''(-2) = 120(-2) - 60(-2)^3 = 240 > 0$

B: Test -1, $f''(-1) = 120(-1) - 60(-1)^3 = -60 < 0$

C: Test 1, $f''(1) = 120(1) - 60(1)^3 = 60 > 0$

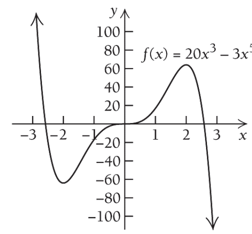
D: Test 2, $f''(2) = 120(2) - 60(2)^3 = -240 < 0$

We see that f is concave up on the intervals $(-\infty, -\sqrt{2})$ and $(0, \sqrt{2})$ and concave down on the intervals $(-\sqrt{2}, 0)$ and $(\sqrt{2}, \infty)$.

Therefore, the points $(-\sqrt{2}, -28\sqrt{2})$, $(0,0)$ and $(\sqrt{2}, 28\sqrt{2})$ are inflection points.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-3	189
-1	-17
1	17
3	-189



33. $f(x) = x^2(1-x)^2 = x^2 - 2x^3 + x^4$

a) $f'(x) = 2x - 6x^2 + 4x^3$

$f''(x) = 2 - 12x + 12x^2$

b) $f'(x)$ exists for all values of x, so the only critical points of f are where $f'(x) = 0$.

$2x - 6x^2 + 4x^3 = 0$

$2x(1 - 3x + 2x^2) = 0$

$2x(1-x)(1-2x) = 0$

$2x = 0$ or $1 - 2x = 0$ or $1 - x = 0$

$x = 0$ or $x = \frac{1}{2}$ or $x = 1$

The critical values are $0, \frac{1}{2}$, and 1.

$f(0) = (0)^2(1-(0))^2 = 0$

$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2\left(1 - \left(\frac{1}{2}\right)\right)^2 = \frac{1}{16}$

$f(1) = (1)^2(1-(1))^2 = 0$

The critical points $(0,0)$, $\left(\frac{1}{2}, \frac{1}{16}\right)$, and $(1,0)$ are on the graph.

c) Applying the Second Derivative Test, we have:

$f''(0) = 2 - 12(0) + 12(0)^2 = 2 > 0$

So, the critical point $(0,0)$ is a relative minimum.

$f''\left(\frac{1}{2}\right) = 2 - 12\left(\frac{1}{2}\right) + 12\left(\frac{1}{2}\right)^2 = -1 < 0$

So, the critical point $\left(\frac{1}{2}, \frac{1}{16}\right)$ is a relative maximum.

$f''(1) = 2 - 12(1) + 12(1)^2 = 2 > 0$

So, the critical point $(1,0)$ is a relative minimum. The solution is continued.

We use the points 0, $\frac{1}{2}$, and 1 to divide the real number line into four intervals, $(-\infty,0)$, $\left(0,\frac{1}{2}\right)$, $\left(\frac{1}{2},1\right)$, and $(1,\infty)$, we know that $f(x)$ is decreasing on the intervals $(-\infty,0)$ and $\left(\frac{1}{2},1\right)$, and $f(x)$ is increasing on the intervals $\left(0,\frac{1}{2}\right)$ and $(1,\infty)$.

d) Find the points of inflection. $f''(x)$ exists for all values of *x*, so the only possible inflection points occur when $f''(x)=0$.

$2-12x+12x^2=0$

$1-6x+6x^2=0$ Dividing by 2

Using the quadratic formula we have:

$x=\frac{3\pm\sqrt{3}}{6}$

$x\approx 0.211$ or $x\approx 0.789$

$f(0.211)\approx 0.028$

$f(0.789)\approx 0.028$

The points, $(0.211,0.028)$ and $(0.789,0.028)$ are possible inflection points on the graph.

e) To determine concavity we use 0.211 and 0.789 to divide the real number line into three intervals,

A : $(-\infty,0.211)$, B: $(0.211,0.789)$, and C: $(0.789,\infty)$

Then test a point in each interval.

A: Test 0, $f''(0)=2-12(0)+12(0)^2$
$=2>0$

B: Test $\frac{1}{2}$, $f''\left(\frac{1}{2}\right)=2-12\left(\frac{1}{2}\right)+12\left(\frac{1}{2}\right)^2$
$=-1<0$

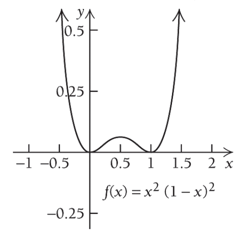
C: Test 1, $f''(1)=2-12(1)+12(1)^2$
$=2>0$

We see that *f* is concave up on the intervals $(-\infty,0.211)$ and $(0.789,\infty)$ and concave down on the interval $(0.211,0.789)$.

Therefore, the points $(0.211,0.028)$ and $(0.789,0.028)$ are inflection points.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-2	36
-1	4
2	4
3	36



34. $f(x)=x^2(3-x)^2$
$=x^2\left(9-6x+x^2\right)$
$=9x^2-6x^3+x^4$

a) $f'(x)=18x-18x^2+4x^3$

$f''(x)=18-36x+12x^2$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all values of *x*, so the only critical points of *f* are where $f'(x)=0$.

$18x-18x^2+4x^3=0$

$2x\left(9-9x+2x^2\right)=0$

$2x(3-2x)(3-x)=0$

$2x=0$ or $3-2x=0$ or $3-x=0$

$x=0$ or $x=\frac{3}{2}$ or $x=3$

The critical values are 0, $\frac{3}{2}$, and 3.

$f(0)=(0)^2(3-(0))^2=0$

$f\left(\frac{3}{2}\right)=\left(\frac{3}{2}\right)^2\left(3-\left(\frac{3}{2}\right)\right)^2=\frac{81}{16}$

$f(3)=(3)^2(3-(3))^2=0$

The critical points $(0,0)$, $\left(\frac{3}{2},\frac{81}{16}\right)$, and $(3,0)$ are on the graph.

c) Applying the Second Derivative Test, we have:

$f''(0)=18-36(0)+12(0)^2=18>0$

So, the critical point $(0,0)$ is a relative minimum.

The solution is continued on the next page.

Applying the Second Derivative test:

$f''\left(\frac{3}{2}\right) = 18 - 36\left(\frac{3}{2}\right) + 12\left(\frac{3}{2}\right)^2 = -9 < 0$

So, the critical point $\left(\frac{3}{2}, \frac{81}{16}\right)$ is a relative maximum.

$f''(3) = 18 - 36(3) + 12(3)^2 = 18 > 0$

So, the critical point $(3,0)$ is a relative minimum.

We use the points 0, $\frac{3}{2}$, and 3 to divide the real number line into four intervals, $(-\infty,0)$, $\left(0,\frac{3}{2}\right)$, $\left(\frac{3}{2},3\right)$, and $(3,\infty)$. We know that $f(x)$ is decreasing on the intervals $(-\infty,0)$ and $\left(\frac{3}{2},3\right)$, and $f(x)$ is increasing on the intervals $\left(0,\frac{3}{2}\right)$ and $(3,\infty)$.

d) Find the points of inflection. $f''(x)$ exists for all values of x, so the only possible inflection points occur when $f''(x) = 0$.

$18 - 36x + 12x^2 = 0$

$3 - 6x + 2x^2 = 0$ Dividing by 6

Using the quadratic formula we have:

$x = \frac{3 \pm \sqrt{3}}{2}$

$x \approx 0.634$ or $x \approx 2.366$

$f(0.634) \approx 2.250$

$f(2.366) \approx 2.250$

The points, $(0.634, 2.250)$ and $(2.366, 2.250)$ are possible inflection points on the graph.

e) To determine concavity we use 0.634 and 2.366 to divide the real number line into three intervals,

A: $(-\infty, 0.634)$, B: $(0.634, 2.366)$, and C: $(2.366, \infty)$

Then test a point in each interval.

A: Test 0,

$f''(0) = 18 - 36(0) + 12(0)^2$
$= 18 > 0$

B: Test 1,

$f''(1) = 18 - 36(1) + 12(1)^2$
$= -6 < 0$

C: Test 3,

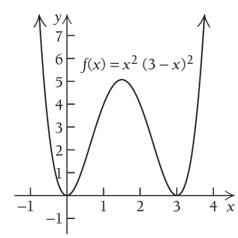
$f''(3) = 18 - 36(3) + 12(3)^2$
$= 18 > 0$

We see that f is concave up on the intervals $(-\infty, 0.634)$ and $(2.366, \infty)$ and concave down on the interval $(0.634, 2.366)$.

Therefore, the points $(0.634, 2.250)$ and $(2.366, 2.250)$ are inflection points.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-2	100
-1	16
1	4
2	4
4	16



35. $f(x) = (x-1)^{2/3}$

a) $f'(x) = \frac{2}{3}(x-1)^{-1/3} = \frac{2}{3(x-1)^{1/3}}$

$f''(x) = -\frac{2}{9}(x-1)^{-4/3} = -\frac{2}{9(x-1)^{4/3}}$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ does not exist for $x = 1$. The equation $f'(x) = 0$ has no solution, therefore, $x = 1$ is the only critical point.

$f(1) = (1-1)^{2/3} = 0$.

So, the critical point, $(1,0)$ is on the graph.

c) We apply the First Derivative Test. We use 1 to divide the real number line into two intervals A: $(-\infty, 1)$ and B: $(1, \infty)$ and then we test a point in each interval at the top of the next page.

Testing a point in each interval, we have

A: Test 0, $f'(0)=\frac{2}{3((0)-1)^{1/3}}=-\frac{2}{3}<0$

B: Test 2, $f'(2)=\frac{2}{3((2)-1)^{1/3}}=\frac{2}{3}>0$

$(1,0)$ is a relative minimum. $f(x)$ is decreasing on the interval $(-\infty,1)$ and increasing on the interval $(1,\infty)$.

d) Find the points of inflection. $f''(x)$ does not exist when $x=1$. The equation $f''(x)=0$ has no solution, so $x=1$ is the only possible inflection point. We know that $f(1)=0$.

e) To determine concavity, we divide the real number line into two intervals, A: $(-\infty,1)$ and B: $(1,\infty)$ and then we test a point in each interval.

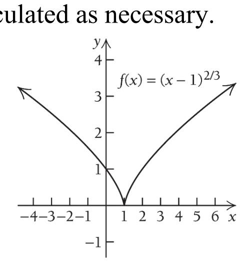
A: Test 0, $f''(0)=-\frac{2}{9((0)-1)^{4/3}}=-\frac{2}{9}<0$

B: Test 2, $f''(2)=-\frac{2}{9((2)-1)^{4/3}}=-\frac{2}{9}<0$

Thus, $f(x)$ is concave down on the intervals $(-\infty,1)$ and $(1,\infty)$. Therefore, the point $(1,0)$ is not an inflection point.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-7	4
0	1
2	1
9	4



36. $f(x)=(x+1)^{2/3}$

a) $f'(x)=\frac{2}{3}(x+1)^{-1/3}=\frac{2}{3(x+1)^{1/3}}$

$f''(x)=-\frac{2}{9}(x+1)^{-4/3}=-\frac{2}{9(x+1)^{4/3}}$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ does not exist for $x=-1$. The equation $f'(x)=0$ has no solution, therefore, $x=-1$ is the only critical value.

$f(-1)=(-1+1)^{2/3}=0$.

So, the critical point, $(-1,0)$, is on the graph.

c) We apply the First Derivative Test. We use -1 to divide the real number line into two intervals A: $(-\infty,-1)$ and B: $(-1,\infty)$ and then we test a point in each interval.

A: Test -2,

$$f'(-2)=\frac{2}{3((-2)+1)^{1/3}}=-\frac{2}{3}<0$$

B: Test 0,

$$f'(0)=\frac{2}{3((0)+1)^{1/3}}=\frac{2}{3}>0$$

Thus, $(-1,0)$ is a relative minimum. We also know that $f(x)$ is decreasing on the interval $(-\infty,-1)$ and increasing on the interval $(-1,\infty)$.

d) Find the points of inflection. $f''(x)$ does not exist when $x=-1$. The equation $f''(x)=0$ has no solution, so $x=-1$ is the only possible inflection point. We know that $f(-1)=0$.

e) To determine concavity, we divide the real number line into two intervals, A: $(-\infty,-1)$ and B: $(-1,\infty)$ and then we test a point in each interval.

A: Test -2,

$$f''(-2)=-\frac{2}{9((-2)+1)^{4/3}}=-\frac{2}{9}<0$$

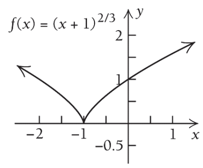
B: Test 0,

$$f''(0)=-\frac{2}{9((0)+1)^{4/3}}=-\frac{2}{9}<0$$

Thus, $f(x)$ is concave down on the interval $(-\infty,-1)$ and on the interval $(-1,\infty)$. Therefore, the point $(-1,0)$ is not an inflection point.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
−9	4
−2	1
0	1
7	4



37. $f(x)=(x-3)^{1/3}-1$

a) $f'(x)=\frac{1}{3}(x-3)^{-2/3}=\frac{1}{3(x-3)^{2/3}}$

$f''(x)=-\frac{2}{9}(x-3)^{-5/3}=-\frac{2}{9(x-3)^{5/3}}$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ does not exist for $x=3$. The equation $f'(x)=0$ has no solution, therefore, $x=3$ is the only critical value.

$f(3)=((3)-3)^{1/3}-1=-1$.

So, the critical point, $(3,-1)$ is on the graph.

c) We apply the First Derivative Test. We use 3 to divide the real number line into two intervals $A:(-\infty,3)$ and B: $(3,\infty)$ and then we test a point in each interval.

A: Test 2, $f'(2)=\frac{1}{3((2)-3)^{2/3}}=\frac{1}{3}>0$

B: Test 4, $f'(4)=\frac{1}{3((4)-3)^{2/3}}=\frac{1}{3}>0$

$f(x)$ is increasing on both intervals $(-\infty,3)$ and $(3,\infty)$, therefore $(3,-1)$ is not a relative extremum.

d) Find the points of inflection. $f''(x)$ does not exist when $x=3$. The equation $f''(x)=0$ has no solution, so at $x=3$ is the only possible inflection point. We know that $f(3)=-1$.

e) To determine concavity, we divide the real number line into two intervals, $A:(-\infty,3)$ and B: $(3,\infty)$ and then we test a point in each interval.

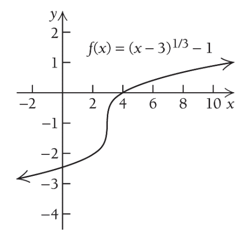
A: Test 2, $f''(2)=-\frac{2}{9((2)-3)^{5/3}}=\frac{2}{9}>0$

B: Test 4, $f''(4)=-\frac{2}{9((4)-3)^{5/3}}=-\frac{2}{9}<0$

Thus, $f(x)$ is concave up on the interval $(-\infty,3)$ and $f(x)$ is concave down on the interval $(3,\infty)$. Therefore, the point $(3,-1)$ is an inflection point.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
−5	−3
2	−2
4	0
11	1



38. $f(x)=(x-2)^{1/3}+3$

a) $f'(x)=\frac{1}{3}(x-2)^{-2/3}=\frac{1}{3(x-2)^{2/3}}$

$f''(x)=-\frac{2}{9}(x-2)^{-5/3}=-\frac{2}{9(x-2)^{5/3}}$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ does not exist for $x=2$. The equation $f'(x)=0$ has no solution, therefore, $x=2$ is the only critical point.

$f(2)=((2)-2)^{1/3}+3=3$.

So, the critical point, $(2,3)$ is on the graph.

c) We apply the First Derivative Test. We use 2 to divide the real number line into two intervals $A:(-\infty,2)$ and B: $(2,\infty)$ and then we test a point in each interval.

A: Test 1, $f'(1)=\frac{1}{3((1)-2)^{2/3}}=\frac{1}{3}>0$

B: Test 3, $f'(3)=\frac{1}{3((3)-2)^{2/3}}=\frac{1}{3}>0$

$f(x)$ is increasing on both intervals $(-\infty,2)$ and $(2,\infty)$, therefore $(2,3)$ is not a relative extremum.

d) Find the points of inflection. $f''(x)$ does not exist when $x=2$. The equation $f''(x)=0$ has no solution, so $x=2$ is the only possible inflection point. We know that $f(2)=3$. To determine concavity, we divide the real number line into two intervals, $A:(-\infty,2)$ and $B:(2,\infty)$ and then we test a point in each interval.

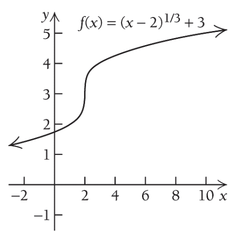
A: Test 1, $f''(1)=-\dfrac{2}{9((1)-2)^{5/3}}=\dfrac{2}{9}>0$

B: Test 3, $f''(3)=-\dfrac{2}{9((3)-2)^{5/3}}=-\dfrac{2}{9}<0$

Thus, $f(x)$ is concave up on the interval $(-\infty,2)$ and $f(x)$ is concave down on the interval $(2,\infty)$. Therefore, the point $(2,3)$ is an inflection point.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-6	1
1	2
3	4
10	5



39. $f(x)=-2(x-4)^{2/3}+5$

a) $f'(x)=-\dfrac{4}{3}(x-4)^{-1/3}=-\dfrac{4}{3(x-4)^{1/3}}$

$f''(x)=\dfrac{4}{9}(x-4)^{-4/3}=\dfrac{4}{9(x-4)^{4/3}}$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ does not exist for $x=4$. The equation $f'(x)=0$ has no solution, therefore, $x=4$ is the only critical point.

$f(4)=-2((4)-4)^{2/3}+5=5$.

So, the critical point $(4,5)$, is on the graph.

c) We apply the First Derivative Test. We use 4 to divide the real number line into two intervals $A:(-\infty,4)$ and B: $(4,\infty)$ and then we test a point in each interval at the top of the next column:

A: Test 3, $f'(3)=-\dfrac{4}{3((3)-4)^{1/3}}=\dfrac{4}{3}>0$

B: Test 5, $f'(5)=-\dfrac{4}{3((5)-4)^{1/3}}=-\dfrac{4}{3}<0$

Thus, $(4,5)$ is a relative maximum.

We also know that $f(x)$ is increasing on the interval $(-\infty,4)$ and decreasing on the interval $(4,\infty)$.

d) Find the points of inflection. $f''(x)$ does not exist when $x=4$. The equation $f''(x)=0$ has no solution, so $x=4$ is the only possible inflection point. We know that $f(4)=5$.

e) To determine concavity, we divide the real number line into two intervals, $A:(-\infty,4)$ and B: $(4,\infty)$ and then we test a point in each interval:

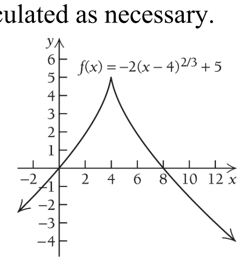
A: Test 3, $f''(3)=\dfrac{4}{9((3)-4)^{4/3}}=\dfrac{4}{9}>0$

B: Test 5, $f''(5)=\dfrac{4}{9((5)-4)^{4/3}}=\dfrac{4}{9}>0$

Thus, $f(x)$ is concave up on both intervals $(-\infty,4)$ and $(4,\infty)$ Therefore, the point $(4,5)$ is not an inflection point.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-4	-3
3	3
5	3
12	-3



40. $f(x)=-3(x-2)^{2/3}+3$

a) $f'(x)=-\dfrac{6}{3}(x-2)^{-1/3}=-\dfrac{2}{(x-2)^{1/3}}$

$f''(x)=\dfrac{6}{9}(x-2)^{-4/3}=\dfrac{2}{3(x-2)^{4/3}}$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ does not exist for $x = 2$. The equation $f'(x) = 0$ has no solution; therefore, $x = 2$ is the only critical point.

$f(2) = -3((2)-2)^{2/3} + 3 = 3$.

So, the critical point, $(2,3)$ is on the graph.

c) We apply the First Derivative Test. We use 2 to divide the real number line into two intervals A: $(-\infty, 2)$ and B: $(2, \infty)$ and then we test a point in each interval.

A: Test 1, $f'(1) = -\dfrac{2}{((1)-2)^{1/3}} = 2 > 0$

B: Test 3, $f'(3) = -\dfrac{2}{((3)-2)^{1/3}} = -2 < 0$

Thus, $(2,3)$ is a relative maximum. We also know that $f(x)$ is increasing on the interval $(-\infty, 2)$ and decreasing on the interval $(2, \infty)$.

d) Find the points of inflection. $f''(x)$ does not exist when $x = 2$. The equation $f''(x) = 0$ has no solution, so $x = 2$ is the only possible inflection point. We know that $f(2) = 3$.

e) To determine concavity, we divide the real number line into two intervals, A: $(-\infty, 2)$ and B: $(2, \infty)$ and then we test a point in each interval.

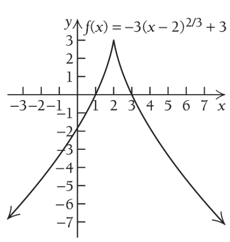
A: Test 1, $f''(1) = \dfrac{2}{3((1)-2)^{4/3}} = \dfrac{2}{3} > 0$

B: Test 3, $f''(3) = \dfrac{2}{3((3)-2)^{4/3}} = \dfrac{2}{3} > 0$

Thus, $f(x)$ is concave up on both intervals $(-\infty, 2)$ and $(2, \infty)$ Therefore, the point $(2,3)$ is not an inflection point.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-6	-9
1	0
3	0
10	-9



41. $f(x) = -x\sqrt{1-x^2} = -x(1-x^2)^{1/2}$

a) $f'(x) = -x \cdot \dfrac{1}{2}(1-x^2)^{-1/2}(-2x) + (1-x^2)^{1/2} \cdot (-1)$

$= \dfrac{2x^2-1}{(1-x^2)^{1/2}} = \dfrac{2x^2-1}{\sqrt{1-x^2}}$

$f''(x) = (2x^2-1)\left(-\dfrac{1}{2}\right)(1-x^2)^{-3/2}(-2x) + (1-x^2)^{-1/2}(4x)$

$= \dfrac{-2x^3+3x}{(1-x^2)^{3/2}}$

The domain of $f(x)$ is $[-1,1]$.

b) $f'(x)$ does not exist when $x = \pm 1$. However, the domain of $f(x)$ is $[-1,1]$. Therefore, relative extrema cannot occur at $x = -1$ or $x = 1$ because there is not an open interval containing -1 or 1 on which the function is defined. The other critical points occur where $f'(x) = 0$.

$$\frac{2x^2-1}{\sqrt{1-x^2}} = 0$$
$$2x^2 - 1 = 0$$
$$x^2 = \frac{1}{2}$$
$$x = \pm\frac{1}{\sqrt{2}}$$

The critical values are $-\dfrac{1}{\sqrt{2}}$ and $\dfrac{1}{\sqrt{2}}$.

$$f\left(-\frac{1}{\sqrt{2}}\right) = -\left(-\frac{1}{\sqrt{2}}\right)\sqrt{1-\left(-\frac{1}{\sqrt{2}}\right)^2}$$
$$= \left(\frac{1}{\sqrt{2}}\right)\sqrt{1-\frac{1}{2}}$$
$$= \left(\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}$$
$$= \frac{1}{2}$$

The solution is continued on the next page.

Evaluating the second critical value, we have:

$$f\left(\frac{1}{\sqrt{2}}\right) = -\left(\frac{1}{\sqrt{2}}\right)\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2}$$
$$= \left(-\frac{1}{\sqrt{2}}\right)\sqrt{1-\frac{1}{2}}$$
$$= \left(\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}$$
$$= -\frac{1}{2}$$

Therefore, $\left(-\frac{1}{\sqrt{2}},\frac{1}{2}\right)$ and $\left(\frac{1}{\sqrt{2}},-\frac{1}{2}\right)$ are critical points on the graph.

c) We use the Second Derivative Test.

$$f''\left(-\frac{1}{\sqrt{2}}\right) = -4 < 0$$

The critical point $\left(-\frac{1}{\sqrt{2}},\frac{1}{2}\right)$ is a relative maximum.

$$f''\left(\frac{1}{\sqrt{2}}\right) = 4 > 0$$

The critical point $\left(\frac{1}{\sqrt{2}},-\frac{1}{2}\right)$ is a relative minimum.

If we use the points $-\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$ to divide the interval $[-1,1]$ into three intervals $\left[-1,-\frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$, and $\left(\frac{1}{\sqrt{2}},1\right]$, we see that $f(x)$ is increasing on the intervals $\left(-1,-\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}},1\right)$ and $f(x)$ is decreasing on the interval $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$.

d) Find the points of inflection. $f''(x)$ does not exist when $x=-1$ and $x=1$. However, inflection points cannot occur at those values because the domain of the function is $[-1,1]$. The remaining possible inflection points occur when $f''(x)=0$. We set the second derivative equal to zero and solve for the possible inflection points.

$$f''(x) = 0$$
$$\frac{-2x^3+3x}{\left(1-x^2\right)^{3/2}} = 0$$
$$-2x^3+3x = 0$$
$$x\left(-2x^2+3\right) = 0$$
$$x = 0 \quad \text{or} \quad 2x^2-3 = 0$$
$$x = 0 \quad \text{or} \quad x^2 = \frac{3}{2}$$
$$x = 0 \quad \text{or} \quad x = \frac{\pm\sqrt{6}}{2}$$

Note that $f(x)$ is not defined for $x = \pm\frac{\sqrt{6}}{2}$.

Therefore, the only possible inflection point is $x=0$. Evaluating the function we have $f(0) = 0\sqrt{1-(0)^2} = 0$.

Therefore, $(0,0)$ is a possible inflection point on the graph.

e) To determine concavity, we use 0 to divide the interval $(-1,1)$ into two intervals, A: $(-1,0)$ and B: $(0,1)$ and then we test a point in each interval.

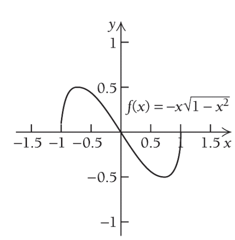
A: Test $-\frac{1}{2}$, $f''\left(-\frac{1}{2}\right) = \frac{-10}{3^{3/2}} < 0$

B: Test $\frac{1}{2}$, $f''\left(\frac{1}{2}\right) = \frac{10}{3^{3/2}} > 0$

Thus, $f(x)$ is concave down on the interval $(-1,0)$ and $f(x)$ is concave up on the interval $(0,1)$. Therefore, the point $(0,0)$ is an inflection point.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-1	0
$-\frac{1}{2}$	$\frac{\sqrt{3}}{4}$
$\frac{1}{2}$	$-\frac{\sqrt{3}}{4}$
1	0



42. $f(x) = x\sqrt{4-x^2} = x\left(4-x^2\right)^{1/2}$

a) $f'(x) = x \cdot \frac{1}{2}\left(4-x^2\right)^{-1/2}(-2x)$
$+\left(4-x^2\right)^{1/2} \cdot (1)$

Next, we simplify the derivative.

$$f'(x) = \frac{-x^2}{\left(4-x^2\right)^{1/2}} + \left(4-x^2\right)^{1/2}$$
$$= \frac{-x^2+4-x^2}{\left(4-x^2\right)^{1/2}}$$
$$= \frac{4-2x^2}{\left(4-x^2\right)^{1/2}}$$
$$= \left(4-2x^2\right)\left(4-x^2\right)^{-1/2}$$

$f''(x)$
$$= \left(4-2x^2\right)\left(-\frac{1}{2}\right)\left(4-x^2\right)^{-3/2}(-2x) +$$
$$\left(4-x^2\right)^{-1/2}(-4x)$$
$$= \frac{x\left(4-2x^2\right)}{\left(4-x^2\right)^{3/2}} - \frac{4x}{\left(4-x^2\right)^{1/2}}$$
$$= \frac{4x-2x^3-4x\left(4-x^2\right)}{\left(4-x^2\right)^{3/2}}$$
$$= \frac{4x-2x^3-16x+4x^3}{\left(4-x^2\right)^{3/2}}$$
$$= \frac{2x^3-12x}{\left(4-x^2\right)^{3/2}}$$

The domain of $f(x)$ is $[-2,2]$.

b) First, we find the critical points.
$f'(x)$ does not exist when $4-x^2=0$.
Solve:
$$4-x^2=0$$
$$x^2=4$$
$$x=\pm 2$$
Since the domain of $f(x)$ is $[-2,2]$, relative extrema cannot occur at $x=-2$ or $x=2$ because there is not an open interval containing -2 or 2 on which the unction is defined. For this reason, we do not consider -2 or 2 in our discussion of relative extrema.
The other critical points occur where
$$f'(x)=0$$
$$\frac{4-2x^2}{\sqrt{4-x^2}}=0$$
$$4-2x^2=0$$
$$x=\pm\sqrt{2}$$
The critical values are $-\sqrt{2}$ and $\sqrt{2}$.
$$f\left(-\sqrt{2}\right) = -\sqrt{2}\sqrt{4-\left(-\sqrt{2}\right)^2} = -\sqrt{2}\sqrt{2} = -2$$
$$f\left(\sqrt{2}\right) = \sqrt{2}\sqrt{4-\left(\sqrt{2}\right)^2} = \sqrt{2}\sqrt{2} = 2$$
Therefore, $\left(-\sqrt{2},-2\right)$ and $\left(\sqrt{2},2\right)$ are critical points on the graph.

c) We use the Second Derivative Test.
$$f''\left(-\sqrt{2}\right) = \frac{2\left(-\sqrt{2}\right)^3 - 12\left(-\sqrt{2}\right)}{\left[4-\left(-\sqrt{2}\right)^2\right]^{3/2}}$$
$$= \frac{-4\sqrt{2}+12\sqrt{2}}{2^{3/2}} = \frac{8\sqrt{2}}{2\sqrt{2}} = 4 > 0$$
The critical point $\left(-\sqrt{2},-2\right)$ is a relative minimum.
$$f''\left(\sqrt{2}\right) = \frac{2\left(\sqrt{2}\right)^3 - 12\left(\sqrt{2}\right)}{\left[4-\left(\sqrt{2}\right)^2\right]^{3/2}}$$
$$= \frac{4\sqrt{2}-12\sqrt{2}}{2^{3/2}} = \frac{-8\sqrt{2}}{2\sqrt{2}} = -4 < 0$$
The critical point $\left(\sqrt{2},2\right)$ is a relative maximum.
If we use the points $-\sqrt{2}$ and $\sqrt{2}$ to divide the interval $[-2,2]$ into three intervals $\left[-2,-\sqrt{2}\right)$, $\left(-\sqrt{2},\sqrt{2}\right)$, and $\left(\sqrt{2},2\right]$, we see that $f(x)$ is decreasing on the intervals $\left(-2,-\sqrt{2}\right)$ and $\left(\sqrt{2},2\right)$ and $f(x)$ is increasing on the interval $\left(-\sqrt{2},\sqrt{2}\right)$.

d) Find the points of inflection. $f''(x)$ does not exist where $4-x^2=0$. We know that this occurs at $x=-2$ and $x=2$. However, just as relative extrema cannot occur at $(-2,0)$ and $(2,0)$, they cannot be inflection points either. Inflection points could occur where $f''(x)=0$.

We set the second derivative equal to zero and solve for *x*.

$$f''(x)=0$$
$$\frac{2x^3-12x}{\left(4-x^2\right)^{3/2}}=0$$
$$2x^3-12x=0$$
$$2x\left(x^2-6\right)=0$$
$$2x=0 \quad \text{or} \quad x^2-6=0$$
$$x=0 \quad \text{or} \quad x^2=6$$
$$x=0 \quad \text{or} \quad x=\pm\sqrt{6}$$

Note that $f(x)$ is not defined for $x=\pm\sqrt{6}$. Therefore, the only possible inflection point is $x=0$.

$f(0)=(0)\sqrt{4-(0)^2}=0$.

Therefore, $(0,0)$ is a possible inflection point on the graph.

e) To determine concavity, we use 0 to divide the interval $(-2,2)$ into two intervals, A: $(-2,0)$ and B: $(0,2)$ and then we test a point in each interval.

A: Test -1,

$$f''(-1)=\frac{2(-1)^3-12(-1)}{\left[4-(-1)^2\right]^{3/2}}=\frac{10}{3^{3/2}}>0$$

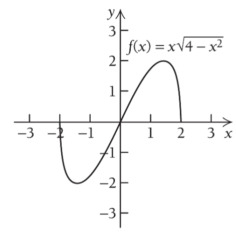
B: Test 1,

$$f''(1)=\frac{2(1)^3-12(1)}{\left[4-(1)^2\right]^{3/2}}=\frac{-10}{3^{3/2}}<0$$

Thus, $f(x)$ is concave up on the interval $(-2,0)$ and $f(x)$ is concave down on the interval $(0,2)$. Therefore, the point $(0,0)$ is an inflection point.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-1	$-\sqrt{3}$
1	$\sqrt{3}$



43. $f(x)=\dfrac{8x}{x^2+1}$

a) $$f'(x)=\frac{\left(x^2+1\right)(8)-(2x)(8x)}{\left(x^2+1\right)^2} \quad \text{Quotient Rule}$$
$$=\frac{8x^2+8-16x^2}{\left(x^2+1\right)^2}$$
$$=\frac{8-8x^2}{\left(x^2+1\right)^2}$$

Next we find the second derivative

$$f''(x)=\frac{\left(x^2+1\right)^2(-16x)-\left(8-8x^2\right)\left[2\left(x^2+1\right)^1(2x)\right]}{\left(\left(x^2+1\right)^2\right)^2}$$
$$=\frac{\left(x^2+1\right)\left[-16x\left(x^2+1\right)-4x\left(8-8x^2\right)\right]}{\left(x^2+1\right)^4}$$
$$=\frac{-16x^3-16x-32x+32x^3}{\left(x^2+1\right)^3}$$
$$=\frac{16x^3-48x}{\left(x^2+1\right)^3}$$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers. Solve:

$$f'(x)=0$$
$$\frac{8-8x^2}{\left(x^2+1\right)^2}=0$$
$$8-8x^2=0$$
$$x^2=1$$
$$x=\pm1$$

The two critical values are $x=-1$ and $x=1$.

The solution is continued on the next page.

Evaluating the function at the critical values, we have:

$f(-1)=\dfrac{8(-1)}{(-1)^2+1}=-\dfrac{8}{2}=-4$

$f(1)=\dfrac{8(1)}{(1)^2+1}=\dfrac{8}{2}=4$

The critical points $(-1,-4)$ and $(1,4)$ are on the graph.

c) We use the Second Derivative Test.

$f''(-1)=4>0$

So the point $(-1,-4)$ is a relative minimum.

$f''(1)=-4<0$

So the point $(1,4)$ is a relative maximum.

$f(x)$ is decreasing on the intervals $(-\infty,1)$ and $(1,\infty)$, and $f(x)$ is increasing on the interval $(-1,1)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so the only possible points of inflection occur when $f''(x)=0$.

$\dfrac{16x^3-48x}{(x^2+1)^3}=0$

$16x^3-48x=0$

$16x(x^2-3)=0$

$16x=0 \quad \text{or} \quad x^2-3=0$

$x=0 \quad \text{or} \quad x^2=3$

$x=0 \quad \text{or} \quad x=\pm\sqrt{3}$

There are three possible inflection points $x=-\sqrt{3}, 0,$ and $\sqrt{3}$.

$f(-\sqrt{3})=-2\sqrt{3}$

$f(0)=0$

$f(\sqrt{3})=2\sqrt{3}$

The points $(-\sqrt{3},-2\sqrt{3})$, $(0,0)$, and $(\sqrt{3},2\sqrt{3})$ are three possible inflection points on the graph.

e) To determine concavity we use $-\sqrt{3}, 0,$ and $\sqrt{3}$ to divide the real number line into four intervals, A: $(-\infty,-\sqrt{3})$, B: $(-\sqrt{3},0)$, C: $(0,\sqrt{3})$, and D: $(\sqrt{3},\infty)$

We will test a point in each interval.

A: Test -2, $f''(-2)=-\dfrac{32}{125}<0$

B: Test -1, $f''(-1)=4>0$

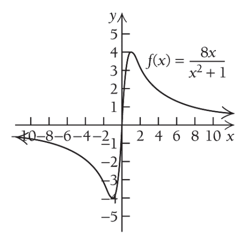
C: Test 1, $f''(1)=-4<0$

D: Test 2, $f''(2)=\dfrac{32}{125}>0$

We see that f is concave down on the intervals $(-\infty,-\sqrt{3})$ and $(0,\sqrt{3})$ and concave up on the intervals $(-\sqrt{3},0)$ and $(\sqrt{3},\infty)$. Therefore the points $(-\sqrt{3},-2\sqrt{3})$, $(0,0)$, and $(\sqrt{3},2\sqrt{3})$ are inflection points.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-3	$-\frac{12}{5}$
-2	$-\frac{16}{5}$
2	$\frac{16}{5}$
3	$\frac{12}{5}$



44. $f(x)=\dfrac{x}{x^2+1}$

a) $f'(x)=\dfrac{(x^2+1)(1)-x(2x)}{(x^2+1)^2}$ Quotient Rule

$=\dfrac{1-x^2}{(x^2+1)^2}$

Next, we find the second derivative.

$$f''(x)=\frac{(x^2+1)^2(-2x)-(1-x^2)\left[2(x^2+1)^1(2x)\right]}{\left((x^2+1)^2\right)^2}$$

$=\dfrac{2x^3-6x}{(x^2+1)^3}$

The domain of f is $\mathbb{R}$.

b) Since $f'(x)$ exists for all real numbers, the only critical values are where $f'(x)=0$.

$$\frac{1-x^2}{\left(x^2+1\right)^2}=0$$

$$1-x^2=0 \quad \text{Multiplying by } \left(x^2+1\right)^2$$

$$x^2=1$$

$$x=\pm\sqrt{1}=\pm 1$$

The two critical values are $x=-1$ and $x=1$.

$$f(-1)=\frac{-1}{(-1)^2+1}=-\frac{1}{2}$$

$$f(1)=\frac{1}{(1)^2+1}=\frac{1}{2}$$

The critical points $\left(-1,-\frac{1}{2}\right)$ and $\left(1,\frac{1}{2}\right)$ are on the graph.

c) We use the Second Derivative Test.

$$f''(-1)=\frac{2(-1)^3-6(-1)}{\left[(-1)^2+1\right]^3}=\frac{4}{8}=\frac{1}{2}>0$$

So the point $\left(-1,-\frac{1}{2}\right)$ is a relative minimum.

$$f''(1)=\frac{2(1)^3-6(1)}{\left[(1)^2+1\right]^3}=\frac{-4}{8}=-\frac{1}{2}<0$$

So the point $\left(1,\frac{1}{2}\right)$ is a relative maximum.

We use -1 and 1 to divide the real number line into three intervals $(-\infty,-1)$, $(-1,1)$, and $(1,\infty)$. $f(x)$ is decreasing on the intervals $(-\infty,1)$ and $(1,\infty)$, and $f(x)$ is increasing on the interval $(-1,1)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so the only possible points of inflection occur when $f''(x)=0$. We find the possible points of inflection at the top of the next column.

$$f''(x)=0$$

$$\frac{2x^3-6x}{\left(x^2+1\right)^3}=0$$

$$2x^3-6x=0$$

$$2x\left(x^2-3\right)=0$$

$$2x=0 \quad \text{or} \quad x^2-3=0$$

$$x=0 \quad \text{or} \quad x^2=3$$

$$x=0 \quad \text{or} \quad x=\pm\sqrt{3}$$

There are three possible inflection points at $x=-\sqrt{3}, 0,$ and $\sqrt{3}$.

$$f\left(-\sqrt{3}\right)=\frac{-\sqrt{3}}{\left(-\sqrt{3}\right)^2+1}=-\frac{\sqrt{3}}{4}$$

$$f(0)=\frac{\sqrt{0}}{\left(\sqrt{0}\right)^2+1}=\frac{0}{1}=0$$

$$f\left(\sqrt{3}\right)=\frac{\sqrt{3}}{\left(\sqrt{3}\right)^2+1}=\frac{\sqrt{3}}{4}$$

The points $\left(-\sqrt{3},-\frac{\sqrt{3}}{4}\right)$, $(0,0)$, and $\left(\sqrt{3},\frac{\sqrt{3}}{4}\right)$ are three possible inflection points on the graph.

e) To determine concavity we use $-\sqrt{3}, 0,$ and $\sqrt{3}$ to divide the real number line into four intervals,

A: $\left(-\infty,-\sqrt{3}\right)$, B: $\left(-\sqrt{3},0\right)$, C: $\left(0,\sqrt{3}\right)$, and D: $\left(\sqrt{3},\infty\right)$.

We test a point in each interval.

A: Test -2, $f''(-2)=-\frac{4}{125}<0$

B: Test -1, $f''(-1)=\frac{1}{2}>0$

C: Test 1, $f''(1)=-\frac{1}{2}<0$

D: Test 2, $f''(2)=\frac{4}{125}>0$

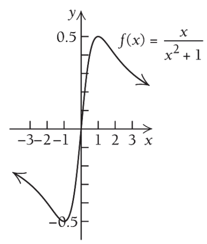
We see that f is concave down on the intervals $\left(-\infty,-\sqrt{3}\right)$ and $\left(0,\sqrt{3}\right)$ and concave up on the intervals $\left(-\sqrt{3},0\right)$ and $\left(\sqrt{3},\infty\right)$.

The solution is continued on the next page.

Using the information from the previous page, we determine the points $\left(-\sqrt{3},-\frac{\sqrt{3}}{4}\right)$, $(0,0)$, and $\left(\sqrt{3},\frac{\sqrt{3}}{4}\right)$ are inflection points.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-3	$-\frac{3}{10}$
-2	$-\frac{2}{5}$
2	$\frac{2}{5}$
3	$\frac{3}{10}$



45. $f(x)=\frac{-4}{x^2+1}=-4\left(x^2+1\right)^{-1}$

a) $f'(x)=-4(-1)\left(x^2+1\right)^{-2}(2x)$ Extended Power Rule

$f'(x)=8x\left(x^2+1\right)^{-2}$

$=\frac{8x}{\left(x^2+1\right)^2}$

Next, we find the second derivative.

$f''(x)$

$=\frac{\left(x^2+1\right)^2(8)-(8x)\left(2\left(x^2+1\right)(2x)\right)}{\left(\left(x^2+1\right)^2\right)^2}$

$=\frac{\left(x^2+1\right)\left[\left(x^2+1\right)(8)-(8x)(2)(2x)\right]}{\left(x^2+1\right)^4}$

$=\frac{8x^2+8-32x^2}{\left(x^2+1\right)^3}$

$=\frac{8-24x^2}{\left(x^2+1\right)^3}$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers Solve:

$f'(x)=0$

$\frac{8x}{\left(x^2+1\right)^2}=0$ Multiplying by $\left(x^2+1\right)^2$

$8x=0$

$x=0$

The critical value is $x=0$.

$f(0)=\frac{-4}{(0)^2+1}=-4$

The critical point $(0,-4)$ is on the graph.

c) We use the Second Derivative Test.

$f''(0)=8>0$

So the point $(0,-4)$ is a relative minimum.

$f(x)$ is decreasing on the interval $(-\infty,0)$, and $f(x)$ is increasing on the interval $(0,\infty)$.

d) $f''(x)$ exists for all real numbers. Solve $f''(x)=0$.

$\frac{8-24x^2}{\left(x^2+1\right)^3}=0$

$8-24x^2=0$ Multiplying by $\left(x^2+1\right)^3$

$x^2=\frac{1}{3}$

$x=\pm\frac{1}{\sqrt{3}}$

There are two possible inflection points $-\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$.

$f\left(\pm\frac{1}{\sqrt{3}}\right)=\frac{-4}{\left(\pm\frac{1}{\sqrt{3}}\right)^2+1}=\frac{-4}{\frac{1}{3}+1}=\frac{-4}{\frac{4}{3}}=-3$

The points $\left(-\frac{1}{\sqrt{3}},-3\right)$ and $\left(\frac{1}{\sqrt{3}},-3\right)$ are possible inflection points on the graph.

e) To determine concavity we use $-\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$ to divide the real number line into three intervals, A: $\left(-\infty,-\frac{1}{\sqrt{3}}\right)$, B: $\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$, and C: $\left(\frac{1}{\sqrt{3}},\infty\right)$. The solution is continued.

Then test a point in each interval.

A: Test -1, $f''(-1) = -2 < 0$

B: Test 0, $f''(0) = 8 > 0$

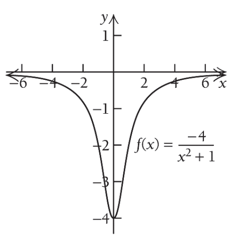
C: Test 1, $f''(1) = -2 < 0$

We see that f is concave down on the intervals $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$ and concave up on the interval $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

Therefore the points $\left(-\frac{1}{\sqrt{3}}, -3\right)$ and $\left(\frac{1}{\sqrt{3}}, -3\right)$ are inflection points.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-4	$-\frac{4}{17}$
-2	$-\frac{4}{5}$
-1	-2
1	-2
2	$-\frac{4}{5}$
4	$-\frac{4}{17}$



46. $f(x) = \dfrac{3}{x^2+1} = 3\left(x^2+1\right)^{-1}$

a) $f'(x) = 3(-1)\left(x^2+1\right)^{-2}(2x)$

$= -6x\left(x^2+1\right)^{-2}$

$= \dfrac{-6x}{\left(x^2+1\right)^2}$

$f''(x)$

$= \dfrac{\left(x^2+1\right)^2(-6) - (-6x)\left(2\left(x^2+1\right)(2x)\right)}{\left(\left(x^2+1\right)^2\right)^2}$

$= \dfrac{\left(x^2+1\right)\left[\left(x^2+1\right)(-6) - (-6x)(2)(2x)\right]}{\left(x^2+1\right)^4}$

$= \dfrac{18x^2 - 6}{\left(x^2+1\right)^3}$

The domain of f is $\mathbb{R}$.

b) Since $f'(x)$ exists for all real numbers, the only critical values are where $f'(x) = 0$.

$\dfrac{-6x}{\left(x^2+1\right)^2} = 0$

$-6x = 0$ Multiplying by $\left(x^2+1\right)^2$

$x = 0$

The critical value is $x = 0$.

$f(0) = \dfrac{3}{(0)^2+1} = 3$

The critical point $(0, 3)$ is on the graph.

c) We use the Second Derivative Test.

$f''(0) = \dfrac{18(0) - 6}{\left((0)^2+1\right)^2} = \dfrac{-6}{1} = -6 < 0$

So the point $(0, 3)$ is a relative maximum.

We use 0 to divide the real number line into two intervals $(-\infty, 0)$ and $(0, \infty)$. $f(x)$ is increasing on the interval $(-\infty, 0)$, and $f(x)$ is decreasing on the interval $(0, \infty)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so the only possible points of inflection occur when $f''(x) = 0$.

$\dfrac{18x^2 - 6}{\left(x^2+1\right)^3} = 0$

$18x^2 - 6 = 0$

$18x^2 = 6$

$x^2 = \dfrac{1}{3}$

$x = \pm\dfrac{1}{\sqrt{3}}$

There are two possible inflection points at $x = -\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$.

$f\left(-\dfrac{1}{\sqrt{3}}\right) = \dfrac{3}{\left(-\frac{1}{\sqrt{3}}\right)^2 + 1}$

$= \dfrac{3}{\frac{1}{3}+1} = \dfrac{3}{\frac{4}{3}} = \dfrac{9}{4}$

The solution is continued on the next page.

Evaluating the function at the second possible inflection point, we have:

$$f\left(\frac{1}{\sqrt{3}}\right)=\frac{3}{\left(\frac{1}{\sqrt{3}}\right)^2+1}$$

$$=\frac{3}{\frac{1}{3}+1}=\frac{3}{\frac{4}{3}}=\frac{9}{4}$$

The points $\left(-\frac{1}{\sqrt{3}},\frac{9}{4}\right)$ and $\left(\frac{1}{\sqrt{3}},\frac{9}{4}\right)$ are possible inflection points on the graph.

e) To determine concavity we use $-\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$ to divide the real number line into three intervals,

A : $\left(-\infty,-\frac{1}{\sqrt{3}}\right)$, B: $\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$, and

C: $\left(\frac{1}{\sqrt{3}},\infty\right)$

Then test a point in each interval.

A: Test -1, $f''(-1)=\dfrac{18(-1)^2-6}{\left((-1)^2+1\right)^3}=\dfrac{3}{2}>0$

B: Test 0, $f''(0)=\dfrac{18(0)^2-6}{\left((0)^2+1\right)^3}=-6<0$

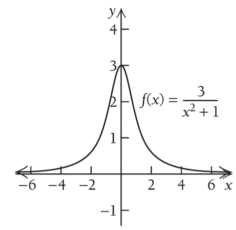
C: Test 1, $f''(1)=\dfrac{18(1)^2-6}{\left((1)^2+1\right)^3}=\dfrac{3}{2}>0$

We see that f is concave up on the intervals $\left(-\infty,-\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}},\infty\right)$ and concave down on the interval $\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$.

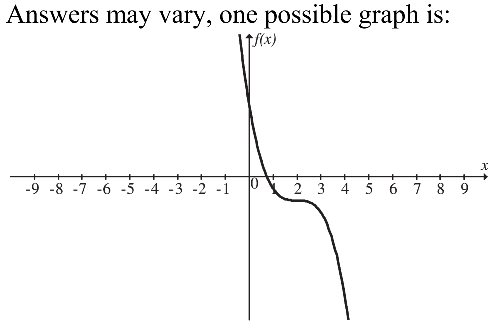
Therefore the points $\left(-\frac{1}{\sqrt{3}},\frac{9}{4}\right)$ and $\left(\frac{1}{\sqrt{3}},\frac{9}{4}\right)$ are inflection points.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

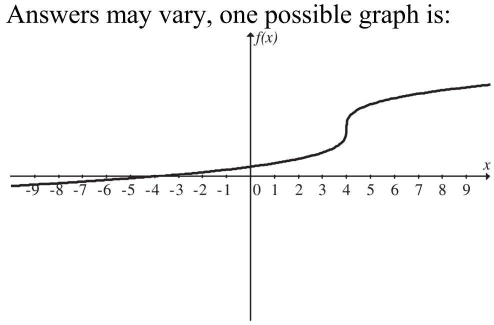
x	$f(x)$
-3	$\frac{3}{10}$
-1	$\frac{3}{2}$
1	$\frac{3}{2}$
3	$\frac{3}{10}$



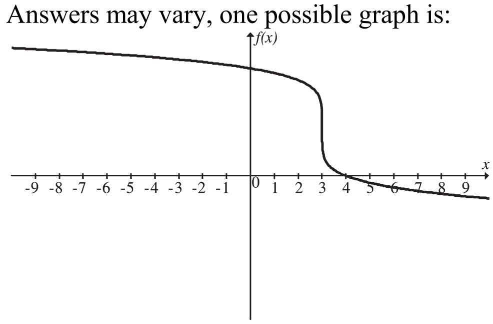
47. Answers may vary, one possible graph is:



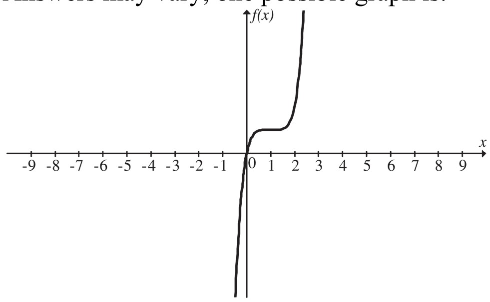
48. Answers may vary, one possible graph is:



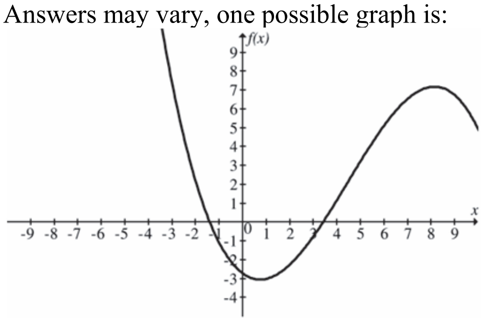
49. Answers may vary, one possible graph is:



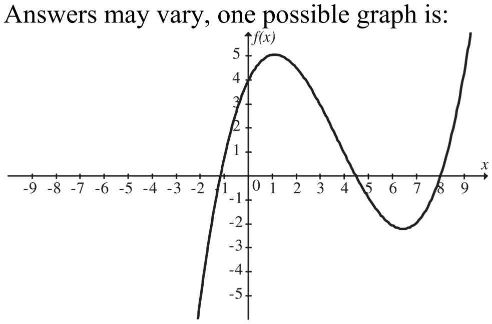
50. Answers may vary, one possible graph is:



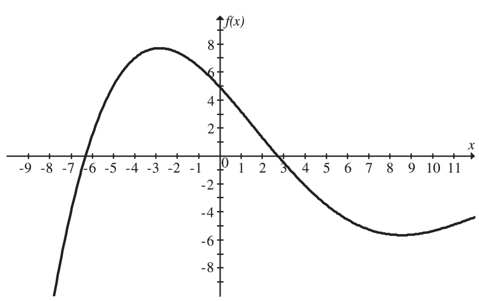
51. Answers may vary, one possible graph is:



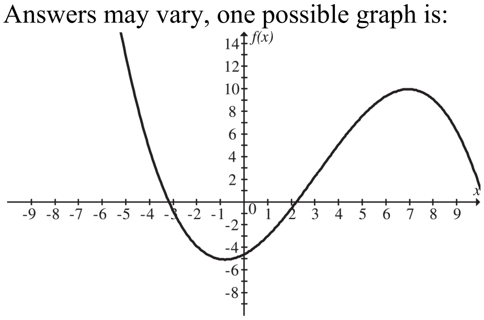
52. Answers may vary, one possible graph is:



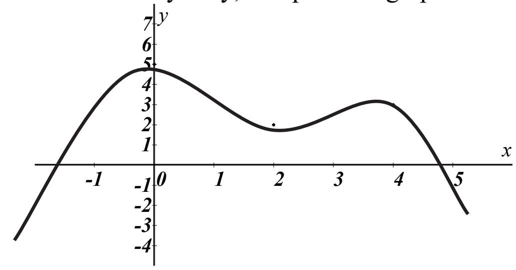
53. Answers may vary, one possible graph is:



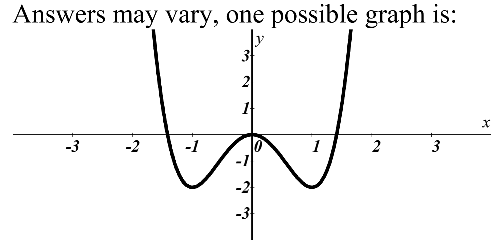
54. Answers may vary, one possible graph is:



55. Answers may vary, one possible graph is:



56. Answers may vary, one possible graph is:



57. $R(x)=50x-0.5x^2$

$C(x)=4x+10$

$P(x)=R(x)-C(x)$

$=(50x-0.5x^2)-(4x+10)$

$=-0.5x^2+46x-10$

We will restrict the domains of all three functions to $x\geq 0$ since a negative number of units cannot be produced and sold.

First graph $R(x)=50x-0.5x^2$

$R'(x)=50-x$

$R''(x)=-1$

Since $R'(x)$ exists for all $x\geq 0$, the only critical points are where $R'(x)=0$.

$50-x=0$

$50=x$ Critical Value

Find the function value at $x=50$.

$R(50)=50(50)-0.5(50)^2$

$=2500-1250$

$=1250$

This critical point $(50,1250)$ is on the graph.

We use the Second Derivative Test:

$R''(50)=-1<0$

The point $(50,1250)$ is a relative maximum.

We use 50 to divide the interval $[0,\infty)$ into two intervals, $[0,50)$ and $(50,\infty)$. We know that R is increasing on $(0,50)$ and decreasing on $(50,\infty)$.

The solution is continued on the next page.

Next, find the inflection points. Since $R''(x)$ exists for all $x \geq 0$, and $R''(x) = -1$, there are no possible inflection points.
Furthermore, since $R''(x) < 0$ for all $x \geq 0$, R is concave down over the interval $(0, \infty)$.
Sketch the graph using the preceding information. The x-intercepts of R are found by solving $R(x) = 0$.

$$50x - 0.5x^2 = 0$$
$$0.5x(100 - x) = 0$$
$$0.5x = 0 \quad \text{or} \quad 100 - x = 0$$
$$x = 0 \quad \text{or} \quad 100 = x$$

The x-intercepts are $(0,0)$ and $(100,0)$.
Next, we graph $C(x) = 4x + 10$. This is a linear function with slope 4 and y-intercept $(0,10)$.
$C(x)$ is increasing over the entire domain $x \geq 0$ and has no relative extrema or points of inflection.
Finally, we graph $P(x) = -0.5x^2 + 46x - 10$

$$P'(x) = -x + 46$$
$$P''(x) = -1$$

Since $P'(x)$ exists for all $x \geq 0$, the only critical points occur when $P'(x) = 0$.

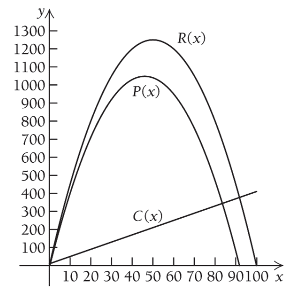
$$-x + 46 = 0$$
$$46 = x \qquad \text{Critical Value}$$

Find the function value at $x = 46$.

$$P(46) = -0.5(46)^2 + 46(46) - 10$$
$$= -1058 + 2116 - 10$$
$$= 1048$$

The critical point $(46,1048)$ is on the graph.
We use the Second Derivative Test:
$P''(46) = -1 < 0$
The point $(46,1048)$ is a relative maximum.
We use 46 to divide the interval $[0, \infty)$ into two intervals, $[0, 46)$ and $(46, \infty)$, we know that P is increasing on $(0, 46)$ and decreasing on $(46, \infty)$.
Next, find the inflection points. Since $P''(x)$ exists for all $x \geq 0$, and $P''(x) = -1$, there are no possible inflection points. Furthermore, since $P''(x) < 0$ for all $x \geq 0$, P is concave down over the interval $(0, \infty)$.

Sketch the graph using the preceding information.



58. $R(x) = 50x - 0.5x^2$

$$C(x) = 10x + 3$$
$$P(x) = R(x) - C(x)$$
$$= (50x - 0.5x^2) - (10x + 3)$$
$$= -0.5x^2 + 40x - 3$$

We will restrict the domains of all three functions to $x \geq 0$.
First graph $R(x) = 50x - 0.5x^2$ as in Exercise 57.
Next, we graph $C(x) = 10x + 3$. This is a linear function with slope 10 and y-intercept $(0,3)$.
$C(x)$ is increasing over the entire domain $x \geq 0$ and has no relative extrema or points of inflection.
Finally, we graph $P(x) = -0.5x^2 + 40x - 3$

$$P'(x) = -x + 40$$
$$P''(x) = -1$$

Since $P'(x)$ exists for all $x \geq 0$, the only critical points occur when $P'(x) = 0$.

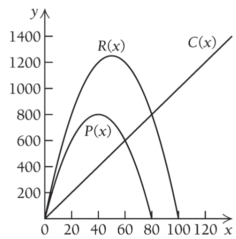
$$-x + 40 = 0$$
$$40 = x \qquad \text{Critical Value}$$
$$P(40) = 797$$

The critical point $(40, 797)$ is on the graph.
We use the Second Derivative Test:
$P''(40) = -1 < 0$
The point $(40, 797)$ is a relative maximum.
We know that P is increasing on $(0, 40)$ and decreasing on $(40, \infty)$.

The solution is continued on the next page.

Next, find the inflection points. Since $P''(x)$ exists for all $x \geq 0$, and $P''(x) = -1$, there are no possible inflection points. Furthermore, since $P''(x) < 0$ for all $x \geq 0$, P is concave down over the interval $(0, \infty)$.

Sketch the graph using the preceding information.



59. $p(x) = \dfrac{13x^3 - 240x^2 - 2460x + 585{,}000}{75{,}000}$

$p'(x) = \dfrac{39x^2 - 480x - 2460}{75{,}000}$

$p''(x) = \dfrac{78x - 480}{75{,}000}$

Since $p'(x)$ exists for all real numbers, the only critical points are where $p'(x) = 0$.

$\dfrac{39x^2 - 480x - 2460}{75{,}000} = 0$

$39x^2 - 480x - 2460 = 0$

Using the quadratic formula, we have:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-480) \pm \sqrt{(-480)^2 - 4(39)(-2460)}}{2(39)}$$

$$= \frac{480 \pm \sqrt{614{,}160}}{78}$$

$x \approx -3.89$ or $x \approx 16.20$ Critical values

Since the domain of the function is $0 \leq x \leq 40$, we consider only $x \approx 16.20$

$p(16.20)$

$= \dfrac{13(16.20)^3 - 240(16.20)^2 - 2460(16.20) - 585{,}000}{75{,}000}$

≈ 7.17

The critical point $(16.20, 7.17)$ is on the graph.

We use the Second Derivative Test:

$p''(x) = \dfrac{78(16.20) - 480}{75{,}000} \approx 0.01 > 0$

The point $(16.20, 7.17)$ is a relative minimum.

If we use the point 16.20 to divide the domain into two intervals, $[0, 16.20)$ and $(16.20, 40]$, we know that p is decreasing on $(0, 16.20)$ and increasing on $(16.20, 40)$.

Next, we find the inflection points. $p''(x)$ exists for all real numbers, so the only possible inflection points are where $p''(x) = 0$

$\dfrac{78x - 480}{75{,}000} = 0$

$78x - 480 = 0$

$78x = 480$

$x \approx 6.15$

$p(6.15)$

$= \dfrac{13(6.15)^3 - 240(6.15)^2 - 2460(6.15) - 585{,}000}{75{,}000}$

≈ 7.52

The point $(6.15, 7.52)$ is a possible inflection point.

To determine concavity, we use 6.15 to divide the domain into two intervals A: $[0, 6.15)$ and B: $(6.15, 40]$ and test a point in each interval.

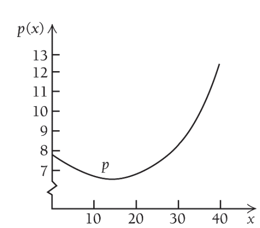
A: Test 1, $p''(1) = \dfrac{78(1) - 480}{75{,}000} = -0.005 < 0$

B: Test 7, $p''(7) = \dfrac{78(7) - 480}{75{,}000} = 0.00088 > 0$

Then p is concave down on $(0, 6.15)$ and concave up on $(6.15, 40)$ and the point $(6.15, 7.52)$ is a point of inflection.

Sketch the graph for $0 \leq x \leq 40$ using the preceding information. Additional function values may be calculated if necessary.

x	$p(x)$
0	7.8
8	7.42
12	7.25
20	7.25
24	7.57
32	9.15
40	12.46



60. $f(x)=0.025x^2-0.71x+20.44$

$f'(x)=0.05x-0.71$

$f''(x)=0.05$

$f'(x)$ exists for all real numbers. Solve

$$f'(x)=0$$
$$0.05x-0.71=0$$
$$0.05x=0.71$$
$$x=14.2$$

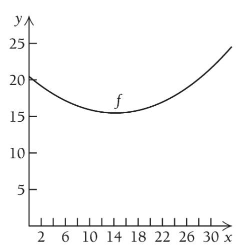
$$f(14.2)=0.025(14.2)^2-0.71(14.2)+20.44$$
$$=15.399$$

$f''(14.2)=0.05$, so $(14.2,15.399)$ is a relative minimum. Then $f(x)$ is decreasing on $(0,14.2)$ and increasing on $(14.2,30)$.

Next, find the points of inflection. Since $f''(x)=0.05$ exists for all real numbers and is always positive, $f(x)$ is concave up on the interval $(0,30)$.

Sketch the graph of $f(x)$ using the information above. Additional values may be calculated as necessary.

x	$f(x)$
0	20.44
5	17.52
10	15.84
15	15.42
20	16.24
25	18.32
30	21.64



61. The monthly rainfall is approximated by

$$R(t)=-0.006t^4+0.213t^3-1.702t^2+0.615t+27.745$$

To find the inflection points we find the first and second derivatives.

$R'(t)=-0.024t^3+0.639t^2-3.404t+0.615$

$R''(t)=-0.072t^2+1.278t-3.404$

Since $R''(t)$ exists for all real values, we solve $R''(t)=0$ By the quadratic formula.

$$t=\frac{-1.278\pm\sqrt{(1.278)^2-4(-0.072)(-3.404)}}{2(-0.072)}$$

$t=3.26$ or $t=14.48$

These are two possible points of inflection.

$R(3.26)\approx 18.36$

$R(14.48)\approx 62.69$

The two points of inflection are $(3.26,18.36)$ and $(14.48,62.69)$. However, $t=14.48$ is not in the domain of this function. The left most inflection point $(3.26,18.36)$ implies that rate of change of the amount of rainfall is decreasing the fastest at this point.

62. $V(r)=k\left(20r^2-r^3\right), \quad 0\le r\le 20$

$V'(r)=k\left(40r-3r^2\right)$

$V''(r)=k(40-6r)$

$V'(r)$ exists for all r in $[0,20]$, so the only critical points occur where $V'(r)=0$.

$$k\left(40r-3r^2\right)=0$$
$$40r-3r^2=0$$
$$r(40-3r)=0$$

$r=0$ or $40-3r=0$

$r=0$ or $40=3r$

$r=0$ or $\frac{40}{3}=r$

Using the Second Derivative Test:

$V''(0)=k(40-6(0))=40k>0 \qquad [k>0]$

$V''\left(\frac{40}{3}\right)=k\left(40-6\left(\frac{40}{3}\right)\right)=-40k<0$

Since $V''\left(\frac{40}{3}\right)<0$, we know that there is a relative maximum at $x=\frac{40}{3}$. Thus, for an object whose radius is $\frac{40}{3}$ mm or 13.33 mm, the maximum velocity is needed to remove the object.

63. Observe that *h* is increasing for all values of *x* for which *g* is positive and *h* is decreasing for all values of *x* for which *g* is negative. Furthermore, for all values of *x* for which *g*=0, *h* has a horizontal tangent. Therefore, *g*=*h'*.

64. Observe that g is increasing for all values of x for which h is positive and g is decreasing for all values of x for which h is negative. Furthermore, for all values of x for which h=0, g has a horizontal tangent. Therefore, $h = g'$.

65. $f(x) = ax^2 + bx + c, \quad a \neq 0$

$f'(x) = 2ax + b$

$f''(x) = 2a$

Since $f'(x)$ exists for all real numbers, the only critical points occur when $f'(x) = 0$. We solve:

$$2ax + b = 0$$
$$2ax = -b$$
$$x = \frac{-b}{2a}$$

So the critical value will occur at $x = \frac{-b}{2a}$.

Applying the second derivative test, we see that

$f''(x) = 2a > 0, \quad$ for $a > 0$

$f''(x) = 2a < 0, \quad$ for $a < 0$

Therefore, a relative maximum occurs at $x = \frac{-b}{2a}$ when $a < 0$ and a relative minimum occurs at $x = \frac{-b}{2a}$ when $a > 0$.

66. The point of inflection will occur when the second derivative is equal to zero or undefined. The derivatives of the function are:

$g(x) = ax^3 + bx^2 + cx + d$

$g'(x) = 3ax^2 + 2bx + c$

$g''(x) = 6ax + 2b.$

Setting the second derivative equal to zero and solving for x, we have:

$$6ax + 2b = 0$$
$$x = -\frac{2b}{6a} = -\frac{b}{3a}.$$

Therefore, the point of inflection must occur at $x = -\frac{b}{3a}$.

67. True.

68. True.

69. True.

70. False, consider $f(x) = \sqrt[3]{x} + \sqrt[3]{x-2}$. This function has points of inflection at $x = 0$, $x = 1$ and $x = 2$ without any critical values between the points of inflection.

71. True.

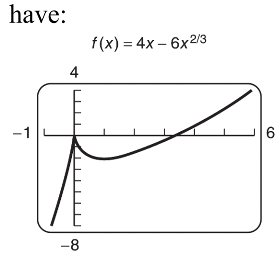
72. False: The function does not switch concavity at the extreme value.

73. True.

74. The rate of change is maximized at the points of inflection. Looking at the graph, we estimate the points of inflection to be 75 days after January first, and 270 days after January first. Therefore, the number of hours of daylight are increasing most rapidly approximately 75 days after January 1st or approximately March 16th and the number of hours of daylight are decreasing most rapidly approximately 270 days after January 1st or approximately September 27th.

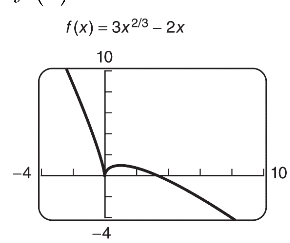
75. $f(x) = 4x - 6x^{2/3}$

Graphing the function on the calculator we have:



Using the minimum/maximum feature on the calculator, we estimate a relative maximum at $(0,0)$ and a relative minimum at $(1,-2)$.

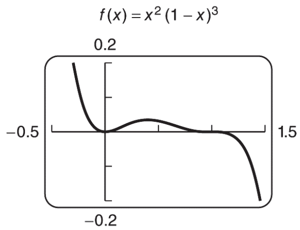
76. $f(x) = 3x^{2/3} - 2x$



We estimate a relative maximum at $(1,1)$ and a relative minimum at $(0,0)$.

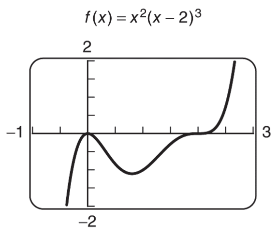
77. $f(x)=x^2(1-x)^3$

Graphing the function on the calculator we have:



Using the minimum/maximum feature on the calculator, we estimate a relative maximum at $(0.4, 0.035)$ and a relative minimum at $(0,0)$.

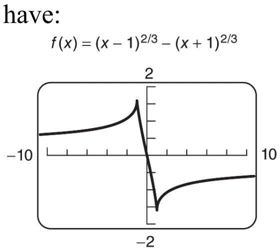
78. $f(x)=x^2(x-2)^3$



We estimate a relative maximum at $(0,0)$ and a relative minimum at $\left(\frac{4}{5},-1.106\right)$.

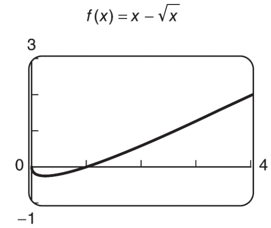
79. $f(x)=(x-1)^{2/3}-(x+1)^{2/3}$

Graphing the function on the calculator we have:



Using the minimum/maximum feature on the calculator, we estimate a relative maximum at $(-1, 1.587)$ and a relative minimum at $(1,-1.587)$.

80. $f(x)=x-\sqrt{x}$



We estimate a relative minimum at $(0.25,-0.25)$.

81.

a) The cubic and quartic functions appear to be equally good fits. The cubic function will ease the computations, where as the quartic function will be a little better fit to the data. Also they quartic function declines sharply after 2011 so the cubic function is probably a better fit.

b) The domain of is the set of non-negative real numbers.

c) The graphs show that the cubic function does not have relative extrema on a reasonable domain. The quartic function has a relative maximum around $x=10.3$. However, judging by the data, it appears that the percentage of households headed by someone with a bachelor's degree or higher is increasing with time.

Exercise Set 2.3

1. $f(x)=\dfrac{x+4}{x-2}$

The expression is in simplified form. We set the denominator equal to zero and solve.

$x-2=0$

$x=2$

The vertical asymptote is the line $x=2$.

2. $f(x)=\dfrac{2x-3}{x-5}$

The expression is in simplified form.
The vertical asymptote is the line $x=5$.

3. $f(x)=\dfrac{5x}{x^2-25}$

First, we write the function in simplified form.

$f(x)=\dfrac{5x}{(x-5)(x+5)}$

Once the expression is in simplified form, we set the denominator equal to zero and solve.

$(x-5)(x+5)=0$

$x-5=0$ or $x+5=0$

$x=5$ or $x=-5$

The vertical asymptotes are the lines $x=-5$ and $x=5$.

4. $f(x)=\dfrac{3x}{x^2-9}=\dfrac{3x}{(x-3)(x+3)}$

Once the expression is in simplified form, we set the denominator equal to zero and solve.

$(x-3)(x+3)=0$

$x-3=0$ or $x+3=0$

$x=3$ or $x=-3$

The vertical asymptotes are the lines $x=-3$ and $x=3$.

5. $f(x)=\dfrac{x+3}{x^3-x}$

First, we write the function in simplified form.

$f(x)=\dfrac{x+3}{x(x^2-1)}$

$=\dfrac{x+3}{x(x-1)(x+1)}$

Once the expression is in simplified form, we set the denominator equal to zero and solve.

$x(x+1)(x-1)=0$

$x=0$ or $x+1=0$ or $x-1=0$

$x=0$ or $x=-1$ or $x=1$

The vertical asymptotes are the lines $x=0,\ x=-1,$ and $x=1$.

6. $f(x)=\dfrac{x+2}{x^3-6x^2+8x}$

First, we write the function in simplified form.

$f(x)=\dfrac{x+2}{x(x^2-6x+8)}=\dfrac{x+2}{x(x-4)(x-2)}$

The vertical asymptotes are the lines $x=0,\ x=2,$ and $x=4$.

7. $f(x)=\dfrac{x+2}{x^2+6x+8}$

First, we write the function in simplified form.

$f(x)==\dfrac{x+2}{(x+2)(x+4)}$

$=\dfrac{1}{x+4}\quad ,\ x\neq -2$ Dividing common factors

Once the expression is in simplified form, we set the denominator equal to zero and solve.

$x+4=0$

$x=-4$

The vertical asymptote is the line $x=-4$.

8. $f(x)=\dfrac{x+6}{x^2+7x+6}$

First, we write the function in simplified form.

$f(x)=\dfrac{x+6}{(x+6)(x+1)}=\dfrac{1}{x+1},\ x\neq -6$

The vertical asymptote is the line $x=-1$.

9. $f(x)=\dfrac{7}{x^2+49}$

The function is in simplified form. The equation $x^2+49=0$ has no real solution; therefore, the function does not have any vertical asymptotes.

10. $f(x)=\dfrac{6}{x^2+36}$

The function is in simplified form. The equation $x^2+36=0$ has no real solution; therefore, the function does not have any vertical asymptotes.

11. $f(x)=\dfrac{6x}{8x+3}$

To find the horizontal asymptote, we consider $\lim\limits_{x\to\infty} f(x)$. To find the limit, we will use some algebra and the fact that as $x\to\infty, \dfrac{b}{ax^n}\to 0$ for any positive integer *n*.

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty}\frac{6x}{8x+3}\\ &= \lim_{x\to\infty}\frac{6x}{8x+3}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} \quad \text{Multiplying by a form of 1}\\ &= \lim_{x\to\infty}\frac{\frac{6x}{x}}{\frac{8x}{x}+\frac{3}{x}}\\ &= \lim_{x\to\infty}\frac{6}{8+\frac{3}{x}}\\ &= \frac{6}{8+0} \qquad \left[\text{as } x\to\infty, \frac{b}{ax^n}\to 0\right]\\ &= \frac{6}{8}=\frac{3}{4}.\end{aligned}$$

In a similar manner, it can be shown that $\lim\limits_{x\to-\infty} f(x)=\dfrac{3}{4}$.

The horizontal asymptote is the line $y=\dfrac{3}{4}$.

12. $f(x)=\dfrac{3x^2}{6x^2+x}$

Find $\lim\limits_{x\to\infty} f(x)$.

$$\lim_{x\to\infty}\frac{3x^2}{6x^2+x}=\lim_{x\to\infty}\frac{3}{6+\frac{1}{x}}=\frac{3}{6+0}=\frac{1}{2}.$$

In a similar manner, it can be shown that $\lim\limits_{x\to-\infty} f(x)=\dfrac{1}{2}$.

The horizontal asymptote is the line $y=\dfrac{1}{2}$.

13. $f(x)=\dfrac{4x}{x^2-3x}$

To find the horizontal asymptote, we consider $\lim\limits_{x\to\infty} f(x)$. To find the limit, we will use some algebra and the fact that as $x\to\infty, \dfrac{b}{ax^n}\to 0$ for any positive integer *n*.

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty}\frac{4x}{x^2-3x}\\ &= \lim_{x\to\infty}\frac{4x}{x^2-3x}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}} \quad \text{Multiplying by a form of 1}\\ &= \lim_{x\to\infty}\frac{\frac{4x}{x^2}}{\frac{x^2}{x^2}-\frac{3}{x^2}}\\ &= \lim_{x\to\infty}\frac{\frac{4}{x}}{1+\frac{3}{x^2}}\\ &= \frac{0}{1+0} \qquad \left[\text{as } x\to\infty, \frac{b}{ax^n}\to 0\right]\\ &= 0.\end{aligned}$$

In a similar manner, it can be shown that $\lim\limits_{x\to-\infty} f(x)=0$.

The horizontal asymptote is the line $y=0$.

14. $f(x)=\dfrac{2x}{3x^3-x^2}$

Find $\lim\limits_{x\to\infty} f(x)$.

$$\lim_{x\to\infty}\frac{2x}{3x^3-x^2}=\lim_{x\to\infty}\frac{\frac{2}{x^2}}{3-\frac{1}{x}}=\frac{0}{3-0}=0.$$

In a similar manner, it can be shown that $\lim\limits_{x\to-\infty} f(x)=0$.

The horizontal asymptote is the line $y=0$.

15. $f(x)=4+\frac{2}{x}$

To find the horizontal asymptote, we consider $\lim_{x\to\infty} f(x)$. To find the limit, we will use the fact that as $x\to\infty, \frac{b}{ax^n}\to 0$ for any positive integer *n*.

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty} 4+\frac{2}{x}$$
$$=4+0 \qquad \left[\text{as } x\to\infty, \frac{b}{ax^n}\to 0\right].$$
$$=4$$

In a similar manner, it can be shown that $\lim_{x\to-\infty} f(x)=4.$

The horizontal asymptote is the line $y=4$.

16. $f(x)=5-\frac{3}{x}$

Find $\lim_{x\to\infty} f(x)$.

$$\lim_{x\to\infty} 5-\frac{3}{x}=5-0=5.$$

In a similar manner, it can be shown that $\lim_{x\to-\infty} f(x)=5.$

The horizontal asymptote is the line $y=5$.

17. $f(x)=\frac{6x^3+4x}{3x^2-x}$

To find the horizontal asymptote, we consider $\lim_{x\to\infty} f(x)$. To find the limit, we will use some algebra and the fact that as $x\to\infty, \frac{b}{ax^n}\to 0$ for any positive integer .

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{6x^3+4x}{3x^2-x}$$
$$=\lim_{x\to\infty}\frac{6x^3+4x}{3x^2-x}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}} \qquad \text{Multiplying by a form of 1}$$
$$=\lim_{x\to\infty}\frac{6x+\frac{4}{x}}{3-\frac{1}{x}}$$
$$=\frac{\lim_{x\to\infty} 6x+\frac{4}{x}}{3-0}=\infty$$

In a similar manner, it can be shown that $\lim_{x\to-\infty} f(x)=-\infty.$

The function increases without bound as $x\to\infty$ and decreases without bound as $x\to-\infty.$ Therefore, the function does not have a horizontal asymptote.

18. $f(x)=\frac{8x^4-5x^2}{2x^3+x^2}$

Find $\lim_{x\to\infty} f(x)$.

$$\lim_{x\to\infty}\frac{8x^4-5x^2}{2x^3+x^2}=\lim_{x\to\infty}\frac{8x-\frac{5}{x}}{2-\frac{1}{x}}=\frac{\lim_{x\to\infty} 8x-\frac{5}{x}}{2-0}=\infty.$$

In a similar manner, it can be shown that $\lim_{x\to-\infty} f(x)=-\infty.$

The function increases without bound as $x\to\infty$ and decreases without bound as $x\to-\infty.$ Therefore, the function does not have a horizontal asymptote.

19. $f(x)=\frac{4x^3-3x+2}{x^3+2x-4}$

To find the horizontal asymptote, we consider $\lim_{x\to\infty} f(x)$. To find the limit, we will use some algebra and the fact that as $x\to\infty, \frac{b}{ax^n}\to 0$ for any positive integer *n*.

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{4x^3-3x+2}{x^3+2x-4}$$
$$=\lim_{x\to\infty}\frac{4x^3-3x+2}{x^3+2x-4}\cdot\frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$
$$=\lim_{x\to\infty}\frac{4-\frac{3}{x^2}+\frac{2}{x^3}}{1+\frac{2}{x^2}-\frac{4}{x^3}}$$
$$=\frac{4}{1} \qquad \left[\text{as } x\to\infty, \frac{b}{ax^n}\to 0\right]$$
$$=4$$

In a similar manner, it can be shown that $\lim_{x\to-\infty} f(x)=4.$

The horizontal asymptote is the line $y=4$.

20. $f(x)=\dfrac{6x^4+4x^2-7}{2x^5-x+3}$

Find $\lim_{x\to\infty} f(x)$.

$$\lim_{x\to\infty}\frac{6x^4+4x^2-7}{2x^5-x+3}=\lim_{x\to\infty}\frac{\frac{6}{x}+\frac{4}{x^3}-\frac{7}{x^5}}{2-\frac{1}{x^4}+\frac{3}{x^5}}$$
$$=\frac{0}{2+0}=0$$

In a similar manner, it can be shown that $\lim_{x\to-\infty} f(x)=0.$

The horizontal asymptote is the line $y=0$.

21. $f(x)=\dfrac{2x^3-4x+1}{4x^3+2x-3}$

To find the horizontal asymptote, we consider $\lim_{x\to\infty} f(x)$. To find the limit, we will use some algebra and the fact that as $x\to\infty, \dfrac{b}{ax^n}\to 0$ for any positive integer n.

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{2x^3-4x+1}{4x^3+2x-3}$$
$$=\lim_{x\to\infty}\frac{2x^3-4x+1}{4x^3+2x-3}\cdot\frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$
$$=\lim_{x\to\infty}\frac{2-\frac{4}{x^2}+\frac{1}{x^3}}{4+\frac{2}{x^2}-\frac{3}{x^3}}$$
$$=\frac{2-0+0}{4+0-0}\qquad\left[\text{as } x\to\infty, \frac{b}{ax^n}\to 0\right]$$
$$=\frac{2}{4}=\frac{1}{2}$$

In a similar manner, it can be shown that $\lim_{x\to-\infty} f(x)=\dfrac{1}{2}.$

The horizontal asymptote is the line $y=\dfrac{1}{2}$.

22. $f(x)=\dfrac{5x^4-2x^3+x}{x^5-x^3+8}$

Find $\lim_{x\to\infty} f(x)$.

$$\lim_{x\to\infty}\frac{5x^4-2x^3+x}{x^5-x^3+8}=\lim_{x\to\infty}\frac{\frac{5}{x}-\frac{2}{x^2}+\frac{1}{x^5}}{1-\frac{1}{x^2}+\frac{8}{x^5}}=\frac{0}{1}=0.$$

In a similar manner, it can be shown that $\lim_{x\to-\infty} f(x)=0.$

The horizontal asymptote is the line $y=0$.

23. $f(x)=\dfrac{4}{x}=4x^{-1}$

a) *Intercepts.* Since the numerator is the constant 4, there are no x-intercepts. The number 0 is not in the domain of the function, so there are no y-intercepts.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x=0$, so the line $x=0$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y=0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x)=-4x^{-2}=-\frac{4}{x^2}$$
$$f''(x)=8x^{-3}=\frac{8}{x^3}$$

The domain of f is $(-\infty,0)\cup(0,\infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 0, but 0 is not in the domain of the function, so $x=0$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use 0 to divide the real number line into two intervals A: $(-\infty,0)$ and B: $(0,\infty)$, and we test a point in each interval.

A: Test -1, $f'(-1)=-\dfrac{4}{(-1)^2}=-4<0$

B: Test 1, $f'(1)=-\dfrac{4}{(1)^2}=-4<0$

The solution is continued on the next page.

Then $f(x)$ is decreasing on both intervals.
Since there are no critical points, there are no relative extrema.

f) *Inflection points*. $f''(x)$ does not exist at 0, but because 0 is not in the domain of the function, there cannot be an inflection point at 0. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

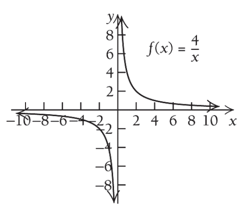
g) *Concavity.* We use 0 to divide the real number line into two intervals
A: $(-\infty,0)$ and B: $(0,\infty)$, and we test a point in each interval.

A: Test -1, $f''(-1)=\dfrac{8}{(-1)^3}=-8<0$

B: Test 1, $f''(1)=\dfrac{8}{(1)^3}=8>0$

Therefore, $f(x)$ is concave down on $(-\infty,0)$ and concave up on $(0,\infty)$.

h) *Sketch*.



24. $f(x)=-\dfrac{5}{x}=-5x^{-1}$

a) *Intercepts.* Since the numerator is the constant -5, there are no *x*-intercepts. The number 0 is not in the domain of the function, so there are no *y*-intercepts.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x=0$, so the line $x=0$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y=0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$f'(x)=5x^{-2}=\dfrac{5}{x^2}$

$f''(x)=-10x^{-3}=-\dfrac{10}{x^3}$

The domain of f is $(-\infty,0)\cup(0,\infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 0, but 0 is not in the domain of the function, so $x=0$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use 0 to divide the real number line into two intervals A: $(-\infty,0)$ and B: $(0,\infty)$, and we test a point in each interval.

A: Test -1, $f'(-1)=\dfrac{5}{(-1)^2}=5>0$

B: Test 1, $f'(1)=\dfrac{5}{(1)^2}=5>0$

Then $f(x)$ is increasing on both intervals.
Since there are no critical points, there are no relative extrema.

f) *Inflection points*. $f''(x)$ does not exist at 0, but because 0 is not in the domain of the function, there cannot be an inflection point at 0. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

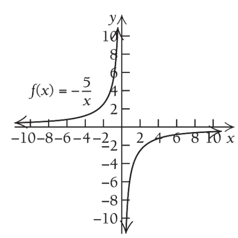
g) *Concavity.* We use 0 to divide the real number line into two intervals
A: $(-\infty,0)$ and B: $(0,\infty)$, and we test a point in each interval.

A: Test -1, $f''(-1)=-\dfrac{10}{(-1)^3}=10>0$

B: Test 1, $f''(1)=-\dfrac{10}{(1)^3}=-10<0$

Therefore, $f(x)$ is concave up on $(-\infty,0)$ and concave down on $(0,\infty)$.

h) *Sketch*. Use the preceding information to sketch the graph. Compute additional function values as needed.



25. $f(x) = \frac{-2}{x-5} = -2(x-5)^{-1}$

a) *Intercepts.* Since the numerator is the constant -2, there are no x-intercepts. To find the y-intercepts we compute $f(0)$.

$$f(0) = \frac{-2}{(0)-5} = \frac{2}{5}$$

The point $\left(0, \frac{2}{5}\right)$ is the y-intercept.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x = 5$, so the line $x = 5$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y = 0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x) = 2(x-5)^{-2} = \frac{2}{(x-5)^2}$$

$$f''(x) = -4(x-5)^{-3} = \frac{-4}{(x-5)^3}$$

The domain of f is $(-\infty, 5) \cup (5, \infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 5, but 5 is not in the domain of the function, so $x = 5$ is not a critical value. The equation $f'(x) = 0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use 5 to divide the real number line into two intervals A: $(-\infty, 5)$ and B: $(5, \infty)$, and we test a point in each interval.

A: Test 4, $f'(4) = \frac{2}{(4-5)^2} = \frac{2}{1} = 2 > 0$

B: Test 6, $f'(6) = \frac{2}{(6-5)^2} = \frac{2}{1} = 2 > 0$

Then $f(x)$ is increasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at 5, but because 5 is not in the domain of the function, there cannot be an inflection point at 5. The equation $f''(x) = 0$ has no solution; therefore, there are no inflection points.

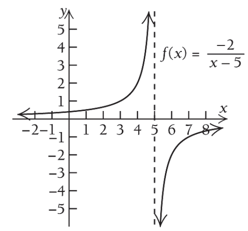
g) *Concavity.* We use 5 to divide the real number line into two intervals
A: $(-\infty, 5)$ and B: $(5, \infty)$, and we test a point in each interval.

A: Test 4, $f''(4) = \frac{-4}{(4-5)^3} = \frac{-4}{-1} = 4 > 0$

B: Test 6, $f''(6) = \frac{-4}{(6-5)^3} = \frac{-4}{1} = -4 < 0$

Therefore, $f(x)$ is concave up on $(-\infty, 5)$ and concave down on $(5, \infty)$.

h) *Sketch.*



26. $f(x) = \frac{1}{x-5} = (x-5)^{-1}$

a) *Intercepts.* Since the numerator is the constant 1, there are no x-intercepts. To find the y-intercepts we compute $f(0)$.

$$f(0) = \frac{1}{(0)-5} = -\frac{1}{5}$$

The point $\left(0, -\frac{1}{5}\right)$ is the y-intercept.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x = 5$, so the line $x = 5$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y = 0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x) = -(x-5)^{-2} = \frac{-1}{(x-5)^2}$$

$$f''(x) = 2(x-5)^{-3} = \frac{2}{(x-5)^3}$$

The domain of f is $(-\infty, 5) \cup (5, \infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 5, but 5 is not in the domain of the function, so $x = 5$ is not a critical value. The equation $f'(x) = 0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use 5 to divide the real number line into two intervals A: $(-\infty, 5)$ and B: $(5, \infty)$, and we test a point in each interval.

A: Test 4, $f'(4) = \dfrac{-1}{(4-5)^2} = -\dfrac{1}{1} = -1 < 0$

B: Test 6, $f'(6) = \dfrac{-1}{(6-5)^2} = -\dfrac{1}{1} = -1 < 0$

Then $f(x)$ is decreasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at 5, but because 5 is not in the domain of the function, there cannot be an inflection point at 5. The equation $f''(x) = 0$ has no solution; therefore, there are no inflection points.

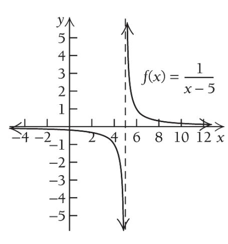
g) *Concavity.* We use 5 to divide the real number line into two intervals A: $(-\infty, 5)$ and B: $(5, \infty)$, and we test a point in each interval.

A: Test 4, $f''(4) = \dfrac{2}{(4-5)^3} = \dfrac{2}{-1} = -2 < 0$

B: Test 6, $f''(6) = \dfrac{2}{(6-5)^3} = \dfrac{2}{1} = 2 > 0$

Therefore, $f(x)$ is concave down on $(-\infty, 5)$ and concave up on $(5, \infty)$.

h) *Sketch.* Use the preceding information to sketch the graph. Compute additional function values as needed.



27. $f(x) = \dfrac{1}{x-3} = (x-3)^{-1}$

a) *Intercepts.* Since the numerator is the constant 1, there are no x-intercepts. To find the y-intercepts we compute $f(0)$.

$$f(0) = \frac{1}{(0)-3} = -\frac{1}{3}$$

The point $\left(0, -\dfrac{1}{3}\right)$ is the y-intercept.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x = 3$, so the line $x = 3$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y = 0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x) = -(x-3)^{-2} = \frac{-1}{(x-3)^2}$$

$$f''(x) = 2(x-3)^{-3} = \frac{2}{(x-3)^3}$$

The domain of f is $(-\infty, 3) \cup (3, \infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 3, but 3 is not in the domain of the function, so $x = 3$ is not a critical value. The equation $f'(x) = 0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use 3 to divide the real number line into two intervals A: $(-\infty, 3)$ and B: $(3, \infty)$, and we test a point in each interval.

A: Test 2, $f'(2) = \dfrac{-1}{((2)-3)^2} = -1 < 0$

B: Test 4, $f'(4) = \dfrac{-1}{((4)-3)^2} = -1 < 0$

Then $f(x)$ is decreasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at 3, but because 3 is not in the domain of the function, there cannot be an inflection point at 3. The equation $f''(x) = 0$ has no solution; therefore, there are no inflection points.

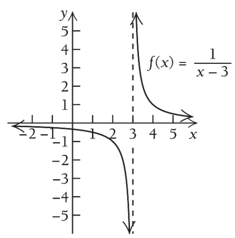
g) *Concavity.* We use 3 to divide the real number line into two intervals
A: $(-\infty,3)$ and B: $(3,\infty)$, and we test a point in each interval.

A: Test 2, $f''(2)=\dfrac{2}{((2)-3)^3}=-2<0$

B: Test 4, $f''(4)=\dfrac{2}{((4)-3)^3}=2>0$

Therefore, $f(x)$ is concave down on $(-\infty,3)$ and concave up on $(3,\infty)$.

h) *Sketch*.



28. $f(x)=\dfrac{1}{x+2}=(x+2)^{-1}$

a) *Intercepts.* Since the numerator is the constant 1, there are no *x*-intercepts. To find the *y*-intercepts we compute $f(0)$.

$$f(0)=\frac{1}{(0)+2}=\frac{1}{2}$$

The point $\left(0,\dfrac{1}{2}\right)$ is the *y*-intercept.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x=-2$, so the line $x=-2$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y=0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x)=-(x+2)^{-2}=\frac{-1}{(x+2)^2}$$

$$f''(x)=2(x+2)^{-3}=\frac{2}{(x+2)^3}$$

The domain of f is $(-\infty,-2)\cup(-2,\infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except -2, but -2 is not in the domain of the function, so $x=-2$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use -2 to divide the real number line into two intervals
A: $(-\infty,-2)$ and B: $(-2,\infty)$, and we test a point in each interval.

A: Test -3, $f'(-3)=\dfrac{-1}{((-3)+2)^2}=-1<0$

B: Test -1, $f'(-1)=\dfrac{-1}{((-1)+2)^2}=-1<0$

Then $f(x)$ is decreasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points*. $f''(x)$ does not exist at -2, but because -2 is not in the domain of the function, there cannot be an inflection point at -2. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

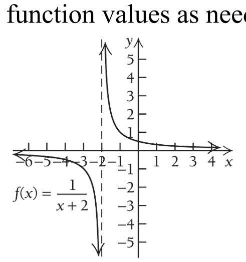
g) *Concavity.* We use -2 to divide the real number line into two intervals
A: $(-\infty,-2)$ and B: $(-2,\infty)$, and we test a point in each interval.

A: Test -3, $f''(-3)=\dfrac{2}{((-3)+2)^3}=-2<0$

B: Test -1, $f''(-1)=\dfrac{2}{((-1)+2)^3}=2>0$

Therefore, $f(x)$ is concave down on $(-\infty,-2)$ and concave up on $(-2,\infty)$.

h) *Sketch*. Use the preceding information to sketch the graph. Compute additional function values as needed.



29. $f(x)=\dfrac{-2}{x+5}=-2(x+5)^{-1}$

a) *Intercepts.* Since the numerator is the constant -2, there are no x-intercepts. To find the y-intercepts we compute $f(0)$.

$$f(0)=\frac{-2}{(0)+5}=\frac{-2}{5}=-\frac{2}{5}$$

The point $\left(0,-\frac{2}{5}\right)$ is the y-intercept.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x=-5$, so the line $x=-5$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y=0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x)=2(x+5)^{-2}=\frac{2}{(x+5)^2}$$

$$f''(x)=-4(x+5)^{-3}=\frac{-4}{(x+5)^3}$$

The domain of f is $(-\infty,-5)\cup(-5,\infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except -5, but -5 is not in the domain of the function, so $x=-5$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use -5 to divide the real number line into two intervals
A: $(-\infty,-5)$ and B: $(-5,\infty)$, and we test a point in each interval.

A: Test -6, $f'(-6)=\dfrac{2}{((-6)+5)^2}=2>0$

B: Test -4, $f'(-4)=\dfrac{2}{((-4)+5)^2}=2>0$

Then $f(x)$ is increasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at -5, but because -5 is not in the domain of the function, there cannot be an inflection point at -5. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

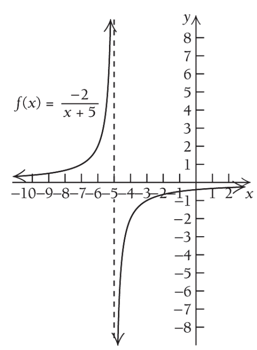
g) *Concavity.* We use -5 to divide the real number line into two intervals
A: $(-\infty,-5)$ and B: $(-5,\infty)$, and we test a point in each interval.

A: Test -6, $f''(-6)=\dfrac{-4}{((-6)+5)^3}=4>0$

B: Test -4, $f''(-4)=\dfrac{-4}{((-4)+5)^3}=-4<0$

Therefore, $f(x)$ is concave up on $(-\infty,-5)$ and concave down on $(-5,\infty)$.

h) *Sketch.*



30. $f(x)=\dfrac{-3}{x-3}=-3(x-3)^{-1}$

a) *Intercepts.* Since the numerator is the constant -3, there are no x-intercepts. To find the y-intercepts we compute $f(0)$.

$$f(0)=\frac{-3}{(0)-3}=\frac{3}{3}=1$$

The point $(0,1)$ is the y-intercept.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x=3$, so the line $x=3$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y=0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x)=3(x-3)^{-2}=\frac{3}{(x-3)^2}$$

$$f''(x)=-6(x-3)^{-3}=\frac{-6}{(x-3)^3}$$

The domain of f is $(-\infty,3)\cup(3,\infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 3, but 3 is not in the domain of the function, so $x=3$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use 3 to divide the real number line into two intervals A: $(-\infty,3)$ and B: $(3,\infty)$, and we test a point in each interval.

A: Test 2, $f'(2)=\dfrac{3}{((2)-3)^2}=3>0$

B: Test 4, $f'(4)=\dfrac{3}{((4)-3)^2}=3>0$

Then $f(x)$ is increasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points*. $f''(x)$ does not exist at 3, but because 3 is not in the domain of the function, there cannot be an inflection point at 3. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

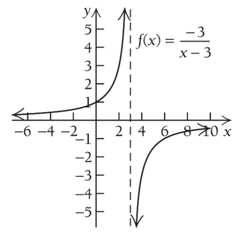
g) *Concavity.* We use 3 to divide the real number line into two intervals A: $(-\infty,3)$ and B: $(3,\infty)$, and we test a point in each interval.

A: Test 2, $f''(2)=\dfrac{-6}{((2)-3)^3}=6>0$

B: Test 4, $f''(4)=\dfrac{-6}{((4)-3)^3}=-6<0$

Therefore, $f(x)$ is concave up on $(-\infty,3)$ and concave down on $(3,\infty)$.

h) *Sketch*. Use the preceding information to sketch the graph. Compute additional function values as needed.



31. $f(x)=\dfrac{2x+1}{x}$

a) *Intercepts.* To find the x-intercepts, solve $f(x)=0$.

$$\frac{2x+1}{x}=0$$

$$2x+1=0$$

$$x=-\frac{1}{2}$$

This value does not make the denominator 0; therefore, the x-intercept is $\left(-\frac{1}{2},0\right)$.

The number 0 is not in the domain of $f(x)$ so there are no y-intercepts.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x=0$, so the line $x=0$ is a vertical asymptote.
Horizontal. The numerator and the denominator have the same degree, so $y=\frac{2}{1}$, or $y=2$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x)=-\frac{1}{x^2}$$

$$f''(x)=2x^{-3}=\frac{2}{x^3}$$

The domain of f is $(-\infty,0)\cup(0,\infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 0, but 0 is not in the domain of the function, so $x=0$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use 0 to divide the real number line into two intervals A: $(-\infty,0)$ and B: $(0,\infty)$, and we test a point in each interval.

A: Test -1, $f'(-1)=-\frac{1}{(-1)^2}=-1<0$

B: Test 1, $f'(1)=-\frac{1}{(1)^2}=-1<0$

Then $f(x)$ is decreasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points*. $f''(x)$ does not exist at 0, but because 0 is not in the domain of the function, there cannot be an inflection point at 0. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

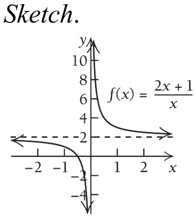
g) *Concavity.* We use 0 to divide the real number line into two intervals A: $(-\infty,0)$ and B: $(0,\infty)$, and we test a point in each interval.

A: Test -1, $f''(-1)=\frac{2}{(-1)^3}=-2<0$

B: Test 1, $f''(1)=\frac{2}{(1)^3}=2>0$

Therefore, $f(x)$ is concave down on $(-\infty,0)$ and concave up on $(0,\infty)$.

h) *Sketch*.



32. $f(x)=\frac{3x-1}{x}$

a) *Intercepts.* To find the *x*-intercepts, solve $f(x)=0$.

$$\frac{3x-1}{x}=0$$
$$3x-1=0$$
$$3x=1$$
$$x=\frac{1}{3}$$

Since $x=\frac{1}{3}$ does not make the denominator 0, the *x*-intercept is $\left(\frac{1}{3},0\right)$.

The number 0 is not in the domain of $f(x)$ so there are no *y*-intercepts.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x=0$, so the line $x=0$ is a vertical asymptote.
Horizontal. The numerator and the denominator have the same degree, so $y=\frac{3}{1}$, or $y=3$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$f'(x)=\frac{1}{x^2}$

$f''(x)=-2x^{-3}=-\frac{2}{x^3}$

The domain of f is $(-\infty,0)\cup(0,\infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 0, but 0 is not in the domain of the function, so $x=0$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use 0 to divide the real number line into two intervals A: $(-\infty,0)$ and B: $(0,\infty)$, and we test a point in each interval.

A: Test -1, $f'(-1)=\frac{1}{(-1)^2}=1>0$

B: Test 1, $f'(1)=\frac{1}{(-1)^2}=1>0$

Then $f(x)$ is increasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points*. $f''(x)$ does not exist at 0, but because 0 is not in the domain of the function, there cannot be an inflection point at 0. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

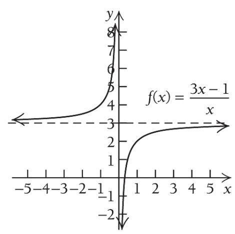
g) *Concavity.* We use 0 to divide the real number line into two intervals A: $(-\infty,0)$ and B: $(0,\infty)$, and we test a point in each interval.

A: Test -1, $f''(-1)=\dfrac{-2}{(-1)^3}=2>0$

B: Test 1, $f''(1)=\dfrac{-2}{(1)^3}=-2<0$

Therefore, $f(x)$ is concave up on $(-\infty,0)$ and concave down on $(0,\infty)$.

h) *Sketch*. Use the preceding information to sketch the graph. Compute additional function values as needed.



33. $f(x)=x+\dfrac{2}{x}=\dfrac{x^2+2}{x}$

a) *Intercepts.* The equation $f(x)=0$ has no real solutions, so there are no x-intercepts. The number 0 is not in the domain of $f(x)$ so there are no y-intercepts.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x=0$, so the line $x=0$ is a vertical asymptote.
Horizontal. The degree of the numerator is greater than the degree of the denominator, so there are no horizontal asymptotes.
Slant. The degree of the numerator is exactly one greater than the degree of the denominator. As $|x|$ approaches ∞,

$f(x)=x+\dfrac{2}{x}$ approaches x. Therefore,

$y=x$ is the slant asymptote.

c) *Derivatives and Domain.*

$$f'(x)=1-2x^{-2}=1-\frac{2}{x^2}$$

$$f''(x)=4x^{-3}=\frac{4}{x^3}$$

The domain of f is $(-\infty,0)\cup(0,\infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 0, but 0 is not in the domain of the function, so $x=0$ is not a critical value. The critical points will occur when $f'(x)=0$.

$$1-\frac{2}{x^2}=0$$
$$1=\frac{2}{x^2}$$
$$x^2=2$$
$$x=\pm\sqrt{2}$$

Thus, $-\sqrt{2}$ and $\sqrt{2}$ are critical values.
$f(-\sqrt{2})=-2\sqrt{2}$ and $f(\sqrt{2})=2\sqrt{2}$, so the critical points $(-\sqrt{2},-2\sqrt{2})$ and $(\sqrt{2},2\sqrt{2})$ are on the graph.

e) *Increasing, decreasing, relative extrema.* We use $-\sqrt{2}$, 0, and $\sqrt{2}$ to divide the real number line into four intervals A: $(-\infty,-\sqrt{2})$ B: $(-\sqrt{2},0)$, C: $(0,\sqrt{2})$, and D: $(\sqrt{2},\infty)$.

A: Test -2, $f'(-2)=1-\dfrac{2}{(-2)^2}=\dfrac{1}{2}>0$

B: Test -1, $f'(-1)=1-\dfrac{2}{(-1)^2}=-1<0$

C: Test 1, $f'(1)=1-\dfrac{2}{(1)^2}=-1<0$

D: Test 2, $f'(2)=1-\dfrac{2}{(2)^2}=\dfrac{1}{2}>0$

Then $f(x)$ is increasing on $(-\infty,-\sqrt{2})$ and $(\sqrt{2},\infty)$ and is decreasing on $(-\sqrt{2},0)$ and $(0,\sqrt{2})$. Therefore, $(-\sqrt{2},-2\sqrt{2})$ is a relative maximum, and $(\sqrt{2},2\sqrt{2})$ is a relative minimum.

f) *Inflection points*. $f''(x)$ does not exist at 0, but because 0 is not in the domain of the function, there cannot be an inflection point at 0. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

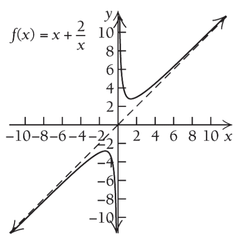
g) *Concavity.* We use 0 to divide the real number line into two intervals A: $(-\infty,0)$ and B: $(0,\infty)$, and we test a point in each interval.

A: Test -1, $f''(-1)=\frac{4}{(-1)^3}=-4<0$

B: Test 1, $f''(1)=\frac{4}{(1)^3}=4>0$

Therefore, $f(x)$ is concave down on $(-\infty,0)$ and concave up on $(0,\infty)$.

h) *Sketch*. Use the preceding information to sketch the graph. Compute additional function values as needed.



34. $f(x)=x+\frac{9}{x}$

a) *Intercepts.* The equation $f(x)=0$ has no real solutions, so there are no x-intercepts. The number 0 is not in the domain of $f(x)$ so there are no y-intercepts.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x=0$, so the line $x=0$ is a vertical asymptote.
Horizontal. The degree of the numerator is greater than the degree of the denominator, so there are no horizontal asymptotes.

Slant. As $|x|$ approaches ∞, $f(x)=x+\frac{9}{x}$ approaches x. Thus, $y=x$ is the slant asymptote.

c) *Derivatives and Domain.*

$f'(x)=1-9x^{-2}=1-\frac{9}{x^2}$

$f''(x)=18x^{-3}=\frac{18}{x^3}$

The domain of f is $(-\infty,0)\cup(0,\infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 0, but 0 is not in the domain of the function, so $x=0$ is not a critical value. The solution to $f'(x)=0$ is $x=\pm3$. Thus, -3 and 3 are critical values.

$f(-3)=-6$ and $f(3)=6$, so the critical points $(-3,-6)$ and $(3,6)$ are on the graph.

e) *Increasing, decreasing, relative extrema.* We use $-3, 0,$ and 3 to divide the real number line into four intervals A: $(-\infty,-3)$ B: $(-3,0)$, C: $(0,3)$, and D: $(3,\infty)$.

A: Test -4, $f'(-4)=\frac{7}{16}>0$

B: Test -1, $f'(-1)=-8<0$

C: Test 1, $f'(1)=-8<0$

D: Test 4, $f'(4)=\frac{7}{16}>0$

Then $f(x)$ is increasing on $(-\infty,-3)$ and $(3,\infty)$ and is decreasing on $(-3,0)$ and $(0,3)$. Therefore, $(-3,-6)$ is a relative maximum, and $(3,6)$ is a relative minimum.

f) *Inflection points*. $f''(x)$ does not exist at 0, but because 0 is not in the domain of the function, there cannot be an inflection point at 0. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

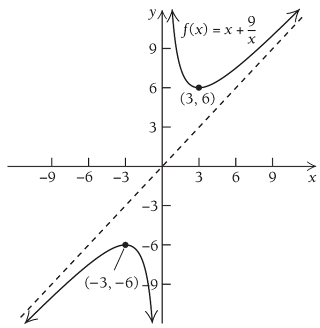
g) *Concavity.* We use 0 to divide the real number line into two intervals A: $(-\infty,0)$ and B: $(0,\infty)$, and we test a point in each interval.

A: Test -1, $f''(-1)=-18<0$

B: Test 1, $f''(1)=18>0$

Therefore, $f(x)$ is concave down on $(-\infty,0)$ and concave up on $(0,\infty)$.

h) *Sketch*.



35. $f(x) = \frac{-1}{x^2} = -x^{-2}$

a) *Intercepts.* Since the numerator is the constant -1, there are no x-intercepts. The number 0 is not in the domain of the function, so there are no y-intercepts.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x = 0$, so the line $x = 0$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y = 0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x) = 2x^{-3} = \frac{2}{x^3}$$

$$f''(x) = -6x^{-4} = -\frac{6}{x^4}$$

The domain of f is $(-\infty, 0) \cup (0, \infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 0, but 0 is not in the domain of the function, so $x = 0$ is not a critical value. The equation $f'(x) = 0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use 0 to divide the real number line into two intervals A: $(-\infty, 0)$ and B: $(0, \infty)$, and we test a point in each interval.

A: Test -1, $f'(-1) = \frac{2}{(-1)^3} = -2 < 0$

B: Test 1, $f'(1) = \frac{2}{(1)^3} = 2 > 0$

Then $f(x)$ is decreasing on $(-\infty, 0)$ and is increasing on $(0, \infty)$. Since there are no critical points, there are no relative extrema.

f) *Inflection points*. $f''(x)$ does not exist at 0, but because 0 is not in the domain of the function, there cannot be an inflection point at 0. The equation $f''(x) = 0$ has no solution; therefore, there are no inflection points.

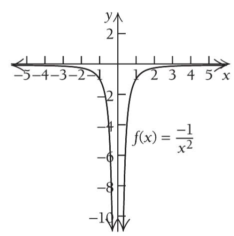
g) *Concavity.* We use 0 to divide the real number line into two intervals A: $(-\infty, 0)$ and B: $(0, \infty)$, and we test a point in each interval.

A: Test -1, $f''(-1) = -\frac{6}{(-1)^4} = -6 < 0$

B: Test 1, $f''(1) = -\frac{6}{(1)^4} = -6 < 0$

Therefore, $f(x)$ is concave down on both intervals.

h) *Sketch*. Use the preceding information to sketch the graph. Compute additional function values as needed.



36. $f(x) = \frac{2}{x^2} = 2x^{-2}$

a) *Intercepts.* Since the numerator is the constant 2, there are no x-intercepts. The number 0 is not in the domain of the function, so there are no y-intercepts.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x = 0$, so the line $x = 0$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y = 0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x) = -4x^{-3} = -\frac{4}{x^3}$$

$$f''(x) = 12x^{-4} = \frac{12}{x^4}$$

The domain of f is $(-\infty, 0) \cup (0, \infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 0, but 0 is not in the domain of the function, so $x = 0$ is not a critical value. The equation $f'(x) = 0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use 0 to divide the real number line into two intervals A: $(-\infty, 0)$ and B: $(0, \infty)$, and we test a point in each interval.

A: Test -1, $f'(-1) = 4 > 0$

B: Test 1, $f'(1) = -4 < 0$

Then $f(x)$ is increasing on $(-\infty, 0)$ and is decreasing on $(0, \infty)$. Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at 0, but because 0 is not in the domain of the function, there cannot be an inflection point at 0. The equation $f''(x) = 0$ has no solution; therefore, there are no inflection points.

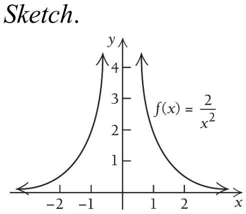
g) *Concavity.* We use 0 to divide the real number line into two intervals A: $(-\infty, 0)$ and B: $(0, \infty)$, and we test a point in each interval.

A: Test -1, $f''(-1) = \frac{12}{(-1)^4} = 12 > 0$

B: Test 1, $f''(1) = \frac{12}{(1)^4} = 12 > 0$

Therefore, $f(x)$ is concave up on both intervals.

h) *Sketch.*



37. $f(x) = \frac{x}{x-3}$

a) *Intercepts.* To find the x-intercepts, solve $f(x) = 0$.

$$\frac{x}{x-3} = 0$$

$$x = 0$$

Since $x = 0$ does not make the denominator 0, the x-intercept is $(0, 0)$. $f(0) = 0$, so the y-intercept is $(0, 0)$ also.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x = 3$, so the line $x = 3$ is a vertical asymptote.
Horizontal. The numerator and the denominator have the same degree, so $y = \frac{1}{1}$, or $y = 1$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x) = \frac{-3}{(x-3)^2}$$

$$f''(x) = 6(x-3)^{-3} = \frac{6}{(x-3)^3}$$

The domain of f is $(-\infty, 3) \cup (3, \infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 3, but 3 is not in the domain of the function, so $x = 3$ is not a critical value. The equation $f'(x) = 0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use 3 to divide the real number line into two intervals A: $(-\infty, 3)$ and B: $(3, \infty)$, and we test a point in each interval.

A: Test 2, $f'(2) = \frac{-3}{((2)-3)^2} = -3 < 0$

B: Test 4, $f'(4) = \frac{-3}{((4)-3)^2} = -3 < 0$

Then $f(x)$ is decreasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points*. $f''(x)$ does not exist at 3, but because 3 is not in the domain of the function, there cannot be an inflection point at 3. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

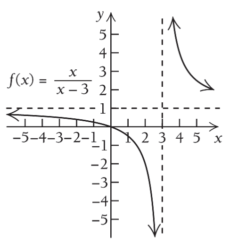
g) *Concavity.* We use 3 to divide the real number line into two intervals A: $(-\infty,3)$ and B: $(3,\infty)$, and we test a point in each interval.

A: Test 2, $f''(2)=\dfrac{6}{((2)-3)^3}=-6<0$

B: Test 4, $f''(4)=\dfrac{6}{((4)-3)^3}=6>0$

Therefore, $f(x)$ is concave down on $(-\infty,3)$ and concave up on $(3,\infty)$.

h) *Sketch*. Use the preceding information to sketch the graph. Compute additional function values as needed.



38. $f(x)=\dfrac{x}{x+2}$

a) *Intercepts.* The numerator is 0 for $x=0$ and since this value of *x* does not make the denominator 0, the *x*-intercept is $(0,0)$.

$f(0)=0$, so the *y*-intercept is $(0,0)$ also.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x=-2$, so the line $x=-2$ is a vertical asymptote.
Horizontal. The numerator and the denominator have the same degree, so $y=\dfrac{1}{1}$, or $y=1$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$f'(x)=\dfrac{2}{(x+2)^2}$

$f''(x)=-4(x+2)^{-3}=-\dfrac{4}{(x+2)^3}$

The domain of f is $(-\infty,-2)\cup(-2,\infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except -2, but -2 is not in the domain of the function, so $x=-2$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use -2 to divide the real number line into two intervals A: $(-\infty,-2)$ and B: $(-2,\infty)$, and we test a point in each interval.

A: Test -3, $f'(-3)=2>0$

B: Test -1, $f'(-1)=2>0$

Then $f(x)$ is increasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points*. $f''(x)$ does not exist at -2, but because -2 is not in the domain of the function, there cannot be an inflection point at -2. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

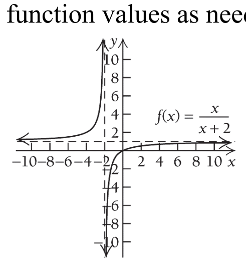
g) *Concavity.* We use -2 to divide the real number line into two intervals A: $(-\infty,-2)$ and B: $(-2,\infty)$, and we test a point in each interval.

A: Test -3, $f''(-3)=4>0$

B: Test -1, $f''(-1)=-4<0$

Therefore, $f(x)$ is concave up on $(-\infty,-2)$ and concave down on $(-2,\infty)$.

h) *Sketch*. Use the preceding information to sketch the graph. Compute additional function values as needed.



39. $f(x)=\dfrac{1}{x^2+3}=\left(x^2+3\right)^{-1}$

a) *Intercepts.* Since the numerator is the constant 1, there are no x-intercepts.

$f(0)=\dfrac{1}{(0)^2+3}=\dfrac{1}{3}$, so the y-intercept is $\left(0,\dfrac{1}{3}\right)$.

b) *Asymptotes.*
Vertical. $x^2+3=0$ has no real solution, so there are no vertical asymptotes.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y=0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$f'(x)=-2x\left(x^2+3\right)^{-2}=-\dfrac{2x}{\left(x^2+3\right)^2}$

$f''(x)=\dfrac{6x^2-6}{\left(x^2+3\right)^3}$

The domain of f is $\mathbb{R}$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all real numbers. Solve $f'(x)=0$

$$-\frac{2x}{\left(x^2+3\right)^2}=0$$
$$2x=0$$
$$x=0$$

The critical value is 0. From step (a) we found $\left(0,\dfrac{1}{3}\right)$ is on the graph.

e) *Increasing, decreasing, relative extrema.* We use 0 to divide the real number line into two intervals A: $(-\infty,0)$ and B: $(0,\infty)$, and we test a point in each interval.

A: Test -1, $f'(-1)=\dfrac{1}{8}>0$

B: Test 1, $f'(1)=-\dfrac{1}{8}<0$

Then $f(x)$ is increasing on $(-\infty,0)$ and is decreasing on $(0,\infty)$. Thus $\left(0,\dfrac{1}{3}\right)$ is a relative maximum.

f) *Inflection points.* $f''(x)$ exists for all real numbers. Solve $f''(x)=0$.

$$\frac{6x^2-6}{\left(x^2+2\right)^3}=0$$
$$6x^2-6=0$$
$$6x^2=6$$
$$x^2=1$$
$$x=\pm 1$$

$f(-1)=\dfrac{1}{4}$ and $f(1)=\dfrac{1}{4}$

So, $\left(-1,\dfrac{1}{4}\right)$ and $\left(1,\dfrac{1}{4}\right)$ are possible points of inflection.

g) *Concavity.* We use -1 and 1 to divide the real number line into three intervals
A: $(-\infty,-1)$ B: $(-1,1)$, and C: $(1,\infty)$

A: Test -2, $f''(-2)=\dfrac{18}{343}>0$

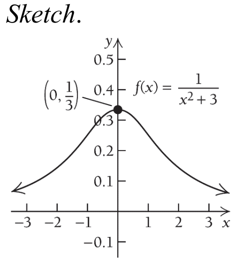
B: Test 0, $f''(0)=-\dfrac{2}{9}<0$

C: Test 2, $f''(2)=\dfrac{18}{343}>0$

Therefore, $f(x)$ is concave up on $(-\infty,-1)$ and $(1,\infty)$, and concave down on $(-1,1)$.

Thus the points $\left(-1,\dfrac{1}{4}\right)$ and $\left(1,\dfrac{1}{4}\right)$ are points of inflection.

h) *Sketch.*



40. $f(x)=\dfrac{-1}{x^2+2}=-\left(x^2+2\right)^{-1}$

a) *Intercepts.* Since the numerator is the constant -1, there are no x-intercepts.

$f(0)=-\dfrac{1}{2}$, so the y-intercept is $\left(0,-\dfrac{1}{2}\right)$.

b) *Asymptotes.*
Vertical. $x^2+2=0$ has no real solution, so there are no vertical asymptotes.

Horizontal. The degree of the numerator is less than the degree of the denominator, so $y = 0$ is the horizontal asymptote.

Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x) = 2x\left(x^2+2\right)^{-2} = \frac{2x}{\left(x^2+2\right)^2}$$

$$f''(x) = \frac{-6x^2+4}{\left(x^2+2\right)^3}$$

The domain of f is $\mathbb{R}$.

d) *Critical Points.* $f'(x)$ exists for all real numbers. $f'(x) = 0$ for $x = 0$, so 0 is a critical value. From step (a) we already know $\left(0, -\frac{1}{2}\right)$ is on the graph.

e) *Increasing, decreasing, relative extrema.* We use 0 to divide the real number line into two intervals A: $(-\infty, 0)$ and B: $(0, \infty)$, and we test a point in each interval.

A: Test -1, $f'(-1) = -\frac{2}{9} < 0$

B: Test 1, $f'(1) = \frac{2}{9} > 0$

Then $f(x)$ is decreasing on $(-\infty, 0)$ and is increasing on $(0, \infty)$. Thus $\left(0, -\frac{1}{2}\right)$ is a relative minimum.

f) *Inflection points.* $f''(x)$ exists for all real numbers. $f''(x) = 0$ for $x = \pm\sqrt{\frac{2}{3}}$, so $-\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{2}{3}}$ are possible inflection points.

$f\left(-\sqrt{\frac{2}{3}}\right) = -\frac{3}{8}$ and $f\left(\sqrt{\frac{2}{3}}\right) = -\frac{3}{8}$

So, $\left(-\sqrt{\frac{2}{3}}, -\frac{3}{8}\right)$ and $\left(\sqrt{\frac{2}{3}}, -\frac{3}{8}\right)$ are points of inflection.

g) *Concavity.* We use $-\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{2}{3}}$ to divide the real number line into three intervals A: $\left(-\infty, -\sqrt{\frac{2}{3}}\right)$ B: $\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$, and C: $\left(\sqrt{\frac{2}{3}}, \infty\right)$

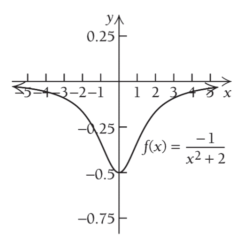
A: Test -1, $f''(-1) = -\frac{2}{27} < 0$

B: Test 0, $f''(0) = \frac{1}{2} > 0$

C: Test 1, $f''(1) = -\frac{2}{27} < 0$

Therefore, $f(x)$ is concave down on $\left(-\infty, -\sqrt{\frac{2}{3}}\right)$ and $\left(\sqrt{\frac{2}{3}}, \infty\right)$ and concave up on $\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$. Therefore the points $\left(-\sqrt{\frac{2}{3}}, -\frac{3}{8}\right)$ and $\left(\sqrt{\frac{2}{3}}, -\frac{3}{8}\right)$ are points of inflection.

h) *Sketch.* Use the preceding information to sketch the graph. Compute additional function values as needed.



41. $f(x) = \dfrac{x+3}{x^2-9} = \dfrac{x+3}{(x+3)(x-3)} = \dfrac{1}{x-3},\ x \neq \pm 3$

We write the expression in simplified form noting that the domain is restricted to all real numbers except for $x = \pm 3$.

a) *Intercepts.* $f(x) = 0$ has no solution. $x = -3$ is not in the domain of the function. Therefore, there are no x-intercepts. To find the y-intercepts we compute $f(0)$.

$$f(0) = \frac{1}{(0)-3} = -\frac{1}{3}$$

The point $\left(0, -\frac{1}{3}\right)$ is the y-intercept.

b) *Asymptotes.*
Vertical. In the original function, the denominator is 0 for $x=-3$ or $x=3$, however, $x=-3$ also made the numerator equal to 0. We look at the limits to determine if there are vertical asymptotes at these points.
$\lim_{x\to -3}\frac{x+3}{x^2-9}=\lim_{x\to -3}\frac{1}{x-3}=\frac{1}{-3-3}=-\frac{1}{6}$.
Because the limit exists, the line $x=-3$ is not a vertical asymptote. Instead, we have a removable discontinuity, or a "hole" at the point $\left(-3,-\frac{1}{6}\right)$.
An open circle is drawn at $\left(-3,-\frac{1}{6}\right)$ to show that it is not part of the graph.
The denominator is 0 for $x=3$ and the numerator is not 0 at this value, so the line $x=3$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y=0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x)=-(x-3)^{-2}=\frac{-1}{(x-3)^2}$$

$$f''(x)=2(x-3)^{-3}=\frac{2}{(x-3)^3}$$

The domain of f as determined in step (b) is $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$.

d) *Critical Points.* $f'(x)$ exists for all values of x except 3, but 3 is not in the domain of the function, so $x=3$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.*
We use -3 and 3 to divide the real number line into three intervals
A: $(-\infty,-3)$ B: $(-3,3)$ and C: $(3,\infty)$.
We notice that $f'(x)<0$ for all real numbers, $f(x)$ is decreasing on all three intervals $(-\infty,-3)$, $(-3,3)$, and $(3,\infty)$.
Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at 3, but because 3 is not in the domain of the function, there cannot be an inflection point at 3. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

g) *Concavity.* We use -3 and 3 to divide the real number line into three intervals
A: $(-\infty,-3)$ B: $(-3,3)$ and C: $(3,\infty)$ and we test a point in each interval.

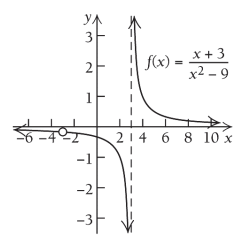
A: Test -4, $f''(-4)=\frac{2}{((-4)-3)^3}=-\frac{2}{343}<0$

B: Test 2, $f''(2)=\frac{2}{((2)-3)^3}=-2<0$

C: Test 4, $f''(4)=\frac{2}{((4)-3)^3}=2>0$

Therefore, $f(x)$ is concave down on $(-\infty,-3)$ and $(-3,3)$ and concave up on $(3,\infty)$.

h) *Sketch.* Use the preceding information to sketch the graph. Compute additional function values as needed.



42. $f(x)=\frac{x-1}{x^2-1}=\frac{1}{x+1},\quad x\neq\pm1$

We write the expression in simplified form noting that the domain is restricted to all real numbers except for $x=\pm1$.

a) *Intercepts.* $f(x)=0$ has no solution, so there are no x-intercepts. To find the y-intercepts we compute $f(0)$

$$f(0)=\frac{1}{(0)+1}=1$$

The point $(0,1)$ is the y-intercept.

b) *Asymptotes.*
Vertical. In the original function, the denominator is 0 for $x=-1$ or $x=1$, however, $x=1$ also made the numerator equal to 0.
The solution is continued on the next page.

We look at the limits to determine if there are vertical asymptotes at these points.

$\lim_{x\to 1}\frac{x-1}{x^2-1}=\lim_{x\to 1}\frac{1}{x+1}=\frac{1}{2}$. Because the limit exists, the line $x=1$ is not a vertical asymptote. Instead, we have a removable discontinuity, or a "hole" at the point $\left(1,\frac{1}{2}\right)$.

An open circle is drawn at this point to show that it is not part of the graph.
The denominator is 0 for $x=-1$ and the numerator is not 0 at this value, so the line $x=-1$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y=0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x)=-(x+1)^{-2}=\frac{-1}{(x+1)^2}$$

$$f''(x)=2(x+1)^{-3}=\frac{2}{(x+1)^3}$$

The domain of f is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$ as determined above.

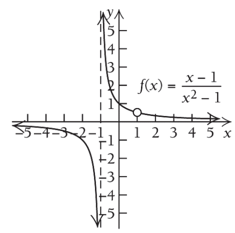
d) *Critical Points.* $f'(x)$ exists for all values of x except -1, but -1 is not in the domain of the function, so $x=-1$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use -1 and 1 to divide the real number line into three intervals
A: $(-\infty,-1)$ B: $(-1,1)$ and C: $(1,\infty)$.
We notice that $f'(x)<0$ for all real numbers, $f(x)$ is decreasing on all three intervals $(-\infty,-1)$, $(-1,1)$, and $(1,\infty)$. Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at -1, but because -1 is not in the domain of the function, there cannot be an inflection point at -1. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

g) *Concavity.* We use -1 and 1 to divide the real number line into three intervals
A: $(-\infty,-1)$ B: $(-1,1)$ and C: $(1,\infty)$.
and we test a point in each interval.
A: Test -2, $f''(-2)=-2<0$
B: Test 0, $f''(0)=2>0$
C: Test 2, $f''(2)=\frac{2}{27}>0$
Therefore, $f(x)$ is concave down on $(-\infty,-1)$ and concave up on $(-1,1)$ and $(1,\infty)$.

h) *Sketch.*



43. $f(x)=\frac{x-1}{x+2}$

a) *Intercepts.* To find the x-intercepts, solve $f(x)=0$.

$$\frac{x-1}{x+2}=0$$
$$x=1$$

Since $x=1$ does not make the denominator 0, the x-intercept is $(1,0)$.

$f(0)=\frac{0-1}{0+2}=-\frac{1}{2}$, so the y-intercept is $\left(0,-\frac{1}{2}\right)$.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x=-2$, so the line $x=-2$ is a vertical asymptote.
Horizontal. The numerator and the denominator have the same degree, so $y=\frac{1}{1}$, or $y=1$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x)=\frac{3}{(x+2)^2}$$

$$f''(x)=-\frac{6}{(x+2)^3}$$

The domain of f is $(-\infty,-2)\cup(-2,\infty)$ as determined in part (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except -2, but -2 is not in the domain of the function, so $x=-2$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use -2 to divide the real number line into two intervals A: $(-\infty,-2)$ and B: $(-2,\infty)$, and we test a point in each interval.

A: Test -3, $f'(-3)=3>0$

B: Test -1, $f'(-1)=3>0$

Then $f(x)$ is increasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points*. $f''(x)$ does not exist at -2, but because -2 is not in the domain of the function, there cannot be an inflection point at -2. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

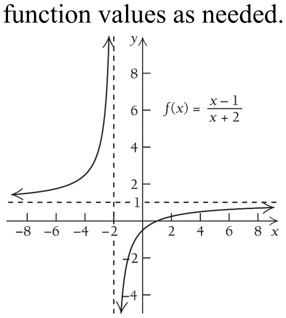
g) *Concavity*. We use -2 to divide the real number line into two intervals A: $(-\infty,-2)$ and B: $(-2,\infty)$, and we test a point in each interval.

A: Test -3, $f''(-3)=6>0$

B: Test -1, $f''(-1)=-6<0$

Therefore, $f(x)$ is concave up on $(-\infty,-2)$ and concave down on $(-2,\infty)$.

h) *Sketch*. Use the preceding information to sketch the graph. Compute additional function values as needed.



44. $f(x)=\frac{x-2}{x+1}$

a) *Intercepts.* $f(x)=0$ for $x=2$ and this value does not make the denominator 0, the x-intercept is $(2,0)$. $f(0)=-2$, so the y-intercept is $(0,-2)$.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x=-1$, so the line $x=-1$ is a vertical asymptote.
Horizontal. The numerator and the denominator have the same degree, so $y=\frac{1}{1}$, or $y=1$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x)=\frac{3}{(x+1)^2}$$

$$f''(x)=-\frac{6}{(x+1)^3}$$

The domain of f is $(-\infty,-1)\cup(-1,\infty)$ as determined in part (b).

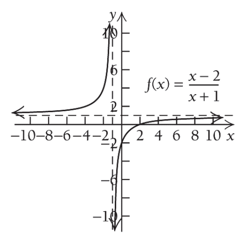
d) *Critical Points.* $f'(x)$ exists for all values of x except -1, but -1 is not in the domain of the function, so $x=-1$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use -1 to divide the real number line into two intervals A: $(-\infty,-1)$ and B: $(-1,\infty)$. We notice that $f'(x)>0$ for all real numbers; therefore, $f(x)$ is increasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points*. $f''(x)$ does not exist at -1, but because -1 is not in the domain of the function, there cannot be an inflection point at -1. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

g) *Concavity.* We use -1 to divide the real number line into two intervals
A: $(-\infty,-1)$ and B: $(-1,\infty)$, and we test a point in each interval.
A: Test -2, $f''(-2)=6>0$
B: Test 0, $f''(0)=-6<0$
Therefore, $f(x)$ is concave up on $(-\infty,-1)$ and concave down on $(-1,\infty)$.

h) *Sketch.*



45. $f(x)=\dfrac{x^2-9}{x+1}$

a) *Intercepts.* To find the x-intercepts, solve $f(x)=0$.

$$\frac{x^2-9}{x+1}=0$$
$$x^2-9=0$$
$$x=\pm 3$$

Neither of these values make the denominator 0, so the x-intercepts are $(-3,0)$ and $(3,0)$.
$f(0)=-9$, so the y-intercept is $(0,-9)$.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x=-1$, so the line $x=-1$ is a vertical asymptote.
Horizontal. The degree of the numerator is greater than the degree of the denominator, so there are no horizontal asymptotes.
Slant. Divide the numerator by the denominator.

$$\begin{array}{r} x \quad -1 \\ x+1\overline{)x^2 \qquad -9} \\ \underline{x^2+x} \\ -x-9 \\ \underline{-x-1} \\ -8 \end{array}$$

By dividing, we get
$$f(x)=x-1-\frac{8}{x+1}$$

As $|x|$ approaches ∞, $f(x)$ approaches $x-1$, so $y=x-1$ is the slant asymptote.

c) *Derivatives and Domain.*
$$f'(x)=\frac{x^2+2x+9}{(x+1)^2}$$
$$f''(x)=-\frac{16}{(x+1)^3}$$
The domain of f is $(-\infty,-1)\cup(-1,\infty)$ as determined in part (b).

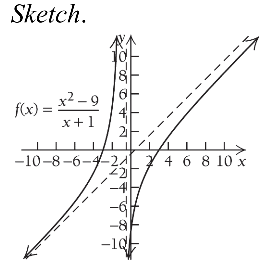
d) *Critical Points.* $f'(x)$ exists for all values of x except -1, but -1 is not in the domain of the function, so $x=-1$ is not a critical value.
$f'(x)=0$ has no real solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use -1 to divide the real number line into two intervals
A: $(-\infty,-1)$ and B: $(-1,\infty)$
We test a point in each interval.
A: Test -2, $f'(-2)=9>0$
B: Test 0, $f'(0)=9>0$
Then $f(x)$ is increasing on both intervals.
Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at -1, but because -1 is not in the domain of the function, there cannot be an inflection point at -1. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

g) *Concavity.* We use -1 to divide the real number line into two intervals
A: $(-\infty,-1)$ and B: $(-1,\infty)$, and we test a point in each interval.
A: Test -2, $f''(-2)=16>0$
B: Test 0, $f''(0)=-16<0$
Therefore, $f(x)$ is concave up on $(-\infty,-1)$ and concave down on $(-1,\infty)$.

h) *Sketch.*



46. $f(x)=\dfrac{x^2-4}{x+3}$

a) *Intercepts.* The numerator is 0 for $x=-2$ or $x=2$ and neither of these values make the denominator 0, so the *x*-intercepts are $(-2,0)$ and $(2,0)$. $f(0)=\dfrac{0^2-4}{0+3}=-\dfrac{4}{3}$, so the *y*-intercept is $\left(0,-\dfrac{4}{3}\right)$.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x=-3$, so the line $x=-3$ is a vertical asymptote.
Horizontal. The degree of the numerator is greater than the degree of the denominator, so there are no horizontal asymptotes.
Slant. By dividing the numerator by the denominator, we get

$$f(x)=x-3+\frac{5}{x+3}$$

As $|x|$ approaches ∞, $f(x)$ approaches $x-3$, so $y=x-3$ is the slant asymptote.

c) *Derivatives and Domain.*

$$f'(x)=\frac{x^2+6x+4}{(x+3)^2}$$

$$f''(x)=\frac{10}{(x+3)^3}$$

The domain of f is $(-\infty,-3)\cup(-3,\infty)$ as determined in part (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except -3, but -3 is not in the domain of the function, so $x=-3$ is not a critical value. Solve $f'(x)=0$.

$$\frac{x^2+6x+4}{(x+3)^2}=0$$

$$x^2+6x+4=0$$

$$x=-3\pm\sqrt{5} \quad \text{Using the Quadratic Formula}$$

$x\approx-5.236$ or $x\approx-0.764$

$f(-5.236)\approx-10.472$ and $f(-0.764)\approx-1.528$, so $(-5.236,-10.472)$ and $(-0.764,-1.528)$ are on the graph.

e) *Increasing, decreasing, relative extrema.*
We use $-5.236,\ -3,$ and -0.764 to divide the real number line into four intervals
A: $(-\infty,-5.236)$, B: $(-5.236,-3)$,
C: $(-3,-0.764)$, and D: $(-0.764,\infty)$

We test a point in each interval.

A: Test -6, $f'(-6)=\dfrac{4}{9}>0$

B: Test -4, $f'(-4)=-4<0$

C: Test -2, $f'(-2)=-4<0$

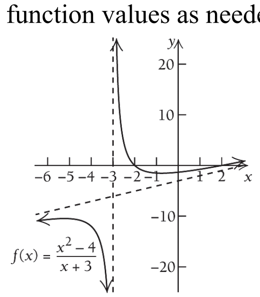
D: Test 0, $f'(0)=\dfrac{4}{9}>0$

Then $f(x)$ is increasing on the intervals $(-\infty,-5.236)$ and $(-0.764,\infty)$, and is decreasing on the intervals $(-5.226,-3)$ and $(-3,-0.764)$. Therefore, $(-5.236,-10.472)$ is a relative maximum and $(-0.764,-1.528)$ is a relative minimum.

f) *Inflection points*. $f''(x)$ does not exist at -3, but because -3 is not in the domain of the function, there cannot be an inflection point at -3. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

g) *Concavity.* We use -3 to divide the real number line into two intervals
A: $(-\infty,-3)$ and B: $(-3,\infty)$, and we test a point in each interval.
A: Test -4, $f''(-4)=-10<0$
B: Test -2, $f''(-2)=10>0$
Therefore, $f(x)$ is concave down on $(-\infty,-3)$ and concave up on $(-3,\infty)$.

h) *Sketch*. Use the preceding information to sketch the graph. Compute additional function values as needed.



47. $f(x)=\dfrac{x-3}{x^2+2x-15}=\dfrac{x-3}{(x-3)(x+5)}=\dfrac{1}{x+5}$, $x\neq 3$.

We write the expression in simplified form noting that the domain is restricted to all real numbers except for $x=-5$ and $x=3$.

a) *Intercepts.* $f(x)=0$ has no solution. $x=3$ is not in the domain of the function. Therefore, there are no x-intercepts. To find the y-intercepts we compute $f(0)$:

$$f(0)=\frac{0-3}{(0)^2+2(0)-15}=\frac{1}{5}$$

The point $\left(0,\frac{1}{5}\right)$ is the y-intercept.

b) *Asymptotes.*
Vertical. In the original function, the denominator is 0 for $x=-5$ or $x=3$; however, $x=3$ also made the numerator equal to 0.
We look at the limits to determine if there are vertical asymptotes at these points.

$$\lim_{x\to 3}\frac{x-3}{x^2+2x-15}=\lim_{x\to 3}\frac{1}{x+5}=\frac{1}{8}$$

Because the limit exists, the line $x=3$ is not a vertical asymptote. Instead, we have a removable discontinuity, or a "hole" at the point $\left(3,\frac{1}{8}\right)$. An open circle is drawn at this point to show that it is not part of the graph.
The denominator is 0 for $x=-5$ and the numerator is not 0 at this value, so the line $x=-5$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y=0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x)=-(x+5)^{-2}=\frac{-1}{(x+5)^2}$$

$$f''(x)=2(x+5)^{-3}=\frac{2}{(x+5)^3}$$

The domain of f as determined in part (b) is $(-\infty,-5)\cup(-5,3)\cup(3,\infty)$.

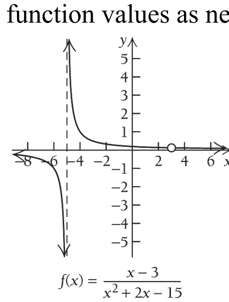
d) *Critical Points.* $f'(x)$ exists for all values of x except -5, but -5 is not in the domain of the function, so $x=-5$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use -5 and 3 to divide the real number line into three intervals A: $(-\infty,-5)$, B: $(-5,3)$, and C: $(3,\infty)$. We notice that $f'(x)<0$ for all real numbers, $f(x)$ is decreasing on all three intervals $(-\infty,-5)$, $(-5,3)$, and $(3,\infty)$. Since there are no critical points, there are no relative extrema.

f) *Inflection points*. $f''(x)$ does not exist at -5, but because -5 is not in the domain of the function, there cannot be an inflection point at -5. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

g) *Concavity.* We use -5 and 3 to divide the real number line into three intervals A: $(-\infty,-5)$ B: $(-5,3)$ and C: $(3,\infty)$, and we test a point in each interval.
A: Test -6, $f''(-6)=-2<0$
B: Test -4, $f''(-4)=2>0$
C: Test 4, $f''(4)=\frac{2}{729}>0$
Therefore, $f(x)$ is concave down on $(-\infty,-5)$ and concave up on $(-5,3)$ and $(3,\infty)$.

h) *Sketch*. Use the preceding information to sketch the graph. Compute additional function values as needed.



48. $f(x)=\dfrac{x+1}{x^2-2x-3}=\dfrac{x+1}{(x-3)(x+1)}=\dfrac{1}{x-3}$,

$x\neq -1$.

We write the expression in simplified form noting that the domain is restricted to all real numbers except for $x=-1$ and $x=3$

a) *Intercepts.* $f(x)=0$ has no solution. $x=-1$ is not in the domain of the function. Therefore, there are no *x*-intercepts. To find the *y*-intercepts we compute $f(0)$:

$$f(0)=\frac{0+1}{(0)^2-2(0)-3}=-\frac{1}{3}$$

The point $\left(0,-\frac{1}{3}\right)$ is the *y*-intercept.

b) *Asymptotes.*
Vertical. In the original function, the denominator is 0 for $x=-1$ or $x=3$, however, $x=-1$ also made the numerator equal to 0. We look at the limits to determine if there are vertical asymptotes at these points.

$$\lim_{x\to -1}\frac{x+1}{x^2-2x-3}=\lim_{x\to -1}\frac{1}{x-3}=\frac{1}{-1-3}=-\frac{1}{4}$$

Because the limit exists, the line $x=-1$ is not a vertical asymptote. Instead, we have a removable discontinuity, or a "hole" at the point $\left(-1,-\frac{1}{4}\right)$. An open circle is drawn at this point to show that it is not part of the graph.
The denominator is 0 for $x=3$ and the numerator is not 0 at this value, so the line $x=3$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y=0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x)=-(x-3)^{-2}=\frac{-1}{(x-3)^2}$$

$$f''(x)=2(x-3)^{-3}=\frac{2}{(x-3)^3}$$

The domain of f as determined in part (b) is $(-\infty,-1)\cup(-1,3)\cup(3,\infty)$.

d) *Critical Points.* $f'(x)$ exists for all values of x except 3, but 3 is not in the domain of the function, so $x=3$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use -1 and 3 to divide the real number line into three intervals
A: $(-\infty,-1)$ B: $(-1,3)$ and C: $(3,\infty)$.
We notice that $f'(x)<0$ for all real numbers, so $f(x)$ is decreasing on all three intervals $(-\infty,-1)$, $(-1,3)$, and $(3,\infty)$.
Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at 3, but because 3 is not in the domain of the function, there cannot be an inflection point at 3. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

g) *Concavity.* We use -1 and 3 to divide the real number line into three intervals
A: $(-\infty,-1)$ B: $(-1,3)$ and C: $(3,\infty)$, and we test a point in each interval.

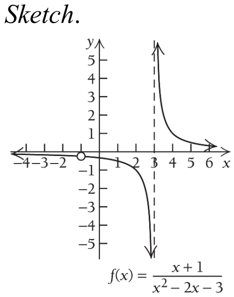
A: Test -2, $f''(-2)=-\dfrac{2}{125}<0$

B: Test 2, $f''(2)=-2<0$

C: Test 4, $f''(4)=2>0$

Therefore, $f(x)$ is concave down on $(-\infty,-1)$ and $(-1,3)$ and concave up on $(3,\infty)$.

h) *Sketch.*



49. $f(x)=\dfrac{2x^2}{x^2-16}$

a) *Intercepts.* The numerator is 0 for $x=0$ and this value does not make the denominator 0, so the *x*-intercept is $(0,0)$.
$f(0)=0$, so the *y*-intercept is $(0,0)$ also.

b) *Asymptotes.*
Vertical. The denominator is 0 when
$x^2-16=0$
$x^2=16$
$x=\pm4$
So the lines $x=-4$ and $x=4$ are vertical asymptotes.
Horizontal. The numerator and the denominator have the same degree, so
$y=\frac{2}{1}$, or $y=2$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*
$$f'(x)=-\frac{64x}{\left(x^2-16\right)^2}$$
$$f''(x)=\frac{192x^2+1024}{\left(x^2-16\right)^3}$$
The domain of f is
$(-\infty,-4)\cup(-4,4)\cup(4,\infty)$ as determined in part (b).

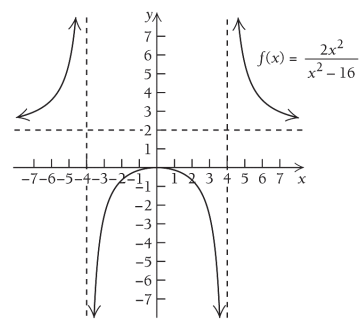
d) *Critical Points.* $f'(x)$ exists for all values of x except $x=-4$ and $x=4$, but -4 and 4 are not in the domain of the function, so $x=-4$ and $x=4$ are not critical values.
$f'(x)=0$ for $x=0$, so $(0,0)$ is the only critical point.

e) *Increasing, decreasing, relative extrema.*
We use $-4,\ 0,$ and 4 to divide the real number line into four intervals
A: $(-\infty,-4)$ B: $(-4,0)$, C: $(0,4)$, and D: $(4,\infty)$.
We test a point in each interval.
A: Test -5, $f'(-5)=\frac{320}{81}>0$
B: Test -1, $f'(-1)=\frac{64}{225}>0$
C: Test 1, $f'(1)=-\frac{64}{225}<0$
D: Test 5, $f'(5)=-\frac{320}{81}<0$
Then $f(x)$ is increasing on the intervals $(-\infty,-4)$ and $(-4,0)$, and is decreasing on the intervals $(0,4)$ and $(4,\infty)$. Thus, there is a relative maximum at $(0,0)$.

f) *Inflection points.* $f''(x)$ does not exist at -4 and 4, but because -4 and 4 are not in the domain of the function, there cannot be an inflection point at -4 or 4. The equation $f''(x)=0$ has no real solution; therefore, there are no inflection points.

g) *Concavity.* We use -4 and 4 to divide the real number line into three intervals
A: $(-\infty,-4)$ B: $(-4,4)$ and C: $(4,\infty)$, and we test a point in each interval.
A: Test -5, $f''(-5)=\frac{5824}{729}>0$
B: Test 0, $f''(0)=-\frac{1}{4}<0$
C: Test 5, $f''(5)=\frac{5824}{729}>0$
Therefore, $f(x)$ is concave up on the intervals $(-\infty,-4)$ and $(4,\infty)$ and concave down on the interval $(-4,4)$.

h) *Sketch*. Use the preceding information to sketch the graph. Compute additional function values as needed.



50. $f(x)=\frac{x^2+x-2}{2x^2-2}=\frac{(x+2)(x-1)}{2(x+1)(x-1)}=\frac{x+2}{2(x+1)}$

We write the expression in simplified form noting that the domain is restricted to all real numbers except for $x=\pm1$.

a) *Intercepts.* $f(x)=0$ for $x=-2$; therefore, the x-intercept is $(-2,0)$.
$$f(0)=\frac{(0)^2+(0)-2}{2(0)^2-2}=1$$
The point $(0,1)$ is the y-intercept.

b) *Asymptotes.*
Vertical. In the original function, the denominator is 0 for $x=-1$ or $x=1$, however, $x=1$ also made the numerator equal to 0. We look at the limits to determine if there are vertical asymptotes at these points.
$\lim_{x\to 1}\frac{x^2+x-2}{2x^2-2}=\lim_{x\to 1}\frac{x+2}{2(x+1)}=\frac{3}{4}$. Because the limit exists, the line $x=1$ is not a vertical asymptote. Instead, we have a removable discontinuity, or a "hole" at the point $\left(1,\frac{3}{4}\right)$.
An open circle is drawn at this point to show that it is not part of the graph.
The denominator is 0 for $x=-1$ and the numerator is not 0 at this value, so the line $x=-1$ is a vertical asymptote.
Horizontal. The numerator and the denominator have the same degree, so $y=\frac{1}{2}$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*
$$f'(x)=-(x+1)^{-2}=\frac{-1}{2(x+1)^2}$$
$$f''(x)=2(x+1)^{-3}=\frac{1}{(x+1)^3}$$
The domain of f is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$ as determined in part (b).

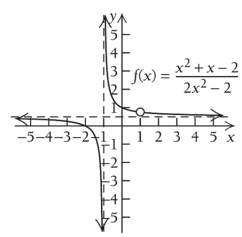
d) *Critical Points.* $f'(x)$ exists for all values of x except -1, but -1 is not in the domain of the function, so $x=-1$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.*
We use -1 and 1 to divide the real number line into three intervals
A: $(-\infty,-1)$ B: $(-1,1)$ and C: $(1,\infty)$.
We notice that $f'(x)<0$ for all real numbers, $f(x)$ is decreasing on all three intervals $(-\infty,-1)$, $(-1,1)$, and $(1,\infty)$. Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at -1, but because -1 is not in the domain of the function, there cannot be an inflection point at -1. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

g) *Concavity.* We use -1 and 1 to divide the real number line into three intervals
A: $(-\infty,-1)$ B: $(-1,1)$ and C: $(1,\infty)$.
and we test a point in each interval.
A: Test -2, $f''(-2)=-1<0$
B: Test 0, $f''(0)=1>0$
C: Test 2, $f''(2)=\frac{1}{27}>0$
Therefore, $f(x)$ is concave down on the interval $(-\infty,-1)$ and concave up on the intervals $(-1,1)$ and $(1,\infty)$.

h) *Sketch.*



51. $f(x)=\frac{10}{x^2+4}$

a) *Intercepts.* Since the numerator is the constant 10, there are no x-intercepts.
$f(0)=\frac{5}{2}$, so the y-intercept is $\left(0,\frac{5}{2}\right)$.

b) *Asymptotes.*
Vertical. $x^2+4=0$ has no real solution, so there are no vertical asymptotes.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y=0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*
$$f'(x)=-20x\left(x^2+4\right)^{-2}=-\frac{20x}{\left(x^2+4\right)^2}$$
$$f''(x)=\frac{20\left(3x^2-4\right)}{\left(x^2+4\right)^3}$$
The domain of f is $\mathbb{R}$.

d) *Critical Points.* $f'(x)$ exists for all real numbers. $f'(x)=0$ for $x=0$, so 0 is a critical value. From step (a) we already know $\left(0,\frac{5}{2}\right)$ is on the graph.

e) *Increasing, decreasing, relative extrema.* We use 0 to divide the real number line into two intervals A: $(-\infty,0)$ and B: $(0,\infty)$, and we test a point in each interval.

A: Test -1, $f'(-1)=\frac{4}{5}>0$

B: Test 1, $f'(1)=-\frac{4}{5}<0$

Then $f(x)$ is increasing on $(-\infty,0)$ and is decreasing on $(0,\infty)$. Thus $\left(0,\frac{5}{2}\right)$ is a relative maximum.

f) *Inflection points*. $f''(x)$ exists for all real numbers. $f''(x)=0$ for $x=\pm\frac{2}{\sqrt{3}}$, which are possible points of inflection. We have:
$f\left(-\frac{2}{\sqrt{3}}\right)=\frac{15}{8}$ and $f\left(\frac{2}{\sqrt{3}}\right)=\frac{15}{8}$.
So, $\left(-\frac{2}{\sqrt{3}},\frac{15}{8}\right)$ and $\left(\frac{2}{\sqrt{3}},\frac{15}{8}\right)$ are inflection points.

g) *Concavity.* We use $-\frac{2}{\sqrt{3}}$ and $\frac{2}{\sqrt{3}}$ to divide the real number line into three intervals
A: $\left(-\infty,-\frac{2}{\sqrt{3}}\right)$ B: $\left(-\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}}\right)$, and C: $\left(\frac{2}{\sqrt{3}},\infty\right)$.

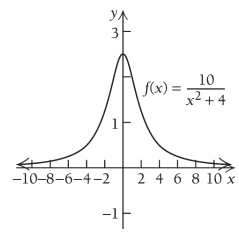
A: Test -2, $f''(-2)=\frac{5}{16}>0$

B: Test 0, $f''(0)=-\frac{5}{4}<0$

C: Test 2, $f''(2)=\frac{5}{16}>0$

Therefore, $f(x)$ is concave up on $\left(-\infty,-\frac{2}{\sqrt{3}}\right)$ and $\left(\frac{2}{\sqrt{3}},\infty\right)$ and concave down on $\left(-\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}}\right)$. Thus, the points $\left(-\frac{2}{\sqrt{3}},\frac{15}{8}\right)$ and $\left(\frac{2}{\sqrt{3}},\frac{15}{8}\right)$ are points of inflection.

h) *Sketch*. Use the preceding information to sketch the graph. Compute additional function values as needed.



52. $f(x)=\frac{1}{x^2-1}$

a) *Intercepts.* Since the numerator is a constant 1, there are no *x*-intercepts.
$f(0)=\frac{1}{(0)^2-1}=-1$
The point $(0,-1)$ is the *y*-intercept.

b) *Asymptotes.*
Vertical. The denominator $x^2-1=(x-1)(x+1)$ is 0 for $x=-1$ or $x=1$, so the lines $x=-1$ and $x=1$ are vertical asymptotes.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y=0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*
$f'(x)=\frac{-2x}{\left(x^2-1\right)^2}$
$f''(x)=\frac{2\left(3x^2+1\right)}{\left(x^2-1\right)^3}$
The domain of f is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$ as determined in part (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except -1 and 1, but these values are not in the domain of the function, so $x=-1$ and $x=1$ are not critical values. $f'(x)=0$ for $x=0$. From step (a) we know $f(0)=-1$, so the critical point is $(0,-1)$.

e) *Increasing, decreasing, relative extrema.* We use $-1,\ 0,$ and 1 to divide the real number line into four intervals A: $(-\infty,-1)$, B: $(-1,0)$, C: $(0,1)$, and D: $(1,\infty)$, and we test a point in each interval.

A: Test -2, $\quad f'(-2)=\frac{4}{9}>0$

B: Test $-\frac{1}{2}$, $f'\left(-\frac{1}{2}\right)=\frac{16}{9}>0$

C: Test $\frac{1}{2}$, $\quad f'\left(\frac{1}{2}\right)=-\frac{16}{9}<0$

D: Test 2, $\quad f'(2)=-\frac{4}{9}<0$

We see that $f(x)$ is increasing on the intervals $(-\infty,-1)$ and $(-1,0)$, and is decreasing on the intervals $(0,1)$ and $(1,\infty)$. Therefore, $(0,-1)$ is a relative maximum.

f) *Inflection points.* $f''(x)$ does not exist at -1 and 1, but because these values are not in the domain of the function, there cannot be an inflection point at -1 or 1. The equation $f''(x)=0$ has no real solution; therefore, there are no inflection points.

g) *Concavity.* We use -1 and 1 to divide the real number line into three intervals A: $(-\infty,-1)$ B: $(-1,1)$ and C: $(1,\infty)$. and we test a point in each interval.

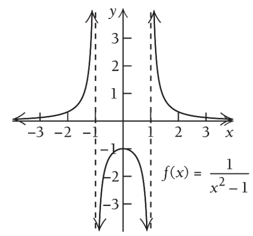
A: Test -2, $f''(-2)=\frac{26}{27}>0$

B: Test 0, $f''(0)=-2<0$

C: Test 2, $f''(2)=\frac{26}{27}>0$

Therefore, $f(x)$ is concave up on $(-\infty,-1)$ and $(1,\infty)$, and concave down on $(-1,1)$.

h) *Sketch.*



53. $f(x)=\frac{x^2+1}{x}$

a) *Intercepts.* The equation $f(x)=0$ has no real solutions, so there are no x-intercepts. The number 0 is not in the domain of $f(x)$ so there are no y-intercepts.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x=0$, so the line $x=0$ is a vertical asymptote.
Horizontal. The degree of the numerator is greater than the degree of the denominator, so there are no horizontal asymptotes.
Slant. The degree of the numerator is exactly one greater than the degree of the denominator. When we divide the numerator by the denominator we have

$f(x)=\frac{x^2+1}{x}=x+\frac{1}{x}$. As $|x|$ approaches ∞, $f(x)=x+\frac{1}{x}$ approaches x. Therefore, $y=x$ is the slant asymptote.

c) *Derivatives and Domain.*

$$f'(x)=\frac{x^2-1}{x^2}$$

$$f''(x)=\frac{2}{x^3}$$

The domain of f is $(-\infty,-0)\cup(0,\infty)$ as determined in part (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 0, but 0 is not in the domain of the function, so $x=0$ is not a critical value. The critical points will occur when $f'(x)=0$.

$$\frac{x^2-1}{x^2}=0$$
$$x^2-1=0$$
$$x^2=1$$
$$x=\pm1$$

The solution is continued on the next page.

From the previous page, we determined -1 and 1 are critical values, thus $f(-1)=-2$ and $f(1)=2$, so the critical points $(-1,-2)$ and $(1,2)$ are on the graph.

e) *Increasing, decreasing, relative extrema.* We use $-1, 0,$ and 1 to divide the real number line into four intervals
A: $(-\infty,-1)$ B: $(-1,0)$, C: $(0,1)$, and
D: $(1,\infty)$. We test a point in each interval.

A: Test -2, $f'(-2)=\frac{3}{4}>0$

B: Test $-\frac{1}{2}$, $f'\left(-\frac{1}{2}\right)=-3<0$

C: Test $\frac{1}{2}$, $f'\left(\frac{1}{2}\right)=-3<0$

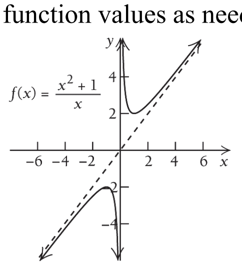
D: Test 2, $f'(2)=\frac{3}{4}>0$

Then $f(x)$ is increasing on $(-\infty,-1)$ and $(1,\infty)$ and is decreasing on $(-1,0)$ and $(0,1)$. Therefore, $(-1,-2)$ is a relative maximum, and $(1,2)$ is a relative minimum.

f) *Inflection points*. $f''(x)$ does not exist at 0, but because 0 is not in the domain of the function, there cannot be an inflection point at 0. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

g) *Concavity*. We use 0 to divide the real number line into two intervals
A: $(-\infty,0)$ and B: $(0,\infty)$, and we test a point in each interval.
A: Test -1, $f''(-1)=-2<0$
B: Test 1, $f''(1)=2>0$
Therefore, $f(x)$ is concave down on $(-\infty,0)$ and concave up on $(0,\infty)$.

h) *Sketch*. Use the preceding information to sketch the graph. Compute additional function values as needed.



54. $f(x)=\dfrac{x^3}{x^2-1}$

a) *Intercepts.* The numerator is 0 for $x=0$ and this value does not make the denominator 0, the x-intercept is $(0,0)$.
$f(0)=0$, so the y-intercept is $(0,0)$ also.

b) *Vertical.* The denominator $x^2-1=(x-1)(x+1)$ is 0 for $x=-1$ or $x=1$, so the lines $x=-1$ and $x=1$ are vertical asymptotes.
Horizontal. The degree of the numerator is greater than the degree of the denominator, so there are no horizontal asymptotes.
Slant. The degree of the numerator is exactly one greater than the degree of the denominator.
When we divide the numerator by the denominator we have
$f(x)=\frac{x^3}{x^2-1}=x+\frac{x}{x^2-1}$. As $|x|$ approaches ∞, $f(x)=x+\frac{x}{x^2-1}$ approaches x. Therefore, $y=x$ is the slant asymptote.

c) *Derivatives and Domain.*

$$f'(x)=\frac{x^4-3x^2}{\left(x^2-1\right)^2}$$

$$f''(x)=\frac{2x^3+6x}{\left(x^2-1\right)^3}$$

The domain of f as determined in part (b) is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$.

d) *Critical Points.* $f'(x)$ exists for all values of x except -1 and 1, but these values are not in the domain of the function, so $x=-1$ and $x=1$ are not a critical values. $f'(x)=0$ for $x=-\sqrt{3}$, $x=0$, and $x=\sqrt{3}$. Therefore, $f\left(-\sqrt{3}\right)=-\frac{3\sqrt{3}}{2}$, $f(0)=0$, and $f\left(\sqrt{3}\right)=\frac{3\sqrt{3}}{2}$. The critical points $\left(-\sqrt{3},-\frac{3\sqrt{3}}{2}\right)$, $(0,0)$ and $\left(\sqrt{3},\frac{3\sqrt{3}}{2}\right)$ are on the graph.

e) *Increasing, decreasing, relative extrema.* We use $-\sqrt{3}, -1, 0, 1,$ and $\sqrt{3}$ to divide the real number line into six intervals
A: $\left(-\infty, -\sqrt{3}\right)$ B: $\left(-\sqrt{3}, -1\right)$, C: $(-1, 0)$, D: $(0, 1)$, E: $\left(1, \sqrt{3}\right)$, and F: $\left(\sqrt{3}, \infty\right)$. We test a point in each interval.

A: Test -2, $f'(-2) = \frac{4}{9} > 0$

B: Test $-\frac{3}{2}$, $f'\left(-\frac{3}{2}\right) = -\frac{27}{25} < 0$

C: Test $-\frac{1}{2}$, $f'\left(-\frac{1}{2}\right) = -\frac{11}{9} < 0$

D: Test $\frac{1}{2}$, $f'\left(\frac{1}{2}\right) = -\frac{11}{9} < 0$

E: Test $\frac{3}{2}$, $f'\left(\frac{3}{2}\right) = -\frac{27}{25} < 0$

F: Test 2, $f'(2) = \frac{4}{9} > 0$

Then $f(x)$ is increasing on the intervals $\left(-\infty, -\sqrt{3}\right)$ and $\left(\sqrt{3}, \infty\right)$ and is decreasing on the intervals $\left(-\sqrt{3}, -1\right)$, $(-1, 0)$, $(0, 1)$, and $\left(1, \sqrt{3}\right)$. Therefore, $(0, 0)$ is not a relative extremum, $\left(-\sqrt{3}, \frac{-3\sqrt{3}}{2}\right)$ is a relative maximum, and $\left(\sqrt{3}, \frac{3\sqrt{3}}{2}\right)$ is a relative minimum.

f) *Inflection points.* $f''(x)$ does not exist at -1 and 1, but because these values are not in the domain of the function, there cannot be an inflection point at -1 or 1. $f''(x) = 0$ when $x = 0$. We know that $f(0) = 0$, so there is a possible inflection point at $(0, 0)$.

g) *Concavity.* We use $-1, 0,$ and 1 to divide the real number line into four intervals
A: $(-\infty, -1)$ B: $(-1, 0)$, C: $(0, 1)$, and D: $(1, \infty)$, and we test a point in each interval.

A: Test -2, $f''(-2) = -\frac{28}{27} < 0$

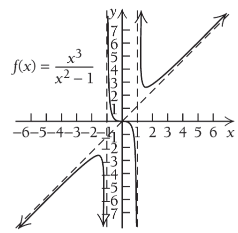
B: Test $-\frac{1}{2}$, $f''\left(-\frac{1}{2}\right) = \frac{208}{27} > 0$

C: Test $\frac{1}{2}$, $f''\left(\frac{1}{2}\right) = -\frac{208}{27} < 0$

D: Test 2, $f''(2) = \frac{28}{27} > 0$

Therefore, $f(x)$ is concave down on $(-\infty, 0)$ and $(0, 1)$ and concave up on $(-1, 0)$ and $(1, \infty)$. Thus, the point $(0, 0)$ is a point of inflection.

h) *Sketch.*



55. $f(x) = \frac{x^2 - 16}{x + 4} = \frac{(x-4)(x+4)}{x+4} = x - 4,\ x \neq -4$

Notice that $f(x) = x - 4$ for all values of x except $x = -4$, where it is undefined. The graph of $f(x)$ will be the graph of $y = x - 4$ except at the point $x = -4$.

a) *Intercepts.* $f(x) = 0$ when $x = 4$, so the x-intercept is $(4, 0)$.
$f(0) = -4.$
The point $(0, -4)$ is the y-intercept.

b) *Asymptotes.*
In simplified form $f(x) = x - 4$, a linear function everywhere except $x = -4$. So there are no asymptotes of any kind.
In the original function, the denominator is 0 for $x = -4$; however, $x = -4$ also made the numerator equal to 0. We look at the limits to determine if there are vertical asymptotes at these points.
$\lim_{x \to -4} \frac{x^2 - 16}{x + 4} = \lim_{x \to -4} (x - 4) = -8$. Because the limit exists, the line $x = -4$ is not a vertical asymptote. Instead, we have a removable discontinuity, or a "hole" at the point $(-4, -8)$. An open circle is drawn at $(-4, -8)$ to show that it is not part of the graph.

c) *Derivatives and Domain.*
$f'(x)=1,\ \ x\neq -4$
$f''(x)=0,\ \ x\neq -4$

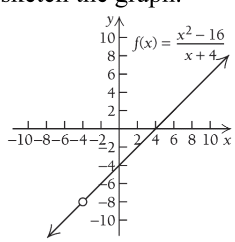
d) *Critical Points.* There are no critical points.

e) *Increasing, decreasing, relative extrema.* We use -4 to divide the real number line into two intervals
A: $(-\infty,-4)$ and B: $(-4,\infty)$.
We notice that $f'(x)>0$ for all real numbers in the domain, $f(x)$ is increasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ is constant; therefore, there are no points of inflection.

g) *Concavity.* $f''(x)$ is 0; therefore, there is no concavity.

h) *Sketch.* Use the preceding information to sketch the graph.



Note: In the preceding problem, we could have noticed that the graph of
$f(x)=\dfrac{x^2-16}{x+4}$ is the graph of $f(x)=x-4$
with the exception of the point $(-4,-8)$ which is a removable discontinuity. We simply need to graph $f(x)=x-4$ with a hole at the point $(-4,-8)$ and determine all other aspects of the graph of $f(x)$ from the linear graph.

56. $f(x)=\dfrac{x^2-9}{x-3}$

We write the expression in simplified form:
$$f(x)=\frac{(x-3)(x+3)}{x-3}=x+3,\quad x\neq 3$$
Note that the domain is restricted to all real numbers except for $x=3$.

a) *Intercepts.* The numerator is 0 when $x=-3$ or $x=3$ however, $x=3$ is not in the domain of $f(x)$, so the *x*-intercept is $(-3,0)$.
$$f(0)=\frac{0^2-9}{(0)-3}=\frac{-9}{-3}=3$$
The point $(0,3)$ is the *y*-intercept.

b) *Asymptotes.*
In simplified form $f(x)=x+3$, a linear function everywhere except $x=3$. So there are no asymptotes of any kind.
In the original function, the denominator is 0 for $x=3$; however, $x=3$ also made the numerator equal to 0. We look at the limits to determine if there is a vertical asymptote at this point.
$$\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}(x+3)=6.$$
The limit exists; therefore, the line $x=3$ is not a vertical asymptote. Instead, we have a removable discontinuity, or a “hole” at the point $(3,6)$. An open circle is drawn at this point to show that it is not part of the graph.

c) *Derivatives and Domain.*
$f'(x)=1,\ \ x\neq 3$
$f''(x)=0,\ \ x\neq 3$
The domain of f is $(-\infty,3)\cup(3,\infty)$ as determined in part (b).

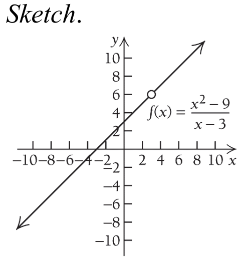
d) *Critical Points.* $f'(x)$ exists for all values of x except 3, but 3 is not in the domain of the function, so $x=3$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use 3 to divide the real number line into two intervals A: $(-\infty,3)$ and B: $(3,\infty)$.
We notice that $f'(x)>0$ for all real numbers in the domain. $f(x)$ is increasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ is constant; therefore, there are no points of inflection.

g) *Concavity.* $f''(x)$ is 0; therefore, there is no concavity.

h) *Sketch.*



57. $C(x)=3x^2+80$

a) $A(x)=\frac{C(x)}{x}=\frac{3x^2+80}{x}=3x+\frac{80}{x}$

b) Using the techniques of this section we find the following information. We will only consider the values of *x* is $(0,\infty)$.

Intercepts. None.

Asymptotes. $x=0$ is the vertical asymptote. There is no horizontal asymptote. As $|x|$ approaches ∞, $A(x)=3x+\frac{80}{x}$ approaches 3*x*. Therefore, $y=3x$ is the slant asymptote.

Increasing, decreasing, relative extrema.

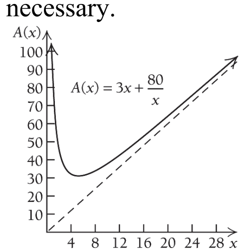
$A'(x)=3-\frac{80}{x^2}$. $A'(x)$ is not defined for $x=0$, however that value is outside the domain of the function. $A'(x)=0$ when $x=\sqrt{\frac{80}{3}}$, and $A\left(\sqrt{\frac{80}{3}}\right)=2\sqrt{240}$.

Using $x=\sqrt{\frac{80}{3}}$ to divide the interval $(0,\infty)$ into two intervals, $\left(0,\sqrt{\frac{80}{3}}\right)$ and $\left(\sqrt{\frac{80}{3}},\infty\right)$, and testing a point in each interval, we find that $A(x)$ is decreasing on $\left(0,\sqrt{\frac{80}{3}}\right)$ and increasing on $\left(\sqrt{\frac{80}{3}},\infty\right)$. Therefore, the point $\left(\sqrt{\frac{80}{3}},2\sqrt{240}\right)$ is a relative minimum.

Inflection points, concavity.

$A''(x)=\frac{160}{x^3}$ exists for all values of *t* in $(0,\infty)$. The equation $A''(x)=0$ has no real solution, so there are no possible points of inflection. Furthermore, $A''(x)>0$ for all *x* in the domain, so $A(x)$ is concave up on $(0,\infty)$.

We use this information to sketch the graph. Additional values may be computed as necessary.



c) The degree of the numerator is exactly one greater than the degree of the denominator. When we divide the numerator by the denominator we have

$A(x)=\frac{3x^2+80}{x}=3x+\frac{80}{x}$. As $|x|$ approaches ∞, $A(x)=3x+\frac{80}{x}$ approaches 3*x*. Therefore, $y=3x$ is the slant asymptote. This means that when a large number of pairs of rocket skates are produced, the average cost can be estimated by multiplying the number of pairs produced by 3 thousand dollars.

58. $V(t)=50-\frac{25t^2}{(t+2)^2}$

a) $V(0)=50-\frac{25(0)^2}{((0)+2)^2}=50-0=50$

The inventory's value after 0 months is \$50 hundreds or \$5000.

$V(5)=50-\frac{25(5)^2}{((5)+2)^2}=50-\frac{625}{49}\approx 37.24$

The inventory's value after 5 months is \$37.24 hundreds or \$3724.

$V(10)=50-\frac{25(10)^2}{((10)+2)^2}=50-\frac{2500}{144}\approx 32.64$

The inventory's value after 10 months is \$32.64 hundreds or \$3264.

$V(70)=50-\frac{25(70)^2}{((70)+2)^2}=50-\frac{122,500}{5184}\approx 26.37$

The inventory's value after 70 months is \$26.37 hundreds or \$2637.

b) Find $V'(t)$ and $V''(t)$.

$$V'(t) = -\frac{(t+2)^2(50t) - 25t^2(2(t+2)(1))}{\left((t+2)^2\right)^2}$$

$$= -\frac{(t+2)\left[(t+2)(50t) - 25t^2(2)\right]}{(t+2)^4}$$

$$= -\frac{100t}{(t+2)^3}$$

$$V''(t) = -\frac{(t+2)^3(100) - 100t\left[3(t+2)^2(1)\right]}{\left((t+2)^3\right)^2}$$

$$= -\frac{(t+2)^2\left[100(t+2) - 100t(3)\right]}{(t+2)^6}$$

$$= -\frac{-200t + 200}{(t+2)^4}$$

$$= \frac{200t - 200}{(t+2)^4}$$

$V'(t)$ exists for all values of t in $[0,\infty)$.

Solve $V'(t) = 0$.

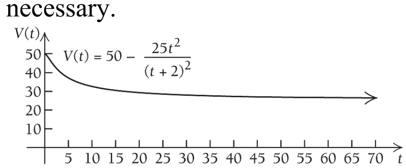
$$-\frac{100t}{(t+2)^3} = 0$$

$$-100t = 0$$

$$t = 0$$

Since $t = 0$ is an endpoint of the domain, there cannot be a relative extrema at $t = 0$. We notice that $V'(t) < 0$ for all x in the domain, therefore, $V(t)$ is decreasing over the interval $(0,\infty)$. Since $V(t)$ is decreasing, the *absolute* maximum value of the inventory will be \$5000 when $t = 0$.

c) Using the techniques of this section, we find the following additional information:
Intercepts. There are no t-intercepts in $[0,\infty)$. The V-intercept is the point $(0,50)$
Asymptotes. There are no vertical asymptotes in $[0,\infty)$.
The line $V = 25$ is a horizontal asymptote.
There are no slant asymptotes.
Increasing, decreasing, relative extrema.
We have already seen that $V(t)$ is decreasing over the interval $(0,\infty)$. There are no relative extrema.
Inflection points, concavity. $V''(t)$ exist for all values of t in $[0,\infty)$. $V''(t) = 0$, when $t = 1$. We use this to split the domain into two intervals. A: $(0,1)$ and B: $(1,\infty)$.
Testing points in each interval, we see $V(t)$ is concave down on $(0,1)$ and concave up on $(1,\infty)$.
We use this information to sketch the graph. Additional values may be computed as necessary.



d) Yes; The value below which V will never fall is $\lim_{t\to\infty} V(t) = 25$. We observed this from the horizontal asymptote on the graph.

59. $C(x) = 5000 + 600x$

$R(x) = -\frac{1}{2}x^2 + 1000x$

a) Profit is revenue minus cost, therefore, the total profit function is:

$$P(x) = R(x) - C(x)$$

$$= -\frac{1}{2}x^2 + 1000x - (5000 + 600x)$$

$$= -\frac{1}{2}x^2 + 400x - 5000$$

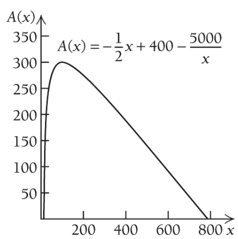
b) $A(x) = \frac{P(x)}{x}$

$$= \frac{-\frac{1}{2}x^2 + 400x - 5000}{x}$$

$$= -\frac{1}{2}x + 400 - \frac{5000}{x}$$

c) As $|x|$ approaches ∞, $A(x)$ approaches $-\frac{1}{2}x + 400$. Therefore, $y = -\frac{1}{2}x + 400$ is the slant asymptote. This represents the average profit for x items, when x is a large number of items.

d) Using the techniques of this section we find the following additional information.
Intercepts. The x-intercepts are $(12.70, 0)$ and $(787.30, 0)$. There is no P-intercept.
Asymptotes. *Vertical:* $x = 0$
Horizonta: None
Slant: $y = -\frac{1}{2}x + 400$
Increasing, decreasing, relative extrema. $A(x)$ is increasing over the interval $(0,100]$ and decreasing over the interval $[100,\infty)$. The point $(100,300)$ is a relative maximum.
Inflection points, concavity. $A(x)$ is concave down on the interval $(0,\infty)$. There are no inflection points. We use this information and compute other function values as necessary to sketch the graph.

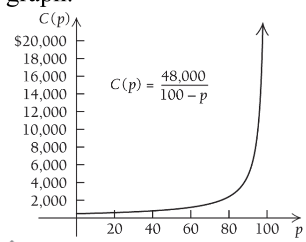


60. $C(p) = \frac{48,000}{100 - p}$

a) $C(0) = \frac{48,000}{100-0} = \frac{48,000}{100} = 480$
The cost of removing 0% of the pollutants from a chemical spill is \$480.
$C(20) = \frac{48,000}{100-(20)} = \frac{48,000}{80} = 600$
The cost of removing 20% of the pollutants from a chemical spill is \$600.
$C(80) = \frac{48,000}{100-(80)} = \frac{48,000}{20} = 2400$
The cost of removing 80% of the pollutants from a chemical spill is \$2400.
$C(90) = \frac{48,000}{100-(90)} = \frac{48,000}{10} = 4800$
The cost of removing 90% of the pollutants from a chemical spill is \$4800.

b) The domain of C is $0 \le p < 100$ since it is not possible to remove less than 0% or more than 100% of the pollutants, and $C(p)$ is not defined for $p = 100$.

c) Using the techniques of this section we find the following additional information.
Intercepts. No p-intercepts. The point $(0,480)$ is the C-intercept.
Asymptotes. *Vertical:* $p = 100$
Horizontal : $C = 0$
Slant: None
Increasing, decreasing, relative extrema. $C(p)$ is increasing over the interval $(0,100)$. There are no relative extrema.
Inflection points, concavity. $C(p)$ is concave up on the interval $(0,100)$. there are no inflection points.
We use this information and compute other function values as necessary to draw the graph.



d) From the result in part (b) we see that there is a vertical asymptote at $p = 100$. This means that the cost of cleaning up the spill increases without bound as the amount of pollutants removed approaches 100%. The company will not be able to afford to clean up 100% of the pollutants.

61. $P(x) = \frac{1}{1 + 0.0362x}$

a)
$P(5) = \frac{1}{1 + 0.0362(5)} = 0.84674$
In 1995, the purchasing power of a dollar was \$0.85.
$P(10) = \frac{1}{1 + 0.0362(10)} = 0.73421439$
In 2000, the purchasing power of a dollar was \$0.73.
$P(25) = \frac{1}{1 + 0.0362(25)} = 0.5249343$
In 2015, the purchasing power of a dollar was \$0.52.

b) Solve $P(x) = 0.50$

$$\frac{1}{1+0.0362x} = 0.50$$
$$1 = 0.50(1+0.0362x)$$
$$1 = 0.50 + 0.0181x$$
$$0.50 = 0.0181x$$
$$27.624 = x$$

27.6 years after 1990, or in 2017, the purchasing power of a dollar will be $0.50.

c) Find $\lim_{x\to\infty} P(x)$.

$$\lim_{x\to\infty} P(x) = \lim_{x\to\infty} \frac{1}{1+0.0362x}$$
$$= \lim_{x\to\infty} \frac{1}{1+0.0362x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$
$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x}+0.0362}$$
$$= \frac{0}{0+0.0362} = 0$$

$\lim_{x\to\infty} P(x) = 0$.

62. $A(t) = \frac{A_0}{t^2+1}$

If we assume 100 cc is the initial amount injected, then $A(t) = \frac{100}{t^2+1}$.

a) $A(0) = \frac{100}{(0)^2+1} = 100$

After 0 hours (at the time of injection) there are 100 cc's of medication in the bloodstream.

$A(1) = \frac{100}{(1)^2+1} = 50$

After 1 hour, there are 50 cc's of the medication in the bloodstream.

$A(2) = \frac{100}{(2)^2+1} = 20$

After 2 hours, there are 20 cc's of the medication in the bloodstream.

$A(7) = \frac{100}{(7)^2+1} = 2$

After 7 hours, there are 2 cc's of the medication in the bloodstream.

$A(10) = \frac{100}{(10)^2+1} \approx 0.99009901$

After 10 hours, there is approximately 0.9901 cc's of the medication in the bloodstream.

b) $A'(t) = \frac{-200t}{(t^2+1)^2}$. Notice that $A'(t) < 0$ for all t in the interval $(0,\infty)$, so there are no relative extrema, and $A(t)$ is decreasing on the interval $(0,\infty)$. Therefore, the maximum value of $A(t)$ is the initial value of 100 cc at the time of injection.

c) Using the techniques of this section we find the following additional information.

Intercepts. There are no t-intercepts. The A-intercept is $(0,100)$.

Asymptotes.	*Vertical:*	None
	Horizontal:	$y = 0$
	Slant:	None

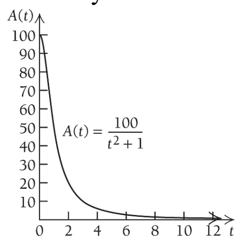
Increasing, decreasing, relative extrema. $A(t)$ is decreasing over the interval $(0,\infty)$. There are no relative extrema.

Inflection points, concavity.

$A''(t) = \frac{200(3t^2-1)}{(t^2+1)^3}$ exists for all t in the interval $[0,\infty)$. $A''(t) = 0$ when $3t^2 - 1 = 0$. The only solution to the equation on $[0,\infty)$ is $t = \frac{1}{\sqrt{3}}$. We divide the interval $(0,\infty)$ into two intervals $\left(0, \frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$ and test a point in each interval. $A(t)$ is concave down on the interval $\left(0, \frac{1}{\sqrt{3}}\right)$ and is concave up on the interval $\left(\frac{1}{\sqrt{3}}, \infty\right)$. The point $\left(\frac{1}{\sqrt{3}}, 75\right)$ is an inflection point.

The solution is continued on the next page.

We use the information on the previous page and compute other function values as necessary to draw the graph.



d) No, the medication does not completely leave the bloodstream. We notice that $\lim_{t\to\infty} A(t)=0$. This means that as t approaches ∞, A approaches 0, but does not actually reach the value 0. We also notice that the equation $A(t)=0$ has no solution, so according to this model, the medication will never completely leave the bloodstream.

63. $E(n)=9\cdot\frac{4}{n}$

a) Calculate each value for the given n.

$E(9)=9\cdot\frac{4}{9}=4.00$

$E(6)=9\cdot\frac{4}{6}=6.00$

$E(3)=9\cdot\frac{4}{3}=12.00$

$E(1)=9\cdot\frac{4}{1}=36.00$

$E\left(\frac{2}{3}\right)=9\cdot\frac{4}{\frac{2}{3}}=9\left(4\cdot\frac{3}{2}\right)=54.00$

$E\left(\frac{1}{3}\right)=9\cdot\frac{4}{\frac{1}{3}}=9\left(4\cdot\frac{3}{1}\right)=108.00$

We complete the table.

Innings Pitched (n)	Earned-Run average (E)
9	4.00
6	6.00
3	12.00
1	36.00
$\frac{2}{3}$	54.00
$\frac{1}{3}$	108.00

b) $\lim_{n\to0} E(n)=\lim_{n\to0} 9\cdot\frac{4}{n}=\lim_{n\to0}\frac{36}{n}=\infty$

If the pitcher gives up one or more runs but gets no one out, the pitcher would be credited with zero innings pitched.

64. Vertical asymptotes occur at values of the variable for which the function is undefined. Thus, they cannot be part of the graph.

65. Asymptotes can be thought of as "limiting lines" for the graph of a function. The graphs and limits in Examples 1, 3 and 6 in section 2.3 of the text book illustrate vertical, horizontal and slant asymptotes.

66. $\lim_{x\to-\infty}\frac{-3x^2+5}{2-x}=\lim_{x\to-\infty}\frac{-3x+\frac{5}{x}}{\frac{2}{x}-1}=$

$\frac{\lim_{x\to-\infty}(-3x)+0}{-1}=-\lim_{x\to-\infty}(-3x)=-\infty$

67. $\lim_{x\to0}\frac{|x|}{x}$

Using $|x|=\begin{cases}-x, & \text{for } x<0\\ x, & \text{for } x\ge0\end{cases}$, we have

$\lim_{x\to0^-}\frac{|x|}{x}=-1$ and $\lim_{x\to0^+}\frac{|x|}{x}=1$.

Therefore, $\lim_{x\to0}\frac{|x|}{x}$ does not exist.

68. $\lim_{x\to-2}\frac{x^3+8}{x^2-4}=\lim_{x\to-2}\frac{(x+2)(x^2-2x+4)}{(x+2)(x-2)}$

$=\lim_{x\to-2}\frac{x^2-2x+4}{x-2}$

$=\frac{(-2)^2-2(-2)+4}{(-2)-2}$.

$=-3$

69. $\lim_{x\to-\infty}\frac{-6x^3+7x}{2x^2-3x-10}=\lim_{x\to-\infty}\frac{-6x+\frac{7}{x}}{2-\frac{3}{x}-\frac{10}{x^2}}$

$=\frac{\lim_{x\to-\infty}(-6x)+0}{2-0-0}$

$=\infty$

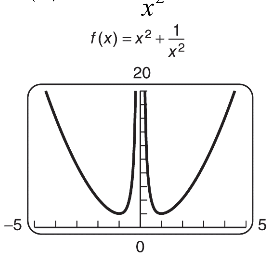
70. $\lim\limits_{x\to 1}\dfrac{x^3-1}{x^2-1}=\lim\limits_{x\to 1}\dfrac{(x-1)(x^2+x+1)}{(x-1)(x+1)}$

$=\lim\limits_{x\to 1}\dfrac{x^2+x+1}{x+1}$

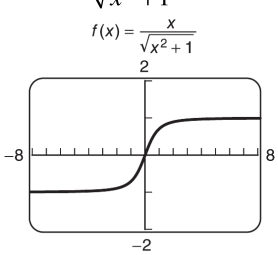
$=\dfrac{(1)^2+(1)+1}{(1)+1}$

$=\dfrac{3}{2}$

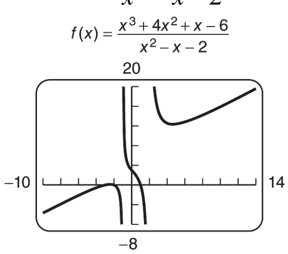
71. $f(x)=x^2+\dfrac{1}{x^2}$



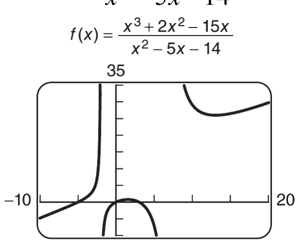
72. $f(x)=\dfrac{x}{\sqrt{x^2+1}}$



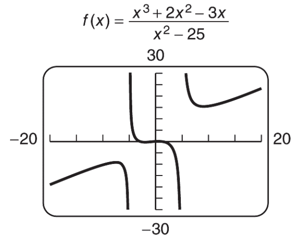
73. $f(x)=\dfrac{x^3+4x^2+x-6}{x^2-x-2}$



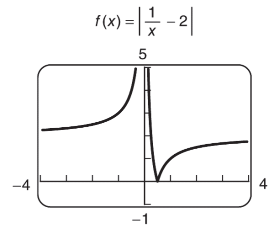
74. $f(x)=\dfrac{x^3+2x^2-15x}{x^2-5x-14}$



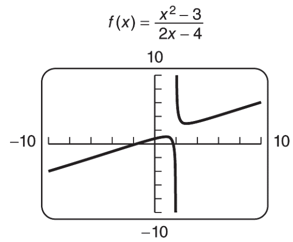
75. $f(x)=\dfrac{x^3+2x^2-3x}{x^2-25}$



76. $f(x)=\left|\dfrac{1}{x}-2\right|$



77. $f(x)=\dfrac{x^2-3}{2x-4}$



a) Solve $f(x)=0$.

$\dfrac{x^2-3}{2x-4}=0$

$x^2-3=0 \qquad x\neq\dfrac{1}{2}$

$x^2=3$

$x=\pm\sqrt{3}\approx\pm 1.732$

The x-intercepts are: $(-1.732,0)$ and $(1.732,0)$.

b) Evaluate at $x=0$.

$f(0)=\dfrac{(0)^2-3}{2(0)-4}=\dfrac{3}{4}$

The y-intercept is $(0,0.75)$.

c) *Vertical.* $x = 2$
Horizontal. None
Slant. Dividing the numerator by the denominator we have

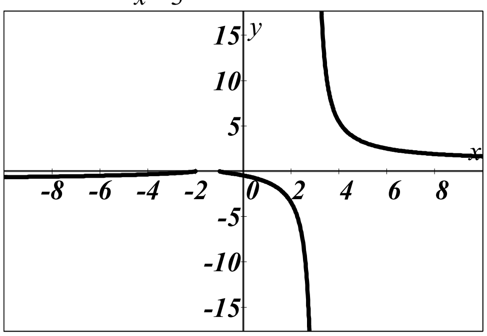
$$f(x) = \frac{x^2 - 3}{2x - 4} = \frac{1}{2}x + 1 + \frac{1}{2x - 4}$$

As $|x|$ approaches ∞,

$f(x)$ approaches $\frac{x}{2} + 1 = 0.5x + 1.$

The slant asymptote is
$y = 0.5x + 1$

78. $f(x) = \dfrac{\sqrt{x^2 + 3x + 2}}{x - 3}$



a) $\lim_{x \to \infty} f(x) = 1$; $\lim_{x \to -\infty} f(x) = -1$

As x increases without bound,

$f(x) = \dfrac{\sqrt{x^2 + 3x + 2}}{x - 3}$ approaches

$\dfrac{\sqrt{x^2}}{x} = \dfrac{|x|}{x} = 1.$

As x decreases without bound,

$f(x) = \dfrac{\sqrt{x^2 + 3x + 2}}{x - 3}$ approaches

$\dfrac{\sqrt{x^2}}{x} = \dfrac{|x|}{x} = -1.$

b) The domain of the function appears to be $(-\infty, -2] \cup [-1, 3) \cup (3, \infty)$. We must throw out values that make the radicand negative, or the denominator 0.

c) From the graph it appears that $\lim_{x \to -2^-} f(x) = 0$ and $\lim_{x \to -1^+} f(x) = 0$.

We verify this algebraically.

$$\lim_{x \to -2^-} \frac{\sqrt{x^2 + 3x + 2}}{x - 3} = \frac{\sqrt{(-2)^2 + 3(-2) + 2}}{(-2) - 3} = 0$$

$$\lim_{x \to -1^+} \frac{\sqrt{x^2 + 3x + 2}}{x - 3} = \frac{\sqrt{(-1)^2 + 3(-1) + 2}}{(-1) - 3} = 0$$

79. $f(x) = \dfrac{x^5 + x - 9}{x^3 + 6x}$

Using long division we have:

$$\begin{array}{r} x^2 \quad -6 \\ x^3 + 6x \overline{\big) x^5 \qquad\qquad + x - 9} \\ \underline{x^5 + 6x^3} \qquad\qquad \\ -6x^3 + 0x^2 + x \\ \underline{-6x^3 \quad + 36x} \\ 37x - 9 \end{array}$$

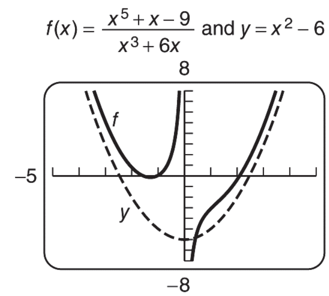
Therefore,

$$f(x) = \frac{x^5 + x - 9}{x^3 + 6x} = x^2 - 6 + \frac{37x - 9}{x^3 + 6x}.$$

As $|x|$ gets large, $f(x)$ approaches $x^2 - 6x$.

Therefore, the nonlinear asymptote is
$y = x^2 - 6$.

Using a calculator, we graph the function, *f,* and the asymptote, *y,* below.



80.

a) Visually inspecting the graph, there appears to be relative maxiumum at $\left(0, \frac{1}{6}\right)$.

However, noticing the graph dips below the horizontal asymptote around $x = 5$, we would think there is a relative minimum for some value $x > 5$. This graph does not give us enough detail to visually determine that point.

b) Calculating the first derivative, we have:

$$f'(x)=\frac{x^2-10x+1}{\left(x^2+x-6\right)^2}.$$

$f'(x)$ does not exist when $x=-3$ or $x=2$; however, those values are not in the domain of $f(x)$. So we set $f'(x)=0$ and solve for x.

$$\frac{x^2-10x+1}{\left(x^2+x-6\right)^2}=0$$

$$x^2-10x+1=0$$

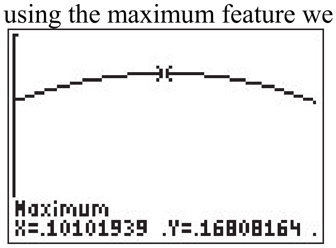
By the quadratic formula, we have:

$$x=\frac{-(-10)\pm\sqrt{(-10)^2-4(1)(1)}}{2(1)}$$
$$=\frac{10\pm\sqrt{100-4}}{2}$$
$$=\frac{10\pm\sqrt{96}}{2}$$

Using the calculator, $f'(x)=0$ when $x\approx 0.10$ and $x\approx 9.90$.

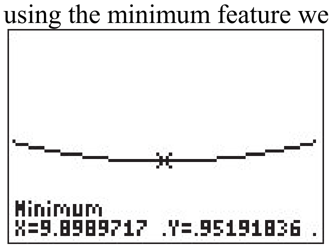
Substititing these values back into the function, we find the critical values occur approximately at $(0.101,0.168)$ and $(9.899,0.952)$.

c) Entering the graph in to the calculator and using the maximum feature we see:



The result is confirmed.

d) Entering the graph in to the calculator and using the minimum feature we see:



The result is confirmed.

e) Our estimates from part (a) are not very close. Even the "obvious" maximum was not correct. It is doubtful that we would have identified the relative minimum without calculus.

81. One possible rational function would be

$$f(x)=\frac{-2x}{x-2}.$$

Using the techniques in this section, we sketch the graph.

Intercepts. $f(x)=0$ when $x=0$. It turns out the x-intercept and the y-intercept is $(0,0)$.

Asymptotes. $x=2$ is the vertical asymptote. The degree of the numerator equals that of the denominator, the line $y=-2$ is the horizontal asymptote.

Increasing, decreasing, relative extrema.

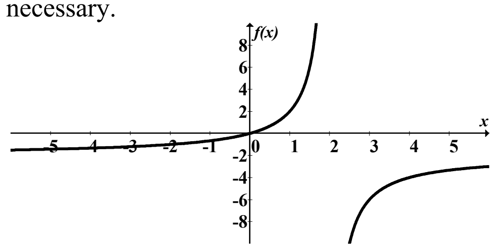
$f'(x)=\frac{4}{(x-2)^2}$. $f'(x)$ is not defined for $x=2$, however that value is outside the domain of the function. $f'(x)>0$ so $f(x)$ is increasing on the intervals $(-\infty,2)$ and $(2,\infty)$. There are no relative extrema.

Inflection points, concavity.

$f''(x)=\frac{-8}{(x-2)^3}$ does not exist when $x=2$.

The equation $f''(x)=0$ has no real solution, so there are no possible points of inflection. Furthermore, $f''(x)>0$ for all x in $(-\infty,2)$, so $f(x)$ is concave up on $(-\infty,2)$, and, $f''(x)<0$ for all x in $(2,\infty)$, so $f(x)$ is concave down on $(2,\infty)$.

We use this information to sketch the graph. Additional values may be computed as necessary.



82. One possible rational function would be

$$f(x)=\frac{3x-1}{x}.$$

Using the techniques in this section, we sketch the graph.

Intercepts. $f(x)=0$ when $x=\frac{1}{3}$. The x-intercept is $\left(\frac{1}{3},0\right)$. When $x=0$, $f(x)$ is undefined, so there is no y-intercept.

Asymptotes. $x = 0$ is the vertical asymptote. The degree of the numerator equals that of the denominator, the line $y = 3$ is the horizontal asymptote.

Increasing, decreasing, relative extrema.

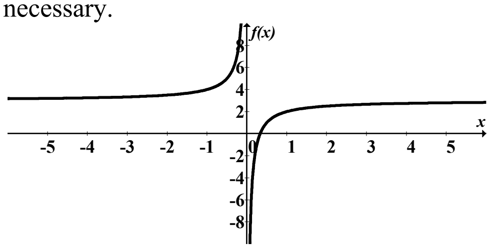
$f'(x) = \frac{1}{x^2}$. $f'(x)$ is not defined for $x = 0$, however that value is outside the domain of the function. $f'(x) > 0$ so $f(x)$ is increasing on the interval $(-\infty, 0)$ and $(0, \infty)$. There are no relative extrema.

Inflection points, concavity.

$f''(x) = \frac{-2}{x^3}$ does not exist when $x = 0$. The equation $f''(x) = 0$ has no real solution, so there are no possible points of inflection.

Furthermore, $f''(x) > 0$ for all x in $(-\infty, 0)$, so $f(x)$ is concave up on $(-\infty, 0)$, and , $f''(x) < 0$ for all x in $(0, \infty)$, so $f(x)$ is concave down on $(0, \infty)$.

We use this information to sketch the graph. Additional values may be computed as necessary.



83. One possible rational function would be

$$g(x) = \frac{x^2 - 2}{x^2 - 1}.$$

Using the techniques in this section, we sketch the graph.

Intercepts. $g(x) = 0$ when $x = \pm\sqrt{2}$. The x-intercepts are $(-\sqrt{2}, 0)$ and $(\sqrt{2}, 0)$ When $x = 0$, $g(x) = 2$, so the y-intercept is $(0, 2)$.

Asymptotes. $x = -1$ and $x = 1$ are the vertical asymptote.

The degree of the numerator equals that of the denominator, the line $y = 1$ is the horizontal asymptote.

Increasing, decreasing, relative extrema.

$g'(x) = \frac{2x}{(x^2 - 1)^2}$. $g'(x)$ is not defined for $x = \pm 1$, however that value is outside the domain of the function. $g'(x) = 0$, when $x = 0$, so there is a critical value at $(0, 2)$.

We notice that $g'(x) < 0$ when $x < -1$ so $g(x)$ is decreasing on the interval $(-\infty, -1)$.

$g'(x) > 0$ when $1 < x$ so $g(x)$ is increasing on the interval $(1, \infty)$. $g'(x) < 0$ when $-1 < x < 0$ so $g(x)$ is decreasing on the interval $(-1, 0)$.

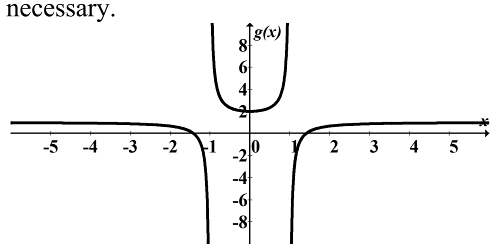
$g'(x) > 0$ when $0 < x < 1$ so $g(x)$ is increasing on the interval $(0, 1)$. There are is a relative minimum at $(0, 2)$.

Inflection points, concavity.

$g''(x) = \frac{-2(3x^2 + 1)}{(x^2 - 1)^3}$. $g''(x)$ does not exist when $x = \pm 1$. The equation $g''(x) = 0$ has no real solution, so there are no possible points of inflection.

Furthermore, $g''(x) > 0$ for all x in $(-1, 1)$, so $g(x)$ is concave up on $(-1, 1)$, and , $g''(x) < 0$ for all x in $(-\infty, -1)$ and $(1, \infty)$, so $g(x)$ is concave down on $(-\infty, -1)$ and $(1, \infty)$.

We use this information to sketch the graph. Additional values may be computed as necessary.



84. One possible rational function would be

$$g(x) = \frac{-3x^2 + 15}{x^2 + 2x}.$$

Using the techniques in this section, we sketch the graph.

Intercepts. $g(x) = 0$ when $x = \pm\sqrt{5}$. The x-intercepts are $(-\sqrt{5}, 0)$ and $(\sqrt{5}, 0)$. When $x = 0$, $g(x)$ is undefined, so there is no y-intercept.

Asymptotes. $x=-2$ and $x=0$ are the vertical asymptote.
The degree of the numerator equals that of the denominator, the line $y=-3$ is the horizontal asymptote.
Increasing, decreasing, relative extrema.
$g'(x)=\dfrac{-6\left(x^2+5x+5\right)}{x^2(x+2)^2}$. $g'(x)$ is not defined for $x=-2$ or $x=0$, however that value is outside the domain of the function. $g'(x)=0$, when $x=\dfrac{-5-\sqrt{5}}{2}$ and $x=\dfrac{-5+\sqrt{5}}{2}$, so there are critical values at $\left(\dfrac{-5+\sqrt{5}}{2},\dfrac{-3\left(\sqrt{5}+5\right)}{2}\right)$ and $\left(\dfrac{-5-\sqrt{5}}{2},\dfrac{3\left(\sqrt{5}-5\right)}{2}\right)$

$g'(x)<0$ when $x<\dfrac{-5-\sqrt{5}}{2}$ so $g(x)$ is decreasing on the interval $\left(-\infty,\dfrac{-5-\sqrt{5}}{2}\right)$.

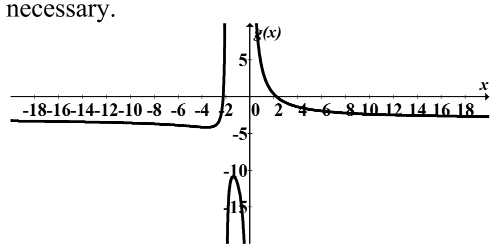
$g'(x)>0$ when $\dfrac{-5-\sqrt{5}}{2}<x<-2$ so $g(x)$ is increasing on the interval $\left(\dfrac{-5-\sqrt{5}}{2},-2\right)$.

$g'(x)>0$ when $-2<x<\dfrac{-5+\sqrt{5}}{2}$ so $g(x)$ is increasing on the interval $\left(-2,\dfrac{-5+\sqrt{5}}{2}\right)$.

There is a relative minimum at $\left(\dfrac{-5-\sqrt{5}}{2},\dfrac{3\left(\sqrt{5}-5\right)}{2}\right)$.

We find that $g'(x)>0$ when $\dfrac{-5+\sqrt{5}}{2}<x<0$ so $g(x)$ is increasing on the interval $\left(\dfrac{-5+\sqrt{5}}{2},0\right)$. There is a relative maximum at $\left(\dfrac{-5+\sqrt{5}}{2},\dfrac{-3\left(\sqrt{5}+5\right)}{2}\right)$.

Inflection points, concavity.
$g''(x)=\dfrac{6\left(2x^3+15x^2+30x+20\right)}{x^3(x+2)^3}$. $g''(x)$ does not exist when $x=-2$ and $x=0$. The equation $g''(x)=0$ has a solution when $x\approx-4.816$.
We find that $g''(x)<0$ when $x<-4.816$ and $g''(x)>0$ when $-4.816<x<-2$, so a point of inflection exists at $(-4.816,-4.025)$
Furthermore, $g''(x)>0$ for all x in $(0,\infty)$, so $g(x)$ is concave up on $(0,\infty)$, and, $g''(x)<0$ for all x in $(-2,0)$, so $g(x)$ is concave down on $(-2,0)$.
We use this information to sketch the graph. Additional values may be computed as necessary.

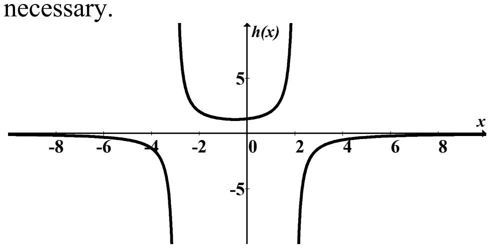


85. One possible rational function would be
$h(x)=\dfrac{-8}{x^2+x-6}$.
Using the techniques in this section, we sketch the graph.
Intercepts. $h(x)=0$ has no real solution. The are no x-intercepts. When $x=0$, $h(x)=\dfrac{4}{3}$, so the y-intercept is $\left(0,\dfrac{4}{3}\right)$.
Asymptotes. $x=-3$ and $x=2$ are the vertical asymptotes.
The degree of the numerator is less than that of the denominator, the line $y=0$ is the horizontal asymptote.

Increasing, decreasing, relative extrema.
$h'(x)=\dfrac{8(2x+1)}{(x-2)^2(x+3)^2}$. $h'(x)$ is not defined for $x=-3$ or $x=2$, however those values are outside the domain of the function. $h'(x)=0$, when $x=\dfrac{-1}{2}$ so there is a critical value at $\left(\dfrac{-1}{2},\dfrac{32}{25}\right)$

$h'(x)<0$ when $-3<x<\dfrac{-1}{2}$ so $h(x)$ is decreasing on the interval $\left(-3,\dfrac{-1}{2}\right)$.

We find $h'(x)>0$ when $\dfrac{-1}{2}<x<2$ so $h(x)$ is increasing on the interval $\left(\dfrac{-1}{2},2\right)$. There is a relative minimum at $\left(\dfrac{-1}{2},\dfrac{32}{25}\right)$. Furthermore, $h'(x)<0$ when $x<-3$ so $h(x)$ is decreasing on the interval $(-\infty,-3)$. $h'(x)>0$ when $2<x$ so $h(x)$ is increasing on the interval $(2,\infty)$.
Inflection points, concavity.
$h''(x)=\dfrac{-16(3x^2+3x+7)}{(x-2)^3(x+3)^3}$. $h''(x)$ does not exist when $x=-2$ and $x=0$. The equation $h''(x)=0$ has no real solution. Therefore $h(x)$ has no points of inflection. $h''(x)<0$ when $x<-3$ and $x>2$.
Therefore, $h(x)$ is concave down on the intervals $(-\infty,-3)$ and $(2,\infty)$. $h''(x)>0$ when $-3<x<2$, so $h(x)$ is concave up on $(-3,2)$.
We use this information to sketch the graph. Additional values may be computed as necessary.



86. One possible rational function would be $h(x)=\dfrac{3}{4x^2-1}$.
Using the techniques in this section, we sketch the graph.
Intercepts. $h(x)=0$ has no real solution. The are no x-intercepts. When $x=0$, $h(x)=-3$, so the y-intercept is $(0,-3)$.
Asymptotes. $x=-\dfrac{1}{2}$ and $x=\dfrac{1}{2}$ are the vertical asymptotes.
The degree of the numerator is less than that of the denominator, the line $y=0$ is the horizontal asymptote.
Increasing, decreasing, relative extrema.
$h'(x)=\dfrac{-24x}{(4x^2-1)^2}$. $h'(x)$ is not defined for $x=-\dfrac{1}{2}$ or $x=\dfrac{1}{2}$, however those values are outside the domain of the function. $h'(x)=0$, when $x=0$ so there is a critical value at $(0,-3)$

$h'(x)>0$ when $-\dfrac{1}{2}<x<0$ so $h(x)$ is increasing on the interval $\left(\dfrac{-1}{2},0\right)$. $h'(x)<0$ when $0<x<\dfrac{1}{2}$ so $h(x)$ is decreasing on the interval $\left(0,\dfrac{1}{2}\right)$. There is a relative maximum at $(0,-3)$.

Furthermore, $h'(x)>0$ when $x<-\dfrac{1}{2}$ so $h(x)$ is increasing on the interval $\left(-\infty,-\dfrac{1}{2}\right)$ and $h'(x)<0$ when $\dfrac{1}{2}<x$ so $h(x)$ is decreasing on the interval $\left(\dfrac{1}{2},\infty\right)$.

Inflection points, concavity.

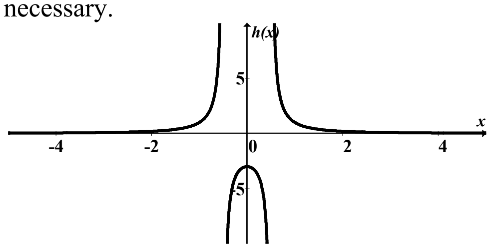
$h''(x)=\frac{-24(12x^2+1)}{(4x^2-1)^3}$. $h''(x)$ does not exist when $x=-\frac{1}{2}$ and $x=\frac{1}{4}$. The equation $h''(x)=0$ has no real solution.

Therefore $h(x)$ has no points of inflection.

$h''(x)<0$ when $-\frac{1}{2}<x<\frac{1}{2}$ and $x>2$.

Therefore, $h(x)$ is concave down on the interval $\left(-\frac{1}{2},\frac{1}{2}\right)$. $h''(x)>0$ when $x<-\frac{1}{2}$ and when $\frac{1}{2}<x$, so $h(x)$ is concave up on the intervals $\left(-\infty,-\frac{1}{2}\right)$ and $\left(\frac{1}{2},\infty\right)$.

We use this information to sketch the graph. Additional values may be computed as necessary.



Exercise Set 2.4

1. a) The absolute maximum gasoline mileage is obtained at a speed of 55 mph.
b) The absolute minimum gasoline mileage is obtained at a speed of 5 mph.
c) At 70 mph, the fuel economy is 25 mpg.

2. Over the interval $[30,70]$, the vehicle's absolute maximum fuel economy is 30 mpg at 55 mph. The vehicle's absolute minimum fuel economy is 25 mpg at 70 mph.

3. $f(x)=5+x-x^2;\quad [0,2]$

a) Find $f'(x)$

$f'(x)=1-2x$

b) Find the Critical values. The derivative exists for all real numbers. Thus, we solve

$$f'(x)=0$$
$$1-2x=0$$
$$1=2x$$
$$\frac{1}{2}=x$$

c) List the critical values and endpoints. These values are $0,\ \frac{1}{2},$ and 2.

d) Evaluate $f(x)$ at each value in step (c).

$$f(0)=5+(0)-(0)^2=5$$
$$f\left(\frac{1}{2}\right)=5+\left(\frac{1}{2}\right)-\left(\frac{1}{2}\right)^2=\frac{21}{4}=5.25$$
$$f(2)=5+(2)-(2)^2=3$$

The largest of these values, $\frac{21}{4}$, is the absolute maximum, it occurs at $x=\frac{1}{2}$. The smallest of these values, 3, is the absolute minimum, it occurs at $x=2$.

4. $f(x)=4+x-x^2;\qquad [0,2]$

a) $f'(x)=1-2x$

b) $f'(x)$ exists for all real numbers. Solve:

$$1-2x=0$$
$$x=\frac{1}{2}$$

c) The critical value and the endpoints are: $0,\ \frac{1}{2},$ and 2.

d) Evaluate $f(x)$ for each value in step (c).

$$f(0)=4+(0)-(0)^2=4$$
$$f\left(\frac{1}{2}\right)=4+\left(\frac{1}{2}\right)-\left(\frac{1}{2}\right)^2=\frac{17}{4}=4.25$$
$$f(2)=4+(2)-(2)^2=2$$

On the interval $[0,2]$, the absolute maximum is $\frac{17}{4}$, which occurs at $x=\frac{1}{2}$. The absolute minimum is 2, which occurs at $x=2$.

5. $f(x)=x^3-\frac{1}{2}x^2-2x+5;\qquad [-2,1]$

a) Find $f'(x)$

$f'(x)=3x^2-x-2$

b) Find the critical values. The derivative exists for all real numbers. Thus, we solve

$$f'(x)=0$$
$$3x^2-x-2=0$$
$$(3x-2)(x-1)=0$$
$$3x+2=0 \text{ or } x-1=0$$
$$x=-\frac{2}{3} \text{ or } x=1$$

c) List the critical values and endpoints. These values are are: $-2,-\frac{2}{3},$ and 1.

d) Evaluate $f(x)$ for each value in step (c).

$$f(-2)=(-2)^3-\frac{1}{2}(-2)^2-2(-2)+5=-9$$
$$f\left(-\frac{2}{3}\right)=\left(-\frac{2}{3}\right)^3+\frac{1}{2}\left(-\frac{2}{3}\right)^2-2\left(-\frac{2}{3}\right)+5=\frac{85}{27}\approx 3.148$$
$$f(1)=(1)^3+\frac{1}{2}(1)^2-2(1)+5=\frac{15}{2}=7.5$$

On the interval $[-2,1]$, the absolute maximum is $\frac{15}{2}$, which occurs at $x=1$. The absolute minimum is -9, which occurs at $x=-2$.

6. $f(x)=x^3-x^2-x+2;\quad [-1,2]$

a) $f'(x)=3x^2-2x-1$

b) $f'(x)$ exists for all real numbers. Solve:

$$f'(x)=0$$
$$3x^2-2x-1=0$$
$$(3x+1)(x-1)=0$$
$$x=-\frac{1}{3}\quad\text{or}\quad x=1$$

c) The critical value and the endpoints are:

$-1,\ -\frac{1}{3},\ 1$, and 2.

d) Evaluate $f(x)$ for each value in step (c).

$$f(-1)=(-1)^3-(-1)^2-(-1)+2=1$$
$$f\left(\frac{-1}{3}\right)=\left(\frac{-1}{3}\right)^3-\left(\frac{-1}{3}\right)^2-\left(\frac{-1}{3}\right)+2=\frac{59}{27}\approx 2.2$$
$$f(1)=(1)^3-(1)^2-(1)+2=1$$
$$f(2)=(2)^3-(2)^2-(2)+2=4$$

On the interval $[-1,2]$, the absolute maximum is 4, which occurs at $x=2$. The absolute minimum is 1, which occurs at $x=-1$ and $x=1$.

7. $f(x)=x^3+\frac{1}{2}x^2-2x+4;\quad [-2,0]$

a) Find $f'(x)$

$f'(x)=3x^2+x-2$

b) Find the critical values. The derivative exists for all real numbers. Thus, we solve

$$3x^2+x-2=0$$
$$(3x-2)(x+1)=0$$
$$3x-2=0\quad\text{or}\quad x+1=0$$
$$x=\frac{2}{3}\quad\text{or}\quad x=-1$$

c) The critical value $x=\frac{2}{3}$ is not in the interval, so we exclude it. We will test the values: $-2,\ -1$, and 0.

d) Evaluate $f(x)$ for each value in step (c).

$$f(-2)=(-2)^3+\frac{1}{2}(-2)^2-2(-2)+4=2$$
$$f(-1)=(-1)^3+\frac{1}{2}(-1)^2-2(-1)+4=\frac{11}{2}=5.5$$
$$f(0)=(0)^3+\frac{1}{2}(0)^2-2(0)+4=4$$

On the interval $[-2,0]$, the absolute maximum is 5.5, which occurs at $x=-1$. The absolute minimum is 2, which occurs at $x=-2$.

8. $f(x)=x^3-x^2-x+3;\quad [-1,0]$

a) $f'(x)=3x^2-2x-1$

b) $f'(x)$ exists for all real numbers. Solve:

$$f'(x)=0$$
$$3x^2-2x-1=0$$
$$(3x+1)(x-1)=0$$
$$x=-\frac{1}{3}\quad\text{or}\quad x=1$$

c) List the critical values and endpoints. The critical value $x=1$ is not in the interval, so we exclude it. We will test the values

$-1,\ -\frac{1}{3}$, and 0.

d) Evaluate $f(x)$ for each value in step (c).

$$f(-1)=(-1)^3-(-1)^2-(-1)+3=2$$
$$f\left(\frac{-1}{3}\right)=\left(\frac{-1}{3}\right)^3-\left(\frac{-1}{3}\right)^2-\left(\frac{-1}{3}\right)+3=\frac{86}{27}\approx 3.2$$
$$f(0)=(0)^3-(0)^2-(0)+3=3$$

On the interval $[-1,0]$, the absolute maximum is $\frac{86}{27}$, which occurs at $x=-\frac{1}{3}$. The absolute minimum is 2, which occurs at $x=-1$.

9. $f(x)=2x+4;\quad [-1,1]$

a) Find $f'(x)$

$f'(x)=2$

b) and c)

The derivative exists and $f'(x)=2$ for all real numbers. Note that the derivative is never 0. Thus, there are no critical values for $f(x)$, and the absolute maximum and absolute minimum will occur at the endpoints of the interval.

d) Evaluate $f(x)$ at the endpoints.

$f(-1)=2(-1)+4=2$

$f(1)=2(1)+4=6$

On the interval $[-1,1]$, the absolute maximum is 6, which occurs at $x=1$. The absolute minimum is 2, which occurs at $x=-1$.

10. $f(x)=5x-7;\qquad [-2,3]$

a) $f'(x)=5$

b) and c)
The derivative exists and is 5 for all real numbers. Note that the derivative is never 0. Thus, there are no critical values for $f(x)$, and the absolute maximum and absolute minimum will occur at the endpoints of the interval.

d) Evaluate $f(x)$ at the endpoints.

$f(-2)=5(-2)-7=-17$

$f(3)=5(3)-7=8$

On the interval $[-2,3]$, the absolute maximum is 8, which occurs at $x=3$. The absolute minimum is -17, which occurs at $x=-2$.

11. $f(x)=7-4x;\qquad [-2,5]$

a) Find $f'(x)$

$f'(x)=-4$

b) and c)
The derivative exists and is -4 for all real numbers. Note that the derivative is never 0. Thus, there are no critical values for $f(x)$, and the absolute maximum and absolute minimum will occur at the endpoints of the interval.

d) Evaluate $f(x)$ at the endpoints.

$f(-2)=7-4(-2)=15$

$f(5)=7-4(5)=-13$

On the interval $[-2,5]$, the absolute maximum is 15, which occurs at $x=-2$. The absolute minimum is -13, which occurs at $x=5$.

12. $f(x)=-2-3x;\qquad [-10,10]$

a) $f'(x)=-3$

b) and c)
$f'(x)=-3$ for all real numbers; therefore, there are no critical points. The absolute maximum and minimum values occur at the endpoints.

d) Evaluate $f(x)$ at the endpoints.

$f(-10)=-2-3(-10)=28$

$f(10)=-2-3(10)=-32$

On the interval $[-10,10]$, the absolute maximum is 28, which occurs at $x=-10$. The absolute minimum is -32, which occurs at $x=10$.

13. $f(x)=-5;\qquad [-1,1]$

Note for all values of x, $f(x)=-5$. Thus, the absolute maximum is -5 for $-1\le x\le 1$ and the absolute minimum is -5 for $-1\le x\le 1$.

14. $f(x)=x^2-6x-3;\qquad [-1,5]$

a) $f'(x)=2x-6$

b) $f'(x)$ exists for all real numbers. Solve:

$2x-6=0$

$2x=6$

$x=3$

c) The critical value and the endpoints are -1, 3, and 5.

d) Evaluate $f(x)$ for each value in step (c).

$f(-1)=(-1)^2-6(-1)-3=4$

$f(3)=(3)^2-6(3)-3=-12$

$f(5)=(5)^2-6(5)-3=-8$

On the interval $[-1,5]$, the absolute maximum is 4, which occurs at $x=-1$. The absolute minimum is -12, which occurs at $x=3$.

15. $g(x)=24;\qquad [4,13]$

Note for all values of x, $g(x)=24$. Thus, the absolute maximum is 24 for $4\le x\le 13$ and the absolute minimum is 24 for $4\le x\le 13$.

16. $f(x)=3-2x-5x^2;\qquad [-3,3]$

a) $f'(x)=-2-10x$

b) $f'(x)$ exists for all real numbers. Solve:

$$-2-10x=0$$
$$x=-\frac{1}{5}$$

c) The critical value and the endpoints are $-3,\ -\frac{1}{5}$, and 3.

d) Evaluate $f(x)$ for each value in step (c).

$$f(-3)=3-2(-3)-5(-3)^2=-36$$
$$f\left(-\frac{1}{5}\right)=3-2\left(-\frac{1}{5}\right)-5\left(-\frac{1}{5}\right)^2=\frac{16}{5}=3.2$$
$$f(3)=3-2(3)-5(3)^2=-48$$

On the interval $[-3,3]$, the absolute maximum is $\frac{16}{5}$, which occurs at $x=-\frac{1}{5}$. The absolute minimum is -48, which occurs at $x=3$.

17. $f(x)=x^2-4x+5;\qquad [-1,3]$

a) $f'(x)=2x-4$

b) $f'(x)$ exists for all real numbers. Solve:

$$2x-4=0$$
$$2x=4$$
$$x=2$$

c) The critical value and the endpoints are $-1,\ 2$, and 3.

d) Evaluate $f(x)$ for each value in step (c).

$$f(-1)=(-1)^2-4(-1)+5=10$$
$$f(2)=(2)^2-4(2)+5=1$$
$$f(3)=(3)^2-4(3)+5=2$$

On the interval $[-1,3]$, the absolute maximum is 10, which occurs at $x=-1$. The absolute minimum is 1, which occurs at $x=2$.

18. $f(x)=x^3-3x^2;\qquad [0,5]$

a) $f'(x)=3x^2-6x$

b) $f'(x)$ exists for all real numbers. Solve:

$$3x^2-6x=0$$
$$3x(x-2)=0$$
$$3x=0 \quad\text{or}\quad x-2=0$$
$$x=0 \quad\text{or}\quad x=2$$

c) The critical value and the endpoints are $0,\ 2$, and 5. Note, since 0 is an endpoint of the interval, $x=0$ is included in this list as an endpoint, not as a critical value.

d) Evaluate $f(x)$ for each value in step (c).

$$f(0)=(0)^3-3(0)^2=0$$
$$f(2)=(2)^3-3(2)^2=-4$$
$$f(5)=(5)^3-3(5)^2=50$$

On the interval $[0,5]$, the absolute maximum is 50, which occurs at $x=5$. The absolute minimum is -4, which occurs at $x=2$.

19. $f(x)=1+6x-3x^2;\qquad [0,4]$

a) $f'(x)=6-6x$

b) $f'(x)$ exists for all real numbers. Solve:

$$6-6x=0$$
$$x=1$$

c) The critical value and the endpoints are $0,\ 1$, and 4. Note, since 0 is an endpoint of the interval, $x=0$ is included in this list as an endpoint, not as a critical value.

d) Evaluate $f(x)$ for each value in step (c).

$$f(0)=1+6(0)-3(0)^2=1$$
$$f(1)=1+6(1)-3(1)^2=4$$
$$f(4)=1+6(4)-3(4)^2=-23$$

On the interval $[0,4]$, the absolute maximum is 4, which occurs at $x=1$. The absolute minimum is -23, which occurs at $x=4$.

20. $f(x)=x^3-3x+6;\qquad [-1,3]$

a) $f'(x)=3x^2-3$

b) $f'(x)$ exists for all real numbers. Solve:

$$3x^2-3=0$$
$$x^2-1=0$$
$$x=\pm 1$$

c) The critical value and the endpoints are $-1,\ 1,$ and 3. Note, since -1 is an endpoint of the interval, $x=-1$ is included in this list as an endpoint, not a critical value.

d) Evaluate $f(x)$ for each value in step (c).

$$f(-1)=(-1)^3-3(-1)+6=8$$
$$f(1)=(1)^3-3(1)+6=4$$
$$f(3)=(3)^3-3(3)+6=24$$

On the interval $[-1,3]$, the absolute maximum is 24, which occurs at $x=3$. The absolute minimum is 4, which occurs at $x=1$.

21. $f(x)=x^3-3x;\qquad[-5,1]$

a) $f'(x)=3x^2-3$

b) $f'(x)$ exists for all real numbers. Solve:

$$3x^2-3=0$$
$$x^2-1=0$$
$$x=\pm 1$$

c) The critical value and the endpoints are $-5,\ -1,$ and 1. Note, since 1 is an endpoint of the interval, $x=1$ is included in this list as an endpoint, not a critical value.

d) Evaluate $f(x)$ for each value in step (c).

$$f(-5)=(-5)^3-3(-5)=-110$$
$$f(-1)=(-1)^3-3(-1)=2$$
$$f(1)=(1)^3-3(1)=-2$$

On the interval $[-5,1]$, the absolute maximum is 2, which occurs at $x=-1$. The absolute minimum is -110, which occurs at $x=-5$.

22. $f(x)=3x^2-2x^3;\qquad[-5,1]$

a) $f'(x)=6x-6x^2$

b) $f'(x)$ exists for all real numbers. Solve:

$$6x-6x^2=0$$
$$6x(1-x)=0$$
$$x=0\quad\text{or}\quad x=1$$

c) The critical value and the endpoints are $-5,\ 0,$ and 1. Note, since 1 is an endpoint of the interval, $x=1$ is included in this list as an endpoint, not a critical value.

d) Evaluate $f(x)$ for each value in step (c).

$$f(-5)=3(-5)^2-2(-5)^3=325$$
$$f(0)=3(0)^2-2(0)^3=0$$
$$f(1)=3(1)^2-2(1)^3=1$$

On the interval $[-5,1]$, the absolute maximum is 325, which occurs at $x=-5$. The absolute minimum is 0, which occurs at $x=0$.

23. $f(x)=1-x^3;\qquad[-8,8]$

a) $f'(x)=-3x^2$

b) $f'(x)$ exists for all real numbers. Solve:

$$-3x^2=0$$
$$x=0$$

c) The critical value and the endpoints are $-8,\ 0,$ and 8.

d) Evaluate $f(x)$ for each value in step (c).

$$f(-8)=1-(-8)^3=513$$
$$f(0)=1-(0)^3=1$$
$$f(8)=1-(8)^3=-511$$

On the interval $[-8,8]$, the absolute maximum is 513, which occurs at $x=-8$. The absolute minimum is -511, which occurs at $x=8$.

24. $f(x)=2x^3;\qquad[-10,10]$

a) $f'(x)=6x^2$

b) $f'(x)$ exists for all real numbers. Solve:

$$6x^2=0$$
$$x=0$$

c) The critical value and the endpoints are $-10,\ 0,$ and 10.

d) Evaluate $f(x)$ for each value in step (c).

$$f(-10)=2(-10)^3=-2000$$
$$f(0)=2(0)^3=0$$
$$f(10)=2(10)^3=2000$$

On the interval $[-10,10]$, the absolute maximum is 2000, which occurs at $x=10$. The absolute minimum is -2000, which occurs at $x=-10$.

25. $f(x)=x^3-6x^2+10; \quad [0,4]$

a) $f'(x)=3x^2-12x$

b) $f'(x)$ exists for all real numbers. Solve:

$$3x^2-12x=0$$
$$3x(x-4)=0$$
$$3x=0 \quad \text{or} \quad x-4=0$$
$$x=0 \quad \text{or} \quad x=4$$

c) The critical values and the endpoints are 0 and 4. Note, since the possible critical values are the endpoints of the interval, they are included in this list as endpoints, not as critical values.

d) Evaluate $f(x)$ for each value in step (c).

$$f(0)=(0)^3-6(0)^2+10=10$$
$$f(4)=(4)^3-6(4)^2+10=-22$$

On the interval $[0,4]$, the absolute maximum is 10, which occurs at $x=0$. The absolute minimum is -22, which occurs at $x=4$.

26. $f(x)=12+9x-3x^2-x^3; \quad [-3,1]$

a) $f'(x)=9-6x-3x^2$

b) $f'(x)$ exists for all real numbers. Solve:

$$9-6x-3x^2=0$$
$$x^2+2x-3=0$$
$$(x+3)(x-1)=0$$
$$x=-3 \quad \text{or} \quad x=1$$

c) The critical values and the endpoints are -3 and 1. Note, since the possible critical values are the endpoints of the interval, they are included in this list as endpoints, not as critical values.

d) Evaluate $f(x)$ for each value in step (c).

$$f(-3)=12+9(-3)-3(-3)^2-(-3)^3=-15$$
$$f(1)=12+9(1)-3(1)^2-(1)^3=17$$

On the interval $[-3,1]$, the absolute maximum is 17, which occurs at $x=1$. The absolute minimum is -15, which occurs at $x=-3$.

27. $f(x)=x^3-x^4; \quad [-1,1]$

a) $f'(x)=3x^2-4x^3$

b) $f'(x)$ exists for all real numbers. Solve:

$$3x^2-4x^3=0$$
$$x^2(3-4x)=0$$
$$x^2=0 \quad \text{or} \quad 3-4x=0$$
$$x=0 \quad \text{or} \quad x=\frac{3}{4}$$

c) The critical values and the endpoints are $-1, 0, \frac{3}{4}$, and 1.

d) Evaluate $f(x)$ for each value in step (c).

$$f(-1)=(-1)^3-(-1)^4=-2$$
$$f(0)=(0)^3-(0)^4=0$$
$$f\left(\frac{3}{4}\right)=\left(\frac{3}{4}\right)^3-\left(\frac{3}{4}\right)^4=\frac{27}{256}\approx 0.105$$
$$f(1)=(1)^3-(1)^4=0$$

e) On the interval $[-1,1]$, the absolute maximum is $\frac{27}{256}$, which occurs at $x=\frac{3}{4}$. The absolute minimum is -2, which occurs at $x=-1$.

28. $f(x)=x^4-2x^3; \quad [-2,2]$

a) $f'(x)=4x^3-6x^2$

b) $f'(x)$ exists for all real numbers. Solve:

$$4x^3-6x^2=0$$
$$2x^2(2x-3)=0$$
$$x=0 \quad \text{or} \quad x=\frac{3}{2}$$

c) The critical values and the endpoints are $-2, 0, \frac{3}{2}$, and 2.

d) Evaluate $f(x)$ for each value in step (c).

$$f(-2)=(-2)^4-2(-2)^3=32$$
$$f(0)=(0)^4-2(0)^3=0$$
$$f\left(\frac{3}{2}\right)=\left(\frac{3}{2}\right)^4-2\left(\frac{3}{2}\right)^3=-\frac{27}{16}=-1.6875$$
$$f(2)=(2)^4-2(2)^3=0$$

The solution is continued on the next page.

From the previous page, we determine on the interval $[-2,2]$, the absolute maximum is 32, which occurs at $x=-2$. The absolute minimum is $-\frac{27}{16}$, which occurs at $x=\frac{3}{2}$.

29. $f(x)=x^4-2x^2+5; \quad [-2,2]$

a) $f'(x)=4x^3-4x$

b) $f'(x)$ exists for all real numbers. Solve:

$$4x^3-4x=0$$
$$4x(x^2-1)=0$$
$$4x=0 \quad \text{or} \quad x^2-1=0$$
$$x=0 \quad \text{or} \quad x=\pm1$$

c) The critical values and the endpoints are $-2, -1, 0, 1,$ and 2.

d) Evaluate $f(x)$ for each value in step (c).

$$f(-2)=(-2)^4-2(-2)^2+5=13$$
$$f(-1)=(-1)^4-2(-1)^2+5=4$$
$$f(0)=(0)^4-2(0)^2+5=5$$
$$f(1)=(1)^4-2(1)^2+5=4$$
$$f(2)=(2)^4-2(2)^2+5=13$$

On the interval $[-2,2]$, the absolute maximum is 13, which occurs at $x=-2$ and $x=2$. The absolute minimum is 4, which occurs at $x=-1$ and $x=1$.

30. $f(x)=x^4-8x^2+3; \quad [-3,3]$

a) $f'(x)=4x^3-16x$

b) $f'(x)$ exists for all real numbers. Solve:

$$4x^3-16x=0$$
$$4x(x^2-4)=0$$
$$4x=0 \quad \text{or} \quad x^2-4=0$$
$$x=0 \quad \text{or} \quad x=\pm2$$

c) The critical values and the endpoints are $-3, -2, 0, 2,$ and 3.

d) Evaluate $f(x)$ for each value in step (c).

$$f(-3)=(-3)^4-8(-3)^2+3=12$$
$$f(-2)=(-2)^4-8(-2)^2+3=-13$$
$$f(0)=(0)^4-8(0)^2+3=3$$
$$f(2)=(2)^4-8(2)^2+3=-13$$
$$f(3)=(3)^4-8(3)^2+3=12$$

On the interval $[-3,3]$, the absolute maximum is 12, which occurs at $x=-3$ and $x=3$. The absolute minimum is -13, which occurs at $x=-2$ and $x=2$.

31. $f(x)=1-x^{2/3}; \quad [-8,8]$

a) $f'(x)=-\frac{2}{3}x^{-1/3}=-\frac{2}{3x^{1/3}}$

b) $f'(x)$ does not exist for $x=0$. The equation $f'(x)=0$ has no solution, so $x=0$ is the only critical value.

c) The critical values and the endpoints are $-8, 0,$ and 8.

d) Evaluate $f(x)$ for each value in step (c).

$$f(-8)=1-(-8)^{2/3}=-3$$
$$f(0)=1-(0)^{2/3}=1$$
$$f(8)=1-(8)^{2/3}=-3$$

On the interval $[-8,8]$, the absolute maximum is 1, which occurs at $x=0$. The absolute minimum is -3, which occurs at $x=-8$ and $x=8$.

32. $f(x)=(x+3)^{2/3}-5; \quad [-4,5]$

a) $f'(x)=\frac{2}{3}(x+3)^{-1/3}=\frac{2}{3(x+3)^{1/3}}$

b) $f'(x)$ does not exist for $x=-3$. The equation $f'(x)=0$ has no solution, so $x=-3$ is the only critical value.

c) The critical values and the endpoints are $-4, -3,$ and 5.

d) Evaluate $f(x)$ for each value in step (c).

$$f(-4)=((-4)+3)^{2/3}-5=-4$$
$$f(-3)=((-3)+3)^{2/3}-5=-5$$
$$f(5)=((5)+3)^{2/3}-5=-1$$

On the interval $[-4,5]$, the absolute maximum is -1, which occurs at $x=5$. The absolute minimum is -5, which occurs at $x=-3$.

33. $f(x)=x+\frac{4}{x}$; $[-8,-1]$

a) $f'(x)=1-4x^{-2}=1-\frac{4}{x^2}$

b) $f'(x)$ does not exist for $x=0$. However, $x=0$ is not in the interval. Solve $f'(x)=0$.

$$1-\frac{4}{x^2}=0$$
$$1=\frac{4}{x^2}$$
$$x^2=4$$
$$x=\pm 2$$

The only critical value in the interval is at $x=-2$.

c) The critical values and the endpoints are $-8,\ -2$ and -1.

d) Evaluate $f(x)$ for each value in step (c).

$$f(-8)=(-8)+\frac{4}{(-8)}=-\frac{17}{2}=-8.5$$
$$f(-2)=(-2)+\frac{4}{(-2)}=-4$$
$$f(-1)=(-1)+\frac{4}{(-1)}=-5$$

On the interval $[-8,-1]$, the absolute maximum is -4, which occurs at $x=-2$. The absolute minimum is $-\frac{17}{2}$, which occurs at $x=-8$.

34. $f(x)=x+\frac{1}{x}$; $[1,20]$

a) $f'(x)=1-x^{-2}=1-\frac{1}{x^2}$

b) $f'(x)$ does not exist for $x=0$. However, $x=0$ is not in the interval. Solve $f'(x)=0$.

$$1-\frac{1}{x^2}=0$$
$$1=\frac{1}{x^2}$$
$$x^2=1$$
$$x=\pm 1$$

The critical value $x=-1$ is not in the interval, and the other critical value is an endpoint.

c) The endpoints are 1 and 20.

d) Evaluate $f(x)$ for each value in step (c).

$$f(1)=1+\frac{1}{1}=2$$
$$f(20)=20+\frac{1}{20}=\frac{401}{20}=20.05$$

On the interval $[1,20]$, the absolute maximum is 20.05, which occurs at $x=20$. The absolute minimum is 2, which occurs at $x=1$.

35. $f(x)=\frac{x^2}{x^2+1}$; $[-2,2]$

a) $f'(x)=\frac{(x^2+1)(2x)-x^2(2x)}{(x^2+1)^2}$ Quotient Rule

$$=\frac{2x^3+2x-2x^3}{(x^2+1)^2}$$
$$=\frac{2x}{(x^2+1)^2}$$

b) $f'(x)$ exists for all real numbers. Solve:

$$f'(x)=0$$
$$\frac{2x}{(x^2+1)^2}=0$$
$$2x=0$$
$$x=0$$

c) The critical values and the endpoints are -2, 0, and 2.

d) Evaluate $f(x)$ for each value in step (c).

$$f(-2)=\frac{(-2)^2}{(-2)^2+1}=\frac{4}{5}$$
$$f(0)=\frac{(0)^2}{(0)^2+1}=0$$
$$f(2)=\frac{(2)^2}{(2)^2+1}=\frac{4}{5}$$

On the interval $[-2,2]$, the absolute maximum is $\frac{4}{5}$, which occurs at $x=-2$ and $x=2$. The absolute minimum is 0, which occurs at $x=0$.

36. $f(x)=\dfrac{4x}{x^2+1}$; $[-3,3]$

a) $f'(x)=\dfrac{(x^2+1)(4)-4x(2x)}{(x^2+1)^2}$

$=\dfrac{4-4x^2}{(x^2+1)^2}$

b) $f'(x)$ exists for all real numbers. Solve:

$\dfrac{4-4x^2}{(x^2+1)^2}=0$

$4-4x^2=0$

$x^2-1=0$

$x=\pm 1$

c) The critical values and the endpoints are $-3,\ -1,\ 1,$ and 3.

d) Evaluate $f(x)$ for each value in step (c).

$f(-3)=\dfrac{4(-3)}{(-3)^2+1}=-\dfrac{6}{5}$

$f(-1)=\dfrac{4(-1)}{(-1)^2+1}=-2$

$f(1)=\dfrac{4(1)}{(1)^2+1}=2$

$f(3)=\dfrac{4(3)}{(3)^2+1}=\dfrac{6}{5}$

On the interval $[-3,3]$, the absolute maximum is 2, which occurs at $x=1$. The absolute minimum is -2, which occurs at $x=-1$.

37. $f(x)=(x+1)^{1/3}$; $[-2,26]$

a) $f'(x)=\dfrac{1}{3}(x+1)^{-2/3}=\dfrac{1}{3(x+1)^{2/3}}$

b) $f'(x)$ does not exist for $x=-1$. The equation $f'(x)=0$ has no solution, so $x=-1$ is the only critical value.

c) The critical values and the endpoints are $-2,\ -1,$ and 26.

d) Evaluate $f(x)$ for each value in step (c).

$f(-2)=((-2)+1)^{1/3}=-1$

$f(-1)=((-1)+1)^{1/3}=0$

$f(26)=((26)+1)^{1/3}=3$

On the interval $[-2,26]$, the absolute minimum is -1, which occurs at $x=-2$. The absolute maximum is 3, which occurs at $x=26$.

38. $f(x)=\sqrt[3]{x}=x^{1/3}$; $[8,64]$

a) $f'(x)=\dfrac{1}{3}x^{-2/3}=\dfrac{1}{3x^{2/3}}$

b) There are no critical values in the interval.

c) The endpoints are 8 and 64.

d) Evaluate $f(x)$ for each value in step (c).

$f(8)=(8)^{1/3}=2$

$f(64)=(64)^{1/3}=4$

On the interval $[8,64]$, the absolute maximum is 4, which occurs at $x=64$. The absolute minimum is 2, which occurs at $x=8$.

39. – 48. Left to the student.

49. $f(x)=30x-x^2$

When no interval is specified, we use the real line $(-\infty,\infty)$.

a) Find $f'(x)$

$f'(x)=30-2x$

b) Find the critical values. The derivative exists for all real numbers. Thus, we solve $f'(x)=0$.

$f'(x)=0$

$30-2x=0$

$-2x=-30$

$x=15$

The only critical value is $x=15$.

c) Since there is only one critical value, we can apply Max-Min Principle 2. First we find $f''(x)$.

$f''(x)=-2$

The solution is continued on the next page.

On the previous page, we determined the second derivative is constant, so $f''(15) = -2$. Since the second derivative is negative at 15, we have a maximum at $x = 15$. Next, we find the function value at $x = 15$.

$f(15) = 30(15) - (15)^2 = 225$

Therefore, the absolute maximum is 225, which occurs at $x = 15$. There is no minimum value.

50. $f(x) = 12x - x^2;$ $(-\infty, \infty)$

a) $f'(x) = 12 - 2x$

b) $f'(x)$ exists for all real numbers. Solve

$$f'(x) = 0$$
$$12 - 2x = 0$$
$$6 = x$$

The only critical value is $x = 6$.

c) $f''(x) = -2$.

$f''(6) = -2 < 0$.

Thus, we have a maximum at $x = 6$.

$f(6) = 12(6) - (6)^2 = 36$

Therefore, the absolute maximum is 36, which occurs at $x = 6$. The function has no minimum value.

51. $f(x) = 2x^2 - 40x + 270$

When no interval is specified , we use the real line $(-\infty, \infty)$.

a) Find $f'(x)$

$f'(x) = 4x - 40$

b) Find the critical values. The derivative exists for all real numbers. Thus, we solve $f'(x) = 0$.

$$4x - 40 = 0$$
$$4x = 40$$
$$x = 10$$

The only critical value is $x = 10$. .

c) Since there is only one critical value, we can apply Max-Min Principle 2. First we find $f''(x)$.

$f''(x) = 4$.

The second derivative is constant, so $f''(10) = 4$.

Since the second derivative is positive at 10, we have a minimum at $x = 10$. Next, we find the function value at $x = 10$.

$f(10) = 2(10)^2 - 40(10) + 270 = 70$

Therefore, the absolute minimum is 70, which occurs at $x = 10$. The function has no maximum value.

52. $f(x) = 2x^2 - 20x + 340;$ $(-\infty, \infty)$

a) $f'(x) = 4x - 20$

b) $f'(x)$ exists for all real numbers. Solve

$$f'(x) = 0$$
$$4x - 20 = 0$$
$$x = 5$$

c) $f''(x) = 4$

$f''(5) = 4 > 0$

Thus, we have a minimum at $x = 5$.

$f(5) = 2(5)^2 - 20(5) + 340 = 290$

Therefore, the absolute minimum is 290, which occurs at $x = 5$. There is no maximum value.

53. $f(x) = 16x - \frac{4}{3}x^3;$ $(0, \infty)$

a) Find $f'(x)$

$f'(x) = 16 - 4x^2$

b) Find the critical values. The derivative exists for all real numbers. Thus, we solve $f'(x) = 0$.

$$16 - 4x^2 = 0$$
$$4x^2 = 16$$
$$x^2 = 4$$
$$x = \pm 2$$

There are two critical values; however, $x = 2$ is the only critical value on the interval $(0, \infty)$.

c) Since there is only one critical value in the interval, we can apply Max-Min Principle 2. First we find $f''(x)$.

$f''(x) = -8x$.

The solution is continued on the next page.

Next evaluate the second derivative at $x = 2$.

$f''(2) = -16 < 0$

Since the second derivative is negative at 2, we have a maximum at $x = 2$.

$f(2) = 16(2) - \frac{4}{3}(2)^3 = \frac{64}{3}$

Therefore, the absolute maximum is $\frac{64}{3}$, which occurs at $x = 2$. There is no minimum value.

54. $f(x) = x - \frac{4}{3}x^3; \quad (0, \infty)$

a) $f'(x) = 1 - 4x^2$

b) $f'(x)$ exists for all real numbers. Solve

$$1 - 4x^2 = 0$$
$$-4x^2 = -1$$
$$x^2 = \frac{1}{4}$$
$$x = \pm\frac{1}{2}$$

There are two critical values; however, the only critical value in $(0, \infty)$ is $x = \frac{1}{2}$.

c) $f''(x) = -8x$.

$f''\left(\frac{1}{2}\right) = -4 < 0$.

Thus, we have a maximum at $x = \frac{1}{2}$.

$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right) - \frac{4}{3}\left(\frac{1}{2}\right)^3 = \frac{1}{3}$

Therefore, the absolute maximum is $\frac{1}{3}$, which occurs at $x = \frac{1}{2}$. The function has no minimum value.

55. $f(x) = x(60 - x) = 60x - x^2$

When no interval is specified , we use the real line $(-\infty, \infty)$.

a) Find $f'(x)$.

$f'(x) = 60 - 2x$.

b) Find the critical values. The derivative exists for all real numbers.

Thus, we solve $f'(x) = 0$.

$$f'(x) = 0$$
$$60 - 2x = 0$$
$$60 = 2x$$
$$30 = x$$

The only critical value is $x = 30$.

c) Since there is only one critical value, we can apply Max-Min Principle 2. First we find $f''(x)$.

$f''(x) = -2$.

The second derivative is constant, so $f''(30) = -2$. Since the second derivative is negative at 30, we have a maximum at $x = 30$. Next, we find the function value at $x = 30$.

$f(30) = 30(60 - 30) = 900$

Therefore, the absolute maximum is 900, which occurs at $x = 30$. The function has no minimum value.

56. $f(x) = x(25 - x) = 25x - x^2; \quad (-\infty, \infty)$

a) $f'(x) = 25 - 2x$

b) $f'(x)$ exists for all real numbers. Solve

$$f'(x) = 0$$
$$25 - 2x = 0$$
$$x = \frac{25}{2}$$

c) $f''(x) = -2$

$f''\left(\frac{25}{2}\right) = -2 < 0$

Thus, we have a maximum at $x = \frac{25}{2}$.

$f\left(\frac{25}{2}\right) = \left(\frac{25}{2}\right)\left(25 - \frac{25}{2}\right) = \frac{625}{4} = 156.25$

Therefore, the absolute maximum is $\frac{625}{4}$, which occurs at $x = \frac{25}{2}$. There is no minimum value.

57. $f(x) = \frac{1}{3}x^3 - 5x; \quad [-3, 3]$

a) Find $f'(x)$.

$f'(x) = x^2 - 5$.

b) Find the critical values. The derivative exists for all real numbers. Thus, we solve $f'(x)=0$.

$x^2-5=0$

$x^2=5$

$x=\pm\sqrt{5}\approx\pm 2.236$

Both critical values are in the interval $[-3,3]$.

c) The interval is closed and there is more than one critical value, so we use Max-Min Principle 1.
The critical points and the endpoints are $-3,\ -\sqrt{5},\ \sqrt{5},$ and $3.$
Next, we find the function values at these points.

$$f(-3)=\frac{1}{3}(-3)^3-5(-3)=6$$

$$f(-\sqrt{5})=\frac{1}{3}(-\sqrt{5})^3-5(-\sqrt{5})=\frac{10\sqrt{5}}{3}\approx 7.454$$

$$f(\sqrt{5})=\frac{1}{3}(\sqrt{5})^3-5(\sqrt{5})=-\frac{10\sqrt{5}}{3}\approx -7.454$$

$$f(3)=\frac{1}{3}(3)^3-5(3)=-6$$

Thus, the absolute maximum over the interval $[-3,3]$, is $\frac{10\sqrt{5}}{3}$, which occurs at $x=-\sqrt{5}$, and the absolute minimum over $[-3,3]$ is $-\frac{10\sqrt{5}}{3}$, which occurs at $x=\sqrt{5}$.

58. $f(x)=\frac{1}{3}x^3-3x;\qquad [-2,2]$

a) $f'(x)=x^2-3$

b) $f'(x)$ exists for all real numbers. Solve

$f'(x)=0$

$x^2-3=0$

$x=\pm\sqrt{3}\approx\pm 1.732.$

Both critical values are in the interval $[-2,2]$.

c) The critical points and the endpoints are $-2,\ -\sqrt{3},\ \sqrt{3},$ and $2.$
Next, we find the function values at these points.

$$f(-2)=\frac{1}{3}(-2)^3-3(-2)=\frac{10}{3}=3.3\overline{3}$$

$$f(-\sqrt{3})=\frac{1}{3}(-\sqrt{3})^3-3(-\sqrt{3})=2\sqrt{3}\approx 3.464$$

$$f(\sqrt{3})=\frac{1}{3}(\sqrt{3})^3-3(\sqrt{3})=-2\sqrt{3}\approx -3.464$$

$$f(2)=\frac{1}{3}(2)^3-3(2)=-\frac{10}{3}=-3.3\overline{3}$$

Thus, the absolute maximum over the interval $[-2,2]$, is $2\sqrt{3}$, which occurs at $x=-\sqrt{3}$, and the absolute minimum over $[-2,2]$ is $-2\sqrt{3}$, which occurs at $x=\sqrt{3}$.

59. $f(x)=-0.001x^2+4.8x-60$

When no interval is specified , we use the real line $(-\infty,\infty)$.

a) Find $f'(x)$

$f'(x)=-0.002x+4.8$

b) Find the critical values. The derivative exists for all real numbers. Thus, we solve $f'(x)=0$.

$-0.002x+4.8=0$

$-0.002x=-4.8$

$x=2400$

The only critical value is $x=2400$.

c) Since there is only one critical value, we can apply Max-Min Principle 2. First we find $f''(x)$.

$f''(x)=-0.002$.

The second derivative is constant, so $f''(2400)=-0.002$.

Since the second derivative is negative at 2400, we have a maximum at $x=2400$.
Next, we find the function value at $x=2400$.

$$f(2400)=-0.001(2400)^2+4.8(2400)-60$$
$$=5700$$

Therefore, the absolute maximum is 5700, which occurs at $x=2400$. The function has no minimum value.

60. $f(x)=-0.01x^2+1.4x-30$

a) $f'(x)=-0.02x+1.4$

b) $f'(x)$ exists for all real numbers. Solve

$$f'(x)=0$$
$$-0.02x+1.4=0$$
$$x=70$$

c) $f''(x)=-0.02$

$f''(70)=-0.02<0$

Thus, we have a maximum at $x=70$.

$f(70)=-0.01(70)^2+1.4(70)-30=19$

Therefore, the absolute maximum is 19, which occurs at $x=70$. There is no minimum value.

61. $f(x)=-x^3+x^2+5x-1;\quad (0,\infty)$

a) Find $f'(x)$.

$f'(x)=-3x^2+2x+5$.

b) Find the critical values. The derivative exists for all real numbers. Thus, we solve $f'(x)=0$.

$$-3x^2+2x+5=0$$
$$-(3x-5)(x+1)=0$$
$$3x-5=0 \quad \text{or} \quad x+1=0$$
$$x=\frac{5}{3} \quad \text{or} \quad x=-1$$

$x=\frac{5}{3}$ is the only critical value on the interval $(0,\infty)$.

c) The interval $(0,\infty)$ is not closed. The only critical value in the interval is $x=\frac{5}{3}$.

Therefore, we can apply Max-Min Principle 2. First, we find the second derivative.

$f''(x)=-6x+2$

$$f''\left(\frac{5}{3}\right)=-6\left(\frac{5}{3}\right)+2=-8<0$$

Since the second derivative is negative when $x=\frac{5}{3}$, there is a maximum at $x=\frac{5}{3}$.

Next, find the function value:

$$f\left(\frac{5}{3}\right)=-\left(\frac{5}{3}\right)^3+\left(\frac{5}{3}\right)^2+5\left(\frac{5}{3}\right)-1=\frac{148}{27}$$

Thus, the absolute maximum over the interval $(0,\infty)$ is $\frac{148}{27}$, which occurs at $x=\frac{5}{3}$. The function has no minimum value.

62. $f(x)=-\frac{1}{3}x^3+6x^2-11x-50;\quad (0,3)$

a) $f'(x)=-x^2+12x-11$

b) $f'(x)$ exists for all real numbers. Solve

$$f'(x)=0$$
$$-x^2+12x-11=0$$
$$x^2-12x+11=0$$
$$(x-11)(x-1)=0$$
$$x-11=0 \quad \text{or} \quad x-1=0$$
$$x=11 \quad \text{or} \quad x=1$$

c) $f''(x)=-2x+12$

$f''(1)=-2(1)+12=10>0$.

Therfore, there is a minimum at $x=1$.

$$f(1)=-\frac{1}{3}(1)^3+6(1)^2-11(1)-50=-\frac{166}{3}$$

Thus, the absolute minimum over the interval $(0,3)$ is $-\frac{166}{3}$, which occurs at $x=1$.

63. $f(x)=15x^2-\frac{1}{2}x^3;\quad [0,30]$

a) Find $f'(x)$.

$f'(x)=30x-\frac{3}{2}x^2$.

b) Find the critical values. The derivative exists for all real numbers. Thus, we solve $f'(x)=0$.

$$30x-\frac{3}{2}x^2=0$$
$$60x-3x^2=0$$
$$3x(20-x)=0$$
$$3x=0 \quad \text{or} \quad 20-x=0$$
$$x=0 \quad \text{or} \quad x=20$$

Both critical values are in the interval $[0,30]$.

c) Since the interval is closed and there is more than one critical value, we apply the Max-Min Principle 1.

The critical values and the endpoints are $0,\ 20,\ \text{and } 30$.

The solution is continued on the next page.

Next, we find the function values.

$f(0)=15(0)^2-\frac{1}{2}(0)^3=0$

$f(20)=15(20)^2-\frac{1}{2}(20)^3=2000$

$f(30)=15(30)^2-\frac{1}{2}(30)^3=0$

The largest of these values, 2000, is the maximum. It occurs at $x=20$. The smallest of these values, 0, is the minimum. It occurs at $x=0$ and $x=30$.

Thus, the absolute maximum over the interval $[0,30]$, is 2000, which occurs at $x=20$, and the absolute minimum over $[0,30]$ is 0, which occurs at $x=0$ and $x=30$.

64. $f(x)=4x^2-\frac{1}{2}x^3$; $[0,8]$

a) $f'(x)=8x-\frac{3}{2}x^2$

b) $f'(x)$ exists for all real numbers. Solve:

$f'(x)=0$

$8x-\frac{3}{2}x^2=0$

$16x-3x^2=0$

$x(16-3x)=0$

$x=0$ or $16x-3=0$

$x=0$ or $x=\frac{3}{16}$

c) The critical values and the endpoints are $0,\ \frac{3}{16}$, and 8.

d) Find the function values.

$f(0)=4(0)^2-\frac{1}{2}(0)^3=0$

$f\left(\frac{16}{3}\right)=4\left(\frac{16}{3}\right)^2-\frac{1}{2}\left(\frac{16}{3}\right)^3=\frac{1024}{27}\approx 37.93$

$f(8)=4(8)^2-\frac{1}{2}(8)^3=0$

Therefore, the absolute maximum over the interval $[0,8]$ is $\frac{1024}{27}$ or $37\frac{25}{27}$, which occurs at $x=\frac{16}{3}$. The absolute minimum is 0, which occurs at $x=0$ and $x=8$.

65. $f(x)=2x+\frac{72}{x}$; $(0,\infty)$

$f(x)=2x+72x^{-1}$

a) Find $f'(x)$.

$f'(x)=2-72x^{-2}=2-\frac{72}{x^2}$.

b) Find the critical values. $f'(x)$ does not exist for $x=0$; however, 0 is not in the interval $(0,\infty)$. Therefore, we solve

$f'(x)=0$

$2-\frac{72}{x^2}=0$

$2=\frac{72}{x^2}$

$2x^2=72$ Multiplying by x^2, since $x\neq 0$.

$x^2=36$

$x=\pm 6$

c) The interval $(0,\infty)$ is not closed. The only critical value in the interval is $x=6$. Therefore, we can apply Max-Min Principle 2. First, we find the second derivative.

$f''(x)=144x^{-3}=\frac{144}{x^3}$

Evaluating the second derivative at $x=6$, we have:

$f''(6)=\frac{144}{(6)^3}=\frac{2}{3}>0$.

Since the second derivative is positive when $x=6$, there is a minimum at $x=6$.

Next, find the function value at $x=6$.

$f(6)=2(6)+\frac{72}{6}=24$

Thus, the absolute minimum over the interval $(0,\infty)$ is 24, which occurs at $x=6$. The function has no maximum value over the interval $(0,\infty)$.

66. $f(x)=x+\frac{3600}{x}$; $(0,\infty)$

a) $f'(x)=1-\frac{3600}{x^2}$

b) Find the critical values. $f'(x)$ does not exists for $x=0$; however, 0 is not in the interval $(0,\infty)$. Therefore, we solve

$$f'(x)=0$$
$$1-\frac{3600}{x^2}=0$$
$$x^2=3600$$
$$x=\pm 60$$

c) The interval $(0,\infty)$ is not closed. The only critical value in the interval is $x=60$.

$$f''(x)=\frac{7200}{x^3}$$
$$f''(60)=\frac{7200}{(60)^3}=\frac{1}{30}>0.$$

Since the second derivative is positive when $x=60$, there is a minimum at $x=60$.

$$f(60)=(60)+\frac{3600}{60}=120$$

Thus, the absolute minimum over the interval $(0,\infty)$ is 120, which occurs at $x=60$. The function has no maximum value over the interval $(0,\infty)$.

67. $f(x)=x^2+\dfrac{432}{x}$; $(0,\infty)$

$f(x)=x^2+432x^{-1}$

a) Find $f'(x)$.

$$f'(x)=2x-432x^{-2}=2x-\frac{432}{x^2}.$$

b) Find the critical values. $f'(x)$ does not exist for $x=0$; however, 0 is not in the interval $(0,\infty)$. Therefore, we solve

$$f'(x)=0$$
$$2x-\frac{432}{x^2}=0$$
$$2x=\frac{432}{x^2}$$
$$2x^3=432 \quad \text{Multiplying by } x^2 \text{, since } x\neq 0.$$
$$x^3=216$$
$$x=6$$

c) The interval $(0,\infty)$ is not closed. The only critical value in the interval is $x=6$. Therefore, we can apply Max-Min Principle 2.

First, we find the second derivative.

$$f''(x)=2+864x^{-3}=2+\frac{864}{x^3}$$

Evaluating the second derivative at $x=6$, we have:

$$f''(6)=2+\frac{864}{(6)^3}=6>0.$$

Since the second derivative is positive when $x=6$, there is a minimum at $x=6$.

Next, find the function value at $x=6$.

$$f(6)=(6)^2+\frac{432}{6}=108$$

Thus, the absolute minimum over the interval $(0,\infty)$ is 108, which occurs at $x=6$.

The function has no maximum value over the interval $(0,\infty)$.

68. $f(x)=x^2+\dfrac{250}{x}$; $(0,\infty)$

a) $f'(x)=2x-\dfrac{250}{x^2}$

b) Find the critical values. $f'(x)$ does not exist for $x=0$; however, 0 is not in the interval $(0,\infty)$. Therefore, we solve

$$f'(x)=0$$
$$2x-\frac{250}{x^2}=0$$
$$2x=\frac{250}{x^2}$$
$$x^3=125$$
$$x=5$$

c) The interval $(0,\infty)$ is not closed. The only critical value in the interval is $x=5$.

$$f''(x)=2+\frac{500}{x^3}$$
$$f''(5)=2+\frac{500}{(5)^3}=6>0.$$

There is a minimum at $x=5$.

$$f(5)=(5)^2+\frac{250}{5}=75$$

Thus, the absolute minimum over the interval $(0,\infty)$ is 75, which occurs at $x=5$.

The function has no maximum value over the interval $(0,\infty)$.

69. $f(x)=2x^4+x; \quad [-1,1]$

a) Find $f'(x)$.

$f'(x)=8x^3+1$.

b) Find the critical values. The derivative exists for all real numbers. Thus, we solve $f'(x)=0$.

$$8x^3+1=0$$
$$8x^3=-1$$
$$x^3=-\frac{1}{8}$$
$$x=-\frac{1}{2}$$

The only critical value $x=-\frac{1}{2}$ is in the interval $[-1,1]$.

c) The interval is closed, and we are looking for both the absolute maximum and absolute minimum values, so we use Max-Min Principle 1.
The critical points and the endpoints are $-1,\ -\frac{1}{2}$, and 1.
Next, we find the function values at these points.

$$f(-1)=2(-1)^4+(-1)=1$$
$$f\left(-\frac{1}{2}\right)=2\left(-\frac{1}{2}\right)^4+\left(-\frac{1}{2}\right)=-\frac{3}{8}$$
$$f(1)=2(1)^4+(1)=3$$

The largest of these values, 3, is the maximum. It occurs at $x=1$. The smallest of these values, $-\frac{3}{8}$, is the minimum. It occurs at $x=-\frac{1}{2}$.
Thus, the absolute maximum over the interval $[-1,1]$, is 3, which occurs at $x=1$, and the absolute minimum over $[-1,1]$ is $-\frac{3}{8}$, which occurs at $x=-\frac{1}{2}$.

70. $f(x)=2x^4-x; \quad [-1,1]$

a) $f'(x)=8x^3-1$

b) Find the critical values. The derivative exists for all real numbers. Thus, we solve $f'(x)=0$.

$$8x^3-1=0$$
$$8x^3=1$$
$$x^3=\frac{1}{8}$$
$$x=\frac{1}{2}$$

The only critical value $x=\frac{1}{2}$ is in the interval $[-1,1]$.

c) The interval is closed, and we are looking for both the absolute maximum and absolute minimum values, so we use Max-Min Principle 1.
The critical values and the endpoints are $-1,\ \frac{1}{2}$, and 1.
Next, we find the function values at these points.

$$f(-1)=2(-1)^4-(-1)=3$$
$$f\left(\frac{1}{2}\right)=2\left(\frac{1}{2}\right)^4-\left(\frac{1}{2}\right)=-\frac{3}{8}$$
$$f(1)=2(1)^4-(1)=1$$

The largest of these values, 3, is the maximum. It occurs at $x=-1$.
The smallest of these values, $-\frac{3}{8}$, is the minimum. It occurs at $x=\frac{1}{2}$.
Thus, the absolute maximum over the interval $[-1,1]$, is 3, which occurs at $x=-1$, and the absolute minimum over $[-1,1]$ is $-\frac{3}{8}$, which occurs at $x=\frac{1}{2}$.

71. $f(x)=\sqrt[3]{x}=x^{1/3}; \quad [0,8]$

a) Find $f'(x)$

$$f'(x)=\frac{1}{3}x^{-2/3}=\frac{1}{3\cdot\sqrt[3]{x^2}}$$

b) Find the critical values. $f'(x)$ does not exist for $x=0$. The equation $f'(x)=0$ has no solution, so the only critical value is 0, which is also an endpoint.

c) The interval is closed, and we are looking for both the absolute maximum and absolute minimum values, so we use Max-Min Principle 1.
The only critical value is an endpoint. The endpoints are 0 and 8.
Next, we find the function values at these points.
$f(0)=\sqrt[3]{0}=0$
$f(8)=\sqrt[3]{8}=2$
The largest of these values, 2, is the maximum. It occurs at $x=8$. The smallest of these values, 0, is the minimum. It occurs at $x=0$.
Thus, the absolute maximum over the interval $[0,8]$, is 2, which occurs at $x=8$, and the absolute minimum over $[0,8]$ is 0, which occurs at $x=0$.

72. $f(x)=\sqrt{x}=x^{1/2}; \quad [0,4]$

a) $f'(x)=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}}$

b) Find the critical values. $f'(x)$ does not exist for $x=0$. The equation $f'(x)=0$ has no solution, so the only critical value is 0, which is also an endpoint.

c) The interval is closed, and we are looking for both the absolute maximum and absolute minimum values, so we use Max-Min Principle 1.
The only critical value is an endpoint. The endpoints are 0 and 4.
Next, we find the function values at these points.
$f(0)=\sqrt{0}=0$
$f(4)=\sqrt{4}=2$
The largest of these values, 2, is the maximum. It occurs at $x=4$. The smallest of these values, 0, is the minimum. It occurs at $x=0$. Thus, the absolute maximum over the interval $[0,4]$ is 2, which occurs at $x=4$, and the absolute minimum over $[0,4]$ is 0, which occurs at $x=0$.

73. $f(x)=(x-1)^3$

When no interval is specified , we use the real line $(-\infty,\infty)$.

a) Find $f'(x)$.
$f'(x)=3(x-1)^2$.

b) Find the critical values. The derivative exists for all real numbers. Thus, we solve $f'(x)=0$.
$3(x-1)^2=0$
$x-1=0$
$x=1$
The only critical value is $x=1$.

c) Since there is only one critical value, we can apply Max-Min Principle 2. First we find $f''(x)$.
$f''(x)=6(x-1)$.
Now,
$f''(1)=6((1)-1)=0$, so the Max-Min Principle 2 fails. We cannot use Max-Min Principle 1, because there are no endpoints.
We note that $f'(x)=3(x-1)^2$ is never negative. Thus, $f(x)$ is increasing everywhere except at $x=1$. Therefore, the function has no maximum or minimum over the interval $(-\infty,\infty)$.
Notice
$f''(0)=-6<0$
$f''(2)=6>0$
and
$f(1)=0$
Therefore, there is a point of inflection at $(1,0)$.

74. $f(x)=(x+1)^3$

When no interval is specified , we use the real line $(-\infty,\infty)$.

a) $f'(x)=3(x+1)^2$

b) $f'(x)$ exists for all real numbers. Solve:
$f'(x)=0$
$3(x+1)^2=0$
$x=-1$
The only critical value is $x=-1$.

c) $f''(x) = 6(x+1)$

$f''(-1) = 6((-1)+1) = 0$.

The Max-Min Principle 2 fails. We cannot use Max-Min Principle 1, because there are no endpoints. We note that

$f'(x) = 3(x+1)^2$ is always positive, except at $x = -1$. Thus, $f(x)$ is increasing everywhere except at $x = -1$. Therefore, the function has no maximum or minimum over the interval $(-\infty, \infty)$.

Notice

$f''(-2) = -6 < 0$

$f''(0) = 6 > 0$

and

$f(-1) = 0$

Therefore, there is a point of inflection at $(-1, 0)$.

75. $f(x) = 2x - 3; \quad [-1, 1]$

a) Find $f'(x)$.

$f'(x) = 2$.

. b) and c)

The derivative exists and is 2 for all real numbers. Therefore, $f'(x)$ is never 0. Thus, there are no critical values. We apply the Max-Min Principle 1. The endpoints are -1 and 1. We find the function values at the endpoints.

$f(-1) = 2(-1) - 3 = -5$

$f(1) = 2(1) - 3 = -1$

Therefore, the absolute maximum over the interval $[-1, 1]$ is -1, which occurs at $x = 1$, and the absolute minimum over the interval $[-1, 1]$ is -5, which occurs at $x = -1$.

76. $f(x) = 9 - 5x; \quad [-10, 10]$

a) $f'(x) = -5$

b) and c)

The derivative exists for all real numbers and is never 0. There are no critical values, so the maximum and minimum occur at the endpoints, -10 and 10. We find the function values at the endpoints.

$f(-10) = 9 - 5(-10) = 59$

$f(10) = 9 - 5(10) = -41$

Therefore, the absolute maximum over the interval $[-10, 10]$ is 59, which occurs at $x = -10$, and the absolute minimum over the interval $[-10, 10]$ is -41, which occurs at $x = 10$.

77. $f(x) = 2x - 3; \quad [-1, 5)$

a) Find $f'(x)$

$f'(x) = 2$

b) and c)

The derivative exists and is 2 for all real numbers. Therefore, $f'(x)$ is never 0. Thus, there are no critical values. We apply the Max-Min Principle 1. There is only one endpoint, $x = -1$. We find the function value at the endpoint.

$f(-1) = 2(-1) - 3 = -5$

We know $f'(x) > 0$ over the interval, so the function is increasing over the interval $(-1, 5)$. Therefore, the minimum value will be the left hand endpoint. The absolute minimum over the interval $[-1, 5)$ is -5, which occurs at $x = -1$. Since the right endpoint is not included in the interval, the function has no maximum value over the interval $[-1, 5)$.

78. $f(x) = x^{2/3}; \quad [-1, 1]$

a) $f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3 \cdot \sqrt[3]{x}}$

b) Find the critical values. $f'(x)$ does not exist for $x = 0$. The equation $f'(x) = 0$ has no solution, so the only critical value is 0.

c) The interval is closed, and we are looking for both the absolute maximum and absolute minimum values, so we use Max-Min Principle 1.

The critical value and the endpoints are -1, 0, and 1.

Next, we find the function values at these points.

$f(-1) = (-1)^{2/3} = 1$

$f(0) = (0)^{2/3} = 0$

$f(1) = (1)^{2/3} = 1$

The solution is continued on the next page.

From the previous page, the largest of these values, 1, is the maximum. It occurs at $x=-1$ and $x=1$. The smallest of these values, 0, is the minimum. It occurs at $x=0$. Thus, the absolute maximum over the interval $[-1,1]$, is 1, which occurs at $x=-1$ and $x=1$, and the absolute minimum over $[-1,1]$ is 0, which occurs at $x=0$.

79. $f(x)=9-5x; \quad [-2,3)$

a) Find $f'(x)$.

$f'(x)=-5$.

b) and c)
The derivative exists for all real numbers and is never 0. There are no critical values. The only endpoint is the left endpoint -2.
$f'(x)<0$ over the interval, so the function is decreasing and a maximum occurs at $x=-2$. We find the function value at $x=-2$.
$f(-2)=9-5(-2)=19$.
The absolute maximum over the interval $[-2,3)$ is 19, which occurs at $x=-2$. The function has no minimum value over the interval $[-2,3)$.

80. $f(x)=\frac{1}{3}x^3-x+\frac{2}{3}$

When no interval is specified, we use the real line $(-\infty,\infty)$.

a) $f'(x)=x^2-1$

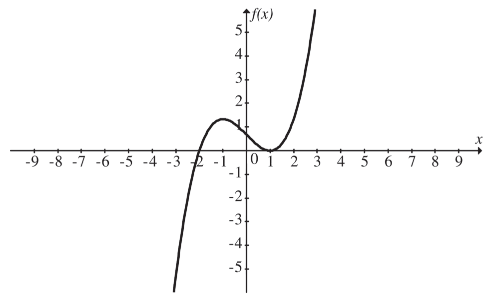
b) Find the critical values. The derivative exists for all real numbers. Thus, we solve $f'(x)=0$.

$x^2-1=0$

$x=\pm 1$

There are two critical values -1 and 1.

c) The interval $(-\infty,\infty)$ is not closed, so the Max-Min Principle 1 does not apply. Since there is more than one critical value, the Max-Min Principle 2 does not apply.
A quick sketch of the graph at the top of the next column will help us determine whether absolute or relative extrema occur at the critical values.



We determine that the function has no absolute extrema over the interval $(-\infty,\infty)$.

81. $g(x)=x^{2/3}$

When no interval is specified, we use the real line $(-\infty,\infty)$.

a) Find $g'(x)$.

$$g'(x)=\frac{2}{3}x^{-1/3}=\frac{2}{3\cdot\sqrt[3]{x}}.$$

b) Find the critical values. $g'(x)$ does not exist for $x=0$. The equation $g'(x)=0$ has no solution, so the only critical value is 0.

c) We apply the Max-Min Principle 2.

$$g''(x)=-\frac{2}{9x^{4/3}}$$

$g''(0)$ does not exist.

Note that $g'(x)<0$ for $x<0$ and $g'(x)>0$ for $x>0$, so $g(x)$ is decreasing on $(-\infty,0)$ and increasing on $(0,\infty)$.
Therefore, the absolute minimum over the interval $(-\infty,\infty)$ is 0, which occurs at $x=0$. The function has no maximum value.

82. $f(x)=\frac{1}{3}x^3-2x^2+x; \quad [0,4]$

a) $f'(x)=x^2-4x+1$

b) $f'(x)$ exists for all real numbers. Solve:

$f'(x)=0$

$x^2-4x+1=0$

We use the quadratic formula to solve the equation.

$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(1)}}{2(1)}$$
$$=\frac{4\pm\sqrt{12}}{2}=2\pm\sqrt{3}$$

The solution is continued on the next page.

Both critical values $x = 2-\sqrt{3} \approx 0.27$ and $x = 2+\sqrt{3} \approx 3.73$ are in the closed interval $[0,4]$.

c) The interval is closed, and there is more than one critical value in the interval, so we use Max-Min Principle 1.
The critical values and the endpoints are $0,\ 2-\sqrt{3},\ 2+\sqrt{3},$ and 4.
Next, we find the function values at these points.

$$f(0) = \frac{1}{3}(0)^3 - 2(0)^2 + (0) = 0$$

$$f\left(2-\sqrt{3}\right) = \frac{1}{3}\left(2-\sqrt{3}\right)^3 - 2\left(2-\sqrt{3}\right)^2 + \left(2-\sqrt{3}\right)$$
$$= -\frac{10}{3} + 2\sqrt{3} \approx 0.131$$

$$f\left(2+\sqrt{3}\right) = \frac{1}{3}\left(2+\sqrt{3}\right)^3 - 2\left(2+\sqrt{3}\right)^2 + \left(2+\sqrt{3}\right)$$
$$= -\frac{10}{3} - 2\sqrt{3} \approx -6.797$$

$$f(4) = \frac{1}{3}(4)^3 - 2(4)^2 + (4)$$
$$= -\frac{20}{3} \approx -6.66\overline{6}$$

Thus, the absolute maximum over the interval $[0,4]$, is $-\frac{10}{3}+2\sqrt{3}$, which occurs at $x = 2-\sqrt{3}$, and the absolute minimum over $[0,4]$ is $-\frac{10}{3}-2\sqrt{3}$, which occurs at $x = 2+\sqrt{3}$.

83. $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 1$

When no interval is specified , we use the real line $(-\infty,\infty)$.

a) Find $f'(x)$

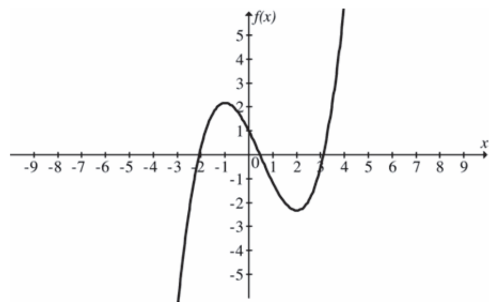
$$f'(x) = x^2 - x - 2$$

b) Find the critical values. The derivative exists for all real numbers. Thus, we solve $f'(x) = 0$.

$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2 \quad \text{or} \quad x = -1$$

There are two critical values -1 and 2.

c) The interval $(-\infty,\infty)$ is not closed, so the Max-Min Principle 1 does not apply. Since there is more than one critical value, the Max-Min Principle 2 does not apply.
A quick sketch of the graph will help us determine absolute or relative extrema occur at the critical values.



We determine that the function has no absolute extrema over the interval $(-\infty,\infty)$.

84. $g(x) = \frac{1}{3}x^3 + 2x^2 + x;\quad [-4,0]$

a) $g'(x) = x^2 + 4x + 1$

b) Find the critical values. The derivative exists for all real numbers. Thus, we solve $g'(x) = 0$.

$$x^2 + 4x + 1 = 0$$

We use the quadratic formula to solve the equation.
The solution to the quadratic equation is $x = -2 \pm \sqrt{3}$.
Both critical values $x = -2-\sqrt{3} \approx -3.73$ and $x = -2+\sqrt{3} \approx -0.27$ are in the closed interval $[-4,0]$.

c) The interval is closed, and there is more than one critical value in the interval, so we use Max-Min Principle 1.
The critical values and the endpoints are $-4,\ -2-\sqrt{3},\ -2+\sqrt{3},$ and 0.
Next, we find the function values at the critical values and the endpoints.

$$g(-4) = \frac{20}{3} \approx 6.66\overline{6}$$
$$g\left(-2-\sqrt{3}\right) = \frac{10}{3} + 2\sqrt{3} \approx 6.797$$
$$g\left(-2+\sqrt{3}\right) = \frac{10}{3} - 2\sqrt{3} \approx -0.131$$
$$g(0) = 0$$

The solution is continued on the next page.

From the previous page, we determine the absolute maximum over the interval $[-4,0]$, is $\frac{10}{3}+2\sqrt{3}$, which occurs at $x=-2-\sqrt{3}$, and the absolute minimum over $[-4,0]$ is $\frac{10}{3}-2\sqrt{3}$, which occurs at $x=-2+\sqrt{3}$.

85. $t(x)=x^4-2x^2$

When no interval is specified , we use the real line $(-\infty,\infty)$.

a) Find $t'(x)$

$t'(x)=4x^3-4x$

b) Find the critical values. The derivative exists for all real numbers. Thus, we solve $t'(x)=0$.

$4x^3-4x=0$

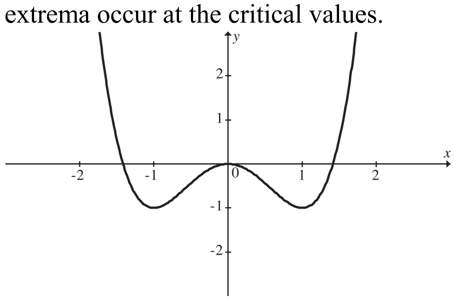
$4x(x^2-1)=0$

$x=0 \quad \text{or} \quad x^2-1=0$

$x=0 \quad \text{or} \quad x=\pm1$

There are three critical values -1, 0, and 1.

c) The interval $(-\infty,\infty)$ is not closed, so the Max-Min Principle 1 does not apply. Since there is more than one critical value, the Max-Min Principle 2 does not apply.
A quick sketch of the graph will help us determine whether absolute or relative extrema occur at the critical values.



We determine that the function has no absolute maximum over the interval $(-\infty,\infty)$. The function's absolute minimum is -1 , which occurs at $x=-1$ and $x=1$.

86. $f(x)=2x^4-4x^2+2$

When no interval is specified , we use the real line $(-\infty,\infty)$.

a) Find $f'(x)$

$f'(x)=8x^3-8x$

b) Find the critical values. The derivative exists for all real numbers. Thus, we solve $f'(x)=0$.

$8x^3-8x=0$

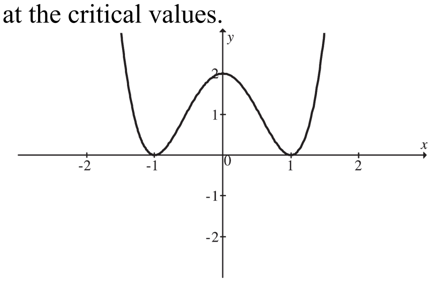
$8x(x^2-1)=0$

$x=0 \quad \text{or} \quad x^2-1=0$

$x=0 \quad \text{or} \quad x=\pm1$

There are three critical values -1, 0, and 1.

c) The interval $(-\infty,\infty)$ is not closed, so the Max-Min Principle 1 does not apply. Since there is more than one critical value, the Max-Min Principle 2 does not apply.
A quick sketch of the graph will help us determine absolute or relative extrema occur at the critical values.



We determine that the function has no absolute maximum over the interval $(-\infty,\infty)$. The function's absolute minimum is 0, which occurs at $x=-1$ and $x=1$.

87. – 96. Left to the student.

97. $M(t)=-2t^2+100t+180, \quad 0\le t\le 40$

a) $M'(t)=-4t+100$

b) $M'(t)$ exists for all real numbers. We solve $M'(t)=0.$

$-4t+100=0$

$4t=100$

$t=25$

c) Since there is only one critical value, we apply the Max-Min Principle 2. First, we find the second derivative.
$M''(t)=-4$. The second derivative is negative for all values of t in the interval, therefore, a maximum occurs at $t=25$.

$M(25)=-2(25)^2+100(25)+180=1430$

The maximum productivity for $0\le t\le 40$ is 1430 units per month, which occurs after $t=25$ years of employment.

98. $N(a) = -a^2 + 300a + 6, \quad 0 \le a \le 300$

a) $N'(a) = -2a + 300$

b) $N'(a)$ exists for all real numbers. Solve:

$$N'(a) = 0$$
$$-2a + 300 = 0$$
$$a = 150$$

c) Since there is only one critical value, we apply the Max-Min Principle 2. First, we find the second derivative.

$N''(a) = -2$. The second derivative is negative for all values of t in the interval, therefore, a maximum occurs at $a = 150$.

$N(150) = -(150)^2 + 300(150) + 6 = 22{,}506$

The maximum number of units that can be sold is 22,506. In order to achieve this maximum, $150,000 must be spent on advertising.

99. $p(x) = -0.039x^3 + 0.594x^2 - 1.967x + 7.555$

We restrict our attention to the years 2003 to 2013. That is, we will look at the x-values $0 \le x \le 10$.

a) Find $p'(x)$.

$p'(x) = -0.117x^2 + 1.188x - 1.967$.

b) Find the critical values. $p'(x)$ exists for all real numbers. Therefore, we solve $p'(x) = 0$.

$-0.117x^2 + 1.188x - 1.967 = 0$

Using the quadratic formula, we have:

$$x = \frac{-(1.188) \pm \sqrt{(1.188)^2 - 4(-0.117)(-1.967)}}{2(-0.117)}$$
$$= \frac{-1.188 \pm \sqrt{0.490788}}{-0.234}$$

$x \approx 2.08$ or $x \approx 8.07$

The critical values are $x \approx 2.08$ and $x \approx 8.07$ is in the interval $[0,10]$.

c) The critical values and the endpoints are 0, 2.08, 8.07, and 10.

d) Using a calculator, we find the function values.

$$p(0) \approx 7.555$$
$$p(2.08) \approx 5.683$$
$$p(8.07) \approx 9.869$$
$$p(10) \approx 8.285$$

The absolute maximum occurs when $x = 8.07$.

According to this model, the maximum percentage of unemployed workers in service occupations occurred in 2011.

100. $f(x) = -0.0135x^2 + 0.265x + 74.6$

We restrict our attention to the years 1992 to 2012. That is, we will look at the x-values $0 \le x \le 20$.

a) $f'(x) = -0.027x + 0.265$

b) $f'(x)$ exists for all real numbers. Solve:

$$f'(x) = 0$$
$$-0.027x + 0.265 = 0$$
$$x = 9.81$$

The critical value 9.81 is in the interval.

c) The critical value and the endpoints are 0, 9.81, and 20.

d) Find the function values.

$$f(0) = -0.0135(0)^2 + 0.265(0) + 74.6$$
$$= 74.6$$
$$f(9.81) = -0.0135(9.81)^2 + 0.265(9.81) + 74.6$$
$$= 75.9$$
$$f(20) = -0.0135(20)^2 + 0.265(20) + 74.6$$
$$= 74.5$$

The maximum occurs when $x = 9.81$. According to this model for the period 1992 to 2012, the percentage of women aged 21-54 in the U.S. Civilian labor forced was at maximum in 2001.

101. We use the model

$$P(t) = 2.69t^4 - 63.941t^3 + 459.895t^2 - 688.692t + 24{,}150.217$$

We consider the interval $[0, \infty)$, where $t = 0$ corresponds to the year 2000.

a) Find $P'(t)$.

$P'(t) = 10.76t^3 - 191.823t^2 + 919.79t - 688.692$

b) $P'(t)$ exists for all real numbers. Solve:

$$P'(t) = 0$$
$$10.76t^3 - 191.823t^2 + 919.79t - 688.692 = 0$$

Using a graphing calculator, we approximate the zeros of $P'(t)$. We find the solutions:

$$t \approx 0.914$$
$$t \approx 7.232$$
$$t \approx 9.681$$

c) The critical values and the endpoints are: $0,\ 0.914,\ 7.232$, and 9.681.

d) Find the function values.

$$P(0)=24{,}150$$
$$P(0.914)\approx 23{,}858$$
$$P(7.232)\approx 26{,}396$$
$$P(9.681)\approx 39{,}533$$

The absolute minimum production of world wide oil was 23,858,000 barrels. The world achieved this production 0.914 years after 2000, or approximately the year 2001.

102. $P(x)=\dfrac{1500}{x^2-6x+10}$

We will restrict our analysis to the nonnegative real numbers $[0,\infty)$, since you cannot produce and sell a negative number of amplifiers.

a) $P'(x)=-\dfrac{3000(x-3)}{\left(x^2-6x+10\right)^2}$

b) $P'(x)$ exists for all real numbers. Solve

$$P'(x)=0$$
$$-\frac{3000(x-3)}{\left(x^2-6x+10\right)^2}=0$$
$$x=3$$

c) Since the interval is open, we apply the Max-Min Principle 2.

$$P''(x)=\frac{3000\left(3x^2-18x+26\right)}{\left(x^2-6x+10\right)^3}$$
$$P''(3)=-3000$$

Therefore, a maximum occurs at $x=3$. We find the function value at $x=3$.

$$P(3)=\frac{1500}{(3)^2-6(3)+10}=1500$$

Producing and selling 3 amplifiers will result in a maximum weekly profit of \$1500.

103. $C(x)=5000+600x$

$R(x)=-\frac{1}{2}x^2+1000x,\ 0\le x\le 600$

a) $P(x)=R(x)-C(x)$

$$=-\frac{1}{2}x^2+1000x-(5000+600x)$$
$$=-\frac{1}{2}x^2+400x-5000$$

b) First, we find the critical values.

$P'(x)=-x+400$

$P'(x)$ exists for all real numbers. Solve:

$$P'(x)=0$$
$$-x+400=0$$
$$x=400$$

The critical value is 400 and the endpoints are 0 and 600. Using the Max-Min Principle 1. We evaluate the function at the endpoints and critical values:

$$P(0)=-\frac{1}{2}(0)^2+400(0)-5000$$
$$=-5000$$
$$P(400)=-\frac{1}{2}(400)^2+400(400)-5000$$
$$=75{,}000$$
$$P(600)=-\frac{1}{2}(600)^2+400(600)-5000$$
$$=55{,}000$$

The total profit is maximized when 400 items are produced.

104. From Exercise 103, we know that

$$P(x)=-\frac{1}{2}x^2+400x-5000$$

a) $A(x)=\dfrac{P(x)}{x}=-\dfrac{1}{2}x+400-\dfrac{5000}{x}$, $0<x\le 600$.

b) $A'(x)=-\dfrac{1}{2}+\dfrac{5000}{x^2}$

$A'(x)$ exists everywhere on $0<x\le 600$.

Solve $A'(x)=0$.

$$-\frac{1}{2}+\frac{5000}{x^2}=0$$
$$x^2=10{,}000$$
$$x=\pm 100$$

$x=100$ is the only critical value in the interval $0<x\le 600$.

The critical value and the endpoint are 100 and 600.

$A(100)=300$

$A(600)=91.67$

The average profit is maximized when 100 items are produced.

105. $B(x)=305x^2-1830x^3,\quad 0\le x\le 0.16$

a) $B'(x)=610x-5490x^2$

b) $B'(x)$ exists for all real numbers. Solve:

$$B'(x) = 0$$
$$610x - 5490x^2 = 0$$
$$610x(1-9x) = 0$$
$$x = 0 \quad \text{or} \quad 1-9x = 0$$
$$x = 0 \quad \text{or} \quad x = \frac{1}{9} \approx 0.11$$

c) The critical points and the endpoints are $0, \frac{1}{9}$, and 0.16.

d) We find the function values.

$$B(0) = 305(0)^2 - 1830(0)^3 = 0$$
$$B\left(\frac{1}{9}\right) = 305\left(\frac{1}{9}\right)^2 - 1830\left(\frac{1}{9}\right)^3$$
$$= \frac{305}{243} \approx 1.255$$
$$B(0.16) = 305(0.16)^2 - 1830(0.16)^3$$
$$\approx 0.312$$

The maximum blood pressure is approximately 1.255, which occurs at a dose of $x = \frac{1}{9}$ cc, or about 0.11 cc of the drug.

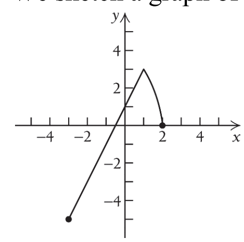
106. In finding the absolute extrema of a function, the second derivative is used when there is exactly one critical value interior to an interval. If there is exactly one critical value, the second derivative will determine the concavity of the function on the interval, and thus determine if there is an absolute maximum or absolute minimum on the interval.

107. We look at the derivative on each piece of the function to determine any critical values. For $-3 < x < 1$, $f'(x) = 2$ so there are no critical values for this part of the function. For $1 < x \le 2$, $f'(x) = -2x$. $f'(x) = 0$ when $x = 0$, which is outside the domain of this piece of the function. Therefore, there are no critical values of $f(x)$. The absolute extrema will occur at one of the endpoints. The function values are:

$$f(-3) = 2(-3) + 1 = -5$$
$$f(1) = 2(1) + 1 = 3$$
$$f(2) = 4 - (2)^2 = 0$$

On the interval $[-3, 2]$, the absolute minimum is -5, which occurs at $x = -3$. The absolute maximum is 3, which occurs at $x = 1$.

We sketch a graph of the function.

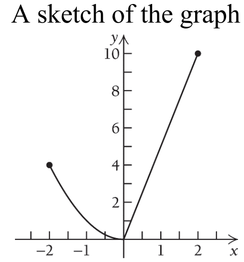


108. We look at the derivative on each piece of the function to determine any critical values. For $-2 < x < 0$, $g'(x) = 2x$. $g'(x) = 0$ when $x = 0$, which is also the endpoint of this part of the domain. For $0 < x < 2$, $g'(x) = 5$. Therefore, there are no critical values of $g(x)$. The absolute extrema will occur at one of the endpoints. The function values are:

$$g(-2) = (-2)^2 = 4$$
$$g(0) = (0)^2 = 0$$
$$g(2) = 5(2) = 10$$

On the interval $[-2, 2]$, the absolute minimum is 0, which occurs at $x = 0$. The absolute maximum is 10, which occurs at $x = 2$

A sketch of the graph is shown below.



109. We look at the derivative on each piece of the function to determine any critical values. For $-4 < x < 0$, $h'(x) = -2x$. $h'(x) = 0$ when $x = 0$, which is also the endpoint of this part of the domain. For $0 < x < 1$, $h'(x) = -1$. Therefore, there are no critical values of $h(x)$ on this part of the domain. For $1 \le x \le 2$, $h'(x) = 1$. Therefore, there are no critical values of $h(x)$ on this part of the domain. The absolute extrema will occur at one of the endpoints. The function values are determined on the next page.

We determine the function values from the previous page:

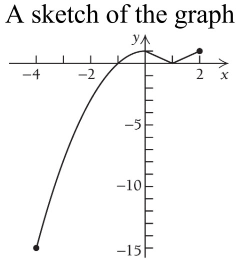
$h(-4) = 1 - (-4)^2 = -15$

$h(0) = 1 - (0) = 1$

$h(1) = (1) - 1 = 0$

$h(2) = (2) - 1 = 1$

On the interval $[-4, 2]$, the absolute minimum is -15, which occurs at $x = -4$. The absolute maximum is 1, which occurs at $x = 0$ and $x = 2$. A sketch of the graph is shown below.



110. We look at the derivative on each piece of the function to determine any critical values. For $-2 < x < 0$, $F'(x) = 2x$. $F'(x) = 0$ when $x = 0$, which is also the endpoint of this part of the domain. For $0 < x < 3$, $F'(x) = -1$. Therefore, there are no critical values of $F(x)$ on this part of the domain. For $3 \le x \le 67$, $F'(x) = \frac{1}{2\sqrt{x-2}}$. Therefore, there are no critical values of $F(x)$ on this part of the domain. The absolute extrema will occur at one of the endpoints. The function values are:

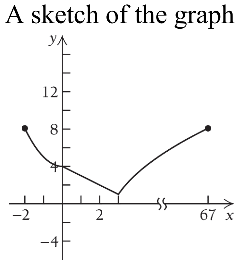
$F(-2) = (-2)^2 + 4 = 8$

$F(0) = 4 - (0) = 4$

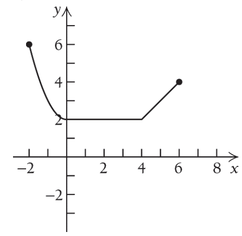
$F(3) = \sqrt{3-2} = 1$

$F(67) = \sqrt{67-2} = \sqrt{65} \approx 8.062$

On the interval $[-2, 67]$, the absolute minimum is 1, which occurs at $x = 3$. The absolute maximum is approximately 8.062, which occurs at $x = 67$.
A sketch of the graph is shown below.



111. a) The sketch of the graph is shown below:



b) From the graph, the absolute maximum is 6 and occurs at $x = -2$.

c) The absolute minimum value for this function is 2. This value occurs over the range $0 \le x \le 4$.

112. a) According to the graph, the maximum elevation is 2290 feet, this elevation occurs at 6.5 miles from the western edge of Rogers Dry Lake.

b) According to the graph, the minimium elevation occurs at 2269 feet. This elevation occurs over the range of 0.5 miles to 5.5 miles from the western edge of Rogers Dry Lake.

113. $g(x) = x\sqrt{x+3}; \qquad [-3, 3]$

a) Find $g'(x)$.

$$g'(x) = x\left[\frac{1}{2}(x+3)^{-1/2}(1)\right] + (1)(x+3)^{1/2}$$

$$= \frac{x}{2(x+3)^{1/2}} + (x+3)^{1/2}$$

$$= \frac{x}{2(x+3)^{1/2}} + \frac{(x+3)^{1/2}}{1} \cdot \frac{2(x+3)^{1/2}}{2(x+3)^{1/2}}$$

Multiplying by a form of 1

$$= \frac{x}{2(x+3)^{1/2}} + \frac{2(x+3)}{2(x+3)^{1/2}}$$

$$= \frac{3x+6}{2(x+3)^{1/2}}, \text{ or } \frac{3x+6}{2\sqrt{x+3}}$$

b) Find the critical values. $g'(x)$ exists for all values in $[-3,3]$ except -3. This is a critical value as well as an endpoint. To find the other critical values, we solve

$$g'(x)=0$$
$$\frac{3x+6}{2\sqrt{x+3}}=0$$
$$3x+6=0$$
$$3x=-6$$
$$x=-2$$

The second critical value on the interval is -2.

c) On a closed interval, the Max-Min Principle 1 can always be used. The critical values and the endpoints are $-3,\ -2$, and 3.

d) Find the function value at each value in step (c)..

$$g(-3)=(-3)\sqrt{(-3)+3}=0$$
$$g(-2)=(-2)\sqrt{(-2)+3}=-2$$
$$g(3)=(3)\sqrt{(3)+3}=3\sqrt{6}$$

Thus, the absolute maximum over the interval $[-3,3]$ is $3\sqrt{6}$, which occurs at $x=3$, and the absolute minimum is -2, which occurs at $x=-2$.

114. $h(x)=x\sqrt{1-x}; \qquad [0,1]$

a) Find $h'(x)$.

$$h'(x)=x\left(\frac{1}{2}\right)(1-x)^{-1/2}(-1)+(1)(1-x)^{1/2}$$
$$=\frac{-x}{2\sqrt{1-x}}+\sqrt{1-x}$$
$$=\frac{-x}{2\sqrt{1-x}}+\frac{2(1-x)}{2\sqrt{1-x}}$$
$$=\frac{-3x+2}{2\sqrt{1-x}}$$

b) $h'(x)$ does not exist for $x=1$. Solve:

$$h'(x)=0$$
$$\frac{-3x+2}{2\sqrt{1-x}}=0$$
$$-3x+2=0$$
$$x=\frac{2}{3}$$

c) The critical values and the endpoints are $0,\ \frac{2}{3}$, and 1.

d) Find the function values.

$$h(0)=(0)\sqrt{1-(0)}=0$$
$$h\left(\frac{2}{3}\right)=\left(\frac{2}{3}\right)\sqrt{1-\left(\frac{2}{3}\right)}=\frac{2}{3}\sqrt{\frac{1}{3}}=\frac{2}{3\sqrt{3}}$$
$$h(1)=(1)\sqrt{1-(1)}=0$$

Thus, the absolute maximum over the interval $[0,1]$ is $\frac{2}{3\sqrt{3}}$, which occurs at $x=\frac{2}{3}$, and the absolute minimum over the interval is 0, which occurs at $x=0$ and $x=1$.

115. $C(x)=(2x+4)+\left(\frac{2}{x-6}\right), \qquad x>6$

$$=2x+4+2(x-6)^{-1}$$

a) Find $C'(x)$.

$$C'(x)=2-2(x-6)^{-2}(1)=2-\frac{2}{(x-6)^2}$$

b) Find the critical values. $C'(x)$ does not exist for $x=6$; however, this value is not in the domain interval, so it is not a critical value. Solve $C'(x)=0$.

$$2-\frac{2}{(x-6)^2}=0$$
$$2=\frac{2}{(x-6)^2}$$
$$2(x-6)^2=2 \qquad \text{Multiplying by } (x-6)^2 \text{ Since } x\neq 6.$$
$$(x-6)^2=1$$
$$x-6=\pm 1 \qquad \text{Taking the square root of both sides.}$$
$$x=6\pm 1$$
$$x=5 \quad \text{or} \quad x=7$$

The only critical value in $(6,\infty)$ is 7.

c) Since there is only one critical value, we apply the Max-Min Principle 2.

$$C''(x)=4(x-6)^{-3}=\frac{4}{(x-6)^3}$$
$$C''(7)=\frac{4}{(7-6)^3}=4>0$$

Therefore, since $C''(7)>0$, there is a minimum at $x=7$.

The Katie’s Clocks should use 7 “quality units” to minimize its total cost of service.

116. $y=(x-a)^2+(x-b)^2$

a) $\frac{dy}{dx}=2(x-a)+2(x-b)=4x-2a-2b$

b) The derivative exists for all real numbers. Solve:

$$\frac{dy}{dx}=0$$
$$4x-2a-2b=0$$
$$4x=2a+2b$$
$$x=\frac{a+b}{2}$$

c) There is only one critical value, we apply the Max-Min Principle 2.

$\frac{d^2y}{dx^2}=4>0$ for all values of x.

Thus, y is a minimum for $x=\frac{a+b}{2}$.

117. From exercise 101, we know that the first derivative is

$$R(t)=P'(t)=10.76t^3-191.823t^2+919.79t-688.692$$

a) We find the maximum rate of change, by finding the critical values of the derivative. $(0,8)$. Taking the derivative we have:

$P''(t)=32.28t^2-383.646t+919.79$

b) $P''(t)$ exists for all real numbers. Solve:

$$P''(t)=0$$
$$32.28t^2-383.646t+919.79=0$$

Using the quadratic formula, we find the zeros of $P''(t)$.

$$t=\frac{-(-383.646)\pm\sqrt{(-383.646)^2-4(32.28)(919.79)}}{2(32.28)}$$

The two solutions are $t\approx 3.33$ and $t\approx 8.554$. Only one of the critical values, $t=3.33$, is in the interval.

c) Since there is only one critical value, we apply the Max-Min Principle 2.

$P'''(t)=64.56t-383.646$

$P'''(3.33)=64.56(3.33)-383.646=-168.6<0$

Therefore, the absolute maximum over the interval $(0,8)$ occurs at $t\approx 3.33$.

$P'(3.33)=644.427\approx 640.$

In the year 2003 – 2004, worldwide oil production was increasing most rapidly. It was increasing at a rate of approximately 640,000 barrels per year.

118. The first derivative is used to find the critical values of the function on an interval. For closed intervals, we know that absolute extrema occur at a critical value in the interval, or at an endpoint of the interval. For open intervals, absolute extrema, if they exists, will occur at a critical value in the interval.

119. $P(t)=0.0000000219t^4-0.0000167t^3+0.00155t^2+0.002t+0.22,\quad 0\le t\le 110$

a) Find $P'(t)$.

$$P'(t)=0.0000000876t^3-0.0000501t^2+0.0031t+0.002$$

$P'(t)$ exists for all real numbers. Solve $P'(t)=0$. We use a calculator to find the zeros of $P'(t)$. We estimate the solutions to be:

$x\approx -0.639$

$x\approx 71.333$

$x\approx 501.223$

Only one of the critical values, $x\approx 71.333$, is in the interval $[0,110]$. We apply the Max-Min Principle 1, to find the absolute maximum.

The critical values and the endpoints are 0, 71.333, and 110.

The function values at these points are

$$P(0)=0.22$$
$$P(71.333)\approx 2.755$$
$$P(110)\approx 0.174$$

Thus, the absolute maximum oil production for the U.S. after 1910 was 2.755 billion barrels per year. This production level occurred 71.333 year after 1910, or in 1981.

b) In 2010, $t=2010-1910=100$. We plug this value into the first derivative to obtain:

$$P'(100)$$
$$=0.0000000876(100)^3-0.0000501(100)^2+0.0031(100)+0.002$$
$$\approx -0.1014.$$

The rate of oil was declining at approximately 0.1014 billion of barrels per year.

The solution is continued on the next page.

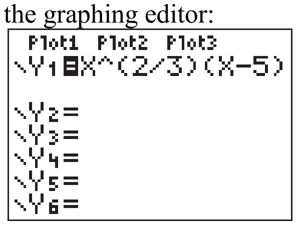
In 2015, $t = 2015 - 1910 = 105$ .We plug this value into the first derivative to obtain:

$$P'(105)$$
$$= 0.0000000876(105)^3 - 0.0000501(105)^2 + 0.0031(105) + 0.002$$
$$\approx -0.1234.$$

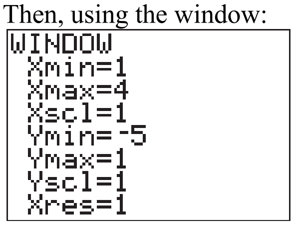
The rate of oil was declining at approximately 0.1234 billion of barrels per year.

120. $f(x) = x^{2/3}(x-5); \quad [1,4]$

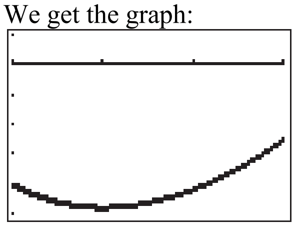
Using a calculator, we enter the equation into the graphing editor:



Then, using the window:



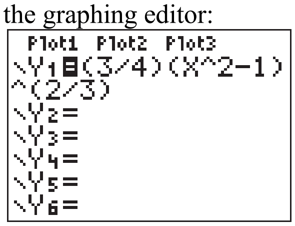
We get the graph:



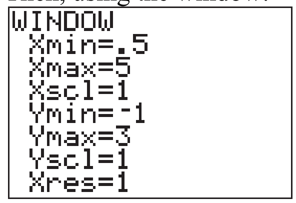
Using the table feature, we locate the extrema. We estimate the absolute maximum to be -2.520 , which occurs at $x = 4$, and the absolute minimum to be -4.762 , which occurs at $x = 2$.

121. $f(x) = \frac{3}{4}(x^2-1)^{2/3}; \quad \left[\frac{1}{2}, \infty\right)$

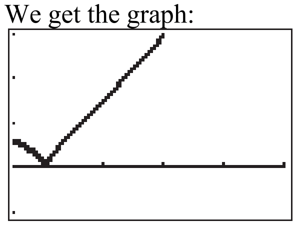
Using a calculator, we enter the equation into the graphing editor:



Then, using the window:



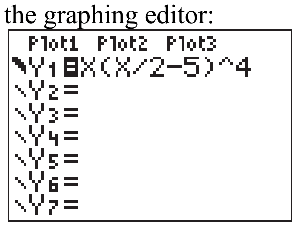
We get the graph:



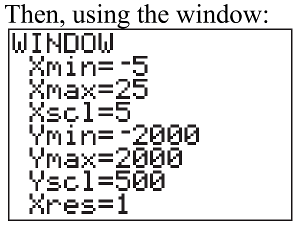
Using the table feature, we locate the extrema. We estimate the absolute minimum to be 0, which occurs at $x = 1$. There is no absolute maximum.

122. $f(x) = x\left(\frac{x}{2} - 5\right)^4; \quad \mathbb{R}$

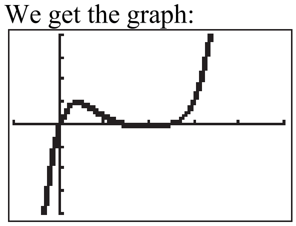
Using a calculator, we enter the equation into the graphing editor:



Then, using the window:



We get the graph:



Looking at the graph, it is clear to see that there are no absolute extrema over the real numbers.

123. a) Using a graphing calculator, we fit the linear equation $y = x + 8.857$. This corresponds to the model $P(t) = t + 8.857$. Where P is the pressure of the contractions and t is the time in minutes. We substitute 7 for t to find the pressure at 7 minutes.

$P(7) = 7 + 8.857 = 15.857$.

The pressure at 7 minutes is 15.857 mm of Hg.

b) Rounding the coefficients to 3 decimal places, we find the quartic regression

$y = 0.117x^4 - 1.520x^3 + 6.193x^2 - 7.018x + 10.009$

Changing the variables we get the model

$P(t) = 0.117t^4 - 1.520t^3 + 6.193t^2 - 7.018t + 10.009$

Using the table feature, when $x = 7$, $y = 24.857$. So the pressure at 7 minutes is 24.86 mm of mercury. (If we use the rounded coefficients above, we get 23.897 mm of Hg.)

Using the trace feature, we estimate the smallest contraction on the interval $[0,10]$ was about 7.62 mm of Hg. This occurred when $x \approx 0.765$ min.

Exercise Set 2.5

1. Express $Q = xy$ as a function of one variable.
First, we solve $x + y = 70$ for y.
$x + y = 70$
$y = 70 - x$
Next, we substitute $70 - x$ for y in $Q = xy$.
$Q = xy$
$Q = x(70 - x)$ Substituting
$= 70x - x^2$
Now that Q is a function of one variable we can find the maximum. First, we find the critical values.
$Q'(x) = 70 - 2x$. Since $Q'(x)$ exists for all real numbers, the only critical value will occur when $Q'(x) = 0$. We solve:
$70 - 2x = 0$
$70 = 2x$
$35 = x$
There is only one critical value. We use the second derivative to determine if the critical value is a maximum. Note that:
$Q''(x) = -2 < 0$. The second derivative is negative for all values of x. Therefore, a maximum occurs at $x = 35$.
Now, $Q(35) = 70(35) - (35)^2 = 1225$
Therefore, the maximum product is 1225, which occurs when $x = 35$. If $x = 35$, then $y = 70 - 35 = 35$. The two numbers are 35 and 35.

2. $x + y = 50$, so $y = 50 - x$.
$Q(x) = xy = x(50 - x) = 50x - x^2$
$Q'(x) = 50 - 2x$
$Q'(x)$ exists for all real numbers. Solve:
$Q'(x) = 0$
$50 - 2x = 0$
$x = 25$
$Q''(x) = -2 < 0$ for all values of x, so a maximum occurs at $x = 25$.
$Q(25) = 50(25) - (25)^2 = 625$
Thus, the maximum product is 625 when $x = 25$ and $y = 50 - 25 = 25$.

3. Let x be one number and y be the other number. Since the difference of the two numbers must be 6, we have $x - y = 6$.
The product, Q, of the two numbers is given by $Q = xy$, so our task is to minimize $Q = xy$, where $x - y = 6$.
First, we express $Q = xy$ as a function of one variable.
Solving $x - y = 6$ for y, we have:
$x - y = 6$
$-y = 6 - x$
$y = x - 6$
Next, we substitute $x - 6$ for y in $Q = xy$.
$Q(x) = x(x - 6) = x^2 - 6x$.
Finding the deriviative, we have:
$Q'(x) = 2x - 6$
The derivative exists for all values of x; thus, the only critical values are where $Q'(x) = 0$.
$2x - 6 = 0$
$2x = 6$
$x = 3$
There is only one critical value. We can use the second derivative to determine whether we have a minimum.
$Q''(x) = 2 > 0$ for all values of x. Therefore, a minimum occurs at $x = 3$.
$Q(3) = (3)^2 - 6(3) = -9$
Thus, the minimum product is -9 when $x = 3$, and $y = 3 - 6 = -3$.

4. Let x be one number and y be the other number. The product, Q, of the two numbers is given by $Q = xy$, so our task is to minimize $Q = xy$, where $x - y = 4$.
First, we express $Q = xy$ as a function of one variable.
Solving $x - y = 4$ for y, we have: $y = x - 4$
Next, we substitute $x - 4$ for y in $Q = xy$.
$Q = x(x - 4) = x^2 - 4x$
$Q'(x) = 2x - 4$
The derivative exists for all values of x; thus, the only critical values are where $Q'(x) = 0$.
$2x - 4 = 0$
$2x = 4$
$x = 2$
The solution is continued on the next page.

There is only one critical value. We can use the second derivative to determine whether we have a minimum.

$Q''(x) = 2 > 0$ for all values of x. Therefore, a minimum occurs at $x = 2$.

$Q(2) = (2)^2 - 4(2) = -4$

Thus, the minimum product is -4 when $x = 2$. Substitute 2 for x in $y = x - 4$ to find y.

$y = 2 - 4 = -2$.

The two numbers which have the minimum product are 2 and -2.

5. Maximize $Q = xy^2$, where x and y are positive numbers such that $x + y^2 = 4$.

Express $Q = xy^2$ as a function of one variable

First, we solve $x + y^2 = 4$ for y^2.

$$x + y^2 = 4$$
$$y^2 = 4 - x$$

Next, we substitute $4 - x$ for y^2 in $Q = xy^2$.

$$Q = xy^2$$
$$Q = x(4 - x)$$
$$= 4x - x^2$$

Now that Q is a function of one variable we can find the maximum. First, we find the critical values.

$Q'(x) = 4 - 2x$.

Since $Q'(x)$ exists for all real numbers, the only critical value will occur when $Q'(x) = 0$. We set $Q'(x) = 0$ and solve for x:

$$4 - 2x = 0$$
$$-2x = -4$$
$$x = 2.$$

There is only one critical value. We use the second derivative to determine if the critical value is a maximum. Note that:

$Q''(x) = -2 < 0$. The second derivative is negative for all values of x. Therefore, a maximum occurs at $x = 2$.

Now,

$Q(2) = (4)(2) - (2)^2 = 4$

Substitute 2 in for x in $x + y^2 = 4$ and solve for y.

$$2 + y^2 = 4$$
$$y^2 = 4 - 2$$
$$y^2 = 2$$
$$y = \pm\sqrt{2}$$
$$y = \sqrt{2} \qquad x \text{ and } y \text{ are positive}$$

Then Q is a maximum when $x = 2$ and $y = \sqrt{2}$.

6. Maximize $Q = xy^2$, where x and y are positive numbers such that $x + y^2 = 1$.

Express $Q = xy^2$ as a function of one variable.

$$x + y^2 = 1$$
$$y^2 = 1 - x$$
$$Q = xy^2$$
$$Q = x(1 - x) = x - x^2 \qquad \text{Substituting}$$

Now that Q is a function of one variable we can find the maximum. First, we find the critical values.

$Q'(x) = 1 - 2x$. Since $Q'(x)$ exists for all real numbers, the only critical value will occur when $Q'(x) = 0$. We solve:

$$1 - 2x = 0$$
$$\frac{1}{2} = x.$$

There is only one critical value. We use the second derivative to determine if the critical value is a maximum. Note that:

$Q''(x) = -2 < 0$. The second derivative is negative for all values of x. Therefore, a maximum occurs at $x = \frac{1}{2}$.

Now, $Q\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

When $x = \frac{1}{2}$, we have:

$$y^2 = 1 - \frac{1}{2}$$
$$y = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$$
$$y = \frac{1}{\sqrt{2}}, \qquad x \text{ and } y \text{ must be positive}$$

The maximum value of Q is $\frac{1}{4}$ when $x = \frac{1}{2}$ and $y = \frac{1}{\sqrt{2}}$.

7. Minimize $Q = x^2 + 2y^2$, where $x + y = 3$.
Express Q as a function of one variable. First, solve $x + y = 3$ for y.

$x + y = 3$
$y = 3 - x$

Then substitute $3 - x$ for y in $Q = x^2 + 2y^2$.

$Q = x^2 + 2(3-x)^2$
$= x^2 + 2(9 - 6x + x^2)$
$= 3x^2 - 12x + 18$

Find $Q'(x)$, where $Q(x) = 3x^2 - 12x + 18$.

$Q'(x) = 6x - 12$

This derivative exists for all values of x; thus the only critical values are where

$Q'(x) = 0$
$6x - 12 = 0$
$6x = 12$
$x = 2$

Since there is only one critical value, we can use the second derivative to determine whether we have a minimum. Note that:
$Q''(x) = 6$, which is positive for all real numbers. Thus $Q''(2) > 0$, so a minimum occurs when $x = 2$. The value of Q is

$Q(2) = 2^2 + 2(3-2)^2$
$= 4 + 2$
$= 6.$

Substitute 2 for x in $y = 3 - x$ to find y.

$y = 3 - x$
$y = 3 - 2$
$y = 1$

Thus, the minimum value of Q is 6 when $x = 2$ and $y = 1$.

8. $x + y = 5$, so $y = 5 - x$.

$Q = 2x^2 + 3y^2 = 2x^2 + 3(5-x)^2$
$= 5x^2 - 30x + 75$
$Q'(x) = 10x - 30$

$Q'(x)$ exists for all real numbers. Solve:

$Q'(x) = 0$
$10x - 30 = 0$
$x = 3$

There is one critical value, we use the second derivative to determine if it is a minimum.
$Q''(x) = 10 > 0$ for all values of x, therefore, $Q''(3) > 0$ and a minimum occurs when $x = 3$.
When $x = 3$, $Q(3) = 2(3)^2 + 3(5-3)^2 = 30.$
When $x = 3$, $y = 5 - 3 = 2$.
Therefore, the minimum value of Q is 30 when $x = 3$ and $y = 2$.

9. Maximize $Q = xy$, where x and y are positive numbers such that $x + \frac{4}{3}y^2 = 1$.
Express Q as a function of one variable. First, solve $x + \frac{4}{3}y^2 = 1$ for x.

$x + \frac{4}{3}y^2 = 1$
$x = 1 - \frac{4}{3}y^2$

Then substitute $1 - \frac{4}{3}y^2$ for x in $Q = xy$.

$Q = xy = \left(1 - \frac{4}{3}y^2\right)y$
$= y - \frac{4}{3}y^3$

Find $Q'(y)$, where $Q(y) = y - \frac{4}{3}y^3$.

$Q'(y) = 1 - 4y^2$

This derivative exists for all values of y; thus the only critical values are where

$Q'(y) = 0$
$1 - 4y^2 = 0$
$-4y^2 = -1$
$y^2 = \frac{1}{4}$
$y = \pm\sqrt{\frac{1}{4}}$
$y = \pm\frac{1}{2}$
$y = \frac{1}{2}$ $\quad$ y must be positive

Since there is only one critical value, we can use the second derivative to determine whether we have a maximum.

The solution is continued on the next page.

Note that:
$Q''(y) = -8y$
and
$Q''\left(\frac{1}{2}\right) = -8\left(\frac{1}{2}\right) = -4 < 0$.

Since $Q''\left(\frac{1}{2}\right)$ is negative, a maximum occurs at $y = \frac{1}{2}$.

Evaluating the function at $y = \frac{1}{2}$ we have:

$$Q\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{4}{3}\left(\frac{1}{2}\right)^3 = \frac{1}{2} - \frac{4}{3}\cdot\frac{1}{8} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}.$$

Substitute $\frac{1}{2}$ for y in $x = 1 - \frac{4}{3}y^2$ to find x.

$$x = 1 - \frac{4}{3}y^2$$
$$x = 1 - \frac{4}{3}\left(\frac{1}{2}\right)^2$$
$$x = 1 - \frac{1}{3}$$
$$x = \frac{2}{3}$$

Thus, the maximum value of Q is $\frac{1}{3}$ when $x = \frac{2}{3}$ and $y = \frac{1}{2}$.

10. $\frac{4}{3}x^2 + y = 16$, so $y = 16 - \frac{4}{3}x^2$.

$Q = xy = x\left(16 - \frac{4}{3}x^2\right) = 16x - \frac{4}{3}x^3$

$Q'(x) = 16 - 4x^2$.

$Q'(x)$ exists for all real numbers.

Solve:

$$Q'(x) = 0$$
$$16 - 4x^2 = 0$$
$$x = \pm 2$$
$$x = 2 \qquad x \text{ must be positive}$$

Only 2 is in the domain of Q.

$Q''(x) = -8x$

When $x = 2$, $Q''(2) = -8(2) = -16$, so a maximum occurs when $x = 2$.

When $x = 2$,

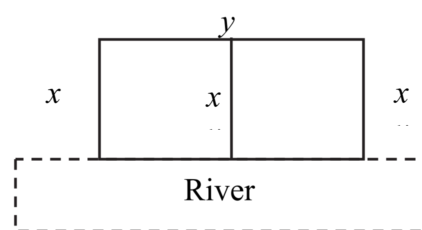
$y = 16 - \frac{4}{3}(2)^2 = 16 - \frac{16}{3} = \frac{32}{3}$.

and

$Q(2) = 2\cdot\frac{32}{3} = \frac{64}{3}$.

Therefore, the maximum value of Q is $\frac{64}{3}$ when $x = 2$ and $y = \frac{32}{3}$.

11. Let x represent the width and y represent the length of the area. It is helpful to draw a picture.



Since the rancher has 240 yards of fence, the perimeter is $3x + y = 240$. Solving this equation for y, we have $y = 240 - 3x$. Since x and y must be positive, we are restricted to the interval $0 < x < 80$.

The objective is to maximize area, which is given by

$A = l \cdot w$

Substituting $y = 240 - 3x$ for the width and x for the length, we have:

$A(x) = x(240 - 3x) = 240x - 3x^2$.

We will maximize the area over the restricted interval by first finding the derivative of the area function.

$A(x) = 240x - 3x^2$

$A'(x) = 240 - 6x$

The derivative exists for all values of x in the interval $(0, 80)$. The only critical values are where:

$$A'(x) = 0$$
$$240 - 6x = 0$$
$$-6x = -240$$
$$x = 40$$

The solution is continued on the next page.

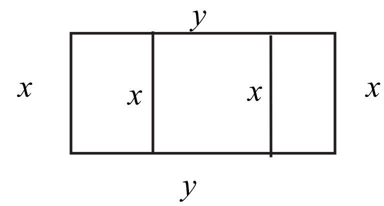
Since there is only one critical value in the interval, we use the second derivative to determine whether we have a maximum. Note, $A''(x) = -6 < 0$ for all values of x, so there is a maximum at $x = 40$.
Next we find the dimensions and the area.
When $x = 40$,
$y = 240 - 3x = 240 - 3(40) = 120$
and
$A(40) = 240(40) - 3(40)^2 = 4800.$
Therefore, the maximum area is 4800 yd^2 when the overall dimensions are 40 yd by 120 yd.

12. Let x represent the length and y represent the width of the swimming area.
Since the life guard has 180 yd of rope and floats, the perimeter of the swimming area is $x + 2y = 180$, or $x = 180 - 2y$
The objective is to maximize area, which is given by
$A = l \cdot w$
Substituting $x = 180 - 2y$ for the length and y for the width, we have:
$A = (180 - 2y)y = 180y - 2y^2$.
We will maximize the area over the interval $0 < y < 90$, because y is the length of one side, and cannot be negative. We find the derivative:
$A'(y) = 180 - 4y$.
This derivative exists for all values of y in $(0, 90)$. Thus the only critical values are where

$$A'(y) = 0$$
$$180 - 4y = 0$$
$$y = 45$$

There is one critical value on the interval.
$A''(y) = -4 < 0$ for all values of y, so a maximum occurs at $y = 45$.
Next, find the dimensions and the area.
When $y = 45$, we have
$x = 180 - 2(45) = 90$
and
$A(45) = 180(45) - 2(45)^2 = 4050.$
Therefore, the maximum area is 4050 yd^2 when the overall dimensions are 45 yd by 90 yd.

13. Let x represent the width and y represent the length of the area. It is helpful to draw a picture.



Since 1200 yards of fence is available, the perimeter is $4x + 2y = 1200$. Solving this equation for y, we have $y = 600 - 2x$. Since x and y must be positive, we are restricted to the interval $0 < x < 300$.
The objective is to maximize area, which is given by
$A = l \cdot w$
Substituting $y = 600 - 2x$ for the width and x for the length, we have:
$A(x) = xy = x(600 - 2x) = 600x - 2x^2$.
We will maximize the area over the restricted interval by first finding the derivative of the area function.
$A(x) = 600x - 2x^2$
$A'(x) = 600 - 4x$
The derivative exists for all values of x in the interval $(0, 300)$. The only critical values are where:

$$A'(x) = 0$$
$$600 - 4x = 0$$
$$-4x = -600$$
$$x = 150$$

Since there is only one critical value in the interval, we use the second derivative to determine whether we have a maximum. Note, $A''(x) = -4 < 0$ for all values of x, so there is a maximum at $x = 150$.
Next we find the dimensions and the area.
When $x = 150$,
$y = 600 - 2x = 600 - 2(150) = 300$
and

$$A(150) = 600(150) - 2(150)^2$$
$$= 45,000.$$

Therefore, the maximum area is $45,000 \text{ yd}^2$ when the overall dimensions are 150 yd by 300 yd.

14. a) Let x represent the width and y represent the length of the area. If the three areas are parallel to the stone wall, only one of the areas will utilize the stone wall as a side. With 600 ft of fencing, the perimeter of the total area will be

$3x + 2y = 600$, or $y = 300 - \frac{3}{2}x$. Since x and y must be positive, we are restricted to the interval $0 < x < 200$. Using this information, we find the area as a function of x.

$$A(x) = xy = x\left(300 - \frac{3}{2}x\right) = 200x - \frac{3}{2}x^2.$$

Therefore,

$$A'(x) = 300 - 3x$$

This derivative exists for all values of x.
Solve:

$$A'(x) = 0$$
$$300 - 3x = 0$$
$$x = 100$$

There is one critical value on the interval.
$A''(x) = -3 < 0$ for all values of x. So a maximum occurs at $x = 100$.
When $x = 100$,

$$y = 300 - \frac{3}{2}(100) = 150$$

and

$$A(100) = 300(100) - \frac{3}{2}(100)^2 = 15,000.$$

Therefore, the maximum area is $15{,}000 \text{ ft}^2$ when the overall dimensions are 100 ft by 150 ft.

b) Let x represent the width and y represent the length of the area. If the three areas are perpendicular to the stone wall, all three areas will utilize the stone wall as a side. With 600 ft of fencing, the perimeter of the total area will be

$4x + y = 600$, or $y = 600 - 4x$. Since x and y must be positive, we are restricted to the interval $0 < x < 150$. Using this information, we find the area as a function of x.

$$A(x) = xy = x(600 - 4x) = 600x - 4x^2.$$

Therefore,

$$A'(x) = 600 - 8x$$

This derivative exists for all values of x.
Solve:

$$A'(x) = 0$$
$$600 - 8x = 0$$
$$x = 75$$

There is one critical value on the interval.
$A''(x) = -8 < 0$ for all values of x. So a maximum occurs at $x = 75$.
When $x = 100$,

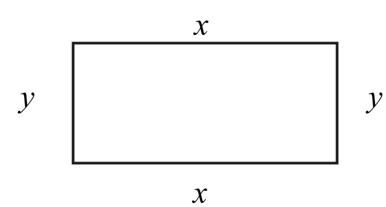
$$y = 600 - 4(75) = 300$$

and

$$A(75) = 600(75) - 4(75)^2 = 22,500.$$

Therefore, the maximum area is $22{,}500 \text{ ft}^2$ when the overall dimensions are 75 ft by 300 ft.

15. Let x represent the length and y represent the width. It is helpful to draw a picture.



The perimeter is found by adding up the length of the sides. Since it is fixed at 42 feet, the equation of the perimeter is $2x + 2y = 42$.

The area is given by $A = xy$.

First, we solve the perimeter equation for y.

$$2x + 2y = 42$$
$$2y = 42 - 2x$$
$$y = 21 - x$$

Then we substitute for y into the area formula.

$$A = xy$$
$$= x(21 - x) = 21x - x^2$$

We want to maximize the area on the interval $(0, 21)$. We consider this interval because x is the length of the rectangle and cannot be negative.
Since the perimeter cannot exceed 42 feet, x cannot be greater than 21, also if x is 21 feet, the width of the rectangle would be 0 feet. We begin by finding $A'(x)$.

$$A'(x) = 21 - 2x.$$

This derivative exists for all values of x in $(0, 21)$. Thus, the only critical values occur where $A'(x) = 0$. We solve the equation on the next page.

Solving $A'(x)=0$, we have:

$$21-2x=0$$
$$-2x=-21$$
$$x=\frac{21}{2}=10.5$$

Since there is only one critical value in the interval, we can use the second derivative to determine whether we have a maximum. Note that

$A''(x)=-2<0$ for all values of x. Thus, $A''(10.5)<0$, so a maximum occurs at $x=10.5$.

Now,

$$A(x)=21x-x^2$$
$$A(10.5)=21(10.5)-(10.5)^2$$
$$=110.25$$

The maximum area is 110.25 ft^2.

Note: when $x=10.5$, $y=21-10.5=10.5$, so the overall dimensions that will achieve the maximum area are 10.5 ft by 10.5 ft.

16. The perimeter is $2l+2w=54$, so $l=27-w$

Since l and w must be positive, we are restricted to the interval $0<w<27$.

$$A(w)=l\cdot w=(27-w)w=27w-w^2$$
$$A'(w)=27-2w$$

This derivative exists for all values of w. Solve:

$$A'(x)=0$$
$$27-2w=0$$
$$w=\frac{27}{2}=13.5$$

There is one critical value on the interval. We find the second derivative.

$A''(w)=-2<0$ for all values of w. So a maximum occurs at $w=13.5$.

When $w=13.5$,

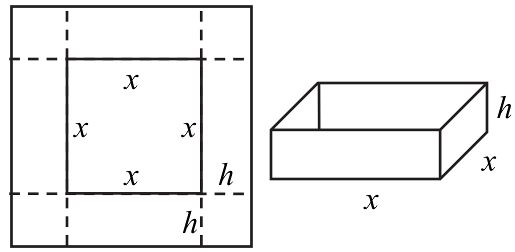
$l=27-13.5=13.5$

and

$A(13.5)=13.5(27-13.5)=182.25.$

Therefore, the maximum area is 182.25 ft^2 when the overall dimensions are 13.5 ft by 13.5 ft.

17. When squares of length h on a side are cut out of the corners, we are left with a square base of length x. A picture will help.



The resulting volume of the box is

$V=lwh=x\cdot x\cdot h=x^2h$.

We want to express V in terms of one variable. Note that the overall length of a side of the cardboard is 20 in. We see from the drawing, that $h+x+h=20$, *or* $x+2h=20$. Solving for h we get:

$$2h=20-x$$
$$h=\frac{1}{2}(20-x)=10-\frac{1}{2}x.$$

Substituting h into the volume equation, we have:

$V=x^2\left(10-\frac{1}{2}x\right)=10x^2-\frac{1}{2}x^3$. The objective is to maximize $V(x)$ on the interval $(0,20)$.

First, we find the derivative.

$$V'(x)=20x-\frac{3}{2}x^2$$

This derivative exists for all x in the interval $(0,20)$. We set the derivative equal to zero and solve for the critical values

$$V'(x)=0$$
$$20x-\frac{3}{2}x^2=0$$
$$x\left(20-\frac{3}{2}x\right)=0$$
$$x=0 \quad\text{or}\quad 20-\frac{3}{2}x=0$$
$$x=0 \quad\text{or}\quad -\frac{3}{2}x=-20$$
$$x=0 \quad\text{or}\quad x=\frac{40}{3}=13\tfrac{1}{3}$$

The solution is continued on the next page.

From the previous page, the only critical value in $(0,20)$ is $\frac{40}{3}$ or about 13.33. Therefore, we can use the second derivative $V''(x)=20-3x$ to determine if we have a maximum. We have

$V''\left(\frac{40}{3}\right)=20-3\left(\frac{40}{3}\right)=-20<0$.

Therefore, there is a maximum at $\frac{40}{3}$.

$$V\left(\frac{40}{3}\right)=10\left(\frac{40}{3}\right)^2-\frac{1}{2}\left(\frac{40}{3}\right)^3$$
$$=\frac{16{,}000}{27}=592\tfrac{16}{27}$$

Now, we find the height of the box.

$h=10-\frac{1}{2}\left(\frac{40}{3}\right)=\frac{10}{3}=3\tfrac{1}{3}$.

Therefore, a box with dimensions $13\frac{1}{3}$ in. by $13\frac{1}{3}$ in. by $3\frac{1}{3}$ in. will yield a maximum volume of $592\frac{16}{27}$ in^3.

18. Using the picture drawn in Exercise 17, the resulting volume of the box is

$V=lwh=x\cdot x\cdot h=x^2h$.

We want to express V in terms of one variable. Note that the overall length of a side of the aluminum is 50 cm. We see from the drawing, that $h+x+h=50$, *or* $x+2h=50$. Solving for h we get:

$h=\frac{1}{2}(50-x)=25-\frac{1}{2}x.$

Substituting h into the volume equation, we have:

$V=x^2\left(25-\frac{1}{2}x\right)=25x^2-\frac{1}{2}x^3$. The objective is to maximize $V(x)$ on the interval $(0,50)$.

First, we find the derivative.

$V'(x)=50x-\frac{3}{2}x^2$

This derivative exists for all x in the interval $(0,50)$, so the critical values will occur when $V'(x)=0$. Solving this equation, we have:

$$50x-\frac{3}{2}x^2=0$$
$$x\left(50-\frac{3}{2}x\right)=0$$
$$x=0 \quad\text{or}\quad x=\frac{100}{3}\approx 33\tfrac{1}{3}.$$

The only critical value in $(0,50)$ is $\frac{100}{3}$, or about 33.33. Therefore, we can use the second derivative $V''(x)=50-3x$ to determine if we have a maximum. We have

$V''\left(\frac{100}{3}\right)=50-3\left(\frac{100}{3}\right)=-50<0$.

Therefore, there is a maximum at $\frac{100}{3}$.

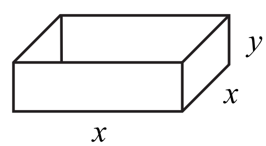
$$V\left(\frac{100}{3}\right)=25\left(\frac{100}{3}\right)^2-\frac{1}{2}\left(\frac{100}{3}\right)^3$$
$$=\frac{250{,}000}{27}=9259\tfrac{7}{27}$$

Now, we find the height of the box.

$h=25-\frac{1}{2}\left(\frac{100}{3}\right)=\frac{25}{3}=8\tfrac{1}{3}$.

Therefore, a box with dimensions $33\frac{1}{3}$ cm by $33\frac{1}{3}$ cm by $8\frac{1}{3}$ cm will yield a maximum volume of $9259\frac{7}{27}$ cm^3.

19. First, we make a drawing.



The surface area of the open-top, square-based, rectangular tank is found by adding the area of the base and the four sides. x^2 is the area of the base, xy is the area of one of the sides and there are four sides, therefore the surface area is given by $S=x^2+4xy$.

The volume must by 32 cubic feet, and is given by $V=l\cdot w\cdot h=x^2y=32$.

To express S in terms of one variable, we solve $x^2y=32$ for y:

$y=\frac{32}{x^2}$.

Substituting, we have:

$$S(x)=x^2+4x\left(\frac{32}{x^2}\right)$$
$$=x^2+\frac{128}{x}=x^2+128x^{-1}$$

Now S is defined only for positive numbers, so we minimize S on the interval $(0,\infty)$.

The solution is continued on the next page.

First, we find $S'(x)$.

$S'(x) = 2x - 128x^{-2}$

$= 2x - \dfrac{128}{x^2}$

Since $S'(x)$ exists for all x in $(0,\infty)$, the only critical values are where $S'(x)=0$. We solve the following equation:

$S'(x) = 0$

$2x - \dfrac{128}{x^2} = 0$

$x^3 = 64$

$x = 4$

Since there is only one critical value, we use the second derivative to determine whether we have a minimum. Note that

$S''(x) = 2 + 256x^{-3} = 2 + \dfrac{256}{x^3}$.

$S''(4) = 2 + \dfrac{256}{4^3} = 6 > 0$. Since the second derivative is positive, we have a minimum at $x = 4$. We find y when $x = 4$.

$y = \dfrac{32}{x^2}$

$= \dfrac{32}{4^2}$

$= 2$

The surface area is minimized when $x = 4$ ft and $y = 2$ ft. We find the minimum surface area by substituting these values into the surface area equation.

$S = x^2 + 4xy$

$= (4)^2 + 4(4)(2)$

$= 16 + 32$

$= 48$

$S(4) = 4^2 + 4\cdot 4\cdot 2 = 48.$

Therefore, when the dimensions are 4 ft by 4 ft by 2 ft, the minimum surface area will be 48 ft^2.

20. Using the drawing in Exercise 19, we see that $S = x^2 + 4xy$ and $V = l\cdot w\cdot h = x^2 y = 62.5$.

Then $y = \dfrac{62.5}{x^2}$, and $S(x) = x^2 + 4x\left(\dfrac{62.5}{x^2}\right)$, or

$S(x) = x^2 + \dfrac{250}{x} = x^2 + 250x^{-1}$.

We restrict the analysis to the interval $(0,\infty)$.

$S'(x) = 2x - 250x^{-2} = 2x - \dfrac{250}{x^2}$

$S'(x)$ exists for all values of x in $(0,\infty)$. Solve:

$S'(x) = 0$

$2x - \dfrac{250}{x^2} = 0$

$x^3 = 125$

$x = 5$

There is one critical value in the interval. We use the second derivative to determine if it is a minimum.

$S''(x) = 2 + 500x^{-3} = 2 + \dfrac{500}{x^3}$

$S''(5) = 2 + \dfrac{500}{5^3} = 6 > 0$, so a minimum occurs when $x = 5$.

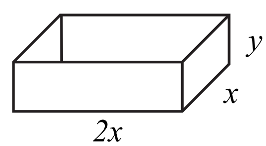
When $x = 5$, $y = \dfrac{62.5}{5^2} = 2.5$.

Therefore,

$S(5) = 5^2 + 4\cdot 5\cdot 2.5 = 75.$

Therefore, when the dimensions are 5 in by 5 in by 2.5 in, the minimum surface area will be 75 in^2.

21. First, we make a drawing.



The surface area of the open-top, rectangular dumpster is found by adding the area of the base and the four sides. $2x^2$ is the area of the base, xy is the area of two of the sides, while $2xy$ is the area of the other two sides. Therefore the surface area is given by

$S = 2x^2 + 2xy + 2(2xy) = 2x^2 + 6xy$.

The volume must by 12 cubic yards, and is given by $V = l\cdot w\cdot h = 2x\cdot x\cdot y = 2x^2 y = 12$.

The solution is continued on the next page.

To express S in terms of one variable, we solve $2x^2y = 12$ for y:

$$y = \frac{6}{x^2}$$

Then

$$S(x) = 2x^2 + 6x\left(\frac{6}{x^2}\right)$$

$$= 2x^2 + \frac{36}{x} = 2x^2 + 36x^{-1}$$

Now S is defined only for positive numbers, so we minimize S on the interval $(0, \infty)$.

First, we find $S'(x)$.

$$S'(x) = 4x - 36x^{-2}$$

$$= 4x - \frac{36}{x^2}$$

Since $S'(x)$ exists for all x in $(0, \infty)$, the only critical values are where $S'(x) = 0$. We solve the following equation:

$$4x - \frac{36}{x^2} = 0$$

$$4x = \frac{36}{x^2}$$

$$x^3 = 9$$

$$x = \sqrt[3]{9} \approx 2.08$$

Since there is only one critical value, we use the second derivative to determine whether we have a minimum.

Note that $S''(x) = 4 + 72x^{-3} = 4 + \frac{72}{x^3}$.

$S''\left(\sqrt[3]{9}\right) = 4 + \frac{72}{\left(\sqrt[3]{9}\right)^3} = 12 > 0$. Since the second derivative is positive, we have a minimum at $x = \sqrt[3]{9} \approx 2.08$. The width is 2.08 yd.; therefore, the length is $2(2.08) \approx 4.16$. We find the height y

$$y = \frac{6}{x^2}$$

$$= \frac{6}{(2.08)^2}$$

$$\approx 1.387$$

The overall dimensions of the dumpster that will minimize surface area are 2.08 yd by 4.16 yd by 1.387 yd.

22. Let x be the width of the container, y be the length of the container and $2x$ be the height of the container.
Since we are including the top and the bottom of the container, the surface area is given by: $S = 4x^2 + 6xy$ and the volume is given by

$$V = y \cdot x \cdot 2x = 2x^2y = 18$$

Then, $y = \frac{9}{x^2}$, and

$$S = 4x^2 + 6x\left(\frac{9}{x^2}\right) = 4x^2 + 54x^{-1}.$$

We are restricted to the interval $(0, \infty)$.

$$S'(x) = 8x - 54x^{-2} = 8x - \frac{54}{x^2}$$

$S'(x)$ exists for all x in the interval. Solve:

$$S'(x) = 0$$

$$8x - \frac{54}{x^2} = 0$$

$$8x^3 = 54$$

$$x^3 = \frac{27}{4}$$

$$x \approx 1.89$$

Since there is only one critical value, we use the second derivative to determine if it is a minimum.

$$S''(x) = 8 + \frac{108}{x^3}$$

$$S''(1.89) \approx 24 > 0$$

Therefore, there is a minimum at $x = 1.89$.
The height is twice that of the width, therefore, the height is $2(1.89) \approx 3.78$. We solve for the length y using:

$$y = \frac{9}{(1.89)^2} \approx 2.52$$

Therefore, the dimensions of the compost container with minimal surface area are 1.89 ft. by 2.52 ft. by 3.78 ft.

23. $R(x) = 50x - 0.5x^2$; $C(x) = 4x + 10$

Profit is equal to revenue minus cost.

$$P(x) = R(x) - C(x)$$

$$= 50x - 0.5x^2 - (4x + 10)$$

$$= -0.5x^2 + 46x - 10$$

The solution is continued on the next page.

Because x is the number of units produced and sold, we are only concerned with the nonnegative values of x. Therefore, we will find the maximum of $P(x)$ on the interval $[0,\infty)$.

First, we find $P'(x)$.

$P'(x) = -x + 46$

The derivative exists for all values of x in $[0,\infty)$.

Thus, we solve $P'(x) = 0$.

$$-x + 46 = 0$$
$$-x = -46$$
$$x = 46$$

There is only one critical value. We can use the second derivative to determine whether we have a maximum.

$P''(x) = -1 < 0$

The second derivative is less than zero for all values of x. Thus, a maximum occurs at $x = 46$.

$$P(46) = -0.5(46)^2 + 46(46) - 10$$
$$= -1058 + 2116 - 10$$
$$= 1048$$

The maximum profit is \$1048 when 46 units are produced and sold.

24. $R(x) = 50x - 0.5x^2;\ C(x) = 10x + 3$

$$P(x) = R(x) - C(x)$$
$$= 50x - 0.5x^2 - (10x + 3)$$
$$= -0.5x^2 + 40x - 3, \qquad 0 \le x < \infty$$

$P'(x) = -x + 40$

The derivative exists for all values of x in $[0,\infty)$.

Thus, we solve $P'(x) = 0$.

$$-x + 40 = 0$$
$$x = 40$$

There is only one critical value.

$P''(x) = -1 < 0$

The second derivative is less than zero for all values of x. Thus, a maximum occurs at $x = 40$.

$$P(40) = -0.5(40)^2 + 40(40) - 3 = 797$$

The maximum profit is \$797 when 40 units are produced and sold.

25. $R(x) = 2x;\ C(x) = 0.01x^2 + 0.6x + 30$

Profit is equal to revenue minus cost.

$$P(x) = R(x) - C(x)$$
$$= 2x - (0.01x^2 + 0.6x + 30)$$
$$= -0.01x^2 + 1.4x - 30$$

Because x is the number of units produced and sold, we are only concerned with the nonnegative values of x. Therefore, we will find the maximum of $P(x)$ on the interval $[0,\infty)$.

First, we find $P'(x)$.

$P'(x) = -0.02x + 1.4$

The derivative exists for all values of x in $[0,\infty)$.

Thus, we solve $P'(x) = 0$.

$$-0.02x + 1.4 = 0$$
$$-0.02x = -1.4$$
$$x = 70$$

There is only one critical value. We can use the second derivative to determine whether we have a maximum.

$P''(x) = -0.02 < 0$

The second derivative is less than zero for all values of x. Thus, a maximum occurs at $x = 70$.

$$P(70) = -0.01(70)^2 + 1.4(70) - 30$$
$$= -49 + 98 - 30$$
$$= 19$$

The maximum profit is \$19 when 70 units are produced and sold.

26. $R(x) = 5x;\ C(x) = 0.001x^2 + 1.2x + 60$

$$P(x) = R(x) - C(x)$$
$$= 5x - (0.001x^2 + 1.2x + 60)$$
$$= -0.001x^2 + 3.8x - 60, \qquad 0 \le x < \infty$$

$P'(x) = -0.002x + 3.8$

The derivative exists for all values of x in $[0,\infty)$.

Thus, we solve $P'(x) = 0$.

$$-0.002x + 3.8 = 0$$
$$x = 1900$$

There is only one critical value.

$P''(x) = -0.002 < 0$

The second derivative is less than zero for all values of x. The maximum occurs at $x = 1900$.

$$P(1900) = -0.001(1900)^2 + 3.8(1900) - 60 = 3550$$

The maximum profit is \$3550 when 1900 units are produced and sold.

27. $R(x) = 9x - 2x^2$

$C(x) = x^3 - 3x^2 + 4x + 1$

$R(x)$ and $C(x)$ are in thousands of dollars and x is in thousands of units.

Profit is equal to revenue minus cost.

$$P(x) = R(x) - C(x)$$
$$= 9x - 2x^2 - \left(x^3 - 3x^2 + 4x + 1\right)$$
$$= -x^3 + x^2 + 5x - 1$$

Because x is the number of units produced and sold, we are only concerned with the nonnegative values of x. Therefore, we will find the maximum of $P(x)$ on the interval $[0, \infty)$.

First, we find $P'(x)$.

$$P'(x) = -3x^2 + 2x + 5$$

The derivative exists for all values of x in $[0, \infty)$.

Thus, we solve $P'(x) = 0$.

$$-3x^2 + 2x + 5 = 0$$
$$3x^2 - 2x - 5 = 0$$
$$(3x - 5)(x + 1) = 0$$
$$3x - 5 = 0 \quad \text{or} \quad x + 1 = 0$$
$$3x = 5 \quad \text{or} \quad x = -1$$
$$x = \frac{5}{3} \quad \text{or} \quad x = -1$$

There is only one critical value in the interval $[0, \infty)$. We can use the second derivative to determine whether we have a maximum.

$$P''(x) = -6x + 2$$

Therefore,

$$P''\left(\frac{5}{3}\right) = -6\left(\frac{5}{3}\right) + 2 = -10 + 2 = -8 < 0$$

The second derivative is less than zero for $x = \frac{5}{3}$. Thus, a maximum occurs at $x = \frac{5}{3}$.

$$P\left(\frac{5}{3}\right) = -\left(\frac{5}{3}\right)^3 + \left(\frac{5}{3}\right)^2 + 5\left(\frac{5}{3}\right) - 1$$
$$= -\frac{125}{27} + \frac{25}{9} + \frac{25}{3} - 1$$
$$= -\frac{125}{27} + \frac{75}{27} + \frac{225}{27} - \frac{27}{27}$$
$$= \frac{148}{27}$$

Note that $x = \frac{5}{3}$ thousand is approximately 1.667 thousand or 1667 units, and that $\frac{148}{27}$ thousand is approximately 5.481 thousand or 5481.

Thus, the maximum profit is approximately \$5481 when approximately 1667 units are produced and sold.

28. $R(x) = 100x - x^2$

$C(x) = \frac{1}{3}x^3 - 6x^2 + 89x + 100$

$R(x)$ and $C(x)$ are in thousands of dollars and x is in thousands of units.

$$P(x) = R(x) - C(x)$$
$$= 100x - x^2 - \left(\frac{1}{3}x^3 - 6x^2 + 89x + 100\right)$$
$$= -\frac{1}{3}x^3 + 5x^2 + 11x - 100, \qquad 0 \le x < \infty$$

$$P'(x) = -x^2 + 10x + 11$$

The derivative exists for all values of x in $[0, \infty)$.

Thus, we solve $P'(x) = 0$.

$$-x^2 + 10x + 11 = 0$$
$$x^2 - 10x - 11 = 0$$
$$(x - 11)(x + 1) = 0$$
$$x - 11 = 0 \quad \text{or} \quad x + 1 = 0$$
$$x = 11 \quad \text{or} \quad x = -1$$

There is only one critical value in the interval $[0, \infty)$. We can use the second derivative to determine whether we have a maximum.

$$P''(x) = -2x + 10$$

Therefore,

$$P''(11) = -2(11) + 10 = -12 < 0$$

The second derivative is less than zero for $x = 11$. Thus, a maximum occurs at $x = 11$.

$$P(11) = -\frac{1}{3}(11)^3 + 5(11)^2 + 11(11) - 100$$
$$\approx 182.333$$

Note that $x = 11$ thousand is 11,000 units, and that 182.333 thousand is 182,333.

Thus, the maximum profit is approximately \$182,333 when 11,000 units are produced and sold.

29. $p = 280 - 0.4x$ Price per unit.

$C(x) = 5000 + 0.6x^2$ Cost per unit.

a) Revenue is price times quantity. Therefore, revenue can be found by multiplying the number of unit sold, x, by the price of the unit, p. Substituting $280 - 0.4x$ for p, we have:

$$R(x) = x \cdot p$$
$$= x(280 - 0.4x)$$
$$R(x) = 280x - 0.4x^2$$

b) Profit is revenue minus cost. Therefore,

$$P(x) = R(x) - C(x)$$
$$= 280x - 0.4x^2 - (5000 + 0.6x^2)$$
$$= -x^2 + 280x - 5000, \qquad 0 \le x < \infty$$

Since x is the number of units produced and sold, we will restrict the domain to the interval $0 \le x < \infty$.

c) To determine the number of suits required to maximize profit, we first find $P'(x)$.

$P'(x) = -2x + 280$.

The derivative exists for all real numbers in the interval $[0, \infty)$. Thus, we solve

$$P'(x) = 0$$
$$-2x + 280 = 0$$
$$-2x = -280$$
$$x = 140$$

Since there is only one critical value, we can use the second derivative to determine whether we have a maximum.

$P''(x) = -2 < 0$

The second derivative is negative for all values of x; therefore, a maximum occurs at $x = 140$.

Riverside Appliances must sell 140 refrigerators to maximize profit.

d) The maximum profit is found by substituting 140 for x in the profit function.

$$P(140) = -(140)^2 + 280(140) - 5000$$
$$= -19,600 + 39,200 - 5000$$
$$= 14,600$$

The maximum profit is \$14,600.

e) The price per refrigerator is given by:

$p = 280 - 0.4x$

Substituting 140 for x, we have:

$p = 280 - 0.4(140) = 224$

The price per refrigerator will be \$224.

30. $p = 150 - 0.5x$, $C(x) = 4000 + 0.25x^2$

a) $R(x) = x \cdot p = x(150 - 0.5x) = 150x - 0.5x^2$

b) $P(x) = R(x) - C(x)$

$$= 150x - 0.5x^2 - (4000 + 0.25x^2)$$

$$P(x) = -0.75x^2 + 150x - 4000, \qquad 0 \le x < \infty$$

c) $P'(x) = -1.5x + 150$

$P'(x)$ exists for all real numbers in the interval $[0, \infty)$. Solve:

$$-1.5x + 150 = 0$$
$$x = 100$$

Since there is only one critical value, we can use the second derivative to determine whether we have a maximum.

$P''(x) = -1.5 < 0$

The second derivative is negative for all values of x; therefore, a maximum occurs at $x = 100$.

Raggs, Ltd. must sell 100 suits to maximize profit.

d) Substitute 100 for x in the profit function.

$$P(100) = -0.75(100)^2 + 150(100) - 4000$$
$$= 3500$$

The maximum profit is \$3500.

e) $p = 150 - 0.5(100) = 150 - 50 = 100$

The price per suit will be \$100.

31. Let x be the amount by which the price of \$80 should be increased. First, we express total revenue R as a function of x. There are two sources of revenue, revenue from tickets and revenue from concessions.

$$R(x) = \begin{pmatrix} \text{Number of} \\ \text{Rooms} \end{pmatrix} \cdot \begin{pmatrix} \text{Price of} \\ \text{Rooms} \end{pmatrix}$$

Note, when the price increases x dollars, the number of rooms occupied falls by x rooms. Thus, the number of rooms is $300 - x$ when price increases x dollars. Therefore, the total revenue function is

$$R(x) = (300 - x)(80 + x)$$
$$= -x^2 + 220x + 24,000$$

The solution is continued on the next page.

The cost of maintining each occupied room is \$22. Therefore the total cost function is:

$$C(x)=\begin{pmatrix}\text{Number of}\\ \text{Rooms}\end{pmatrix}\cdot 22$$
$$=(300-x)22$$
$$=6600-22x.$$

Now we can find the profit function for the hotel.

$$P(x)=R(x)-C(x)$$
$$=-x^2+220x+24{,}000-(6600-22x)$$
$$=-x^2+242x+17{,}400.$$

To find x such that $P(x)$ is a maximum, we first find $P'(x)$:

$$P'(x)=-2x+242$$

This derivative exists for all real numbers x. thus, the only critical values are where $P'(x)=0$; so we solve that equation:

$$P'(x)=0$$
$$-2x+242=0$$
$$-2x=-242$$
$$x=121$$

The second derivative $P''(x)=-2<0$ is negative for all values of x, therefore a maximum occurs at $x=121$.

The charge per unit should be \$80 + \$121, or \$201.

32. Let x be the amount by which the price of \$18 should be decreased (if x is negative, the price would be increased to maximize revenue). First, we express total revenue R as a function of x. There are two sources of revenue, revenue from tickets and revenue from concessions.

$$R(x)=\begin{pmatrix}\text{Revenue from}\\ \text{tickets}\end{pmatrix}+\begin{pmatrix}\text{Revenue from}\\ \text{concessions}\end{pmatrix}$$
$$=\begin{pmatrix}\text{Number of}\\ \text{People}\end{pmatrix}\cdot\begin{pmatrix}\text{Ticket}\\ \text{Price}\end{pmatrix}+\begin{pmatrix}\text{Number of}\\ \text{People}\end{pmatrix}\cdot 4.50$$

Note, the increase in ticket sales is 10,000 x, when price drops $3x$ dollars. Therefore, the increase in ticket sales is $\frac{10{,}000}{3}x$ when price drops x dollars.

$$R(x)=\left(40{,}000+\frac{10{,}000}{3}x\right)(18-x)+\left(40{,}000+\frac{10{,}000}{3}x\right)(4.50)$$
$$=-\frac{10{,}000}{3}x^2+35{,}000x+900{,}000$$

Therefore, the total revenue function is

$$R(x)=-\frac{10{,}000}{3}x^2+35{,}000x+900{,}000$$
$$R'(x)=-\frac{20{,}000}{3}x+35{,}000$$

This derivative exists for all real numbers x. thus, the only critical values are where $R'(x)=0$; so we solve that equation:

$$-\frac{20{,}000}{3}x+35{,}000=0$$
$$-\frac{20{,}000}{3}x=-35{,}000$$
$$x=5.25$$

The second derivative $R''(x)=-\frac{20{,}000}{3}<0$ is negative, for all values of x, therefore a maximum occurs at $x=5.25$. In order to maximize revenue, the university should charge \$18 − \$5.25, or \$12.75.

We can find the attendance using

$$40{,}000+\frac{10{,}000}{3}(5.25)=57{,}500$$

The average attendance when ticket price is \$12.75 is 57,500 people.

33. Let x be the amount the number of new officers Oak Glen should place on patrol. The total number of parking tickets written per day is

$$p(x)=\begin{pmatrix}\text{Number of}\\ \text{officers}\end{pmatrix}\cdot\begin{pmatrix}\text{Avg. Tickets}\\ \text{per day}\end{pmatrix}$$
$$=(8+x)(24-4x)$$
$$=-4x^2-8x+192.$$

To find x such that $p(x)$ is a maximum, we first find $p'(x)$:

$$p'(x)=-8x-8.$$

This derivative exists for all real numbers x. thus, the only critical values are where $P'(x)=0$.

The solution is continued on the next page.

We solve the equation:

$$P'(x) = 0$$
$$-8x - 8 = 0$$
$$-8x = 8$$
$$x = -1$$

The second derivative $P''(x) = -8 < 0$ is negative for all values of x, therefore a maximum occurs at $x = -1$.
This means that Oak Glean should place one fewer officer on patrol in order to maximize the number of parking tickets written in a day.

34. Let x equal the number of additional trees per acre which should be planted. The total yield per acre is equal to the yield per tree times the number of trees so, we have:

$$Y(x) = (30 - x)(20 + x)$$
$$= 600 + 10x - x^2$$

Find $Y'(x)$:

$$Y'(x) = 10 - 2x.$$

This derivative exists for all real numbers x. Thus, the only critical values are where $Y'(x) = 0$; so we solve that equation:

$$10 - 2x = 0$$
$$x = 5$$

This corresponds to planting 5 trees.
Since this is the only critical value, we can use the second derivative,
$R''(x) = -2 < 0$, which is negative for all values of x. A maximum occurs at $x = 5$.
Therefore, in order to maximize yield, the apple farm should plant 20 + 5, or 25 trees per acre.

35. a) First find the slope of the line.

$$m = \frac{1.12 - 1}{0.59 - 1} = -\frac{12}{41}$$
$$y - 1 = -\frac{12}{41}(x - 1)$$
$$y = -\frac{12}{41}x + \frac{53}{41}$$

So the demand function is:

$$q(x) = -\frac{12}{41}x + \frac{53}{41}$$

b) First, we express total revenue R as a function of x. Revenue is price times quantity demanded, therefore,

$$R(x) = x \cdot q(x)$$
$$R(x) = x\left(-\frac{12}{41}x + \frac{53}{41}\right)$$
$$= -\frac{12}{41}x^2 + \frac{53}{41}x$$

To find x such that $R(x)$ is a maximum, we first find $R'(x)$:

$$R'(x) = -\frac{24}{41}x + \frac{53}{41}$$

$R'(x)$ exists for all real numbers. Solve:

$$R'(x) = 0$$
$$-\frac{24}{41}x + \frac{53}{41} = 0$$
$$x = \frac{53}{24}$$
$$x \approx 2.21$$

The second derivative $R''(x) = -\frac{24}{41} < 0$, is negative, for all values of x, therefore a maximum occurs at $x \approx 2.21$.
To maximize revenue, the price of nitrogen should increase 221% from the January 2001 price.

36. a) When $x = 25$, $q = 2.13$. When $x = 25 + 1$, or 26, then $q = 2.13 - 0.04 = 2.09$. We use the points $(25, 2.13)$ and $(26, 2.09)$ to find the linear demand function $q(x)$. First, we find the slope:

$$m = \frac{2.13 - 2.09}{25 - 26} = \frac{0.04}{-1} = -0.04.$$

Next, we use the point-slope equation:

$$q - 2.13 = -0.04(x - 25)$$
$$q - 2.13 = -0.04x + 1$$
$$q = -0.04x + 3.13$$

Therefore, the linear demand function is:

$$q(x) = -0.04x + 3.13.$$

b) Revenue is price times quantity; therefore, the revenue function is:

$$R(x) = x \cdot q(x)$$
$$= x(-0.04x + 3.13)$$
$$= -0.04x^2 + 3.13x$$

The solution is continued on the next page.

Find $R'(x)$:

$R'(x) = -0.08x + 3.13$.

This derivative exists for all values of x. So the only critical values occur when $R'(x) = 0$; so we solve that equation:

$$-0.08x + 3.13 = 0$$
$$-0.08x = -3.13$$
$$x = 39.125$$

Since this is the only critical value, we can use the second derivative,

$R''(x) = -0.08 < 0$,

to determine whether we have a maximum. Since $R''(39.125)$ is negative, $R(39.125)$ is a maximum.

In order to maximize revenue, the State of Maryland should charge \$39.125 or rounding up to \$39.13 per license plate.

37. Let x be the number of \$0.10 increase that should be made. Then,

$$R(x) = (\text{Attendance}) \cdot (\text{Admission Price})$$
$$= (180 - x)(5 + 0.1x)$$
$$= -0.1x^2 + 13x + 900$$

To find x such that $R(x)$ is a maximum, we first find $R'(x)$:

$R'(x) = -0.2x + 13$

$R'(x)$ exists for all real numbers. Solve:

$$R'(x) = 0$$
$$-0.2x + 13 = 0$$
$$x = 65$$

Since there is only one critical value, we can use the second derivative,

$R''(x) = -0.2 < 0$,

to determine whether we have a maximum. Since $R''(65)$ is negative, a maximum occurs at $x = 65$.

When $x = 65$ the admission price that will maximize revenue is given by:

$\$5 + 0.1(65) = 11.50$

Therefore, the theater owner should charge \$11.50 per ticket.

38. The volume of the box is given by

$V = x \cdot x \cdot y = x^2 y = 320.$

The area of the base is x^2. The cost of the base is $15x^2$ cents.

The area of the top is x^2. The cost of the top is $10x^2$ cents.

The area of each side is xy. The cost of the four sides is $2.5(4xy)$ cents.

The total costs in cents is given by

$C = 15x^2 + 10x^2 + 2.5(4xy) = 25x^2 + 10xy.$

To express C in terms of one variable, we solve $x^2 y = 320$ for y:

$$y = \frac{320}{x^2}.$$

Then,

$$C(x) = 25x^2 + 10x\left(\frac{320}{x^2}\right) = 25x^2 + \frac{3200}{x}$$

The function is defined only for positive numbers, so we are minimizing C on the interval $(0, \infty)$.

First, we find $C'(x)$.

$$C'(x) = 50x - 3200x^{-2} = 50x - \frac{3200}{x^2}$$

Since $C'(x)$ exists for all x in $(0, \infty)$, the only critical values are where $C'(x) = 0$. Thus, we solve the following equation:

$$50x - \frac{3200}{x^2} = 0$$
$$50x = \frac{3200}{x^2}$$
$$x^3 = 64$$
$$x = 4$$

This is the only critical value, so we can use the second derivative to determine whether we have a minimum.

$$C''(x) = 50 + 6400x^{-3} = 50 + \frac{6400}{x^3}$$

Note that the second derivative is positive for all positive values of x, therefore we have a minimum at $x = 4$. We find y when $x = 4$.

$$y = \frac{320}{x^2} = \frac{320}{(4)^2} = \frac{320}{16} = 20$$

The cost is minimized when the dimensions are 4 ft by 4 ft by 20 ft.

39. The area of the parking area is given by $A = x \cdot y = 5000$.

Since three sides are chain link fencing, the total cost of the chain link fencing is given by $4.50(2x + y)$ dollars.

One side is wooden fencing, the cost of the wooden fence is $7y$ dollars.

The total costs in dollars is given by

$$C = 4.50(2x+y)+7y$$
$$= 9x+11.5y.$$

To express C in terms of one variable, we solve $xy = 5000$ for y:

$$y = \frac{5000}{x}.$$

Then,

$$C(x) = 9x + 11.5\left(\frac{5000}{x}\right)$$
$$= 9x + \frac{57,500}{x}$$

The function is defined only for positive numbers, so we are minimizing C on the interval $(0, \infty)$.

First, we find $C'(x)$.

$$C'(x) = 9 - 57,500x^{-2} = 9 - \frac{57,500}{x^2}$$

Since $C'(x)$ exists for all x in $(0, \infty)$, the only critical values are where $C'(x) = 0$. Thus, we solve the following equation:

$$9 - \frac{57,500}{x^2} = 0$$
$$9 = \frac{57,500}{x^2}$$
$$9x^2 = 57,500$$
$$x^2 = \frac{57,500}{9}$$
$$x = \sqrt{\frac{57,500}{9}}$$
$$x \approx 79.93$$

This is the only critical value, so we can use the second derivative to determine whether we have a minimum.

$$C''(x) = 172,500x^{-3} = \frac{172,500}{x^3}$$

Note that the second derivative is positive for all positive values of x, therefore we have a minimum at $x = 79.93$.

We find y when $x = 79.93$.

$$y = \frac{5000}{x} = \frac{5000}{(79.93)} \approx 62.55.$$

The cost is minimized when the dimensions are 62.55 ft by 79.92 ft. (Note, the wooden fence side should be 62.55 feet.)
Substituting these dimensions into the cost function we have:

$$C = 9x + 11.5y$$
$$= 9(79.93) + 11.5(62.55)$$
$$= 1438.695 \approx 1439.$$

The total cost of fencing the parking area is approximately \$1439.

40. $A = x \cdot y = 1200$

Cost of Stone $= 35(2x)$

Cost of Wood $= 28(2y)$

Therefore the total cost function is:
$C = 70x + 56y.$

To express C in terms of one variable, we solve $xy = 1200$ for y:

$$y = \frac{1200}{x}.$$

Then,

$$C(x) = 70x + 56\left(\frac{1200}{x}\right) = 70x + \frac{67,200}{x}$$

The function is defined only for positive numbers, so we are minimizing C on the interval $(0, \infty)$.

First, we find $C'(x)$.

$$C'(x) = 70 - \frac{67,200}{x^2}$$

Since $C'(x)$ exists for all x in $(0, \infty)$, the only critical values are where $C'(x) = 0$. Thus, we solve the following equation:

$$70 - \frac{67,200}{x^2} = 0$$
$$70x^2 = 67,200$$
$$x \approx 30.98$$

This is the only critical value, so we can use the second derivative to determine whether we have a minimum.

$$C''(x) = \frac{134,400}{x^3}$$

Note that the second derivative is positive for all positive values of x, therefore we have a minimum at $x = 30.98$.
The solution is continued on the next page.

We find y when $x = 30.98$.

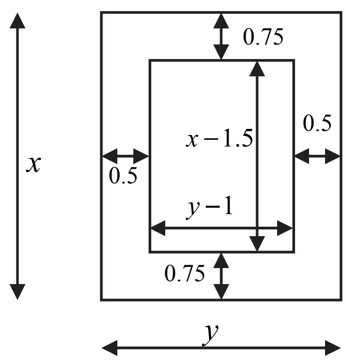
$$y = \frac{1200}{30.98} = 38.73.$$

Therefore, the cost will be minimized when the stone wall is 30.98 yards long and the wooden fencing is 38.73 yards long. The minimum cost is given by:

$$C = 70(30.98) + 56(38.73) = 4337.48.$$

The minimum cost will be approximately \$4337.

41. Let x and y represent the outside length and width, respectively.



We know that $xy = 73.125$, so $y = \frac{73.125}{x}$.

We want to maximize the print area:

$$A = (x-1.5)(y-1)$$
$$= xy - x - 1.5y + 1.5.$$

Substituting for y, we get:

$$A(x) = x\left(\frac{73.125}{x}\right) - x - 1.5\left(\frac{73.125}{x}\right) + 1.5$$
$$= 73.125 - x - \frac{109.6875}{x} + 1.5$$
$$= 74.625 - x - \frac{109.6875}{x}$$

$$A'(x) = -1 + 109.6875x^{-2} = -1 + \frac{109.6875}{x^2}$$

$A'(x)$ exists for all values in the domain of A.

Solve:

$$A'(x) = 0$$
$$-1 + \frac{109.6875}{x^2} = 0$$
$$x^2 = 109.6875$$
$$x = \pm\sqrt{109.6875}$$

The only critical value in the domain of A is $x = \sqrt{109.6875}$; Therefore, we can use the second derivative to determine if it is a maximum.

$$A''(x) = -219.375x^{-3} = -\frac{219.375}{x^3}.$$

$A''\left(\sqrt{109.6875}\right) < 0$, so $A\left(\sqrt{109.6875}\right)$ is a maximum.

When

$$x = \sqrt{109.6875} \approx 10.47,$$
$$y = \frac{73.125}{\sqrt{109.6875}} \approx 6.98.$$

Therefore, the outside dimensions should be approximately 10.47 in. by 6.98 in. to maximize the print area.

42. Let x equal the lot size. Now the inventory costs are given by:

$$C(x) = \text{Yearly Carrying Cost} + \text{Yearly reorder Cost}$$

Yearly carrying costs:

$$C_c(x) = 20 \cdot \frac{x}{2}$$
$$= 10x$$

Yearly reorder costs:

$$C_r(x) = (40 + 16x)\left(\frac{100}{x}\right)$$
$$= \frac{4000}{x} + 1600$$

Hence, the total inventory cost is:

$$C(x) = C_c(x) + C_r(x)$$
$$= 10x + \frac{4000}{x} + 1600.$$

We want to find the minimum value of C on the interval $[1,100]$. First, we find $C'(x)$:

$$C'(x) = 10 - 4000x^{-2} = 10 - \frac{4000}{x^2}.$$

The derivative exists for all x in $[1,100]$, so the only critical values are where $C'(x) = 0$.

$$C'(x) = 0$$
$$10 - \frac{4000}{x^2} = 0$$
$$10 = \frac{4000}{x^2}$$
$$10x^2 = 4000$$
$$x^2 = 400$$
$$x = \pm 20$$

The only critical value in the interval is $x = 20$.

The solution is continued on the next page.

We can use the second derivative to determine whether we have a minimum.

$C''(x) = 8000x^{-3} = \frac{8000}{x^3}$

Notice that $C''(x)$ is positive for all values of x in $[1,100]$, we have a minimum at $x = 20$. Thus, to minimize inventory costs, the store should order pool tables $\frac{100}{20} = 5$ times per year. The lot size will be 20 tables.

43. Let x be the lot size.

$C(x) = \text{Yearly Carrying Cost} + \text{Yearly reorder Cost}$

We consider each cost separately.

Yearly carrying costs, $C_c(x)$: Can be found by multiplying the cost to store the items by the number of items in storage. The average amount held in stock is $\frac{x}{2}$, and it cost \$4 per bowling ball for storage. Thus:

$$C_c(x) = 4 \cdot \frac{x}{2} = 2x$$

Yearly reorder costs, $C_r(x)$: Can be found by multiplying the cost of each order by the number of reorders. The cost of each order is $1 + 0.5x$, and the number of orders per year is $\frac{200}{x}$. Therefore,

$$C_r(x) = (1 + 0.5x)\left(\frac{200}{x}\right) = \frac{200}{x} + 100$$

The total inventory cost is given by:

$$C(x) = C_c(x) + C_r(x) = 2x + \frac{200}{x} + 100, \quad 1 \le x \le 200$$

We want to find the minimum value of C on the interval $[1,200]$. First, we find $C'(x)$:

$$C'(x) = 2 - 200x^{-2} = 2 - \frac{200}{x^2}$$

$C'(x)$ exists for all x in $[1,200]$. $[1,100]$, so the only critical values are where $C'(x) = 0$.

Solving the equation, we have:

$$C'(x) = 0$$
$$2 - \frac{200}{x^2} = 0$$
$$x = \pm 10$$

The only critical value in the domain is $x = 10$. Therefore, we use the second derivative,

$$C''(x) = 400x^{-3} = \frac{400}{x^3}$$

to determine whether we have a minimum. $C''(10) = 0.4 > 0$, so $C(10)$ is a minimum. In order to minimize inventory costs. The store should order $\frac{200}{10} = 20$ times per year. The lot size will be 10 bowling balls.

44. Let x equal the lot size.

$C(x) = \text{Yearly Carrying Cost} + \text{Yearly reorder Cost}$

Yearly carrying costs, $C_c(x)$:

$$C_c(x) = 2 \cdot \frac{x}{2} = x$$

Yearly reorder costs, $C_r(x)$:

$$C_r(x) = (5 + 2.50x)\left(\frac{720}{x}\right) = \frac{3600}{x} + 1800$$

Hence, the total inventory cost is:

$$C(x) = C_c(x) + C_r(x) = x + \frac{3600}{x} + 1800.$$

Then,

$$C'(x) = 1 - 3600x^{-2} = 1 - \frac{3600}{x^2}.$$

The derivative exists for all x in $[1,720]$. Solve:

$$C'(x) = 0$$
$$1 - \frac{3600}{x^2} = 0$$
$$x^2 = 3600$$
$$x = \pm 60$$

The only critical value in the interval is $x = 60$, so we can use the second derivative to determine whether we have a minimum.

$$C''(x) = 7200x^{-3} = \frac{7200}{x^3}.$$

The solution is continued on the next page.

Notice that $C''(x)$ is positive for all values of x in $[1, 720]$, we have a minimum at $x = 60$. Thus, to minimize inventory costs, the store should order calculators $\frac{720}{60} = 12$ times per year. The lot size will be 60 calculators.

45. Let x equal the lot size. Now the inventory costs are given by:

$C(x) = \underset{\text{Cost}}{\text{Yearly Carrying}} + \underset{\text{Cost}}{\text{Yearly reorder}}$

We consider each cost separately.

Yearly carrying costs, $C_c(x)$: Can be found by multiplying the cost to store the items by the number of items in storage. The average amount held in stock is $\frac{x}{2}$, and it cost \$2 per calculator for storage. Thus:

$$C_c(x) = 8 \cdot \frac{x}{2} = 4x$$

Yearly reorder costs, $C_r(x)$: Can be found by multiplying the cost of each order by the number of reorders. The cost of each order is $10 + 5x$, and the number of orders per year is $\frac{360}{x}$. Therefore,

$$C_r(x) = (10 + 5x)\left(\frac{360}{x}\right) = \frac{3600}{x} + 1800$$

Hence, the total inventory cost is:

$$C(x) = C_c(x) + C_r(x) = 4x + \frac{3600}{x} + 1800, \quad 1 \le x \le 360$$

We want to find the minimum value of C on the interval $[1, 360]$. First, we find $C'(x)$:

$$C'(x) = 4 - 3600x^{-2} = 4 - \frac{3600}{x^2}$$

The derivative exists for all x in $[1, 360]$, so the only critical values are where $C'(x) = 0$. We solve the equation at the top of the next column.

$$C'(x) = 0$$
$$4 - \frac{3600}{x^2} = 0$$
$$4 = \frac{3600}{x^2}$$
$$4x^2 = 3600$$
$$x^2 = 900$$
$$x = \pm 30$$

The only critical value in the domain is $x = 30$. Therefore, we use the second derivative,

$$C''(x) = 7200x^{-3} = \frac{7200}{x^3}$$

to determine whether we have a minimum. $C''(30) = 0.27 > 0$, so $C(30)$ is a minimum. In order to minimize inventory costs. The store should order $\frac{360}{30} = 12$ times per year. The lot size will be 30 surf boards.

46. Let x equal the lot size.

Yearly carrying costs:

$$C_c(x) = 2 \cdot \frac{x}{2} = x$$

Yearly reorder costs:

$$C_r(x) = (4 + 2.50x)\left(\frac{256}{x}\right) = \frac{1024}{x} + 640$$

Hence, the total inventory cost is:

$$C(x) = C_c(x) + C_r(x) = x + \frac{1024}{x} + 640.$$

We want to find the minimum value of C on the interval $[1, 256]$. First, we find $C'(x)$:

$$C'(x) = 1 - 1024x^{-2} = 1 - \frac{1024}{x^2}.$$

The derivative exists for all x in $[1, 256]$, so the only critical values are where $C'(x) = 0$. We solve that equation:

$$1 - \frac{1024}{x^2} = 0$$
$$1 = \frac{1024}{x^2}$$
$$x^2 = 1024$$
$$x = \pm 32$$

The solution is continued on the next page.

The only critical value in the interval is $x = 32$, so we can use the second derivative to determine whether we have a minimum.

$C''(x) = 2048x^{-3} = \frac{2048}{x^3}$

Notice that $C''(x)$ is positive for all values of x in $[1, 256]$, so we have a minimum at $x = 32$.

Thus, to minimize inventory costs, the store should order calculators $\frac{256}{32} = 8$ times per year. The lot size will be 32 calculators.

47. Let x be the lot size.

Yearly carrying cost: $C_c(x) = 8 \cdot \frac{x}{2} = 4x$

Yearly reorder cost: $C_r(x) = (10 + 6x)\left(\frac{360}{x}\right) = \frac{3600}{x} + 2160$

Therefore,

$$C(x) = C_c(x) + C_r(x) = 4x + \frac{3600}{x} + 2160, \quad 1 \le x \le 360$$

$C'(x) = 4 - 3600x^{-2} = 4 - \frac{3600}{x^2}$

$C'(x)$ exists for all x in $[1, 360]$. Solve:

$$C'(x) = 0$$
$$4 - \frac{3600}{x^2} = 0$$
$$x = \pm 30$$

The only critical value in the domain is $x = 30$. Therefore, we use the second derivative,

$C''(x) = 7200x^{-3} = \frac{7200}{x^3}$

to determine whether we have a minimum.

$C''(30) = 0.27 > 0$, so $C(30)$ is a minimum.

In order to minimize inventory costs. The store should order $\frac{360}{30} = 12$ times per year. The lot size will be 30 surf boards.

48. The volume of the container must be 250 in^3. Therefore, we use formula for the volume cylinder to obtain $250 = \pi r^2 h$.

Solving for h we have:

$h = \frac{250}{\pi r^2}$.

The container consists of a circular top and a circular bottom. The area for each of the top and bottom of the container is given by $A = \pi r^2$. The side material that when laid out is a rectangle of height h and whose length is the same as the circumference of the circular ends, $2\pi r$. Therfore, the surface area of the side material is $A = 2\pi rh$. Therefore, the total surface area is the sum of the area of the top and the bottom plus the the side material:

$A = 2\pi r^2 + 2\pi rh$.

Substituting for h, we have area as a function of the radius, r.

$A(r) = 2\pi r^2 + 2\pi r\left(\frac{250}{\pi r^2}\right)$

$A(r) = 2\pi r^2 + \frac{500}{r}$

The nature of this problem requires $r > 0$. We differentiate the area function with respect to r:

$A'(r) = 4\pi r - \frac{500}{r^2}$.

We find the critical values by setting the derivative equal to zero and solving for r. Remember, $r > 0$.

$$A'(r) = 0$$
$$4\pi r - \frac{500}{r^2} = 0$$
$$4\pi r = \frac{500}{r^2}$$
$$4\pi r^3 = 500$$
$$r^3 = \frac{500}{4\pi}$$
$$r = \sqrt[3]{\frac{125}{\pi}}$$
$$r \approx 3.414$$

This is the only critical value in the interval $r > 0$. We will calculate the second derivative to determine the concavity of the function.

$A''(r) = 4\pi + \frac{1000}{r^3}$.

Evaluating the second derivative at the critical value, we have:

$$A''(3.414) = 4\pi + \frac{1000}{(3.414)^3} \approx 37.697 > 0.$$

Since the second derivative is positive at the critical value, the critical value represents a relative minimum.

The solution is continued on the next page.

We determine the height of the container by substituting back into

$$h = \frac{250}{\pi r^2}.$$

$$h = \frac{250}{\pi(3.414)^2} \approx 6.828$$

Therefore, the dimensions of the container that will minimize the surface area are a height of 6.828 inches and a radius of 3.414 inches.

49. The volume of the container must be 400 cm^3. Therefore, we use formula for the volume cylinder to obtain $400 = \pi r^2 h$.
Solving for h we have:

$$h = \frac{400}{\pi r^2}.$$

Therefore, the total surface area is the sum of the area the bottom plus the the side material:
$A = \pi r^2 + 2\pi r h$.
Substituting for h in area formula we have area as a function of the radius, r.

$$A(r) = \pi r^2 + 2\pi r\left(\frac{400}{\pi r^2}\right)$$

$$A(r) = \pi r^2 + \frac{800}{r}$$

The nature of this problem requires $r > 0$. We differentiate the area function with respect to r:

$$A'(r) = 2\pi r - \frac{800}{r^2}.$$

We find the critical values by setting the derivative equal to zero and solving for r. Remember, $r > 0$.

$$A'(r) = 0$$

$$2\pi r - \frac{800}{r^2} = 0$$

$$2\pi r = \frac{800}{r^2}$$

$$r^3 = \frac{800}{2\pi}$$

$$r = \sqrt[3]{\frac{400}{\pi}}$$

$$r \approx 5.03$$

This is the only critical value in the interval $r > 0$. We will calculate the second derivative to determine the concavity of the function.

$$A''(r) = 2\pi + \frac{1600}{r^3}.$$

We evaluate the second derivative at the critical value at the top of the nexg column.

$$A''(5.03) = 2\pi + \frac{1600}{(5.03)^3} \approx 18.855 > 0.$$

Since the second derivative is positive at the critical value, the critical value represents a relative minimum. We determine the height of the container by substituting back into

$$h = \frac{400}{\pi r^2}.$$

$$h = \frac{400}{\pi(5.03)^2} \approx 5.03$$

Therefore, the dimensions of the container that will minimize the surface area are a height of 5.03 cm and a radius of 5.03 cm.

50. The cost function for the container would be is given by:

$$C = 0.005\begin{pmatrix}\text{area of the}\\ \text{top and bottom}\end{pmatrix} + 0.003\begin{pmatrix}\text{area of the}\\ \text{side material}\end{pmatrix}$$

Using the information from problem 48, we know the total area for the top and bottom of the can was given by $2\pi r^2$ and the area of the side material was given by $\frac{500}{r^r}$. Therefore, the cost function is:

$$C(r) = 0.005\left(2\pi r^2\right) + 0.003\left(\frac{500}{r}\right)$$

$$= 0.01\pi r^2 + \frac{1.5}{r}.$$

The nature of the problem still requires $r > 0$. Calculating the derivative of the cost function we have:

$$C'(r) = 0.01\left(2\pi r^{2-1}\right) + 1.5\left(-1r^{-1-1}\right)$$

$$= 0.02\pi r - \frac{1.5}{r^2}$$

Find the critical values by setting the derivative equal to zero and solving for r.

$$C'(r) = 0$$

$$0.02\pi r - \frac{1.5}{r^2} = 0$$

$$r^3 = \frac{75}{\pi}$$

$$r = \sqrt[3]{\frac{75}{\pi}}$$

$$r \approx 2.879$$

The solution is continued on the next page.

From the previous page, we determine $r \approx 2.879$ is the only critical value in the interval $r > 0$.
Next, we find the second derivative:

$$C''(r) = 0.02\pi + \frac{3}{r^3}.$$

Substituting the critical value into the second derivative, we have:

$$C''(2.879) = 0.022\pi + \frac{3}{(2.879)^3}$$
$$\approx 0.19 > 0.$$

Since the second derivative is positive at the critical value, the critical value represents a relative minimum. We substitute the critical value back into $h = \frac{250}{\pi r^2}$ to determine the value for h. Notice, we substitute before the rounding of r.

$$h = \frac{250}{\pi(2.879411)^2}$$
$$\approx 9.598$$

The dimensions of the container that will minimize the cost of the container are a radius of 2.879 in and a height of 9.589 in.

51. The cost function for the container would be is given by:

$$C = 0.0015\begin{pmatrix}\text{area of}\\ \text{the base}\end{pmatrix} + 0.0008\begin{pmatrix}\text{area of}\\ \text{the side}\end{pmatrix}$$

Using the information from problem 49, the cost function is:

$$C(r) = 0.0015(\pi r^2) + 0.0008\left(\frac{800}{r}\right)$$
$$= 0.0015\pi r^2 + \frac{0.64}{r}.$$

The nature of the problem still requires $r > 0$.
Calculating the derivative of the cost function we have:

$$C'(r) = \frac{d}{dr}\left[0.0015\pi r^2 + \frac{0.64}{r}\right]$$
$$= 0.003\pi r - \frac{0.64}{r^2}$$

Find the critical values by setting the derivative equal to zero and solving for r.

$$C'(r) = 0$$
$$0.003\pi r - \frac{0.64}{r^2} = 0$$
$$0.003\pi r = \frac{0.64}{r^2}$$
$$r^3 = \frac{0.64}{0.003\pi}$$
$$r = \sqrt[3]{\frac{0.64}{0.003\pi}}$$
$$r \approx 4.08$$

This is the only critical value in the interval $r > 0$.
Next, we find the second derivative:

$$C''(r) = \frac{d}{dr}\left(0.003\pi r - \frac{0.64}{r^2}\right)$$
$$= 0.003\pi + \frac{1.28}{r^3}.$$

Substituting the critical value into the second derivative, we have:

$$C''(4.08) = 0.003\pi + \frac{1.28}{(4.08)^3}$$
$$\approx 0.028 > 0.$$

Since the second derivative is positive at the critical value, the critical value represents a relative minimum. We substitute the critical value back into $h = \frac{400}{\pi r^2}$ to determine the value for h. Notice, we substitute before the rounding of r.

$$h = \frac{400}{\pi(4.08)^2}$$
$$\approx 7.65$$

The dimensions of the container that will minimize the cost of the container are a radius of 4.08 cm and a height of 7.65 cm.

52. Case I.
If y is the length, the girth is $x + x + x + x$, or $4x$.
Case II.
If x is the length, the girth is $x + y + x + y$, or $2x + 2y$.
Case I.
The combined length and girth is $y + 4x = 84$.
The volume is $V = x \cdot x \cdot y = x^2 y$.
The solution is continued on the next page.

We want express V in terms of one variable.
We solve $y+4x=84$, for y.

$y+4x=84$

$y=84-4x$

Thus,

$V=x^2(84-4x)=84x^2-4x^3$.

To maximize $V(x)$ we first find $V'(x)$.

$V'(x)=168x-12x^2$

This derivative exists for all x, so the critical values will occur when $V'(x)=0$; therefore, we solve that equation.

$168x-12x^2=0$

$12x(14-x)=0$

$12x=0$ or $14-x=0$

$x=0$ or $x=14$

Since $x\neq 0$, the only critical value is $x=14$.
We can use the second derivative,
$V''(x)=168-24x$,
to determine whether we have a maximum.

$V''(14)=168-24(14)=-168<0$

Therefore, we have a maximum at $x=14$.
If $x=14$, then $y=84-4(14)=28$.

Therefore, the dimensions that will maximize the volume of the package are 14 in. by 14 in. by 28 in. The volume is $14\times14\times28=5488\text{ in}^3$.

Case II.
The combine length and girth is
$x+2x+2y=3x+2y=84$.

The volume is $V=x\cdot x\cdot y=x^2y$.

We want express V in terms of one variable.
We solve $3x+2y=84$, for y.

$3x+2y=84$

$2y=84-3x$

$y=42-\dfrac{3}{2}x$

Thus,

$V=x^2\left(42-\dfrac{3}{2}x\right)=42x^2-\dfrac{3}{2}x^3$.

To maximize $V(x)$ we first find $V'(x)$.

$V'(x)=84x-\dfrac{9}{2}x^2$

This derivative exists for all x, so the critical values will occur when $V'(x)=0$. We solve the equation at the top of the next column.

$84x-\dfrac{9}{2}x^2=0$

$3x\left(28-\dfrac{3}{2}x\right)=0$

$3x=0$ or $28-\dfrac{3}{2}x=0$

$x=0$ or $-\dfrac{3}{2}x=-28$

$x=0$ or $x=\dfrac{56}{3}$

Since $x\neq 0$, the only critical value is $x=\dfrac{56}{3}$

We can use the second derivative,
$V''(x)=84-9x$,
to determine whether we have a maximum.

$V''\left(\dfrac{56}{3}\right)=84-9\left(\dfrac{56}{3}\right)=-84<0$

Therefore, we have a maximum at $x=\dfrac{56}{3}$.

If $x=\dfrac{56}{3}\approx 18.67$, then $y=42-\dfrac{3}{2}\left(\dfrac{56}{3}\right)=14$.

Therefore, the dimensions that will maximize the volume of the package are 18.67 in. by 18.67 in. by 14 in. The volume is

$\dfrac{56}{3}\times\dfrac{56}{3}\times14\approx 4878.2\text{ in}^3$.

Comparing Case I and Case II, we see that the maximum volume is 5488 in^3 when the dimensions are 14 in. by 14 in. by 28 in.

53. Let y represent the dimension on the lot line and let x represent the other dimension. Then the length of fencing that the person must pay for is

$\dfrac{1}{2}y+x+y+x=2x+\dfrac{3}{2}y$.

We know $xy=48$; therefore, $y=\dfrac{48}{x}$.

The length of fencing as a function of x is:

$F(x)=2x+\dfrac{3}{2}\left(\dfrac{48}{x}\right)=2x+\dfrac{72}{x},\quad 0<x<\infty$.

$F'(x)=2-72x^{-2}=2-\dfrac{72}{x^2}$

$F'(x)$ exists for all x in $(0,\infty)$. Solve:

$F'(x)=0$

$2-\dfrac{72}{x^2}=0$

$x=\pm6$

The solution is continued on the next page.

Only the critical value $x=6$ is in $(0,\infty)$. Since there is only one critical value, we use the second derivative to determine whether we have a minimum.

$F''(x)=144x^{-3}=\dfrac{144}{x^3}$

$F''(6)=\dfrac{2}{3}>0$, so $F(6)$ is a minimum.

When $x=6,\ y=\dfrac{48}{6}=8$. The dimensions that minimize the cost are 6 yd by 8 yd. Where the longer side of the lot is adjacent to the neighbor's yard.

54. Use the figure in the text book. Since the radius of the window is *x*, the diameter of the window is $2x$, which is also the length of the base of the window.
The circumference of a circle whose radius is *x* is given by:

$C=2\pi x. \qquad (C=2\pi r)$

Therefore, the perimeter of the semicircle is

$\dfrac{1}{2}C=\dfrac{2\pi x}{2}=\pi x.$

The perimeter of the three sides of the rectangle which form the remaining part of the total perimeter of the window is given by:

$2x+y+y=2x+2y.$

The total perimeter of the window is:

$\pi x+2x+2y=24$.

Maximizing the amount of light is the same as maximizing the area of the window. The area of the circle with radius *x* is:

$A=\pi x^2, \qquad \left(A=\pi r^2\right).$

Therefore, the area of the semicircle is:

$\dfrac{1}{2}A=\dfrac{1}{2}\pi x^2.$

The area of the rectangle is $2x\cdot y$.

The total area of the Norman window is

$A=\dfrac{1}{2}\pi x^2+2xy.$

To express *A* in terms of one variable, we solve $\pi x+2x+2y=24$ for *y*:

$$\begin{aligned}\pi x+2x+2y&=24\\ 2y&=24-2x-\pi x\\ y&=12-x-\frac{\pi}{2}x\end{aligned}$$

Then,

$$\begin{aligned}A(x)&=\frac{1}{2}\pi x^2+2x\left(12-x-\frac{\pi}{2}x\right)\\ &=\frac{1}{2}\pi x^2+24x-2x^2-\pi x^2\\ &=\left(-\frac{1}{2}\pi-2\right)x^2+24x\end{aligned}$$

We maximize *A* on the interval $(0,24)$. We first find $A'(x)$.

$A'(x)=(-\pi-4)x+24$.

Since $A'(x)$ exists for all *x* in $(0,24)$, the only critical points are where $A'(x)=0$. Thus, we solve the following equation:

$$\begin{aligned}A'(x)&=0\\ (-\pi-4)x+24&=0\\ (-\pi-4)x&=-24\\ x&=\frac{-24}{(-\pi-4)}\\ x&=\frac{-24}{-(\pi+4)}\\ x&=\frac{24}{\pi+4}\approx 3.36\end{aligned}$$

This is the only critical value, so we can use the second derivative to determine whether we have a maximum.

$A''(x)=-\pi-4<0$

Since $A''(x)$ is negative for all values of *x*, we have a maximum at $x=\dfrac{24}{\pi+4}$. We find *y* when $x=\dfrac{24}{\pi+4}$:

$$\begin{aligned}y&=12-\frac{\pi}{2}x-x\\ &=12-\frac{\pi}{2}\left(\frac{24}{\pi+4}\right)-\frac{24}{\pi+4}\\ &=12\left(\frac{\pi+4}{\pi+4}\right)-\frac{12\pi}{\pi+4}-\frac{24}{\pi+4}\\ &=\frac{12\pi+48-12\pi-24}{\pi+4}\\ &=\frac{24}{\pi+4}\approx 3.36\end{aligned}$$

The solution is continued on the next page.

To maximize the amount of light through the window, the dimensions must be $x = \frac{24}{\pi + 4}$ ft and $y = \frac{24}{\pi + 4}$ ft, or approximately $x \approx 3.36$ ft and $y \approx 3.36$ ft.

55. Since the stained glass transmits only half as much light as the semicircle in Exercise 54, we express the function A as:

$$A = \frac{1}{2} \cdot \frac{1}{2}\pi x^2 + 2xy$$
$$= \frac{1}{4}\pi x^2 + 2xy$$

The perimeter is still the same, so we can substitute $12 - \frac{\pi}{2}x - x$ for y to get:

$$A(x) = \frac{1}{4}\pi x^2 + 2x\left(12 - \frac{\pi}{2}x - x\right)$$
$$= \frac{1}{4}\pi x^2 + 24x - 2x^2 - \pi x^2$$
$$= \left(-\frac{3}{4}\pi - 2\right)x^2 + 24x, \quad 0 < x < 24.$$

Find $A'(x)$.

$$A'(x) = \left(-\frac{3}{2}\pi - 4\right)x + 24$$

Since $A'(x)$ exists for all x in $(0, 24)$, the only critical points are where $A'(x) = 0$. Thus, we solve the following equation:

$$\left(-\frac{3}{2}\pi - 4\right)x + 24 = 0$$
$$(3\pi + 8)x - 48 = 0$$
$$x = \frac{48}{3\pi + 8} \approx 2.75$$

This is the only critical value, so we can use the second derivative to determine whether we have a maximum.

$$A''(x) = -\frac{3}{2}\pi - 4 < 0$$

Since $A''(x)$ is negative for all values of x, we have a maximum at $x = \frac{48}{3\pi + 8}$. We find y when $x = \frac{48}{3\pi + 8}$ at the top of the next column.

$$y = 12 - \frac{\pi}{2}x - x$$
$$= 12 - \frac{\pi}{2}\left(\frac{48}{3\pi + 8}\right) - \frac{48}{3\pi + 8}$$
$$= \frac{12\pi + 48}{3\pi + 8} \approx 4.92$$

To maximize the amount of light through the window, the dimensions must be $x = \frac{48}{3\pi + 8}$ ft and $y = \frac{12\pi + 48}{3\pi + 8}$ ft or approximately $x \approx 2.75$ ft and $y \approx 4.92$ ft.

56-110. Left to the Student.

111. Let x represent a positive number. Then, $\frac{1}{x}$ is the reciprocal of the number, and x^2 is the square of the number. The sum, S, of the reciprocal and five times the square is given by:

$$S(x) = \frac{1}{x} + 5x^2.$$

We want to minimize $S(x)$ on the interval $(0, \infty)$. First, we find $S'(x)$

$$S'(x) = -x^{-2} + 10x = -\frac{1}{x^2} + 10x$$

Since $S'(x)$ exists for all values of x in $(0, \infty)$, the only critical values occur when $S'(x) = 0$. We solve the following equation:

$$-\frac{1}{x^2} + 10x = 0$$
$$10x = \frac{1}{x^2}$$
$$10x^3 = 1$$
$$x^3 = \frac{1}{10}$$
$$x = \sqrt[3]{\frac{1}{10}} = \frac{1}{\sqrt[3]{10}}$$

Since there is only one critical value, we use the second derivative,

$$S''(x) = 2x^{-3} + 10 = \frac{2}{x^3} + 10,$$

to determine whether it is a minimum. The second derivative is positive for all x in $(0, \infty)$; therefore, the sum is a minimum when $x = \frac{1}{\sqrt[3]{10}}$.

112. Let x represent a positive number. The sum, S, of the reciprocal and four times the square is given by:

$$S(x)=\frac{1}{x}+4x^2.$$

We want to minimize $S(x)$ on the interval $(0,\infty)$. First, we find $S'(x)$

$$S'(x)=-x^{-2}+8x=-\frac{1}{x^2}+8x$$

$S'(x)$ exists for all values of x in $(0,\infty)$. Solve:

$$S'(x)=0$$

$$-\frac{1}{x^2}+8x=0$$

$$x=\sqrt[3]{\frac{1}{8}}=\frac{1}{2}$$

Since there is only one critical value, we use the second derivative,

$$S''(x)=2x^{-3}+8=\frac{2}{x^3}+8,$$

to determine whether it is a minimum. The second derivative is positive for all x in $(0,\infty)$; therefore, the sum is a minimum when $x=\frac{1}{2}$.

113. Let A represent the amount deposited in savings account and i represent the interest rate paid on the money deposited. If A is directly proportional to i, then there is some positive constant k such that $A=ki$. The interest earned by the bank is represented by $18\%A$, or $0.18A$. The interest paid by the bank is represented by iA. Thus the profit received by the bank is given by

$$P=0.18A-iA.$$

We express P as a function of the interest the bank pays on the money deposited, i, by substituting ki for A.

$$P=0.18(ki)-i(ki)$$

$$=0.18ki-ki^2$$

We maximize P on the interval $(0,\infty)$. First, we find $P'(i)$.

$$P'(i)=0.18k-2ki$$

Since $P'(i)$ exists for all i in $(0,\infty)$, the only critical values are where $P'(i)=0$.

We solve the following equation:

$$0.18k-2ki=0$$

$$-2ki=-0.18k$$

$$i=\frac{-0.18k}{-2k}$$

$$i=0.09$$

Since there is only one critical point, we can use the second derivative to determine whether we have a minimum. Notice that $P''(i)=-2k$, which is a negative constant $(k>0)$. Thus, $P''(0.09)$ is negative, so $P(0.09)$ is a maximum. To maximize profit, the bank should pay 9% on its savings accounts.

114. The circumference of the circle is $x=2\pi r$.

Solving the equation for r we get, $r=\frac{x}{2\pi}$.

The area of the circle is $A=\pi r^2$, thus, substituting for r we have $A=\pi\left(\frac{x}{2\pi}\right)^2=\frac{x^2}{4\pi}$.

The perimeter of the square is $24-x$.

The length of a side of the square is $\frac{24-x}{4}$.

Therefore, the area of the square is:

$$A_s=\left(\frac{24-x}{4}\right)^2=\frac{x^2-48x+576}{16}.$$

The total area is:

$$A=A_c+A_s$$

$$=\frac{x^2}{4\pi}+\frac{x^2-48x+576}{16}$$

$$=\frac{x^2}{4\pi}+\frac{1}{16}x^2-3x+36$$

$$=\left(\frac{1}{4\pi}+\frac{1}{16}\right)x^2-3x+36$$

$$=\left(\frac{4+\pi}{16\pi}\right)x^2-3x+36$$

We minimize the area on the interval $(0,24)$. First, we find $A'(x)$.

$$A'(x)=2\left(\frac{4+\pi}{16\pi}\right)x-3$$

$$=\left(\frac{4+\pi}{8\pi}\right)x-3$$

$A'(x)$ exists for all x in $(0,24)$; therefore, the only critical value occurs when $A'(x)=0$.

The solution is continued on the next page.

Setting the derivative equal to zero, we solve for the critical value.

$$\left(\frac{4+\pi}{8\pi}\right)x-3=0$$

$$x=3\left(\frac{8\pi}{4+\pi}\right)=\frac{24\pi}{4+\pi}\approx 10.56$$

Since there is only one critical value, we use the second derivative,

$$A''(x)=\frac{4+\pi}{8\pi}>0$$

to determine whether we have a minimum. The second derivative is positive for all values of x in the interval; therefore a minimum occurs at

$$x=\frac{24}{4+\pi}\approx 10.56.$$

The wire should be cut to $x\approx 10.56$ in. in order to form the circle and $24-10.56\approx 13.44$ in. in order to form the square.

There is no maximum if the string is to be cut. One would interpret the maximum to be at the endpoint, with the string uncut and used to form a circle.

115. Using the drawing in the text, we write a function that gives the cost of the power line. The length of the power line on the land is given by $4-x$, so the cost of laying the power line underground is given by:

$$C_L(x)=3000(4-x)=12{,}000-3000x.$$

The length of the power line that will be under water is $\sqrt{1+x^2}$, so the cost of laying the power line underwater is given by:

$$C_W(x)=5000\sqrt{1+x^2}$$

Therefore, the total cost of laying the power line is:

$$C(x)=C_L(x)+C_W(x)$$

$$=12{,}000-3000x+5000\sqrt{1+x^2}.$$

We want to minimize $C(x)$ over the interval $0\le x\le 4$. First, we find the derivative.

$$C'(x)=-3000+5000\left(\frac{1}{2}\right)\left(1+x^2\right)^{-1/2}(2x)$$

$$=-3000+5000x\left(1+x^2\right)^{-1/2}$$

$$=-3000+\frac{5000x}{\sqrt{1+x^2}}$$

Since the derivative exists for all x, we find the critical values by solving the equation:

$$C'(x)=0$$

$$-3000+\frac{5000x}{\sqrt{1+x^2}}=0$$

$$-3000\sqrt{1+x^2}+5000x=0$$

$$5000x=3000\sqrt{1+x^2}$$

$$\frac{5}{3}x=\sqrt{1+x^2}$$

$$\left(\frac{5}{3}x\right)^2=\left(\sqrt{1+x^2}\right)^2$$

$$\frac{25}{9}x^2=1+x^2$$

$$\frac{16}{9}x^2=1$$

$$x^2=\frac{9}{16}$$

$$x=\pm\sqrt{\frac{9}{16}}$$

$$x=\pm\frac{3}{4}$$

The only critical value in the interval $[0,4]$ is $x=\frac{3}{4}$, so we can use the second derivative to determine if we have a minimum.

$$C''(x)=\frac{\left(1+x^2\right)^{1/2}(5000)-5000x\left[\frac{1}{2}\left(1+x^2\right)^{-1/2}(2x)\right]}{\left[\left(1+x^2\right)^{1/2}\right]^2}$$

$$=\frac{5000\sqrt{1+x^2}-\frac{5000x^2}{\sqrt{1+x^2}}}{\left(1+x^2\right)}$$

$$=\frac{5000}{\sqrt{1+x^2}}-\frac{5000x^2}{\left(1+x^2\right)^{3/2}}$$

$$=\frac{5000\left(1+x^2\right)-5000x^2}{\left(1+x^2\right)^{3/2}}$$

$$=\frac{5000}{\left(1+x^2\right)^{3/2}}$$

The solution is continued on the next page.

$C''(x)$ is positive for all x in $[0,4]$; therefore, a minimum occurs at $x=\frac{3}{4}$. When $x=\frac{3}{4}$,

$4-\frac{3}{4}=\frac{13}{4}=3.25$.

Therefore, S should be 3.25 miles down shore from the power station.

Note: since we are minimizing cost over a closed interval, we could have used Max-Min Principle 1 to determine the minimum, and avoided finding the second derivative. The critical value and the endpoints are $0, \frac{3}{4},$ and 4.

The function values at these three points are:

$$C(0)=12{,}000-3000(0)+5000\left(\sqrt{1+(0)^2}\right)$$
$$=17{,}000$$
$$C\left(\frac{3}{4}\right)=12{,}000-3000\left(\frac{3}{4}\right)+5000\left(\sqrt{1+\left(\frac{3}{4}\right)^2}\right)$$
$$=16{,}000$$
$$C(4)=12{,}000-3000(4)+5000\left(\sqrt{1+(4)^2}\right)$$
$$\approx 20{,}615.53$$

Therefore, the minimum occurs when $x=\frac{3}{4}$, or when S is 3.25 miles down shore from the power station.

116. The distance from point C to point S is given by $\sqrt{9+x^2}$, and the distance from point S to point A is $8-x$. Therefore, the total energy expended by the pigeon is:

$E(x)=1.28r\sqrt{9+x^2}+r(8-x)$, where r is a positive constant measuring the rate of energy the pigeon uses.

We want to minimize $E(x)$ on the interval $[0,8]$.

$$E'(x)=\frac{1.28rx}{\left(9+x^2\right)^{1/2}}-r$$

$E'(x)$ exists for all x in $[0,8]$. Solve:

$$E'(x)=0$$
$$\frac{1.28rx}{\left(9+x^2\right)^{1/2}}-r=0$$
$$x\approx\pm3.755$$

The only critical value in $[0,8]$ is $x\approx3.755$, so we can use the second derivative to determine whether we have a maximum.

Note that $E''(x)=\frac{11.52r}{\left(9+x^2\right)^{3/2}}$ and that $E''(x)>0$ for all x in the interval $[0,8]$.

Therefore, a minimum occurs when $x\approx3.755$. The pigeon should reach land about $8-3.755$ or 4.245 miles down shore from A.

117. The objective is to maximize the parallel areas subject to the constraint of having k units of fencing. Using the figure in the book, we can let x equal the length of the two parallel sides and y equal the length of the three parallel sides. Since the two areas are the identical, we can either maximize the area of one of the smaller areas or maximize the area of the entire field. That is, we can either maximize

$A_1=x\cdot y$ or $A_2=2x\cdot y$.

First we will maximize A_1. The perimeter of the fields is given by $P=4x+3y$. Since we have k units of fencing, we know that

$4x+3y=k$.

Solve for y and substitute into the area function to get the area as a function of one variable.

$$4x+3y=k$$
$$3y=k-4x$$
$$y=\frac{k-4x}{3}$$
$$y=\frac{1}{3}k-\frac{4}{3}x$$

Substituting into the area function we have:

$$A_1=x\cdot\left(\frac{1}{3}k-\frac{4}{3}x\right)=\frac{1}{3}kx-\frac{4}{3}x^2.$$

We find the first derivative with respect to x.

$$\frac{dA_1}{dx}=\frac{1}{3}k-\frac{8}{3}x.$$

The deriviative exists for all values of x. We find the critical values by solving:

$$\frac{dA_1}{dx}=0$$
$$\frac{1}{3}k-\frac{8}{3}x=0$$
$$-\frac{8}{3}x=-\frac{1}{3}k$$
$$x=\frac{1}{8}k.$$

The solution is continued on the next page.

There is only one critical value, so we can use the second derivative to determine if it is a maximum. Notice that the second derivative, $\frac{d^2A_1}{dx^2} = -\frac{8}{3}$, is negative for all values of x. Therefore the maximum area will occur when the width of the fields are $\frac{1}{8}k$ units long. We substitute in to find the length of the fields y.

$$y = \frac{1}{3}k - \frac{4}{3}x$$
$$= \frac{1}{3}k - \frac{4}{3}\left(\frac{1}{8}k\right)$$
$$= \frac{1}{3}k - \frac{1}{6}k$$
$$= \frac{1}{6}k = \frac{k}{6}.$$

Therefore the length of the rectangular areas that will maximize the area given k units of fencing is $y = \frac{k}{6}$ units.

Notice, if we maximized $A_2 = 2x \cdot y$. The constraint is the same:

$$4x + 3y = k \Rightarrow y = \frac{1}{3}k - \frac{4}{3}x.$$
$$A_2 = 2x \cdot \left(\frac{1}{3}k - \frac{4}{3}x\right) = \frac{2}{3}kx - \frac{8}{3}x^2.$$
$$\frac{dA_2}{dx} = \frac{2}{3}k - \frac{16}{3}x.$$
$$\frac{dA_2}{dx} = 0$$
$$\frac{2}{3}k - \frac{16}{3}x = 0$$
$$x = \frac{1}{8}k.$$

There is only one critical value, so we can use the second derivative to determine if it is a maximum. Notice that the second derivative, $\frac{d^2A_1}{dx^2} = -\frac{16}{3}$, is negative for all values of x. Therefore the maximum area will occur when the width of the fields are $\frac{1}{8}k$ units long. We substitute in to find the length of the fields y.

$$y = \frac{1}{3}k - \frac{4}{3}\left(\frac{1}{8}k\right) = \frac{k}{6}.$$

Therefore the length of the rectangular areas that will maximize the area given k units of fencing is $y = \frac{k}{6}$ units as determined previously.

118. Using the drawing in the text, we write a function which gives the total distance between the cities.
The distance from C_1 to the bridge can be given by $\sqrt{a^2 + (p-x)^2}$. The distance over the bridge is r. The distance from the bridge to C_2 can be given by $\sqrt{b^2 + x^2}$. Therefore, the total distance between the two cities is given by:

$$D(x) = \sqrt{a^2 + (p-x)^2} + r + \sqrt{x^2 + b^2}$$

To minimize the distance, we find the derivative of the function first.

$$D'(x) = \frac{1}{2}\left[a^2 + (p-x)^2\right]^{-1/2} \cdot 2(p-x)(-1) + \frac{1}{2}\left[b^2 + x^2\right]^{-1/2}(2x)$$
$$= \frac{x-p}{\sqrt{a^2 + (p-x)^2}} + \frac{x}{\sqrt{b^2 + x^2}}$$

The derivative exists for all values of x in the interval $[0, p]$. Therefore, the only critical values occur when $D'(x) = 0$. We solve this equation.

$$\frac{x-p}{\sqrt{a^2 + (p-x)^2}} + \frac{x}{\sqrt{b^2 + x^2}} = 0.$$

The solution to this equation is

$$x = \frac{bp}{b-a} \text{ or } x = \frac{bp}{b+a}.$$

Only $x = \frac{bp}{b+a}$ is in $[0, p]$.

Since there is only one critical value, we can use the second derivative to determine if there is a minimum. The second derivative is given by:

$$D''(x) = \frac{a^2}{\left[a^2 + (p-x)^2\right]^{3/2}} + \frac{b^2}{\left[x^2 + b^2\right]^{3/2}}.$$

$D''(x) > 0$ for all values of x; therefore, a minimum occurs at $x = \frac{bp}{b+a}$. The bridge should be located such that the distance x is $\frac{bp}{b+a}$ units.

119. $C(x)=8x+20+\frac{x^3}{100}$

a) To determine the average cost, we divide the total cost function by the number of units produced:

$$A(x)=\frac{C(x)}{x}.$$

$$A(x)=\frac{8x+20+\frac{x^3}{100}}{x}$$

$$=8+\frac{20}{x}+\frac{x^2}{100}$$

b) Taking the derivative of the total cost function and the average cost function we have:

$$C'(x)=\frac{d}{dx}\left(8x+20+\frac{x^3}{100}\right)$$

$$=8+\frac{3x^2}{100}.$$

$$A'(x)=\frac{d}{dx}\left(8+\frac{20}{x}+\frac{x^2}{100}\right)$$

$$=\frac{d}{dx}\left(8+20x^{-1}+\frac{1}{100}x^2\right)$$

$$=-20x^{-2}+\frac{1}{100}(2x)$$

$$=-\frac{20}{x^2}+\frac{x}{50}.$$

c) The derivative exists for all x in $(0,\infty)$; therefore, the critical values occur when $A'(x)=0$. Solve:

$$-\frac{20}{x^2}+\frac{x}{50}=0$$

$$\frac{x}{50}=\frac{20}{x^2}$$

$$x\cdot x^2=20\cdot 50$$

$$x^3=1000$$

$$x=10$$

There is only one critical value, so we use the second derivative to determine whether we have a minimum.

$$A''(x)=\frac{40}{x^3}+\frac{1}{50}$$

$A''(10)=\frac{3}{50}>0.$ Thus $A(10)$ is a minimum.

Find the function value when $x=10$:

$$A(10)=8+\frac{20}{10}+\frac{10^2}{100}=11.$$

The minimum average cost is \$11 when 10 units are produced.

$$C'(10)=8+\frac{3}{100}\left(10^2\right)=11.$$

The marginal cost is \$11 when 10 units are produced.

d) The two values are equal:
$A(10)=C'(10)=11.$

120. $A(x)=\frac{C(x)}{x}$

a) Taking the derivative of $A(x)$ we have:

$$A'(x)=\frac{d}{dx}\left[\frac{C(x)}{x}\right]$$

$$=\frac{x\cdot C'(x)-C(x)\cdot 1}{x^2}\quad \text{Quotient Rule}$$

$$=\frac{x\cdot C'(x)-C(x)}{x^2}$$

b) The derivative exists for all x in $(0,\infty)$, therefore, the critical values will occur when $A'(x_0)=0$. We solve the equation:

$$\frac{x_0\cdot C'(x_0)-C(x_0)}{x_0{}^2}=0$$

$$x_0\cdot C'(x_0)-C(x_0)=0\quad \text{Multiplying by } x_0{}^2\neq 0.$$

$$x_0\cdot C'(x_0)=C(x_0)$$

$$C'(x_0)=\frac{C(x_0)}{x_0}=A(x_0)$$

121. Express Q as a function of one variable. First, solve $x+y=1$ for y. We have:

$y=1-x$.

Substituting we have:

$$Q=x^3+2(1-x)^3$$

$$=x^3+2\left(1-3x+3x^2-x^3\right)$$

$$=-x^3+6x^2-6x+2$$

Next, we find $Q'(x)$.

$$Q'(x)=-3x^2+12x-6$$

The derivative exists for all values of x in the interval $(0,\infty)$.

The solution is continued on the next page.

Note the constraint $y = 1 - x$ actually limits us to look at the interval $(0,1)$.

The only critical values are where $Q'(x) = 0$.

We solve the equation:

$-3x^2 + 12x - 6 = 0$

$x^2 - 4x + 2 = 0$

Using the quadratic formula, we have:

$x = 2 \pm \sqrt{2}$.

When $x = 2 + \sqrt{2}$, $y = 1 - (2 + \sqrt{2}) = -1 - \sqrt{2}$.

When $x = 2 - \sqrt{2}$, $y = 1 - (2 - \sqrt{2}) = -1 + \sqrt{2}$.

Since x and y must be positive, we only consider $x = 2 - \sqrt{2}$ and $y = -1 + \sqrt{2}$.

Note, $Q''(x) = -6x + 12$ and

$Q''(2 - \sqrt{2}) = -6(2 - \sqrt{2}) + 12 \approx 8.48 > 0$, so we have a minimum at $x = 2 - \sqrt{2}$.

The minimum value of Q is found by substituting.

$$Q = x^3 + 2y^3$$
$$= (2 - \sqrt{2})^3 + 2(-1 + \sqrt{2})^3$$
$$= 6 - 4\sqrt{2}$$

122. Express Q as a function of one variable. First, solve $x^2 + y^2 = 2$ for y. We have:

$$y^2 = 2 - x^2$$
$$y = \pm\sqrt{2 - x^2}$$

y is a real number if x is in the interval $[-\sqrt{2}, \sqrt{2}]$.

If $y = -\sqrt{2 - x^2}$, we substitute for y to get:

$$Q = 3x + y^3$$
$$Q = 3x + \left(-\sqrt{2 - x^2}\right)^3$$
$$= 3x - (2 - x^2)^{3/2}$$

Next, we find $Q'(x)$.

$$Q'(x) = 3 - \left(\frac{3}{2}\right)(2 - x^2)^{1/2}(-2x)$$
$$= 3 + 3x(2 - x^2)^{1/2}$$
$$= 3 + 3x\sqrt{2 - x^2}$$

The derivative exists for all values of x in the interval $[-\sqrt{2}, \sqrt{2}]$; thus, the only critical values are where $Q'(x) = 0$. We solve:

$$Q'(x) = 0$$
$$3 + 3x\sqrt{2 - x^2} = 0$$
$$3 = -3x\sqrt{2 - x^2}$$
$$1 = -x\sqrt{2 - x^2}$$
$$1^2 = \left(-x\sqrt{2 - x^2}\right)^2$$
$$1 = x^2(2 - x^2)$$
$$1 = 2x^2 - x^4$$
$$x^4 - 2x^2 + 1 = 0$$
$$(x^2 - 1)^2 = 0$$
$$x^2 - 1 = 0$$
$$x^2 = 1$$
$$x = \pm 1$$

We notice that $x = 1$ is an extraneous solution which does not work.

$3 + 3(1)\sqrt{2 - (1)^2} = 3 + 3 = 6 \neq 0$.

Therefore, the only critical value is $x = -1$. The critical value and the endpoints are $-\sqrt{2}$, -1, and $\sqrt{2}$.

$$Q(-\sqrt{2}) = 3(-\sqrt{2}) - \left(2 - (-\sqrt{2})^2\right)^{3/2} = -3\sqrt{2}$$
$$Q(-1) = 3(-1) - \left(2 - (-1)^2\right)^{3/2} = -4$$
$$Q(\sqrt{2}) = 3(\sqrt{2}) - \left(2 - (\sqrt{2})^2\right)^{3/2} = 3\sqrt{2}$$

The minimum value of Q is $-3\sqrt{2}$ and occurs when $x = -\sqrt{2}$ and $y = -\sqrt{2 - (-\sqrt{2})^2} = 0$

Next, we repeat the process for $y = \sqrt{2 - x^2}$.

We notice that:

$$Q = 3x + y^3$$
$$Q = 3x + \left(\sqrt{2 - x^2}\right)^3$$
$$= 3x + (2 - x^2)^{3/2}$$

The solution is continued on the next page.

Next, we find $Q'(x)$.

$$Q'(x) = 3 + \left(\frac{3}{2}\right)\left(2 - x^2\right)^{1/2}(-2x)$$
$$= 3 - 3x\left(2 - x^2\right)^{1/2}$$
$$= 3 - 3x\sqrt{2 - x^2}$$

The derivative exists for all values of x in the interval $\left[-\sqrt{2}, \sqrt{2}\right]$; thus, the only critical values are where $Q'(x) = 0$. We solve the equation:

$$3 - 3x\sqrt{2 - x^2} = 0$$
$$3 = 3x\sqrt{2 - x^2}$$
$$1 = x\sqrt{2 - x^2}$$
$$1^2 = \left(x\sqrt{2 - x^2}\right)^2$$
$$1 = x^2\left(2 - x^2\right)$$
$$1 = 2x^2 - x^4$$
$$x^4 - 2x^2 + 1 = 0$$
$$\left(x^2 - 1\right)^2 = 0$$
$$x^2 - 1 = 0$$
$$x^2 = 1$$
$$x = \pm 1$$

We notice that $x = -1$ is an extraneous solution which does not work.

$3 - 3(-1)\sqrt{2 - (-1)^2} = 3 + 3 = 6 \neq 0$.

Therefore, the only critical value is $x = 1$. The critical value and the endpoints are $-\sqrt{2},\ 1,$ and $\sqrt{2}.$ We evaluate the function:

$$Q\left(-\sqrt{2}\right) = 3\left(-\sqrt{2}\right) + \left(2 - \left(-\sqrt{2}\right)^2\right)^{3/2} = -3\sqrt{2}$$

$$Q(1) = 3(1) + \left(2 - (1)^2\right)^{3/2} = 4$$

$$Q\left(\sqrt{2}\right) = 3\left(\sqrt{2}\right) + \left(2 - \left(\sqrt{2}\right)^2\right)^{3/2} = 3\sqrt{2}$$

The minimum value of Q is $-3\sqrt{2}$ occurs when $x = -\sqrt{2}$ and $y = -\sqrt{2 - \left(-\sqrt{2}\right)^2} = 0$.

Regardless of what value of y we chose, we see that the minimum of Q, is $-3\sqrt{2}$, when $x = -\sqrt{2}$ and $y = 0.$

123. Let x be the lot size.

Yearly carrying cost: $C_c(x) = a \cdot \frac{x}{2} = \frac{ax}{2}$

Yearly reorder cost: $C_r(x) = (b + cx)\left(\frac{Q}{x}\right)$

$$= \frac{bQ}{x} + cQ$$

Then,

$$C(x) = C_c(x) + C_r(x)$$
$$= \frac{ax}{2} + \frac{bQ}{x} + cQ, \quad 1 \le x \le Q$$

To find the minimum, we take the first deriviative:

$$C'(x) = \frac{a}{2} - bQx^{-2} = \frac{a}{2} - \frac{bQ}{x^2}$$

$C'(x)$ exists for all x in $[1, Q]$. So the only critical values occur when $C'(x) = 0$. Solve:

$$C'(x) = 0$$
$$\frac{a}{2} - \frac{bQ}{x^2} = 0$$
$$\frac{a}{2} = \frac{bQ}{x^2}$$
$$ax^2 = 2bQ$$
$$x^2 = \frac{2bQ}{a}$$
$$x = \pm\sqrt{\frac{2bQ}{a}}$$

The only critical value in the domain is $x = \sqrt{\frac{2bQ}{a}}$.

Since there is only one critical value in the domain, we use the second derivative,

$$C''(x) = 2bQx^{-3} = \frac{2bQ}{x^3}$$

to determine whether we have a minimum. $C''(x) > 0$ for all x in $[1, Q]$, so a minimum occurs at $x = \sqrt{\frac{2bQ}{a}}$.

In order to minimize inventory costs. The store should order $\frac{Q}{\sqrt{\frac{2bQ}{a}}} = \sqrt{\frac{aQ}{2b}}$ times per year.

The lot size will be $\sqrt{\frac{2bQ}{a}}$ units.

124. From Exercise 123, we know that the store should order a lot size of $\sqrt{\frac{2bQ}{a}}$ units, $\sqrt{\frac{aQ}{2b}}$ times per year.
When $Q = 2500,\ a = 10,\ b = 20,\ c = 9$, the store should order:

$$\sqrt{\frac{aQ}{2b}} = \sqrt{\frac{10(2500)}{2(20)}} = 25 \text{ times per year.}$$

The lot size of each order should be:

$$\sqrt{\frac{2(20)(2500)}{10}} = 100 \text{ units.}$$

125-128. The starting value and step size were chosen to limit the amount of space used . Approaches can vary.

125. Using a spreadsheet we numerically estimate the maximum:

Starting Value:	24	
Step Size:	0.25	
x	$y = \frac{100-2x}{3}$	$q = x \cdot y$
24	17.33333333	416
24.25	17.16666667	416.2917
24.5	17	416.5
24.75	16.83333333	416.625
25	16.66666667	416.6667
25.25	16.5	416.625
25.5	16.33333333	416.5
25.75	16.16666667	416.2917
26	16	416

We determine the maximum of $Q \approx 416.67$ to occur when $x = 25$ and $y \approx 16.67$.

126. Using a spreadsheet we numerically estimate the maximum:

Starting Value:	3.22	
Step Size:	0.01	
x	$y = \sqrt{16-x^2}$	$Q = x^2 \cdot y$
3.22	2.373099239	24.60524215
3.23	2.35947028	24.61611748
3.24	2.345719506	24.62442508
3.25	2.331844763	24.63011031
3.26	2.317843826	24.63311704
3.27	2.303714392	24.63338762
3.28	2.289454083	24.63086281
3.29	2.275060439	24.6254817
3.3	2.260530911	24.61718162

We determine the maximum of $Q \approx 24.63$ to occur when $x \approx 3.27$ and $y \approx 2.30$.

127. Using a spreadsheet we numerically estimate the maximum:

Starting Value:	2.2	
Step Size:	0.01	
x	$y = \sqrt{10-0.5x^2}$	$Q = x \cdot y^3$
2.2	2.75317998	45.91202934
2.21	2.749172603	45.91962105
2.22	2.745141162	45.92477319
2.23	2.741085551	45.92748186
2.24	2.737005663	45.92774328
2.25	2.732901389	45.9255538
2.26	2.728772618	45.92090987
2.27	2.724619239	45.91380804
2.28	2.720441141	45.90424501

We determine the maximum of $Q \approx 45.93$ to occur when $x \approx 2.24$ and $y \approx 2.74$.

128. Using a spreadsheet we numerically estimate the maximum:

Starting Value:	4.6	
Step Size:	0.1	
x	$y=\sqrt{50-x^2}$	$Q=x^2\cdot y^2$
4.6	5.370288633	610.2544
4.7	5.282991577	616.5319
4.8	5.192301994	621.1584
4.9	5.098038839	624.0199
5	5	625
5.1	4.897958759	623.9799
5.2	4.79165942	620.8384
5.3	4.680811895	615.4519
5.4	4.565084884	607.6944

We determine the maximum of $Q=625$ to occur when $x=5$ and $y=5$.

Exercise Set 2.6

1. $R(x)=50x-0.5x^2;\ C(x)=4x+10$

a) Total profit is revenue minus cost.

$P(x)=R(x)-C(x)$

$$P(x)=50x-0.5x^2-(4x+10)$$
$$=50x-0.5x^2-4x-10$$
$$=-0.5x^2+46x-10$$

b) Substituting 20 for x into the three functions, we have:

$R(20)=50(20)-0.5(20)^2=800$

The total revenue from the sale of the first 20 units is \$800.

$C(20)=4(20)+10=90$

The total cost of producing the first 20 units is \$90.

$$P(20)=R(20)-C(20)$$
$$=800-90$$
$$=710$$

The total profit is \$710 when the first 20 units are produced and sold.

Note, we could have also used the profit function, $P(x)$, from part (a) to find the profit.

$P(20)=-0.5(20)^2+46(20)-10=710$

c) Finding the derivative for each of the functions, we have:

$R'(x)=50-x$

$C'(x)=4$

$P'(x)=-x+46$

d) Substituting 20 for x in each of the three marginal functions, we have:

$R'(20)=50-20=30$

Once 20 units have been sold, the approximate revenue for the 21st unit is \$30.

$C'(20)=4$

Once 20 units have been produced, the approximate cost for the 21st unit is \$4.

$P'(20)=-20+46=26$

Once 20 units have been produced and sold, the approximate profit from the sale of the 21st unit is \$26.

2. $R(x)=5x;\ C(x)=0.001x^2+1.2x+60$

a) Total profit is revenue minus cost.

$$P(x)=R(x)-C(x)$$
$$=5x-(0.001x^2+1.2x+60)$$
$$=5x-0.001x^2-1.2x-60$$
$$=-0.001x^2+3.8x-60$$

b) $R(100)=5(100)=500$

The total revenue from the sale of the first 100 units is \$500.

$C(100)=0.001(100)^2+1.2(100)+60=190$

The total cost of producing the first 100 units is \$190.

$P(100)=-0.001(100)^2+3.8(100)-60=310$

The total profit is \$310 when the first 100 units are produced and sold.

c) $R'(x)=5$

$C'(x)=0.002x+1.2$

$P'(x)=-0.002x+3.8$

d) $R'(100)=5$

Once 100 units have been sold, the approximate revenue for the 101st unit is \$5.

$C'(100)=1.4$

Once 100 units have been produced, the approximate cost for the 101st unit is \$1.40.

$P'(100)=-0.002(100)+3.8=3.6$

Once 100 units have been produced and sold, the approximate profit from the sale of the 101st unit is \$3.60.

e) In part (b), we are observing the total revenue, cost and profit from the production and sale of the first 100 items. In part (d), we are observing the approximate revenue, cost and profit from the production and sale of the 101st unit only. These quantities are also known as the marginal revenue, marginal cost and marginal profit.

3. $C(x)=0.002x^3+0.1x^2+42x+300$

a) Substituting 40 for x into the cost function, we have:

$$C(40)=0.002(40)^3+0.1(40)^2+42(40)+300$$
$$=2268\ \text{(hundreds of dollars)}$$

The current daily cost of producing 40 security systems is \$226,800.

b) In order to find the additional cost of producing 41 chairs monthly, we first find the total cost of producing 41 security systems in a day.

$C(41) = 0.002(41)^3 + 0.1(41)^2 + 42(41) + 300$
$= 2327.94$ (hundreds of dollars)

Next, we subtract the cost of producing 40 security systsems daily found in part (a) from the cost of producing 41 security systems daily.

$C(41) - C(40) = 2327.94 - 2268 = 59.94$

The additional daily cost of increasing production to 41 security systems daily is \$5994.

c) First, we find the marginal cost function,

$C'(x) = 0.006x^2 + 0.2x + 42$

Next, substituting 40 for *x,* we have:

$C'(40) = 59.6$

The marginal cost when 40 security systems are produced daily is \$5960.

d) In part (a) we found that it cost \$2268 hundred to produce 40 security systems per day. The additional cost of producing 2 additional security systems is 2(\$59.60) = \$119.20 hundreds. Therefore, the estimated daily cost of producing 42 security systems per day is

$C(42) \approx \$2268 + \$119.20 = \$2387.20$

hundreds or \$238,720 per day.

4. $C(x) = 0.001x^3 + 0.07x^2 + 19x + 700$

a) Substituting 25 for *x* into the cost function, we have:

$C(25) = 0.001(25)^3 + 0.07(25)^2 + 19(25) + 700$
$= 1234.375$

The current monthly cost of producing 25 daypacks is \$1234.38.

b) We first find the total cost of producing 26 daypacks in a month.

$C(26) = 0.001(26)^3 + 0.07(26)^2 + 19(26) + 700$
$= 1258.896$

Therefore,

$C(26) - C(25) = 1258.896 - 1234.375$
$= 24.521$

The additional cost of increasing production to 26 daypacks monthly is \$24.52.

c) $C'(x) = 0.003x^2 + 0.14x + 19.$

$C'(25) = 0.003(25)^2 + 0.14(25) + 19$
$= 24.375$

The marginal cost when 25 daypacks have been produced is \$24.38.

d) Using the marginal cost from part (c), the additional cost required to produce 2 additional daypacks monthly is:

$2(24.375) = 48.75.$

Therefore, the difference in cost between producing 25 and 27 daypacks per month is approximately \$48.75.

e) In part (a) we found that it cost \$1234.38 to produce 25 daypacks per month. In part (d) we found that the difference in cost between 25 daypacks and 27 daypacks per month was \$48.75. Therefore, the approximate total cost of producing 27 daypacks per month is

$C(27) \approx 1234.38 + 48.75 = 1283.13.$

We predict the cost of producing 27 daypacks monthly will be \$1283.13.

5. $R(x) = 0.005x^3 + 0.01x^2 + 0.5x$

a) Substituting 70 for *x*, we have:

$R(70) = 0.005(70)^3 + 0.01(70)^2 + 0.5(70)$
$= 1715 + 49 + 35$
$= 1799$

The currently daily revenue from selling 70 lawn chairs per day is \$1799.

b) Substituting 73 for *x*, we have:

$R(73) = 0.005(73)^3 + 0.01(73)^2 + 0.5(73)$
$= 2034.875$
≈ 2034.88

Therefore, the increase in revenue from increasing sales to 73 chairs per day is:

$R(73) - R(70) = 2034.88 - 1799$
$= 235.88$

Revenue will increase \$235.88 per day if the number of chairs sold increases to 73 per day.

c) First we find the marginal revenue function by finding the derivative of the revenue function.

$R'(x) = 0.015x^2 + 0.02x + 0.5$

Substituting 70 for *x*, we have:

$R'(70) = 0.015(70)^2 + 0.02(70) + 0.5$
$= 75.40$

The marginal revenue when 70 lawn chairs are sold daily is \$75.40.

d) In part (a) we found that selling 70 lawn chairs per day resulted in a revenue of \$1799. In part (c) we found that the marginal revenue when 70 chairs were sold is \$75.40. Using these two numbers, we estimate the daily revenue generated by selling 71 chairs is

$$R(71) \approx R(70) + R'(70)$$
$$= \$1799 + \$75.40 = \$1874.40.$$

Similarly, the daily revenue generated by selling 72 chairs, or 2 additional chairs, daily is approximately

$$R(72) \approx R(70) + 2 \cdot R'(70)$$
$$\approx \$1799 + 2(\$75.40) \approx \$1949.80.$$

The daily revenue generated by selling 73 chairs, or 3 additional chairs, daily is approximately

$$R(73) \approx R(70) + 3 \cdot R'(70)$$
$$\approx \$1799 + 3(\$75.40) \approx \$2025.20.$$

6. $P(x) = -0.006x^3 - 0.2x^2 + 900x - 1200$

a) $P(60) = \$50,784$

b) $P(60) - P(59) = 50,784 - 49,971.53 = 812.47$

The dealership would lose \$812.47 per week if it were only able to sell 59 cars weekly.

c) $P'(x) = -0.018x^2 - 0.4x + 900$

$$P'(60) = -0.018(60)^2 - 0.4(60) + 900$$
$$= 811.20$$

The marginal profit is \$811.20 when 60 cars are sold each week.

d) $P(61) \approx P(60) + P'(60) = \$51,595.20$

The estimated profit of selling 61 cars per week is \$51,595.20.

7. $R(x) = 0.007x^3 - 0.5x^2 + 150x$

a) Substituting 26 for x, we have:

$$R(26) = 0.007(26)^3 - 0.5(26)^2 + 150(26)$$
$$= 123.032 - 338 + 3900$$
$$= 3685.03$$

The current monthly revenue is \$3685.03.

b) First we find the total montly revenue for selling 28 suitcases

$$R(28) = 0.007(28)^3 - 0.5(28)^2 + 150(28)$$
$$= 3961.66$$

The difference in monthly revenue from selling 26 suitcases and 28 suitcases a month is:

$$R(28) - R(26) = \$3961.66 - \$3685.03$$
$$= \$276.63.$$

If sales increased from 26 to 28 suitcases, monthly revenue would increase \$276.63

c) First we find marginal revenue by taking the derivative of the revenue function.

$$R'(x) = 0.021x^2 - x + 150$$

Next we substitute 26 in for x.

$$R'(26) = 0.021(26)^2 - (26) + 150 = 138.196$$

Marginal revenue is 138.20 when 26 suitcases are sold.

d) From part (a), we know that when 26 suitcases are sold, total monthly revenue is \$3685.03. From part (c), we know that when 26 suitcases are sold, marginal revenue is \$138.20. Therefore, we estimate:

$$R(27) \approx R(26) + R'(26)$$
$$R(27) \approx \$3685.03 + \$138.20 \approx \$3823.23$$

We estimate the revenue from selling 27 suitcases per month to be \$3823.23.

8. $P(x) = -0.004x^3 - 0.3x^2 + 600x - 800$

a) Substituting 9 for x, we have:

$$P(9) = -0.004(9)^3 - 0.3(9)^2 + 600(9) - 800$$
$$= 4572.784$$

The currently weekly profit is \$4572.78.

b) The difference in weekly profit from selling 8 laptops and 9 laptops per week is

$$P(9) - P(8) = 4572.78 - 3978.75$$
$$= 594.03$$

Therefore, Crawford Computing would lose \$594.03 each week if 8 laptops were sold each week instead of 9.

c) $P'(x) = -0.012x^2 - 0.6x + 600$

$$P'(9) = -0.012(9)^2 - 0.6(9) + 600$$
$$= 593.628$$

The marginal profit is \$593.63 when 9 laptops are sold weekly.

d) We estimate:

$$P(10) \approx P(9) + P'(9)$$
$$\approx 4572.78 + 593.63$$
$$\approx 5166.41$$

The total weekly profit is approximately \$5166.41 when 10 laptops are built and sold weekly.

9. $N(1000) = 500,000$ means that 500,000 computers will be sold annually when the price of the computer is \$1000. $N'(1000) = -100$ means that when the price is increased \$1 to \$1001, sales will decrease by 100 computers per year.

10. $N(1025) \approx N(1000) + 25 \cdot N'(1000)$

$N(1025) \approx 500,000 + 25(-100) \approx 497,500$

We estimate that 497,500 computers will be sold annually if the price is increased to \$1025.

11. $C(x) = 0.01x^2 + 1.6x + 100$

$\Delta C = C(x + \Delta x) - C(x)$

Substituting $x = 80,$ and $\Delta x = 1$ we have

$$\Delta C = C(80+1) - C(80)$$
$$= 0.01(81)^2 + 1.6(81) + 100 - \left[0.01(80)^2 + 1.6(80) + 100\right]$$
$$= 3.21$$

The additional cost of producing the 81st unit is \$3.21.

Finding the derivative of $C(x)$ we have:

$C'(x) = 0.02x + 1.6$

Substituting 80 for x, we have:

$C'(80) = 0.02(80) + 1.6 = 3.20$

The marginal cost when 80 units are produced is \$3.20.

12. $C(x) = 0.01x^2 + 0.6x + 30$

$$\Delta C = C(x + \Delta x) - C(x)$$
$$\Delta C = C(70+1) - C(70)$$
$$= C(71) - C(70)$$
$$= 0.01(71)^2 + 0.6(71) + 30 - \left[0.01(70)^2 + 0.6(70) + 30\right]$$
$$= 2.01$$

The additional cost of producing the 71st unit is \$2.01.

$C'(x) = 0.02x + 0.6$

$C'(70) = 0.02(70) + 0.6 = 2.00$

The marginal cost when 70 units are produced is \$2.00.

13. $R(x) = 2x$

$\Delta R = R(x + \Delta x) - R(x)$

Substituting $x = 70,$ and $\Delta x = 1$ we have

$$\Delta R = R(70+1) - R(70)$$
$$= R(71) - R(70)$$
$$= 2(71) - [2(70)]$$
$$= 2$$

The additional revenue from selling the 71st unit is \$2.00.

Finding the derivative of $R(x)$ we have:

$R'(x) = 2$.

The derivative is constant; therefore,

$R'(70) = 2$

The marginal revenue when 70 units are produced is \$2.00.

14. $R(x) = 3x$

$$\Delta R = R(x + \Delta x) - R(x)$$
$$\Delta R = R(80+1) - R(80)$$
$$= 3(81) - [3(80)]$$
$$= 3$$

The additional revenue from selling the 81st unit is \$3.00.

$R'(x) = 3$.

$R'(80) = 3$

The marginal revenue when 80 units are produced is \$3.00.

15. $C(x)=0.01x^2+1.6x+100;\ R(x)=3x$

a) Finding the profit function we have:

$$P(x)=R(x)-C(x)$$
$$=3x-(0.01x^2+1.6x+100)$$
$$=-0.01x^2+1.4x-100$$

b) $\Delta P=P(x+\Delta x)-P(x)$

Substituting $x=80,$ and $\Delta x=1$ we have

$$\Delta P=P(80+1)-P(80)$$
$$=-0.01(81)^2+1.4(81)-100-\left[-0.01(80)^2+1.4(80)-100\right]$$
$$=-0.21$$

The additional profit of producing and selling the 81st unit is $-\$0.21$.

Finding the derivative of $P(x)$ we have:

$$P'(x)=-0.02x+1.4$$

Substituting 80 for x, we have:

$$P'(80)=-0.02(80)+1.4=-0.20$$

The marginal profit when 80 units are produced and sold is $-\$0.20$.

Note: We notice that $\Delta P=\Delta R-\Delta C$ and $P'(x)=R'(x)-C'(x)$. We could have used this knowledge and our work from Exercises 11 and 14 to simplify our work. We have:

$$\Delta P=\Delta R-\Delta C$$
$$=3-3.21=-0.21$$
$$P'(80)=R'(80)-C'(80)$$
$$=3-3.20$$
$$=-0.20$$

16. $C(x)=0.01x^2+0.6x+30;\ R(x)=2x$

a) Finding the profit function we have:

$$P(x)=R(x)-C(x)$$
$$=2x-(0.01x^2+0.6x+30)$$
$$=-0.01x^2+1.4x-30$$

b) $\Delta P=P(x+\Delta x)-P(x)$

$$\Delta P=P(70+1)-P(70)$$
$$=P(71)-P(70)$$
$$=-0.01(71)^2+1.4(71)-30-\left[-0.01(70)^2+1.4(70)-30\right]$$
$$=-0.01$$

The additional profit of producing and selling the 71st unit is $-\$0.01$.

$$P'(x)=-0.02x+1.4$$
$$P'(70)=-0.02(70)+1.4=0.00$$

The marginal profit when 70 units are produced and sold is \$0.00.

17. $S=0.007p^3-0.5p^2+150p$

a) We take the derivative of the supply function with respect to price.

$$\frac{dS}{dp}=0.021p^2-p+150$$

b) Substituting 25 for p in the supply function we have:

$$S=0.007(25)^3-0.5(25)^2+150(25)$$
$$=109.375-312.50+3750$$
$$=3546.875$$

Producers will want to supply 3547 units when price is \$25 per unit.

c) Substituting 25 for p into the answer from part (a) we have:

$$\left.\frac{dS}{dp}\right|_{p=25}=0.021(25)^2-25+150=138.125$$

This result implies when the price is \$25, a \$1 increase in price will lead to an increase in supply of approximately 138 pens.

d) We would expect the rate of change of quantity with respect to price to be positive. All things being equal, it is reasonable to assume as the price of a good or service increases, the supply for that good or service will increase.

18. $A(x)=\dfrac{13x+100}{x}$

$$A'(x)=\frac{x(13)-(13x+100)(1)}{x^2}$$
$$=-\frac{100}{x^2}$$

Therefore,

$$\Delta A\approx A'(x)\Delta x$$
$$\approx A'(100)\Delta x \qquad (x=100)$$
$$\approx -\frac{100}{(100)^2}(1) \qquad (\Delta x=1)$$
$$\approx -0.01$$

The average cost changes by about $-\$0.01$. (We see an approximate decrease in average cost of one cent.)

19. $M(t) = -2t^2 + 100t + 180$

a) Substituting for *t*, we have:

$M(5) = -2(5)^2 + 100(5) + 180 = 630$

$M(10) = -2(10)^2 + 100(10) + 180 = 980$

$M(25) = -2(25)^2 + 100(25) + 180 = 1430$

$M(45) = -2(45)^2 + 100(45) + 180 = 630$

b) To find marginal productivity, we take the derivative of the productivity function: $M'(t) = -4t + 100$.

c) Substituting for *t*, we have:

$M'(5) = -4(5) + 100 = 80$

$M'(10) = -4(10) + 100 = 60$

$M'(25) = -4(25) + 100 = 0$

$M'(45) = -4(45) + 100 = -80$

We see that the additional monthly output per years of service decreases each year the employee is with the company.

d) The employees *marginal productivity* is at its highest point when the employee is new to the company. The employee is still learning how to do the job and will make the greatest gains. As the employee gains experience, the *marginal productivity* begins to decrease. The employee is still being more productive each month, but just doesn't increase total output as much as the previous month's increase. Eventually, age catches up to the employee and they cannot produce the output they once did. *Marginal productivity* becomes negative as total output starts to fall.

20. $S(p) = 0.08p^3 + 2p^2 + 10p + 11$

$p = 18.00,\ \Delta p = 18.20 - 18.00 = 0.20$

$S'(p) = 0.24p^2 + 4p + 10$

$$\begin{aligned}\Delta S &\approx S'(p)\,\Delta p\\ &\approx S'(18.00)\,\Delta p\\ &\approx \left(0.24(18)^2 + 4(18) + 10\right)(0.2)\\ &\approx 31.952\end{aligned}$$

The supplier will supply approximately 32 more units.

21. $P(x) = 567 + x\left(36x^{0.6} - 104\right)$

$\quad = 567 + 36x^{1.6} - 104x$

x is the number of years since 1960; therefore, the year 2014 corresponds to $x = 2014 - 1960 = 54$, and the year 2015 corresponds to $x = 2015 - 1960 = 55$. To estimate the increase in gross domestic product from 2014 to 2015, we establish that $x = 54$ and $\Delta x = 1$. Next, we find the derivative of $P(x)$:

$P'(x) = 36(1.6)\,x^{0.6} - 104 = 57.6x^{0.6} - 104$.

Therefore,

$$\begin{aligned}\Delta P &\approx P'(x)\Delta x\\ &\approx P'(54)\Delta x && [x = 54]\\ &\approx \left(57.6(54)^{0.6} - 104\right)\Delta x\\ &\approx (526.7521497)(1) && [\Delta x = 1]\\ &\approx 526.75\end{aligned}$$

The gross domestic product should increase about \$526.75 billion between 2014 and 2015.

22. $N(x) = -x^2 + 300x + 6$

x is in thousands of dollars so, $x = 100,\ \Delta x = 1$.

$N'(x) = -2x + 300$

$$\begin{aligned}\Delta N &\approx N'(x)\,\Delta x\\ &\approx N'(100)\,\Delta x\\ &\approx \left[-2(100) + 300\right](1)\\ &\approx 100\end{aligned}$$

Norris will sell approximately 100 more units by increasing its advertising expenditure from \$100,000 to \$101,000.

23. Yes, the taxation in 2014 was progressive. The 25,001st dollar is taxed at a rate of 15%, the 80,001st dollar is taxed at a rate of 25%, and the 140,001st dollar is taxed at a rate of 33%.

24. Marcy's marginal tax rate is 28%, while Tyrone marginal tax rate is also 28%. However, an increase in \$5000 will push Tyrone into the next tax bracket putting his marginal tax at 33%.Therefore, since Marcy is in a lower marginal tax bracket, she will keep more of the \$5000 after taxes.

25. Alan's marginal tax rate is currently 25%. If he earns another \$10,000, dollars, this will push him into the 28% tax bracket and he will pay about \$0.28 per dollar earned in taxes.

26. The marginal tax rate at $50,000 is 25%. By earning an extra $3000 it will not push her out of the 25% tax bracket. Therefore, her tax liability will not grow if she takes the extra work.

27. $y = f(x) = x^3,\ x = 2,$ and $\Delta x = 0.01$

$$\begin{aligned}\Delta y &= f(x+\Delta x) - f(x)\\ &= f(2+0.01) - f(2) \quad \text{Substituting 2 for } x \text{ and 0.01 for } \Delta x\\ &= f(2.01) - f(2)\\ &= (2.01)^3 - (2)^3\\ &= 0.1206\end{aligned}$$

$$f'(x)\Delta x = 3x^2 \cdot \Delta x \quad \left[f(x) = x^3; f'(x) = 3x^2\right]$$

$$\begin{aligned}f'(2)\Delta x &= 3(2)^2(0.01) \quad \text{Substituting 2 for } x \text{ and 0.01 for } \Delta x\\ &= 12(0.01)\\ &= 0.12\end{aligned}$$

28. $y = f(x) = x^2,\ x = 2,$ and $\Delta x = 0.01$

$$\begin{aligned}\Delta y &= f(x+\Delta x) - f(x)\\ &= f(2+0.01) - f(2)\\ &= (2.01)^2 - (2)^2\\ &= 0.0401\end{aligned}$$

$$f'(x)\Delta x = 2x \cdot \Delta x$$

$$\begin{aligned}f'(2)\Delta x &= 2(2)\cdot(0.01)\\ &= 4(0.01)\\ &= 0.04\end{aligned}$$

29. $y = f(x) = x + x^2,\ x = 3,$ and $\Delta x = 0.04$

$$\begin{aligned}\Delta y &= f(x+\Delta x) - f(x)\\ &= f(3+0.04) - f(3) \quad \text{Substituting 3 for } x \text{ and 0.04 for } \Delta x\\ &= f(3.04) - f(3)\\ &= \left[(3.04) + (3.04)^2\right] - \left[(3) + (3)^2\right]\\ &= [12.2816] - [12]\\ &= 0.2816\end{aligned}$$

$$f'(x)\Delta x = (1+2x)\cdot \Delta x \quad \begin{bmatrix} f(x) = x + x^2; \\ f'(x) = 1 + 2x \end{bmatrix}$$

$$\begin{aligned}f'(3)\Delta x &= \left[1 + 2(3)\right]\cdot(0.04) \quad \text{Substituting 3 for } x \text{ and 0.04 for } \Delta x\\ &= [7](0.04)\\ &= 0.28\end{aligned}$$

30. $y = f(x) = x - x^2,\ x = 3,$ and $\Delta x = 0.02$

$$\begin{aligned}\Delta y &= f(x+\Delta x) - f(x)\\ &= f(3+0.02) - f(3)\\ &= \left[(3.02) - (3.02)^2\right] - \left[(3) - (3)^2\right]\\ &= -0.1004\end{aligned}$$

$$f'(x)\Delta x = (1-2x)\cdot \Delta x$$

$$\begin{aligned}f'(3)\Delta x &= (1 - 2(3))(0.02)\\ &= -0.10\end{aligned}$$

31. $y = f(x) = \dfrac{1}{x} = x^{-1},\ x = 1,$ and $\Delta x = 0.2$

$$\begin{aligned}\Delta y &= f(x+\Delta x) - f(x)\\ &= f(1+0.2) - f(1) \quad \text{Substituting 1 for } x \text{ and 0.2 for } \Delta x\\ &= f(1.2) - f(1)\\ &= \left[\frac{1}{1.2}\right] - \left[\frac{1}{1}\right]\\ &= -0.1667\end{aligned}$$

$$f'(x)\Delta x = \left(-x^{-2}\right)\cdot \Delta x \quad \begin{bmatrix} f(x) = x^{-1}; \\ f'(x) = -x^{-2} \end{bmatrix}$$

$$\begin{aligned}f'(1)\Delta x &= \left(-(1)^{-2}\right)(0.2) \quad \text{Substituting 1 for } x \text{ and 0.2 for } \Delta x\\ &= -1(0.2)\\ &= -0.2\end{aligned}$$

32. $y = f(x) = \dfrac{1}{x^2} = x^{-2},\ x = 1,$ and $\Delta x = 0.5$

$$\begin{aligned}\Delta y &= f(x+\Delta x) - f(x)\\ &= f(1+0.5) - f(1)\\ &= \left[\frac{1}{(1.5)^2}\right] - \left[\frac{1}{(1)^2}\right]\\ &= -0.5556\end{aligned}$$

$$f'(x)\Delta x = -2x^{-3}\cdot \Delta x$$

$$\begin{aligned}f'(1)\Delta x &= \left[-2(1)^{-3}\right]\cdot(0.5)\\ &= [-2](0.5)\\ &= -1\end{aligned}$$

33. $y = f(x) = 3x - 1,\ x = 4,$ and $\Delta x = 2$

$$\begin{aligned}\Delta y &= f(x+\Delta x) - f(x)\\ &= f(4+2) - f(4) \quad \text{Substituting 4 for } x \text{ and 2 for } \Delta x\\ &= f(6) - f(4)\\ &= [3(6)-1] - [3(4)-1]\\ &= [17] - [11]\\ &= 6\end{aligned}$$

$f'(x)\Delta x = (3)\cdot\Delta x \quad [f(x) = 3x - 1; f'(x) = 3]$

$f'(4)\Delta x = (3)\cdot(2)$ Substituting 4 for x and 2 for Δx

$= 6$

34. $y = f(x) = 2x - 3,\ x = 8,$ and $\Delta x = 0.5$

$$\begin{aligned}\Delta y &= f(x+\Delta x) - f(x)\\ &= f(8+0.5) - f(8)\\ &= [2(8.5)-3] - [2(8)-3]\\ &= 1\end{aligned}$$

$f'(x)\Delta x = (2)\cdot\Delta x$

$f'(8)\Delta x = (2)\cdot(0.5)$

$= 1$

35. We first think of the number closest to 26 that is a perfect square. This is 25. What we will do is approximate how $y = \sqrt{x}$, changes when x changes from 25 to 26. Let

$y = f(x) = \sqrt{x} = x^{1/2}$

Then $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

Using, $\Delta y \approx f'(x)\Delta x$, we have

$$\begin{aligned}\Delta y &\approx f'(x)\Delta x\\ &\approx \frac{1}{2\sqrt{x}}\cdot\Delta x\end{aligned}$$

We are interested in Δy as x changes from 25 to 26, so

$$\begin{aligned}\Delta y &\approx \frac{1}{2\sqrt{x}}\cdot\Delta x\\ &\approx \frac{1}{2\sqrt{25}}\cdot 1 \quad \text{Replacing } x \text{ with 25 and } \Delta x \text{ with 1}\\ &\approx \frac{1}{2\cdot 5}\\ &\approx \frac{1}{10} = 0.100\end{aligned}$$

We can now approximate $\sqrt{26}$;

$$\begin{aligned}\sqrt{26} &= \sqrt{25} + \Delta y\\ &= 5 + \Delta y\\ &\approx 5 + 0.100\\ &\approx 5.100\end{aligned}$$

To five decimal places $\sqrt{26} = 5.09902$. Thus, our approximation is reasonably accurate.

36. Let $y = f(x) = \sqrt{x},\ x = 8,\ \Delta x = 1$

$$\begin{aligned}\Delta y &\approx f'(x)\Delta x\\ &\approx \frac{1}{2\sqrt{x}}\cdot\Delta x\\ &\approx \frac{1}{2\sqrt{9}}\cdot(1)\\ &\approx \frac{1}{6} = 0.167\end{aligned}$$

We can now approximate $\sqrt{8}$;

$$\begin{aligned}\sqrt{8} &= \sqrt{9} - \Delta y\\ &= 3 - \Delta y\\ &\approx 3 - 0.167 \approx 2.833\end{aligned}$$

To five decimal places $\sqrt{8} = 2.82843$. Thus, our approximation is reasonably accurate.

37. We first think of the number closest to 102 that is a perfect square. This is 100. What we will do is approximate how $y = \sqrt{x}$, changes when x changes from 100 to 102. Let

$y = f(x) = \sqrt{x} = x^{1/2}$

Then $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

Using, $\Delta y \approx f'(x)\Delta x$, we have

$$\begin{aligned}\Delta y &\approx f'(x)\Delta x\\ &\approx \frac{1}{2\sqrt{x}}\cdot\Delta x\end{aligned}$$

We are interested in Δy as x changes from 100 to 102, so

$$\begin{aligned}\Delta y &\approx \frac{1}{2\sqrt{x}}\cdot\Delta x\\ &\approx \frac{1}{2\sqrt{100}}\cdot 2 \quad \text{Replacing } x \text{ with 100 and } \Delta x \text{ with 2}\\ &\approx \frac{1}{2\cdot 10}\cdot 2\\ &\approx \frac{1}{10} = 0.100\end{aligned}$$

The solution is continued on the next page.

We can now approximate $\sqrt{102}$;

$$\sqrt{102} = \sqrt{100} + \Delta y$$
$$= 10 + \Delta y$$
$$\approx 10 + 0.100$$
$$\approx 10.100$$

To five decimal places $\sqrt{102} = 10.09950$. Thus, our approximation is reasonably accurate.

38. Let $y = f(x) = \sqrt{x},\ x = 100,\ \Delta x = 3$

$$\Delta y \approx f'(x)\Delta x$$
$$\approx \frac{1}{2\sqrt{x}} \cdot \Delta x$$
$$\approx \frac{1}{2\sqrt{100}} \cdot (3)$$
$$\approx \frac{3}{20} = 0.150$$

We can now approximate $\sqrt{103}$;

$$\sqrt{103} = \sqrt{100} + \Delta y$$
$$\approx 10 + 0.15$$
$$\approx 10.150$$

To five decimal places $\sqrt{103} = 10.14889$. Thus, our approximation is reasonably accurate.

39. We first think of the number closest to 1005 that is a perfect cube. This is 1000. What we will do is approximate how $y = \sqrt[3]{x}$, changes when x changes from 1000 to 1005. Let

$$y = f(x) = \sqrt[3]{x} = x^{1/3}$$

Then $f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$

Using, $\Delta y \approx f'(x)\Delta x$, we have

$$\Delta y \approx f'(x)\Delta x$$
$$\approx \frac{1}{3\sqrt[3]{x^2}} \cdot \Delta x$$

We are interested in Δy as x changes from 1000 to 1005, so

$$\Delta y \approx \frac{1}{3\sqrt[3]{x^2}} \cdot \Delta x$$

Replacing x with 1000 and Δx with 5, we have

$$\approx \frac{1}{3 \cdot \sqrt[3]{(1000)^2}} \cdot 5$$
$$\approx \frac{1}{3 \cdot 100} \cdot 5$$
$$\approx \frac{1}{60} = 0.017$$

We can now approximate $\sqrt[3]{1005}$;

$$\sqrt[3]{1005} = \sqrt[3]{1000} + \Delta y$$
$$= 10 + \Delta y$$
$$\approx 10 + 0.017$$
$$\approx 10.017$$

To five decimal places $\sqrt[3]{1005} = 10.01664$ Thus, our approximation is reasonably accurate.

40. Let $y = f(x) = \sqrt[3]{x},\ x = 27,\ \Delta x = 1$

$$\Delta y \approx f'(x)\Delta x$$
$$\approx \frac{1}{3 \cdot \sqrt[3]{x^2}} \cdot \Delta x$$
$$\approx \frac{1}{3 \cdot \sqrt[3]{(27)^2}} \cdot (1)$$
$$= \frac{1}{27} \approx 0.037$$

We can now approximate $\sqrt[3]{28}$;

$$\sqrt[3]{28} = \sqrt[3]{27} + \Delta y$$
$$\approx 3 + 0.037$$
$$\approx 3.037$$

To five decimal places $\sqrt[3]{28} = 3.03659$ Thus, our approximation is reasonably accurate.

41. $y = \sqrt{3x-2} = (3x-2)^{1/2}$

First, we find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{1}{2}(3x-2)^{-1/2}(3) = \frac{3}{2\sqrt{3x-2}}.$$

Then

$$dy = \frac{3}{2\sqrt{3x-2}}dx.$$

Note that the expression for dy contains two variables x and dx.

42. $y=\sqrt{x+1}=(x+1)^{1/2}$

First, we find $\frac{dy}{dx}$:

$\frac{dy}{dx}=\frac{1}{2}(x+1)^{-1/2}(1)=\frac{1}{2\sqrt{x+1}}$.

Then

$dy=\frac{1}{2\sqrt{x+1}}dx$.

43. $y=(2x^3+1)^{3/2}$

First, we find $\frac{dy}{dx}$:

$\frac{dy}{dx}=\frac{3}{2}(2x^3+1)^{1/2}(6x^2)$ By the extended power rule

$=9x^2(2x^3+1)^{1/2}$

$=9x^2\sqrt{2x^3+1}.$

Then

$dy=9x^2\sqrt{2x^3+1}\,dx$.

Note that the expression for dy contains two variables x and dx.

44. $y=x^3(2x+5)^2$

First, we find $\frac{dy}{dx}$:

$\frac{dy}{dx}=x^3\left[2(2x+5)(2)\right]+3x^2(2x+5)^2$

$=4x^3(2x+5)+3x^2(2x+5)^2$

$=\left(4x^3+3x^2(2x+5)\right)(2x+5)$.

$=(4x^3+6x^3+15x^2)(2x+5)$

$=(10x^3+15x^2)(2x+5)$

$=5x^2(2x+3)(2x+5).$

Then

$dy=5x^2(2x+3)(2x+5)\,dx$.

45. $y=\frac{x^3+x+2}{x^2+3}$

First, we find $\frac{dy}{dx}$. By the quotient rule we have:

$\frac{dy}{dx}=\frac{(x^2+3)(3x^2+1)-(x^3+x+2)(2x)}{(x^2+3)^2}$

$=\frac{(3x^4+10x^2+3)-(2x^4+2x^2+4x)}{(x^2+3)^2}$.

$=\frac{x^4+8x^2-4x+3}{(x^2+3)^2}.$

Then

$dy=\frac{x^4+8x^2-4x+3}{(x^2+3)^2}dx$.

Note that the expression for dy contains two variables x and dx.

46. $y=\sqrt[5]{x+27}=(x+27)^{1/5}$

First, we find $\frac{dy}{dx}$:

$\frac{dy}{dx}=\frac{1}{5}(x+27)^{-4/5}(1)=\frac{1}{5\cdot\sqrt[5]{(x+27)^4}}$.

Then

$dy=\frac{1}{5\cdot\sqrt[5]{(x+27)^4}}dx$.

47. $y=x^4-2x^3+5x^2+3x-4$

First, we find $\frac{dy}{dx}$:

$\frac{dy}{dx}=4x^3-6x^2+10x+3$.

Then

$dy=(4x^3-6x^2+10x+3)\,dx$.

Note that the expression for dy contains two variables x and dx.

48. $y=(7-x)^8$

First, we find $\frac{dy}{dx}$:

$\frac{dy}{dx}=8(7-x)^7(-1)=-8(7-x)^7$.

Then

$dy=-8(7-x)^7\,dx$.

49. From Exercise 48, we know:

$dy = -8(7-x)^7\,dx$,

when $x=1$ and $dx = 0.01$, we have:

$$dy = -8(7-1)^7(0.01)$$
$$= -8(6)^7(0.01)$$
$$= -22{,}394.88.$$

50. From Exercise 47, we know:

$dy = \left(4x^3 - 6x^2 + 10x + 3\right)dx$.

When $x = 2$ and $dx = 0.1$ we have:

$$dy = \left(4(2)^3 - 6(2)^2 + 10(2) + 3\right)(0.1)$$
$$= (31)(0.1)$$
$$= 3.1$$

51. $y = (3x-10)^5$

First, we find $\dfrac{dy}{dx}$:

$$\frac{dy}{dx} = 5(3x-10)^4(3) \quad \text{By the extended power rule}$$
$$= 15(3x-10)^4$$

Then

$dy = 15(3x-10)^4\,dx$.

When $x = 4$ and $dx = 0.03$ we have:

$$dy = 15(3(4)-10)^4(0.03)$$
$$= 15(2)^4(0.03)$$
$$= 7.2$$

52. $y = x^5 - 2x^3 - 7x$

First, we find $\dfrac{dy}{dx}$:

$$\frac{dy}{dx} = 5x^4 - 6x^2 - 7.$$

Then

$dy = \left(5x^4 - 6x^2 - 7\right)dx$.

When $x = 3$ and $dx = 0.02$ we have:

$$dy = \left(5(3)^4 - 6(3)^2 - 7\right)(0.02)$$
$$= 6.88.$$

53. Let $y = f(x) = x^4 - x^2 + 8$

First we find $f'(x)$:

$f'(x) = 4x^3 - 2x$.

Then

$$dy = f'(x)\,dx$$
$$= \left(4x^3 - 2x\right)dx$$

To approximate $f(5.1)$, we will use $x = 5$ and $dx = 0.1$ to determine the differential dy.

Substituting 5 for x and 0.1 for dx we have:

$$dy = f'(5)\,dx$$
$$= \left(4(5)^3 - 2(5)\right)(0.1)$$
$$= \left(4(125) - 10\right)(0.1)$$
$$= (500-10)(0.1)$$
$$= (490)(0.1)$$
$$= 49$$

Next, we find

$$f(5) = (5)^4 - (5)^2 + 8$$
$$= 625 - 25 + 8$$
$$= 608$$

Now,

$$f(5.1) \approx f(5) + f'(5)\,dx$$
$$\approx 608 + 49$$
$$\approx 657$$

54. Let $y = f(x) = x^3 - 5x + 9$

First we find $f'(x)$:

$f'(x) = 3x^2 - 5$.

Then

$$dy = f'(x)\,dx$$
$$= \left(3x^2 - 5\right)dx$$

To approximate $f(3.2)$, we will use $x = 3$ and $dx = 0.2$ to determine the differential dy.

$$dy = f'(3)\,dx$$
$$= \left(3(3)^2 - 5\right)(0.2)$$
$$= (27-5)(0.2)$$
$$= (22)(0.2)$$
$$= 4.4$$

The solution is continued on the next page.

Next, we find

$$f(3) = (3)^3 - 5(3) + 9$$
$$= 27 - 15 + 9$$
$$= 21$$

Now,

$$f(3.2) \approx f(3) + f'(3)\,dx$$
$$\approx 21 + 4.4$$
$$\approx 25.4$$

55. $S = 0.02235h^{0.42246}w^{0.51456}$

We begin by noticing that we are wanting to estimate the change in surface area due to a change in weight w; therefore, we will first find $\frac{dS}{dw}$. Since $h = 160$, we have:

$$S = 0.02235(160)^{0.42246}\ w^{0.51456}$$
$$= 0.02235(8.53399783)\,w^{0.51456}$$
$$= 0.19073485w^{0.51456}$$

Now we can take the derivative of S with respect to w.

$$\frac{dS}{dw} = 0.19073485(0.51456)\,w^{-0.48544}$$
$$= 0.09814452w^{-0.48544}$$

Therefore,

$$dS = \left(0.09814452w^{-0.48544}\right)dw$$

Now that we have the differential, we can use her weight of 60 kg to approximate how much her surface area changes when her weight drops 1 kg.

We substitute 60 for w and -1 for dw to get:

$$dS \approx \left(0.09814452(60)^{-0.48544}\right)(-1)$$
$$\approx -0.01345$$

The patient's surface area will change by -0.01345 m^2.

56. $N(t) = \dfrac{0.8t + 1000}{5t + 4}$

First we find $N'(t)$ by the quotient rule.

$$N'(t) = \frac{(5t+4)(0.8) - (0.8t+1000)(5)}{(5t+4)^2}$$
$$= \frac{4t + 3.2 - 4t - 5000}{(5t+4)^2}$$
$$= -\frac{4996.8}{(5t+4)^2}$$

The differential is:

$$dN = N'(t)\,dt$$
$$= -\frac{4996.8}{(5t+4)^2}\cdot dt$$

We approximate the change in bodily concentration from 1.0 hr to 1.1 hr by using 1.0 for t and 0.1 for dt.

$$dN = -\frac{4996.8}{\left(5(1.0)+4\right)^2}\cdot(0.1)$$
$$= -\frac{4996.8}{(9)^2}(0.1)$$
$$\approx -6.16889$$

Next, we approximate the change in bodily concentration from 2.8 hr to 2.9 hr by using 2.8 for t and 0.1 for dt.

$$dN = -\frac{4996.8}{\left(5(2.8)+4\right)^2}\cdot(0.1)$$
$$= -\frac{4996.8}{(18)^2}(0.1)$$
$$\approx -1.54222$$

The concentration changes more from 1.0 hr to 1.1 hr.

57. $p(x) = 0.06x^3 - 0.5x^2 + 1.64x + 24.76$

First we find $p'(x)$.

$$p'(x) = 0.06(3x^2) - 0.5(2x) + 1.64$$
$$= 0.18x^2 - x + 1.64.$$

The differential is:

$$dp = p'(x)\,dx$$
$$= \left(0.18x^2 - x + 1.64\right)dx$$

Since x is the number of years since 2008, we have 2010 implies $x = 2$ and 2012 implies $x = 4$. To estimate the change in ticket prices from 2010 and 2012, we substitute 2 for x and 2 for dx.

$$dp = \left(0.18(2)^2 - (2) + 1.64\right)(2)$$
$$= (0.36)(2) = 0.72$$

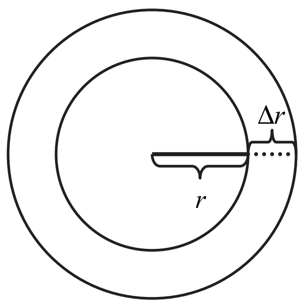
To estimate the change in ticket prices from 2014 and 2016, we substitute 6 for x and 2 for dx.

$$dp = \left(0.18(6)^2 - (6) + 1.64\right)(2)$$
$$= (2.12)(2) = 4.24$$

Ticket prices will increase more between 2014 and 2016.

58. The circumference of the earth, which is the original length of the rope, is given by $C(r) = 2\pi r$, where r is the radius of the earth.

We need to find the change in the length of the radius, Δr, when the length of the rope is increased 10 feet.



Using differentials, $\Delta C \approx C'(r)\Delta r$ represents the change in the length of the rope. Therefore, $\Delta C = 10$, and we have:

$10 = C'(r)\Delta r$

Noticing that $C'(r) = 2\pi$, we have:

$10 = 2\pi \cdot \Delta r$

$\frac{10}{2\pi} = \Delta r$

Therefore, the rope is raised approximately

$\Delta r = \frac{5}{\pi} \approx 1.59$ feet above the earth.

59. $A(x) = \frac{C(x)}{x}$

To find Marginal Average Cost, we take the derivative of the average cost function. By the quotient rule we have:

$$A'(x) = \frac{x \cdot C'(x) - C(x)(1)}{x^2}$$
$$= \frac{x \cdot C'(x) - C(x)}{x^2}.$$

60. a) The surface area of the water tank is given by $A = 2\pi r^2$.

Calculating the differential we have:

$dA = A'(r)dr = 4\pi r dr.$

The tolerance is in feet is ± 0.5.

$dA = 4(3.14)(100)(\pm 0.5) \approx \pm 628$

The approximate difference in surface area when the tolerance is taken into consideration is ± 628 square feet.

b) The possible extra area is 628 square feet. Since each additional can will cover 300 square feet, they will need to bring 3 extra cans to account for any extra area.

c) 3 additional cans will cost the painters $90.

61. The volume of the spherical cavern is given by:

$V(r) = \frac{4}{3}\pi r^3$

First we find the derivatve of the volume function.

$V'(r) = \frac{4}{3}\pi(3r^2) = 4\pi r^2.$

Therefore, the differential is:

$dV = 4\pi r^2 dr$

Substituting the given information, we have:

$dV = 4(3.14)(400)^2(2) \approx 4,019,200.$

The enlarged cavern will contain an additional 4,019,200 cubic feet.

62. $p = 100 - \sqrt{x}$

$$R(x) = p \cdot x$$
$$= (100 - \sqrt{x})x$$
$$= 100x - x^{3/2}.$$
$$R'(x) = 100 - \frac{3}{2}x^{1/2} = 100 - \frac{3\sqrt{x}}{2}.$$

63. $p = 400 - x$

Since revenue is price times quantity, the revenue function is given by:

$$R(x) = p \cdot x$$
$$= (400 - x)x$$
$$= 400x - x^2$$

To find the marginal revenue, we take the derivative of the revenue function. Thus:

$R'(x) = 400 - 2x$

64. $p = 500 - x$

$$R(x) = p \cdot x$$
$$= (500 - x)x$$
$$= 500x - x^2$$
$$R'(x) = 500 - 2x.$$

65. $p=\frac{4000}{x}+3$

Since revenue is price times quantity, the revenue function is given by:

$$R(x)=p\cdot x$$
$$=\left(\frac{4000}{x}+3\right)x$$
$$=4000+3x$$

To find the marginal revenue, we take the derivative of the revenue function. Thus:

$$R'(x)=3$$

66. $p=\frac{3000}{x}+5$

$$R(x)=p\cdot x$$
$$=\left(\frac{3000}{x}+5\right)x$$
$$=3000+5x.$$
$$R'(x)=5.$$

67. Answers will vary. Calculus in its present form was essentially developed independently in the 17th century by Isaac Newton and Gottfried Wilhelm von Leibniz. During the 18^{th} century calculus was challenged by some philosophers and religious leaders who argued that the infinitely small quantities represented by differentials were meaningless. These critics were silenced when the concept of “limit” was introduced. In it, a differential was not thought of as an infinitely small quantity; rather, a derivative was considered to be the limit approached by two differentials as each becomes infinitely small.

68. For a function $y=f(x)$, the differential, *dy,* can be used to approximate the true change in the value of $f(x)$ when a small change is made in the value of *x*.

Exercise Set 2.7

1. a) The demand function is
$q = D(x) = 400 - x$.
The definition of the elasticity of demand is given by: $E(x) = -\frac{x \cdot D'(x)}{D(x)}$. In order to find the elasticity of demand, we need to find the derivative of the demand function first.
$\frac{dq}{dx} = D'(x) = \frac{d}{dx}(400 - x) = -1$.
Next, we substitute -1 for $D'(x)$, and $400 - x$ for $D(x)$ into the expression for elasticity.
$$E(x) = -\frac{x \cdot (-1)}{400 - x} = \frac{x}{400 - x}$$

b) Substituting $x = 125$ into the expression found in part (a) we have:
$$E(125) = \frac{(125)}{400 - (125)} = \frac{125}{275} = \frac{5}{11}$$
Since $E(125) = \frac{5}{11}$ is less than one, the demand is inelastic.

c) The values of x for which $E(x) = 1$ will maximize total revenue. We solve:
$$E(x) = 1$$
$$\frac{x}{400 - x} = 1$$
$$x = 400 - x$$
$$2x = 400$$
$$x = 200$$
A price of \$200 will maximize total revenue.

2. $q = D(x) = 500 - x;\ x = 38$

a) $D'(x) = -1$
$$E(x) = -\frac{x \cdot D'(x)}{D(x)} = -\frac{x \cdot (-1)}{500 - x} = \frac{x}{500 - x}$$

b) $E(38) = \frac{38}{500 - 38} = \frac{19}{231}$
$E(38) < 1$, so demand is inelastic.

c) Solve $E(x) = 1$
$$\frac{x}{500 - x} = 1$$
$$x = 500 - x$$
$$2x = 500$$
$$x = 250$$
A price of \$250 will maximize total revenue.

3. $q = D(x) = 200 - 4x;\ x = 46$

a) $D'(x) = -4$
$$E(x) = -\frac{x \cdot D'(x)}{D(x)} = -\frac{x \cdot (-4)}{200 - 4x} = \frac{4x}{200 - 4x}$$
$$= \frac{x}{50 - x}$$

b) Substituting $x = 46$ into the expression found in part (a) we have:
$$E(46) = \frac{46}{50 - 46} = \frac{46}{4} = \frac{23}{2} = 11.5$$
Since $E(46) > 1$, demand is elastic.

c) We solve $E(x) = 1$
$$\frac{x}{50 - x} = 1$$
$$x = 50 - x$$
$$2x = 50$$
$$x = 25$$
A price of \$25 will maximize total revenue.

4. $q = D(x) = 500 - 2x;\ x = 57$

a) $D'(x) = -2$
$$E(x) = -\frac{x \cdot D'(x)}{D(x)} = -\frac{x \cdot (-2)}{500 - 2x} = \frac{2x}{500 - 2x}$$
$$= \frac{x}{250 - x}$$

b) $E(57) = \frac{57}{250 - 57} = \frac{57}{193}$
$E(57) < 1$, so demand is inelastic.

c) Solve $E(x) = 1$
$$\frac{x}{250 - x} = 1$$
$$x = 250 - x$$
$$2x = 250$$
$$x = 125$$
A price of \$125 will maximize total revenue.

5. $q = D(x) = \dfrac{400}{x}$; $x = 50$

a) First, we rewrite the demand function.

$D(x) = \dfrac{400}{x} = 400x^{-1}$.

Next, we take the derivative of the demand function, using the Power Rule.

$D'(x) = 400(-1)x^{-2} = -400x^{-2}$

Making the appropriate substitutions into the elasticity function, we have

$$E(x) = -\frac{x \cdot D'(x)}{D(x)} = -\frac{x \cdot (-400x^{-2})}{400x^{-1}} =$$
$$= \frac{400x^{-1}}{400x^{-1}} = 1$$

Therefore, $E(x) = 1$ for all values of x.

b) $E(50) = 1$, so demand is unit elastic.

c) $E(x) = 1$ for all values of x. Therefore, total revenue is maximized for all values of x. In other words, total revenue is the same regardless of the price.

6. $q = D(x) = \dfrac{3000}{x}$; $x = 60$

a) $D(x) = \dfrac{3000}{x} = 3000x^{-1}$

$D'(x) = -3000x^{-2}$

Making the appropriate substitutions into the elasticity function, we have

$$E(x) = -\frac{x \cdot D'(x)}{D(x)} = -\frac{x \cdot (-3000x^{-2})}{3000x^{-1}} =$$
$$= \frac{3000x^{-1}}{3000x^{-1}} = 1$$

Therefore, $E(x) = 1$ for all values of x.

b) $E(60) = 1$, so demand is unit elastic.

c) $E(x) = 1$ for all values of x. Therefore, total revenue is maximized for all values of x. In other words, total revenue is the same regardless of the price.

7. $q = D(x) = \sqrt{600 - x}$; $x = 100$

a) First rewrite the demand function:

$D(x) = (600 - x)^{1/2}$.

Next, we take the derivative of the demand function, using the Chain Rule:

$$D'(x) = \frac{1}{2}(600 - x)^{-1/2} \cdot \frac{d}{dx}(600 - x)$$
$$= \frac{-1}{2\sqrt{600 - x}}$$

Making the appropriate substitutions into the elasticity function, we have

$$E(x) = -\frac{x \cdot D'(x)}{D(x)} = -\frac{x \cdot \left(\frac{-1}{2\sqrt{600 - x}}\right)}{\sqrt{600 - x}} =$$
$$= \frac{\frac{x}{2\sqrt{600 - x}}}{\sqrt{600 - x}} = \frac{x}{2(600 - x)}$$
$$= \frac{x}{1200 - 2x}$$

b) Substituting $x = 100$ into the expression found in part (a) we have:

$$E(100) = \frac{100}{1200 - 2(100)} = \frac{100}{1000} = \frac{1}{10}$$

Since $E(100) < 1$, demand is inelastic.

c) Solve $E(x) = 1$

$$\frac{x}{1200 - 2x} = 1$$
$$x = 1200 - 2x$$
$$3x = 1200$$
$$x = 400$$

A price of \$400 will maximize total revenue.

8. $q = D(x) = \sqrt{300 - x}$; $x = 250$

a) $D(x) = (300 - x)^{1/2}$.

$$D'(x) = \frac{1}{2}(300 - x)^{-1/2}\left(\frac{d}{dx}(300 - x)\right)$$
$$= \frac{-1}{2\sqrt{300 - x}}$$

$$E(x) = -\frac{x \cdot D'(x)}{D(x)} = -\frac{x \cdot \left(\frac{-1}{2\sqrt{300 - x}}\right)}{\sqrt{300 - x}} =$$
$$= \frac{\frac{x}{2\sqrt{300 - x}}}{\sqrt{300 - x}} = \frac{x}{2(300 - x)}$$
$$= \frac{x}{600 - 2x}$$

b) $E(250)=\dfrac{250}{600-2(250)}=\dfrac{5}{2}$

Since $E(250)>1$, demand is elastic.

c) Solve $E(x)=1$

$$\frac{x}{600-2x}=1$$
$$x=600-2x$$
$$3x=600$$
$$x=200$$

A price of \$200 will maximize total revenue.

9. $q=D(x)=\dfrac{100}{(x+3)^2}$; $x=1$

a) First, we rewrite the demand function:

$D(x)=100(x+3)^{-2}$.

Next, we take the derivative of the demand function, using the Chain Rule:

$$D'(x)=100(-2)(x+3)^{-3}\left(\frac{d}{dx}(x+3)\right)$$
$$=-200(x+3)^{-3}$$
$$=-\frac{200}{(x+3)^3}$$

Making the appropriate substitutions into the elasticity function, we have:

$$E(x)=-\frac{x\cdot D'(x)}{D(x)}=-\frac{x\cdot\left(-\dfrac{200}{(x+3)^3}\right)}{\dfrac{100}{(x+3)^2}}$$
$$=x\cdot\left(\frac{200}{(x+3)^3}\right)\frac{(x+3)^2}{100}$$
$$=\frac{2x}{x+3}$$

b) Substituting $x=1$ into the expression found in part (a) we have:

$E(1)=\dfrac{2\cdot(1)}{1+3}=\dfrac{2}{4}=\dfrac{1}{2}$

Since $E(1)<1$, demand is inelastic.

c) Solve $E(x)=1$

$$\frac{2x}{x+3}=1$$
$$2x=x+3$$
$$x=3$$

A price of \$3 will maximize total revenue.

10. $q=D(x)=\dfrac{500}{(2x+12)^2}$; $x=8$

a) $D(x)=500(2x+12)^{-2}$.

$$D'(x)=500(-2)(2x+12)^{-3}(2)$$
$$=-2000(2x+12)^{-3}$$
$$=-\frac{2000}{(2x+12)^3}$$

Making the appropriate substitutions into the elasticity function, we have

$$E(x)=-\frac{x\cdot D'(x)}{D(x)}=-\frac{x\cdot\left(-\dfrac{2000}{(2x+12)^3}\right)}{\dfrac{500}{(2x+12)^2}}=$$
$$=x\cdot\left(\frac{2000}{(2x+12)^3}\right)\frac{(2x+12)^2}{500}=\frac{2x}{x+6}$$

b) $E(8)=\dfrac{2\cdot(8)}{(8)+6}=\dfrac{16}{14}=\dfrac{8}{7}$

Since $E(8)>1$, demand is elastic.

c) Solve $E(x)=1$

$$\frac{2x}{x+6}=1$$
$$2x=x+6$$
$$x=6$$

A price of \$6 will maximize total revenue.

11. $q=D(x)=50{,}000+300x-3x^2$

a) $D'(x)=300-6x$

$$E(x)=-\frac{x\cdot D'(x)}{D(x)}$$
$$=-\frac{x\cdot(300-6x)}{50{,}000+300x-3x^2}$$
$$=-\frac{300x-6x^2}{50{,}000+300x-3x^2}$$
$$=\frac{6x^2-300x}{50{,}000+300x-3x^2}$$

b) Substituting in to the elasticity of demand we have:

$$E(75)=\frac{6(75)^2-300(75)}{50,000+300(75)-3(75)^2}$$
$$=\frac{11,250}{55,625}$$
$$\approx 0.20.$$

Since $E(75)<1$, the demand for oil is inelastic at $75 a barrel.

c) Substituting in to the elasticity of demand we have:

$$E(100)=\frac{6(100)^2-300(100)}{50,000+300(100)-3(100)^2}$$
$$=\frac{30,000}{50,000}$$
$$\approx 0.6.$$

Since $E(100)<1$, the demand for oil is inelastic at $100 a barrel.

d) Substituting in to the elasticity of demand we have:

$$E(125)=\frac{6(125)^2-300(125)}{50,000+300(125)-3(125)^2}$$
$$=\frac{56,250}{40,625}$$
$$\approx 1.38.$$

Since $E(125)>1$, the demand for oil is elastic at $125 a barrel.

e) Revenue will be maximized when $E(x)=1$.

Therefore, we solve:

$$\frac{6x^2-300x}{50,000+300x-3x^2}=1$$
$$6x^2-300x=50,000+300x-3x^2$$
$$9x^2-600x-50,000=0$$

Using the quadratic formula, we find that the solutions to the equation.

$$x=\frac{-(-600)\pm\sqrt{(-600)^2-4(9)(-50000)}}{2(9)}$$
$$x=\frac{600\pm\sqrt{2,160,000}}{18}$$
$$x\approx -48.316 \quad \text{or} \quad x\approx 114.983.$$

The only solution that is feasible is $x\approx 114.98$. Thus, oil revenues will be maximized when price is $114.98 a barrel.

f) Substituting the answer found in part (e) into the demand function we have:

$$D(114.98)=50,000+300(114.98)-3(114.98)^2$$
$$=44,832.80.$$

The demand for oil is about 44,832 million barrels per day or 44.8 billion barrels per day at a price of $114.98 a barrel. At the time this solution was created, the price of oil was $100.83 per barrel. Thus according to this model, by increasing the price, oil producers could increase revenue.

g) The demand for oil is inelastic at $110 a barrel. Therefore an increase in price will result in an increase in total revenue.

12. $q=D(x)=967-25x$

a) $D'(x)=-25$

$$E(x)=-\frac{x\cdot D'(x)}{D(x)}=-\frac{x\cdot(-25)}{967-25x}$$
$$=\frac{25x}{967-25x}$$

b) We set $E(x)=1$ and solve for x.

$$\frac{25x}{967-25x}=1$$
$$25x=967-25x$$
$$50x=967$$
$$x=19.34$$

Demand is unitary elastic when price is 19.34 cents.

c) Demand is elastic when $E(x)>1$.

Testing a value on each side of 19.34 cents, we have:

$$E(19)=\frac{25\cdot 19}{967-25\cdot 19}\approx 0.97<1$$
$$E(20)=\frac{25\cdot 20}{967-25\cdot 20}\approx 1.07>1$$

Therefore, the demand for cookies is elastic for prices greater than 19.34 cents.

d) Demand is inelastic when $E(x)<1$.

Using the calculations from part (c), we see that the demand for cookies is inelastic for prices less than 19.34 cents.

e) Total revenue is maximized when $E(x)=1$.

In part (b) we showed that $E(x)=1$ when price was 19.34 cents. Therefore, revenue will be maximized when price is 19.34 cents.

f) We have shown that the demand for cookies is elastic when the price of cookies is 20 cents. Therefore a small increase in price will cause total revenue to decrease.

13. $q = D(x) = \sqrt{200 - x^3}$

a) First, we rewrite the demand function to make it easier to find the derivative.

$D(x) = \left(200 - x^3\right)^{1/2}$.

Next, using the Chain Rule, we have:

$$D'(x) = \frac{1}{2}\left(200 - x^3\right)^{-1/2}\left(-3x^2\right)$$
$$= \frac{-3x^2}{2\sqrt{200 - x^3}}$$

Now, substituting in the elasticity function we get:

$$E(x) = -\frac{x \cdot D'(x)}{D(x)} = -\frac{x \cdot \left(\frac{-3x^2}{2\sqrt{200 - x^3}}\right)}{\sqrt{200 - x^3}}$$
$$= \frac{3x^3}{2\left(\sqrt{200 - x^3}\right)^2}$$
$$= \frac{3x^3}{2\left(200 - x^3\right)}$$
$$= \frac{3x^3}{400 - 2x^3}$$

b) $$E(3) = \frac{3(3)^3}{400 - 2(3)^3}$$
$$= \frac{81}{346}$$
$$\approx 0.2341$$

Since $E(3) < 1$, the demand for computer games is inelastic when price is \$3.

c) From part (b) we know that the demand for computer games is inelastic at a price of \$3. Therefore an increase in the price of computer games will lead to an increase in the total revenue.

14. $q = D(x) = \dfrac{2x + 300}{10x + 11}$

a) Using the Quotient Rule, we have:

$$D'(x) = \frac{(10x + 11) \cdot 2 - (2x + 300) \cdot 10}{(10x + 11)^2}$$
$$= \frac{20x + 22 - (20x + 3000)}{(10x + 11)^2}$$
$$= \frac{-2978}{(10x + 11)^2}$$

Therefore,

$$E(x) = -\frac{x \cdot D'(x)}{D(x)} = -\frac{x \cdot \frac{-2978}{(10x + 11)^2}}{\frac{2x + 300}{10x + 11}}$$
$$= \frac{2978x}{(10x + 11)^2} \cdot \frac{10x + 11}{2x + 300}$$
$$= \frac{1489x}{(10x + 11)(x + 150)}$$

b) $$E(3) = \frac{1489(3)}{(10(3) + 11)((3) + 150)}$$
$$= \frac{1489}{2091}$$
$$\approx 0.7121$$

Since $E(3) < 1$, the demand for tomato plants is inelastic when the price is \$3 per plant.

c) We determined in part (b) that the demand for tomato plants was inelastic at \$3. Therefore, an increase in the price of tomato plants will lead to an increase in revenue.

15. $q = D(x) = 180 - 10x$

a) First we find the derivative of the demand function.

$D'(x) = -10$

Therefore, the elasticity of demand is:

$$E(x) = -\frac{x \cdot D'(x)}{D(x)}$$
$$= -\frac{x \cdot (-10)}{180 - 10x}$$
$$= -\frac{-10x}{180 - 10x}$$
$$= \frac{10x}{180 - 10x}$$

b) Substituting in to the elasticity of demand we have:

$$E(8) = \frac{10(8)}{180 - 10(8)} = \frac{80}{100} = 0.8.$$

c) Revenue will be maximized when $E(x) = 1$.

Therefore, we solve:

$$\frac{10x}{180 - 10x} = 1$$
$$10x = 180 - 10x$$
$$20x = 180$$
$$x = 9.$$

When the price of sunglass cases is \$9 a case, revenue will be maximized.

d) A 20% increase in the price would increase the price from \$8 a case to \$9.60 a case.

At \$8 a case, demand is:

$q = D(8) = 180 - 10(8) = 100$ cases.

Therefore revenue is price times quantity or $R = p \cdot q = \$8 \cdot 100 = \$800$.

At \$9.60 a case, demand is:

$q = D(9.60) = 180 - 10(9.60) = 84$ cases.

Therefore revenue is price times quantity or $R = p \cdot q = \$9.60 \cdot 84 = \$806.40$. Therefore, revenue will increase if Tipton Industries raises the price 20%.

We know increasing from \$8 to \$9 would increase revenue, and increasing from \$9 to \$9.60 would decrease revenue, we just didn't know how much the increase and decrease would be. Therefore, the answer to part (d) does not contradict the answers to parts (b) and (c). In fact it confirms them.

16. $q = D(x) = \dfrac{k}{x^n}$

a) First, we rewrite the demand function to make it easier to find the derivative.

$D(x) = k \cdot x^{-n}$

Using the Power Rule, we have:

$D'(x) = k \cdot (-n)x^{-n-1} = -nkx^{-n-1}$

Substituting into the elasticity function we have:

$$E(x) = -\frac{x \cdot D'(x)}{D(x)} = -\frac{x \cdot \left(-nkx^{-n-1}\right)}{kx^{-n}} = \frac{nk\left(x \cdot x^{-n-1}\right)}{kx^{-n}} = \frac{nk\left(x^{-n}\right)}{kx^{-n}} = n$$

b) No, the elasticity of demand is constant for all prices. $E(x) = n$.

c) Total revenue is maximized when $E(x) = 1$. Since $E(x) = n$, total revenue will be maximized when $n = 1$.

17. Answers will vary. The elasticity of demand is a measure of the responsiveness of quantity demanded to changes in price. This measure allows economists to determine how sensitive quantity demanded is to price changes and help predict the effect of price changes on total revenue.

18. Answers will vary. In general, the greater the availability of substitutes and the better the substitutes are will cause goods to have a higher elasticity of demand. For example, the demand for tea is relatively elastic because of the wide range of substitutes available, such as coffee, soda, or water. However, a diabetic's demand for insulin is very inelastic because there are no close substitutes for insulin.

Exercise Set 2.8

1. Differentiate implicitly to find $\frac{dy}{dx}$.

We have $3x^3 - y^2 = 8$.

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(3x^3 - y^2\right) = \frac{d}{dx}(8)$$

$$\frac{d}{dx}3x^3 - \frac{d}{dx}y^2 = \frac{d}{dx}8$$

$$9x^2 - 2y \cdot \frac{dy}{dx} = 0 \qquad \text{Next, we isolate } \frac{dy}{dx}.$$

$$-2y\frac{dy}{dx} = -9x^2$$

$$\frac{dy}{dx} = \frac{9x^2}{2y}$$

Find the slope of the tangent line to the curve at $(2,4)$.

Replacing x with 2 and y with 4, we have:

$$\frac{dy}{dx} = \frac{9x^2}{2y} = \frac{9(2)^2}{2(4)} = \frac{36}{8} = \frac{9}{2}.$$

The slope of the tangent line to the curve at $(2,4)$ is $\frac{9}{2}$.

2. $x^3 + 2y^3 = 6$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(x^3 + 2y^3\right) = \frac{d}{dx}(6)$$

$$3x^2 + 6y^2 \cdot \frac{dy}{dx} = 0 \qquad \text{Next, we isolate } \frac{dy}{dx}$$

$$6y^2 \cdot \frac{dy}{dx} = -3x^2$$

$$\frac{dy}{dx} = \frac{-x^2}{2y^2}$$

Find the slope of the tangent line to the curve at $(2,-1)$.

$$\frac{dy}{dx} = \frac{-(2)^2}{2(-1)^2} = \frac{-4}{2} = -2.$$

The slope of the tangent line to the curve at $(2,-1)$ is -2.

3. $2x^3 + 4y^2 = -12$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(2x^3 + 4y^2\right) = \frac{d}{dx}(-12)$$

$$6x^2 + 8y \cdot \frac{dy}{dx} = 0 \qquad \text{Next, we isolate } \frac{dy}{dx}.$$

$$8y \cdot \frac{dy}{dx} = -6x^2$$

$$\frac{dy}{dx} = \frac{-6x^2}{8y}$$

$$\frac{dy}{dx} = -\frac{3x^2}{4y}$$

Find the slope of the tangent line to the curve at $(-2,-1)$.

Replacing x with -2 and y with -1, we have:

$$\frac{dy}{dx} = -\frac{3x^2}{4y} = \frac{-3(-2)^2}{4(-1)} = \frac{-12}{-4} = 3.$$

The slope of the tangent line to the curve at $(-2,-1)$ is 3.

4. $2x^2 - 3y^3 = 5$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(2x^2 - 3y^3\right) = \frac{d}{dx}(5)$$

$$4x - 9y^2 \cdot \frac{dy}{dx} = 0 \qquad \text{Next, we isolate } \frac{dy}{dx}.$$

$$-9y^2 \cdot \frac{dy}{dx} = -4x$$

$$\frac{dy}{dx} = \frac{4x}{9y^2}$$

Find the slope of the tangent line to the curve at $(-2,1)$.

$$\frac{dy}{dx} = \frac{4(-2)}{9(1)^2} = \frac{-8}{9} = -\frac{8}{9}.$$

The slope of the tangent line to the curve at $(-2,1)$ is $-\frac{8}{9}$.

5. $x^2 + y^2 = 1$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(1)$$

$$2x + 2y \cdot \frac{dy}{dx} = 0 \qquad \text{Next, we isolate } \frac{dy}{dx}.$$

$$2y \cdot \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Find the slope of the tangent line to the curve at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

Replacing x with $\frac{1}{2}$ and y with $\frac{\sqrt{3}}{2}$, we have:

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{-1}{\sqrt{3}}.$$

The slope of the tangent line to the curve at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is $-\frac{1}{\sqrt{3}}$.

6. $x^2 - y^2 = 1$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(x^2 - y^2\right) = \frac{d}{dx}(1)$$

$$2x - 2y \cdot \frac{dy}{dx} = 0 \qquad \text{Next, we isolate } \frac{dy}{dx}.$$

$$-2y \cdot \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{x}{y}$$

Find the slope of the tangent line to the curve at $\left(\sqrt{3}, \sqrt{2}\right)$.

$$\frac{dy}{dx} = \frac{\sqrt{3}}{\sqrt{2}} = \sqrt{\frac{3}{2}}.$$

The slope of the tangent line to the curve at $\left(\sqrt{3}, \sqrt{2}\right)$ is $\sqrt{\frac{3}{2}}$.

7. $2x^3y^2 = -18$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(2x^3y^2\right) = \frac{d}{dx}(-18)$$

$$2x^3 \frac{d}{dx} y^2 + y^2 \frac{d}{dx}\left(2x^3\right) = 0 \qquad \text{Product Rule}$$

$$2x^3\left(2y \cdot \frac{dy}{dx}\right) + y^2\left(6x^2\right) = 0$$

$$4x^3y \cdot \frac{dy}{dx} + 6x^2y^2 = 0 \qquad \text{Next, we isolate } \frac{dy}{dx}.$$

$$4x^3y \cdot \frac{dy}{dx} = -6x^2y^2$$

$$\frac{dy}{dx} = \frac{-6x^2y^2}{4x^3y}$$

$$\frac{dy}{dx} = -\frac{3y}{2x}$$

Find the slope of the tangent line to the curve at $(-1, 3)$.

Replacing x with -1 and y with 3, we have:

$$\frac{dy}{dx} = -\frac{3y}{2x} = -\frac{3(3)}{2(-1)} = \frac{9}{2}$$

The slope of the tangent line to the curve at $(-1, 3)$ is $\frac{9}{2}$.

8. $3x^2y^4 = 12$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(3x^2y^4\right) = \frac{d}{dx}(12)$$

$$3x^2\left(4y^3 \cdot \frac{dy}{dx}\right) + y^4 \cdot (3 \cdot 2x) = 0 \qquad \text{Product Rule}$$

$$12x^2y^3 \cdot \frac{dy}{dx} + 6xy^4 = 0 \qquad \text{Next, we isolate } \frac{dy}{dx}.$$

$$12x^2y^3 \cdot \frac{dy}{dx} = -6xy^4$$

$$\frac{dy}{dx} = -\frac{y}{2x}$$

Find the slope of the tangent line to the curve at $(2, -1)$.

$$\frac{dy}{dx} = -\frac{(-1)}{2(2)} = \frac{1}{4}.$$

The slope of the tangent line to the curve at $(2, -1)$ is $\frac{1}{4}$.

9. $x^4 - x^2y^3 = 12$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(x^4 - x^2y^3\right) = \frac{d}{dx}(12)$$

$$\frac{d}{dx}\left(x^4\right) - \frac{d}{dx}\left(x^2y^3\right) = \frac{d}{dx}(12)$$

$$4x^3 - \left[x^2\left(3y^2 \cdot \frac{dy}{dx}\right) + y^3(2x)\right] = 0 \quad \text{Product Rule}$$

$$4x^3 - 3x^2y^2 \cdot \frac{dy}{dx} - 2xy^3 = 0$$

Next, we isolate $\frac{dy}{dx}$.

$$-3x^2y^2 \cdot \frac{dy}{dx} = 2xy^3 - 4x^3$$

$$\frac{dy}{dx} = \frac{2xy^3 - 4x^3}{-3x^2y^2}$$

$$\frac{dy}{dx} = \frac{4x^2 - 2y^3}{3xy^2}$$

Find the slope of the tangent line to the curve at $(-2,1)$.

Replacing x with -2 and y with 1, we have:

$$\frac{dy}{dx} = \frac{4x^2 - 2y^3}{3xy^2} = \frac{4(-2)^2 - 2(1)^3}{3(-2)(1)^2} = -\frac{7}{3}$$

The slope of the tangent line to the curve at $(-2,1)$ is $-\frac{7}{3}$.

10. $x^3 - x^2y^2 = -9$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(x^3 - x^2y^2\right) = \frac{d}{dx}(-9)$$

$$3x^2 - \left[x^2\left(2y \cdot \frac{dy}{dx}\right) + y^2(2x)\right] = 0$$

$$3x^2 - 2x^2y \cdot \frac{dy}{dx} - 2xy^2 = 0$$

$$-2x^2y \cdot \frac{dy}{dx} = 2xy^2 - 3x^2$$

$$\frac{dy}{dx} = \frac{2xy^2 - 3x^2}{-2x^2y}$$

$$\frac{dy}{dx} = \frac{3x - 2y^2}{2xy}$$

Find the slope of the tangent line to the curve at $(3,-2)$.

$$\frac{dy}{dx} = \frac{3(3) - 2(-2)^2}{2(3)(-2)} = \frac{9-8}{-12} = -\frac{1}{12}.$$

The slope of the tangent line to the curve at $(3,-2)$ is $-\frac{1}{12}$.

11. $xy + y^2 - 2x = 0$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(xy + y^2 - 2x\right) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(xy) + \frac{d}{dx}\left(y^2\right) - \frac{d}{dx}(2x) = \frac{d}{dx}(0)$$

$$\left[x\left(\frac{dy}{dx}\right) + y(1)\right] + 2y \cdot \frac{dy}{dx} - 2(1) = 0$$

$$x \cdot \frac{dy}{dx} + y + 2y \cdot \frac{dy}{dx} - 2 = 0$$

$$x \cdot \frac{dy}{dx} + 2y \cdot \frac{dy}{dx} = 2 - y$$

$$(x + 2y) \cdot \frac{dy}{dx} = 2 - y$$

$$\frac{dy}{dx} = \frac{2-y}{x+2y}$$

Find the slope of the tangent line to the curve at $(1,-2)$.

Replacing x with 1 and y with -2, we have:

$$\frac{dy}{dx} = \frac{2-y}{x+2y} = \frac{2-(-2)}{(1)+2(-2)} = \frac{4}{-3} = -\frac{4}{3}$$

The slope of the tangent line to the curve at $(1,-2)$ is $-\frac{4}{3}$.

12. $xy - x + 2y = 3$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}(xy - x + 2y) = \frac{d}{dx}(3)$$

$$\left[x\left(\frac{dy}{dx}\right) + y(1)\right] - 1 + 2 \cdot \frac{dy}{dx} = 0$$

$$x \cdot \frac{dy}{dx} + y - 1 + 2 \cdot \frac{dy}{dx} = 0$$

We isolate $\frac{dy}{dx}$ at the top of the next page.

Continued from the previous page.

$$(x+2)\cdot\frac{dy}{dx}=1-y$$

$$\frac{dy}{dx}=\frac{1-y}{x+2}$$

Find the slope of the tangent line to the curve at $\left(-5,\frac{2}{3}\right)$.

$$\frac{dy}{dx}=\frac{1-\left(\frac{2}{3}\right)}{(-5)+2}=\frac{\frac{1}{3}}{-3}=-\frac{1}{9}.$$

The slope of the tangent line to the curve at $\left(-5,\frac{2}{3}\right)$ is $-\frac{1}{9}$.

13. $4x^3-y^4-3y+5x+1=0$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(4x^3-y^4-3y+5x+1\right)=\frac{d}{dx}(0)$$

$$\frac{d}{dx}4x^3-\frac{d}{dx}y^4-\frac{d}{dx}3y+\frac{d}{dx}5x+\frac{d}{dx}1=0$$

$$12x^2-4y^3\cdot\frac{dy}{dx}-3\cdot\frac{dy}{dx}+5+0=0$$

$$-4y^3\cdot\frac{dy}{dx}-3\cdot\frac{dy}{dx}=-12x^2-5$$

$$\left(-4y^3-3\right)\cdot\frac{dy}{dx}=-12x^2-5$$

$$\frac{dy}{dx}=\frac{-12x^2-5}{-4y^3-3}$$

$$\frac{dy}{dx}=\frac{12x^2+5}{4y^3+3}$$

Find the slope of the tangent line to the curve at $(1,-2)$.

Replacing x with 1 and y with -2, we have:

$$\frac{dy}{dx}=\frac{12x^2+5}{4y^3+3}=\frac{12(1)^2+5}{4(-2)^3+3}=-\frac{17}{29}$$

The slope of the tangent line to the curve at $(1,-2)$ is $-\frac{17}{29}$.

14. $x^2y-2x^3-y^3+1=0$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(x^2y-2x^3-y^3+1\right)=\frac{d}{dx}(0)$$

$$\left[x^2\left(\frac{dy}{dx}\right)+y(2x)\right]-2\left(3x^2\right)-\left(3y^2\right)\cdot\frac{dy}{dx}=0$$

$$x^2\cdot\frac{dy}{dx}+2xy-6x^2-3y^2\cdot\frac{dy}{dx}=0$$

$$\left(x^2-3y^2\right)\cdot\frac{dy}{dx}=6x^2-2xy$$

$$\frac{dy}{dx}=\frac{6x^2-2xy}{x^2-3y^2}$$

Find the slope of the tangent line to the curve at $(2,-3)$.

$$\frac{dy}{dx}=\frac{6(2)^2-2(2)(-3)}{(2)^2-3(-3)^2}=\frac{24+12}{4-27}=\frac{36}{-23}=-\frac{36}{23}$$

The slope of the tangent line to the curve at $(2,-3)$ is $-\frac{36}{23}$.

15. $x^2+2xy=3y^2$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(x^2+2xy\right)=\frac{d}{dx}\left(3y^2\right)$$

$$\frac{d}{dx}x^2+\frac{d}{dx}2xy=\frac{d}{dx}3y^2$$

$$2x+2\left[x\left(\frac{dy}{dx}\right)+y(1)\right]=3\left(2y\cdot\frac{dy}{dx}\right)$$

$$2x+2x\cdot\frac{dy}{dx}+2y=6y\cdot\frac{dy}{dx}$$

$$2x\cdot\frac{dy}{dx}-6y\cdot\frac{dy}{dx}=-2x-2y$$

$$(2x-6y)\cdot\frac{dy}{dx}=-2(x+y)$$

$$\frac{dy}{dx}=\frac{-2(x+y)}{2x-6y}$$

$$\frac{dy}{dx}=\frac{-2(x+y)}{-2(-x+3y)}$$

$$\frac{dy}{dx}=\frac{x+y}{3y-x}$$

16. $2xy+3=0$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}(2xy+3)=\frac{d}{dx}(0)$$
$$2\cdot\left[x\left(\frac{dy}{dx}\right)+y(1)\right]+0=0$$
$$2x\cdot\frac{dy}{dx}+2y=0$$
$$2x\cdot\frac{dy}{dx}=-2y$$
$$\frac{dy}{dx}=-\frac{y}{x}$$

17. $x^2-y^2=16$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(x^2-y^2\right)=\frac{d}{dx}(16)$$
$$\frac{d}{dx}x^2-\frac{d}{dx}y^2=\frac{d}{dx}(16)$$
$$2x-2y\cdot\frac{dy}{dx}=0$$
$$-2y\cdot\frac{dy}{dx}=-2x$$
$$\frac{dy}{dx}=\frac{-2x}{-2y}$$
$$\frac{dy}{dx}=\frac{x}{y}$$

18. $x^2+y^2=25$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(x^2+y^2\right)=\frac{d}{dx}(25)$$
$$2x+2y\cdot\frac{dy}{dx}=0$$
$$2y\cdot\frac{dy}{dx}=-2x$$
$$\frac{dy}{dx}=\frac{-2x}{2y}$$
$$\frac{dy}{dx}=-\frac{x}{y}$$

19. $y^3=x^5$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(y^3\right)=\frac{d}{dx}\left(x^5\right)$$
$$3y^2\cdot\frac{dy}{dx}=5x^4$$
$$\frac{dy}{dx}=\frac{5x^4}{3y^2}$$

20. $y^5=x^3$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(y^5\right)=\frac{d}{dx}\left(x^3\right)$$
$$5y^4\cdot\frac{dy}{dx}=3x^2$$
$$\frac{dy}{dx}=\frac{3x^2}{5y^4}$$

21. $x^2y^3+x^3y^4=11$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(x^2y^3+x^3y^4\right)=\frac{d}{dx}(11)$$
$$\frac{d}{dx}\left(x^2y^3\right)+\frac{d}{dx}\left(x^3y^4\right)=0$$

Notice:

$$\frac{d}{dx}\left(x^2y^3\right)=x^2\left(3y^2\cdot\frac{dy}{dx}\right)+y^3(2x)$$
$$=3x^2y^2\cdot\frac{dy}{dx}+2xy^3$$

and

$$\frac{d}{dx}\left(x^3y^4\right)=x^3\left(4y^3\cdot\frac{dy}{dx}\right)+y^4\left(3x^2\right)$$
$$=4x^3y^3\cdot\frac{dy}{dx}+3x^2y^4$$

Therefore,

$$\frac{d}{dx}\left(x^2y^3\right)+\frac{d}{dx}\left(x^3y^4\right)=0$$
$$3x^2y^2\cdot\frac{dy}{dx}+2xy^3+4x^3y^3\cdot\frac{dy}{dx}+3x^2y^4=0$$

We isolate $\frac{dy}{dx}$, on the next page.

Continued from the previous page.

$$\left(4x^3y^3+3x^2y^2\right)\cdot\frac{dy}{dx}=-3x^2y^4-2xy^3$$
$$\frac{dy}{dx}=\frac{-3x^2y^4-2xy^3}{4x^3y^3+3x^2y^2}$$
$$\frac{dy}{dx}=\frac{xy^2\left(-3xy^2-2y\right)}{xy^2\left(4x^2y+3x\right)}$$
$$\frac{dy}{dx}=-\frac{2y+3xy^2}{4x^2y+3x}$$

22. $x^3y^2-x^5y^3=-19$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(x^3y^2-x^5y^3\right)=\frac{d}{dx}(-19)$$
$$2x^3y\cdot\frac{dy}{dx}+3x^2y^2-3x^5y^2\cdot\frac{dy}{dx}-5x^4y^3=0$$
$$\left(2x^3y-3x^5y^2\right)\frac{dy}{dx}=5x^4y^3-3x^2y^2$$
$$\frac{dy}{dx}=-\frac{5x^4y^3-3x^2y^2}{3x^5y^2-2x^3y}$$
$$\frac{dy}{dx}=\frac{-5x^2y^2+3y}{3x^3y-2x}$$

23. $p^2+p+2x=40$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(p^2+p+2x\right)=\frac{d}{dx}(40)$$
$$2p\cdot\frac{dp}{dx}+\frac{dp}{dx}+2\cdot 1=0$$
$$(2p+1)\cdot\frac{dp}{dx}=-2$$
$$\frac{dp}{dx}=\frac{-2}{2p+1}$$

24. $p^3+p-3x=50$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(p^3+p-3x\right)=\frac{d}{dx}(50)$$
$$3p^2\cdot\frac{dp}{dx}+\frac{dp}{dx}-3\cdot 1=0$$

We isolate $\frac{dp}{dx}$ at the top of the next column.

$$\left(3p^2+1\right)\cdot\frac{dp}{dx}=3$$
$$\frac{dp}{dx}=\frac{3}{3p^2+1}$$

25. $xp^3=24$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(xp^3\right)=\frac{d}{dx}(24)$$
$$x\left(3p^2\cdot\frac{dp}{dx}\right)+p^3(1)=0 \qquad \text{Product Rule}$$
$$3xp^2\cdot\frac{dp}{dx}=-p^3$$
$$\frac{dp}{dx}=\frac{-p^3}{3xp^2}$$
$$\frac{dp}{dx}=-\frac{p}{3x}$$

26. $x^3p^2=108$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(x^3p^2\right)=\frac{d}{dx}(108)$$
$$x^3\left(2p\cdot\frac{dp}{dx}\right)+p^2\left(3x^2\right)=0$$
$$2x^3p\cdot\frac{dp}{dx}=-3x^2p^2$$
$$\frac{dp}{dx}=\frac{-3x^2p^2}{2x^3p}$$
$$\frac{dp}{dx}=-\frac{3p}{2x}$$

27. $\frac{x^2p+xp+1}{2x+p}=1$

Multiply both sides by $2x+p$ to clear the fraction.

$$(2x+p)\left(\frac{x^2p+xp+1}{2x+p}\right)=(1)(2x+p)$$
$$x^2p+xp+1=2x+p$$

The solution is continued on the next page.

Differentiating both sides of the equation on the previous page with respect to x yields:

$$\frac{d}{dx}\left(x^2p+xp+1\right)=\frac{d}{dx}(2x+p)$$
$$\frac{d}{dx}x^2p+\frac{d}{dx}xp+\frac{d}{dx}1=\frac{d}{dx}2x+\frac{d}{dx}p$$
$$x^2\cdot\frac{dp}{dx}+p\cdot 2x+x\cdot\frac{dp}{dx}+p\cdot 1+0=2+\frac{dp}{dx}$$
$$x^2\cdot\frac{dp}{dx}+x\cdot\frac{dp}{dx}-1\cdot\frac{dp}{dx}=2-2xp-p$$
$$\left(x^2+x-1\right)\frac{dp}{dx}=2-2xp-p$$
$$\frac{dp}{dx}=\frac{2-2xp-p}{x^2+x-1}$$

28. $\dfrac{xp}{x+p}=2$

Multiply both sides by $x+p$ to clear the fraction.

$$(x+p)\left(\frac{xp}{x+p}\right)=(2)(x+p)$$
$$xp=2x+2p$$

Differentiate both sides with respect to x.

$$\frac{d}{dx}(xp)=\frac{d}{dx}(2x+2p)$$
$$x\cdot\frac{dp}{dx}+p\cdot 1=2\cdot 1+2\cdot\frac{dp}{dx}$$
$$x\cdot\frac{dp}{dx}-2\cdot\frac{dp}{dx}=2-p$$
$$(x-2)\cdot\frac{dp}{dx}=2-p$$
$$\frac{dp}{dx}=\frac{2-p}{x-2}$$

29. $(p+4)(x+3)=48$

Expanding the left hand side of the equation we have:

$$px+3p+4x+12=48$$
$$px+3p+4x=36$$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}(px+3p+4x)=\frac{d}{dx}(36)$$
$$p\cdot 1+x\cdot\frac{dp}{dx}+3\cdot\frac{dp}{dx}+4\cdot 1=0$$
$$(x+3)\cdot\frac{dp}{dx}=-p-4$$
$$\frac{dp}{dx}=\frac{-p-4}{x+3}$$

30. $1000-300p+25p^2=x$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(1000-300p+25p^2\right)=\frac{d}{dx}(x)$$
$$-300\cdot\frac{dp}{dx}+25\cdot 2p\cdot\frac{dp}{dx}=1$$
$$(50p-300)\cdot\frac{dp}{dx}=1$$
$$\frac{dp}{dx}=\frac{1}{50p-300}$$

31. $G^2+H^2=25$

We differentiate both sides with respect to t.

$$\frac{d}{dt}\left(G^2+H^2\right)=\frac{d}{dt}(25)$$
$$2G\cdot\frac{dG}{dt}+2H\frac{dH}{dt}=0$$
$$2H\cdot\frac{dH}{dt}=-2G\cdot\frac{dG}{dt}$$
$$\frac{dH}{dt}=-\frac{G}{H}\cdot\frac{dG}{dt}$$

We find H when $G=0$:

$$(0)^2+H^2=25$$
$$H^2=25$$
$$H=5,\qquad H\text{ is nonnegative}$$

Next, we substitute 5 in for H, 0 in for G, and 3 in for $\dfrac{dG}{dt}$ to determine $\dfrac{dH}{dt}$.

$$\frac{dH}{dt}=-\frac{G}{H}\cdot\frac{dG}{dt}$$
$$=-\frac{(0)}{(5)}\cdot(3)=0$$

We find H when $G=1$:

$$(1)^2+H^2=25$$
$$H^2=24$$
$$H=\sqrt{24}=2\sqrt{6},\qquad H\text{ is nonnegative}$$

Next, we substitute $2\sqrt{6}$ in for H, 1 in for G, and 3 in for $\dfrac{dG}{dt}$ to determine $\dfrac{dH}{dt}$.

$$\frac{dH}{dt}=-\frac{G}{H}\cdot\frac{dG}{dt}$$
$$=-\frac{(1)}{\left(2\sqrt{6}\right)}\cdot(3)=-\frac{3}{2\sqrt{6}}$$

The solution is continued on the next page.

We find H when $G=3$:

$$(3)^2+H^2=25$$
$$H^2=16$$
$$H=4, \quad H \text{ is nonnegative}$$

Next, we substitute 4 in for H, 3 in for G, and 3 in for $\frac{dG}{dt}$ to determine $\frac{dH}{dt}.$

$$\frac{dH}{dt}=-\frac{G}{H}\cdot\frac{dG}{dt}$$
$$=-\frac{(3)}{(4)}\cdot(3)=-\frac{9}{4}$$

32. $A^3+B^3=9$

We differentiate both sides with respect to *t.*

$$\frac{d}{dt}\left(A^3+B^3\right)=\frac{d}{dt}(9)$$
$$3A^2\cdot\frac{dA}{dt}+3B^2\cdot\frac{dB}{dt}=0$$
$$3A^2\cdot\frac{dA}{dt}=-3B^2\cdot\frac{dB}{dt}$$
$$\frac{dA}{dt}=\frac{-3B^2}{3A^2}\cdot\frac{dB}{dt}$$
$$\frac{dA}{dt}=\frac{-B^2}{A^2}\cdot\frac{dB}{dt}$$

We find B when $A=2$:

$$A^3+B^3=9$$
$$(2)^3+B^3=9$$
$$8+B^3=9$$
$$B^3=1$$
$$B=1$$

Next, we substitute 2 for A, 1 for B, and 3 for $\frac{dB}{dt}$ into the formula for $\frac{dA}{dt}$:

$$\frac{dA}{dt}=\frac{-(1)^2}{(2)^2}\cdot(3)=-\frac{3}{4}$$

33. $R(x)=50x-0.5x^2$

Differentiating with respect to time we have:

$$\frac{d}{dt}R(x)=\frac{d}{dt}\left(50x-0.5x^2\right)$$
$$\frac{dR}{dt}=50\cdot\frac{dx}{dt}-x\cdot\frac{dx}{dt}$$
$$\frac{dR}{dt}=(50-x)\cdot\frac{dx}{dt}$$

Next, we substitute 10 for x and 5 for dx/dt.

$$\frac{dR}{dt}=(50-10)\cdot 5=(40)\cdot 5=200$$

The rate of change of total revenue with respect to time is \$200 per day.

$$C(x)=10x+3$$

Differentiating with respect to time we have:

$$\frac{d}{dt}C(x)=\frac{d}{dt}(10x+3)$$
$$\frac{dC}{dt}=10\cdot\frac{dx}{dt}$$

Next, we substitute 10 for x and 5 for dx/dt.

$$\frac{dC}{dt}=10\cdot(5)=50$$

The rate of change of total cost with respect to time is \$50 per day.

Profit is revenue minus cost. Therefore;

$$P(x)=R(x)-C(x)$$
$$=50x-0.5x^2-(10x+3)$$
$$=-0.5x^2+40x-3$$

Differentiating with respect to time we have:

$$\frac{d}{dt}P(x)=\frac{d}{dt}\left(-0.5x^2+40x-3\right)$$
$$\frac{dP}{dt}=-x\cdot\frac{dx}{dt}+40\frac{dx}{dt}$$
$$\frac{dP}{dt}=(40-x)\cdot\frac{dx}{dt}$$

Next, we substitute 10 for x and 5 for dx/dt.

$$\frac{dP}{dt}=(40-(10))\cdot(5)=(30)(5)=150$$

The rate of change of total profit with respect to time is \$150 per day.

34. $R(x)=50x-0.5x^2$

Differentiating with respect to time we have:

$$\frac{d}{dt}R(x)=\frac{d}{dt}\left(50x-0.5x^2\right)$$
$$\frac{dR}{dt}=50\cdot\frac{dx}{dt}-x\cdot\frac{dx}{dt}$$
$$\frac{dR}{dt}=(50-x)\cdot\frac{dx}{dt}$$

Next, we substitute 30 for x and 20 for dx/dt.

$$\frac{dR}{dt}=(50-30)\cdot 20=(20)\cdot 20=400$$

The rate of change of total revenue with respect to time is \$400 per day.

The solutioin is continued on the next page.

Looking at the cost function
$C(x) = 4x + 10$.
Differentiating with respect to time we have:
$$\frac{d}{dt}C(x) = \frac{d}{dt}(4x+10)$$
$$\frac{dC}{dt} = 4\cdot\frac{dx}{dt}$$
Next, we substitute 30 for *x* and 20 for *dx/dt.*
$$\frac{dC}{dt} = 4\cdot(20) = 80$$
The rate of change of total cost with respect to time is \$80 per day.
Profit is revenue minus cost. Therefore;
$$P(x) = R(x) - C(x)$$
$$= 50x - 0.5x^2 - (4x+10)$$
$$= -0.5x^2 + 46x - 10$$
Differentiating with respect to time we have:
$$\frac{d}{dt}P(x) = \frac{d}{dt}\left(-0.5x^2 + 46x - 10\right)$$
$$\frac{dP}{dt} = -x\cdot\frac{dx}{dt} + 46\frac{dx}{dt}$$
$$\frac{dP}{dt} = (46 - x)\cdot\frac{dx}{dt}$$
Next, we substitute 30 for *x* and 20 for *dx/dt.*
$$\frac{dP}{dt} = (46-(30))\cdot(20) = (16)(20) = 320$$
The rate of change of total profit with respect to time is \$320 per day.

35. $R(x) = 280x - 0.4x^2$

Differentiating with respect to time we have:
$$\frac{d}{dt}R(x) = \frac{d}{dt}\left(280x - 0.4x^2\right)$$
$$\frac{dR}{dt} = 280\cdot\frac{dx}{dt} - 0.8x\cdot\frac{dx}{dt}$$
$$\frac{dR}{dt} = (280 - 0.8x)\cdot\frac{dx}{dt}$$
Next, we substitute 200 for *x* and 300 for *dx/dt.*
$$\frac{dR}{dt} = (280 - 0.8(200))\cdot 300 = 36,000$$
The rate of change of total revenue with respect to time is \$36,000 per day.
$$C(x) = 5000 + 0.6x^2$$
Differentiating with respect to time we have:
$$\frac{d}{dt}C(x) = \frac{d}{dt}\left(5000 + 0.6x^2\right)$$
$$\frac{dC}{dt} = 1.2x\cdot\frac{dx}{dt}$$
Next, we substitute 200 for *x* and 300 for *dx/dt.*
$$\frac{dC}{dt} = 1.2(200)\cdot 300 = 72,000$$
The rate of change of total cost with respect to time is \$72,000 per day.
Profit is revenue minus cost. Therefore;
$$P(x) = R(x) - C(x)$$
$$= 280x - 0.4x^2 - \left(5000 + 0.6x^2\right)$$
$$= -x^2 + 280x - 5000$$
Differentiating with respect to time we have:
$$\frac{d}{dt}P(x) = \frac{d}{dt}\left(-x^2 + 280x - 5000\right)$$
$$\frac{dP}{dt} = -2x\cdot\frac{dx}{dt} + 280\frac{dx}{dt}$$
$$\frac{dP}{dt} = (280 - 2x)\cdot\frac{dx}{dt}$$
Next, we substitute 200 for *x* and 300 for *dx/dt.*
$$\frac{dP}{dt} = (280 - 2(200))\cdot(300) = -36,000$$
The rate of change of total profit with respect to time is $-\$36,000$ per day.

36. $R(x) = 2x$

Differentiating with respect to time we have:
$$\frac{d}{dt}R(x) = \frac{d}{dt}(2x)$$
$$\frac{dR}{dt} = 2\cdot\frac{dx}{dt}$$
Next, we substitute 20 for *x* and 8 for *dx/dt.*
$$\frac{dR}{dt} = 2\cdot 8 = 16$$
The rate of change of total revenue with respect to time is \$16 per day.
$$C(x) = 0.01x^2 + 0.6x + 30$$
Differentiating with respect to time we have:
$$\frac{d}{dt}C(x) = \frac{d}{dt}\left(0.01x^2 + 0.6x + 30\right)$$
$$\frac{dC}{dt} = 0.02x\cdot\frac{dx}{dt} + 0.6\cdot\frac{dx}{dt}$$
$$\frac{dC}{dt} = (0.02x + 0.6)\cdot\frac{dx}{dt}$$
Next, we substitute 20 for *x* and 8 for *dx/dt.*
$$\frac{dC}{dt} = (0.02(20) + 0.6)\cdot 8 = 8$$
The rate of change of total cost with respect to time is \$8 per day.

The solution is continued on the next page.

Profit is revenue minus cost. Therefore;

$$P(x) = R(x) - C(x)$$
$$= 2x - \left(0.01x^2 + 0.6x + 30\right)$$
$$= -0.01x^2 + 1.4x - 30$$

Differentiating with respect to time we have:

$$\frac{d}{dt}P(x) = \frac{d}{dt}\left(-0.01x^2 + 1.4x - 30\right)$$
$$\frac{dP}{dt} = -0.02x \cdot \frac{dx}{dt} + 1.4\frac{dx}{dt}$$
$$\frac{dP}{dt} = (1.4 - 0.02x) \cdot \frac{dx}{dt}$$

Next, we substitute 20 for x and 8 for dx/dt.

$$\frac{dP}{dt} = (1.4 - 0.02(20)) \cdot (8) = 8$$

The rate of change of total profit with respect to time is \$8 per day.

37. $5p + 4x + 2px = 60$

First, we take the derivative of both sides of the equation with respect to t.

$$\frac{d}{dt}[5p + 4x + 2px] = \frac{d}{dt}[60]$$
$$5\frac{dp}{dt} + 4\frac{dx}{dt} + 2\left(\underbrace{p \cdot \frac{dx}{dt} + \frac{dp}{dt} \cdot x}_{\text{Product Rule}}\right) = 0$$
$$5\frac{dp}{dt} + 4\frac{dx}{dt} + 2p \cdot \frac{dx}{dt} + 2x \cdot \frac{dp}{dt} = 0$$

Next, we solve for $\frac{dx}{dt}$.

$$4\frac{dx}{dt} + 2p \cdot \frac{dx}{dt} = -5\frac{dp}{dt} - 2x \cdot \frac{dp}{dt}$$
$$(4 + 2p)\frac{dx}{dt} = -(5 + 2x) \cdot \frac{dp}{dt}$$
$$\frac{dx}{dt} = \frac{-(5 + 2x)}{(4 + 2p)} \cdot \frac{dp}{dt}$$

Substituting 3 for x, 5 for p, and 1.5 for $\frac{dp}{dt}$, we have:

$$\frac{dx}{dt} = \frac{-(5 + 2(3))}{(4 + 2(5))} \cdot (1.5)$$
$$= \frac{-(11)}{14} \cdot (1.5)$$
$$= \frac{-16.5}{14}$$
$$\approx -1.18$$

Sales are changing at a rate of -1.18 sales per day.

38. $R = xp$

From Exercise 37, we know that $\frac{dx}{dt} = \frac{-16.5}{14}$

$$\frac{dR}{dt} = x \cdot \frac{dp}{dt} + p \cdot \frac{dx}{dt}$$

Substituting the appropriate values, we have:

$$\frac{dR}{dt} = (3)(1.5) + (5)\left(\frac{-16.5}{14}\right) \approx -1.39$$

Total revenue is changing at a rate of $-\$1.39$ per day.

39. $A = \pi r^2$

To find the rate of change of the area of the Arctic ice cap with respect to time, we take the derivative of both sides of the equation with respect to t.

$$\frac{d}{dt}A = \frac{d}{dt}\left[\pi r^2\right]$$
$$\frac{dA}{dt} = \pi \frac{d}{dt}\left[r^2\right] \qquad \text{Constant Multiple Rule}$$
$$\frac{dA}{dt} = \pi\left[2r \cdot \frac{dr}{dt}\right] \qquad \text{Chain Rule}$$
$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

In 2013, the r was 792 miles, and $\frac{dr}{dt} = -4.7$ miles per year. Substituting these values into the derivative, we have:

$$\frac{dA}{dt} = 2\pi(792)(-4.7)$$
$$\approx -23{,}388.52899$$
$$\approx -23{,}389$$

Therefore, in 2013 the Arctic ice cap was changing at a rate of $-23{,}389\ \text{mi}^2$ per year. Another way of stating this is to say that the Arctic ice cap was *shrinking* at a rate of 23,389 mi^2/yr.

40. $A = \pi r^2$

$$\frac{d}{dt}A = \frac{d}{dt}\left[\pi r^2\right]$$
$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

Substituting, we have:

$$\frac{dA}{dt} = 2\pi(25)(-1) = -50\pi \approx -157.0796$$

The area of the wound is decreasing at a rate of 157.08 mm^2/day.

41. $S = \frac{\sqrt{hw}}{60}$

First, we substitute 165 for *h*, and then we take the derivative of both sides with respect to *t*.

$$S = \frac{\sqrt{165w}}{60} = \frac{\sqrt{165}}{60} \cdot w^{1/2}$$

$$\frac{d}{dt}[S] = \frac{d}{dt}\left[\frac{\sqrt{165}}{60} \cdot w^{1/2}\right]$$

$$\frac{dS}{dt} = \frac{\sqrt{165}}{60} \cdot \frac{1}{2} w^{-1/2} \cdot \frac{dw}{dt}$$

$$= \frac{\sqrt{165}}{120} \frac{1}{w^{1/2}} \cdot \frac{dw}{dt}$$

$$= \frac{\sqrt{165}}{120\sqrt{w}} \cdot \frac{dw}{dt}$$

Now, we will substitute 70 for *w* and -2 for $\frac{dw}{dt}$.

$$\frac{dS}{dt} = \frac{\sqrt{165}}{120\sqrt{70}} \cdot (-2)$$

$$\approx -0.0256$$

Therefore, Kim's surface area is changing at a rate of $-0.0256\ \text{m}^2$/month. We could also say that Kim's surface area is *decreasing* by 0.0256 m^2/month.

42. $V = \frac{p}{4Lv}\left(R^2 - r^2\right)$

We assume that *r, p, L* and *v* are constants.

a) Taking the derivative of both sides with respect to *t*, we have:

$$\frac{dV}{dt} = \frac{d}{dt}\left[\frac{p}{4Lv}\left(R^2 - r^2\right)\right]$$

$$= \frac{p}{4Lv}\left[\frac{d}{dt}R^2 - \frac{d}{dt}r^2\right]$$

$$= \frac{p}{4Lv}\left[2R \cdot \frac{dR}{dt}\right]$$

$$= \frac{pR}{2Lv} \cdot \frac{dR}{dt}$$

Substituting, we have:

$$\frac{dV}{dt} = \frac{500R}{2(80)(0.003)} \cdot \frac{dR}{dt}$$

$$= \frac{500R}{0.48} \cdot \frac{dR}{dt}$$

b) Using the derivative found in part (a), and substituting the values for *R* and *dR/dt*, we have:

$$\frac{dV}{dt} = \frac{500(0.075)}{0.48} \cdot (-0.0002)$$

$$\approx -0.0156$$

The speed of the blood is changing at a rate of $-0.0156\ \text{mm}/\text{sec}^2$.

43. $V = \frac{p}{4Lv}\left(R^2 - r^2\right)$

We assume that *r, p, L* and *v* are constants.

a) Taking the derivative of both sides with respect to *t*, we have:

$$\frac{dV}{dt} = \frac{d}{dt}\left[\frac{p}{4Lv}\left(R^2 - r^2\right)\right]$$

$$= \frac{p}{4Lv}\left[\frac{d}{dt}R^2 - \frac{d}{dt}r^2\right]$$

$$= \frac{p}{4Lv}\left[2R \cdot \frac{dR}{dt} - 0\right]$$

$$= \frac{pR}{2Lv} \cdot \frac{dR}{dt}$$

Substituting 70 for *L*, 400 for *p* and 0.003 for *v*, we have:

$$\frac{dV}{dt} = \frac{400R}{2(70)(0.003)} \cdot \frac{dR}{dt}$$

$$= \frac{400R}{0.42} \cdot \frac{dR}{dt}$$

$$\approx 952.38R \cdot \frac{dR}{dt}$$

b) Using the derivative in part (a), we substitute 0.00015 for *dR/dt* and 0.1 for *R* to get:

$$\frac{dV}{dt} = 952.38(0.1) \cdot (0.00015)$$

$$\approx 0.0143$$

The speed of the person's blood will be increasing at a rate of 0.0143 mm/sec^2.

44. $D^2 = x^2 + y^2$

After 1 hour,

$$D^2 = 25^2 + 60^2$$

$$D^2 = 4225$$

$$D = 65$$

$$\frac{dx}{dt} = 25 \text{ and } \frac{dy}{dt} = 60$$

The solution is continued on the next page.

Differentiating both sides of the distance equation on the previous page with respect to *t*, we have:

$$\frac{d}{dt}D^2 = \frac{d}{dt}\left[x^2 + y^2\right]$$

$$2D \cdot \frac{dD}{dt} = 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt}$$

$$\frac{dD}{dt} = \frac{x \cdot \frac{dx}{dt} + y \cdot \frac{dy}{dt}}{D}$$

Substituting the appropriate information, we have:

$$\frac{dD}{dt} = \frac{(25)\cdot(25) + (60)\cdot(60)}{(65)} = 65$$

One hour after the cars leave, the distance between the two cars is increasing at a rate of 65 mph.

45. Since the ladder forms a right triangle with the wall and the ground, we know that:

$$x^2 + y^2 = 26^2$$

$$x^2 + y^2 = 676$$

We are looking for $\frac{dy}{dt}$.

Differentiating both sides of the equation with respect to *t*, we have:

$$\frac{d}{dt}\left[x^2 + y^2\right] = \frac{d}{dt}[676]$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2y\frac{dy}{dt} = -2x\frac{dx}{dt} \quad \text{Subtracting}$$

$$\frac{dy}{dt} = \frac{-2x}{2y} \cdot \frac{dx}{dt} \quad \text{Dividing by } 2y$$

$$\frac{dy}{dt} = \frac{-x}{y} \cdot \frac{dx}{dt}$$

The lower end of the wall is being pulled away from the wall at a rate of 5 feet per second; therefore, $\frac{dx}{dt} = 5$.

When the lower end is 10 feet away from the wall, $x = 10$, we substitute and solve for *y* at the top of the next column.

$$(10)^2 + y^2 = 676$$

$$100 + y^2 = 676$$

$$y^2 = 676 - 100$$

$$y^2 = 576$$

$$y = \pm\sqrt{576}$$

$$y = \pm 24$$

$$y = 24 \quad y \text{ must be positive}$$

We substitute 10 for *x*, 24 for *y* and 5 for $\frac{dx}{dt}$ into the derivative to get:

$$\frac{dy}{dt} = \frac{-x}{y} \cdot \frac{dx}{dt}$$

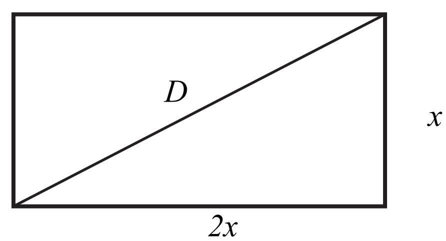
$$= -\frac{(10)}{(24)} \cdot (5)$$

$$= -\frac{25}{12}$$

$$= -2\tfrac{1}{12}$$

When the lower end of the ladder is 10 feet from the wall, the top of the ladder is moving down the wall at a rate of $-2\tfrac{1}{12}$ feet per second.

46. First we draw a picture.



First, from the picture we know that:

$$x^2 + (2x)^2 = D^2$$

$$5x^2 = D^2$$

When *x* is 440 m wide, we have

$$D^2 = 5(440)^2$$

$$D^2 = 968{,}000$$

$$D = 440\sqrt{5}$$

Differentiating both sides of $5x^2 = D^2$ with respect to *t*, we have:

$$10x\frac{dx}{dt} = 2D\frac{dD}{dt}$$

$$\frac{dx}{dt} = \frac{D}{5x}\frac{dD}{dt}$$

The solution is continued on the next page.

Now, when $\frac{dD}{dt} = 90$, $x = 440$, and $D = 440\sqrt{5}$ we have:

$$\frac{dx}{dt} = \frac{\left(440\sqrt{5}\right)}{5(440)}(90)$$
$$= \frac{\sqrt{5}}{5}(90)$$
$$= 18\sqrt{5}$$

We are looking to find how fast the area is changing.

$$A = 2x^2$$
$$\frac{dA}{dt} = 4x\frac{dx}{dt}$$

Substituting, we have:

$$\frac{dA}{dt} = 4(440)\left(18\sqrt{5}\right)$$
$$= 31,680\sqrt{5}$$
$$\approx 70,838.63$$

The area is changing at a rate of 70,838.63 m^2/yr.

47. $V = \frac{4}{3}\pi r^3$

Differentiating both sides with respect to *t*, we have:

$$\frac{dV}{dt} = \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right]$$
$$= \frac{4}{3}\pi \cdot \frac{d}{dt}\left[r^3\right]$$
$$= \frac{4}{3}\pi\left[3r^2\frac{dr}{dt}\right]$$
$$= 4\pi r^2 \cdot \frac{dr}{dt}$$

Next, substituting 0.7 for *dr/dt* and 7.5 for *r*, we have:

$$\frac{dV}{dt} = 4\pi(7.5)^2(0.7)$$
$$= 4\pi(56.25)(0.7)$$
$$= 157.5\pi$$
$$\approx 494.8$$

The cantaloupe's volume is changing approximately at the rate of 494.8 cm^3/week.

48. $\sqrt{x} + \sqrt{y} = 1$

$$\frac{d}{dx}\left(x^{1/2} + y^{1/2}\right) = \frac{d}{dx}[1]$$
$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2} \cdot \frac{dy}{dx} = 0$$
$$\frac{1}{2}y^{-1/2} \cdot \frac{dy}{dx} = -\frac{1}{2}x^{-1/2}$$
$$\frac{dy}{dx} = \frac{-\frac{1}{2}x^{-1/2}}{\frac{1}{2}y^{-1/2}}$$
$$\frac{dy}{dx} = \frac{-x^{-1/2}}{y^{-1/2}}$$
$$\frac{dy}{dx} = \frac{-y^{1/2}}{x^{1/2}} = \frac{-\sqrt{y}}{\sqrt{x}}$$

49. $\frac{1}{x^2} + \frac{1}{y^2} = 5$

$$x^{-2} + y^{-2} = 5$$

Differentiating both sides with respect to *x*, we have:

$$\frac{d}{dx}\left[x^{-2}\right] + \frac{d}{dx}\left[y^{-2}\right] = \frac{d}{dx}[5]$$
$$-2x^{-3} - 2y^{-3}\frac{dy}{dx} = 0$$
$$-2y^{-3}\frac{dy}{dx} = 2x^{-3}$$
$$\frac{dy}{dx} = -\frac{x^{-3}}{y^{-3}}$$
$$\frac{dy}{dx} = -\frac{y^3}{x^3}$$

50. $y^3 = \frac{x-1}{x+1}$

$$\frac{d}{dx}\left[y^3\right] = \frac{d}{dx}\left[\frac{x-1}{x+1}\right]$$
$$3y^2\frac{dy}{dx} = \frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}$$
$$3y^2\frac{dy}{dx} = \frac{2}{(x+1)^2}$$
$$\frac{dy}{dx} = \frac{2}{3y^2(x+1)^2}$$

51. $y^2 = \dfrac{x^2-1}{x^2+1}$

Differentiating both sides with respect to x, we have:

$$\frac{d}{dx}\left[y^2\right] = \frac{d}{dx}\left[\frac{x^2-1}{x^2+1}\right]$$

$$2y\frac{dy}{dx} = \frac{\left(x^2+1\right)(2x)-\left(x^2-1\right)(2x)}{\left(x^2+1\right)^2}$$

$$2y\frac{dy}{dx} = \frac{2x^3+2x-2x^3+2x}{\left(x^2+1\right)^2}$$

$$2y\frac{dy}{dx} = \frac{4x}{\left(x^2+1\right)^2}$$

$$\frac{dy}{dx} = \frac{4x}{2y\left(x^2+1\right)^2}$$

$$\frac{dy}{dx} = \frac{2x}{y\left(x^2+1\right)^2}$$

52. $x^{3/2} + y^{2/3} = 1$

$$\frac{d}{dx}\left[x^{3/2}+y^{2/3}\right] = \frac{d}{dx}(1)$$

$$\frac{3}{2}x^{1/2} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\frac{2}{3}y^{-1/3}\frac{dy}{dx} = -\frac{3}{2}x^{1/2}$$

$$\frac{dy}{dx} = \frac{-\frac{3}{2}x^{1/2}}{\frac{2}{3}y^{-1/3}}$$

$$\frac{dy}{dx} = \frac{-9}{4}\cdot\frac{x^{1/2}}{y^{-1/3}} = \frac{-9}{4}\cdot x^{1/2}y^{1/3}$$

53. $(x-y)^3 + (x+y)^3 = x^5 + y^5$

Differentiating both sides with respect to x, we have:

$$\frac{d}{dx}\left[(x-y)^3+(x+y)^3\right] = \frac{d}{dx}\left[x^5+y^5\right]$$

$$\frac{d}{dx}(x-y)^3 + \frac{d}{dx}(x+y)^3 = \frac{d}{dx}x^5 + \frac{d}{dx}y^5$$

$$3(x-y)^2\cdot\frac{d}{dx}(x-y) + 3(x+y)^2\frac{d}{dx}(x+y) = 5x^4 + 5y^4\frac{dy}{dx}$$

Continued at the top of the next column.

$$3(x-y)^2\left(1-\frac{dy}{dx}\right) + 3(x+y)^2\left(1+\frac{dy}{dx}\right) = 5x^4 + 5y^4\frac{dy}{dx}$$

$$3(x-y)^2 - 3(x-y)^2\frac{dy}{dx} + 3(x+y)^2 + 3(x+y)^2\frac{dy}{dx} = 5x^4 + 5y^4\frac{dy}{dx}$$

$$\left[3(x+y)^2 - 3(x-y)^2 - 5y^4\right]\frac{dy}{dx} = 5x^4 - 3(x-y)^2 - 3(x+y)^2$$

$$\frac{dy}{dx} = \frac{5x^4-3(x-y)^2-3(x+y)^2}{3(x+y)^2-3(x-y)^2-5y^4}$$

Simplification will yield:

$$\frac{dy}{dx} = \frac{5x^4-6x^2-6y^2}{12xy-5y^4}$$

54. $xy + x - 2y = 4$

Differentiate implicitly to find $\dfrac{dy}{dx}$.

$$\frac{d}{dx}[xy+x-2y] = \frac{d}{dx}[4]$$

$$x\frac{dy}{dx} + y\cdot 1 + 1 - 2\frac{dy}{dx} = 0$$

$$(x-2)\frac{dy}{dx} = -1-y$$

$$\frac{dy}{dx} = \frac{-1-y}{x-2} = \frac{1+y}{2-x}$$

Differentiate $\dfrac{dy}{dx}$ implicitly to find $\dfrac{d^2y}{dx^2}$.

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{1+y}{2-x}\right]$$

$$\frac{d^2y}{dx^2} = \frac{(2-x)\left(\frac{dy}{dx}\right)-(1+y)(-1)}{(2-x)^2}$$

$$= \frac{(2-x)\left(\frac{1+y}{2-x}\right)+(1+y)}{(2-x)^2} \quad \text{Substituting } \frac{1+y}{2-x} \text{ for } \frac{dy}{dx}$$

$$= \frac{1+y+1+y}{(2-x)^2}$$

$$= \frac{2+2y}{(2-x)^2}$$

55. $y^2 - xy + x^2 = 5$

Differentiate implicitly to find $\frac{dy}{dx}$.

$$\frac{d}{dx}\left[y^2 - xy + x^2\right] = \frac{d}{dx}[5]$$

$$2y\frac{dy}{dx} - \left[x\frac{dy}{dx} + y\cdot 1\right] + 2x = 0$$

$$(2y - x)\frac{dy}{dx} - y + 2x = 0$$

$$(2y - x)\frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

Differentiate $\frac{dy}{dx}$ implicitly to find $\frac{d^2y}{dx^2}$.

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{y - 2x}{2y - x}\right]$$

$$\frac{d^2y}{dx^2} = \frac{(2y - x)\left(\frac{dy}{dx} - 2\right) - (y - 2x)\left(2\frac{dy}{dx} - 1\right)}{(2y - x)^2}$$

Simplifying the numerator we have:

$$\frac{d^2y}{dx^2} = \left[\left(2y\frac{dy}{dx} - 4y - x\frac{dy}{dx} + 2x\right) - \left(2y\frac{dy}{dx} - y - 4x\frac{dy}{dx} + 2x\right)\right] \div (2y - x)^2$$

$$\frac{d^2y}{dx^2} = \frac{-3y + 3x\frac{dy}{dx}}{(2y - x)^2}$$

Substituting $\frac{y - 2x}{2y - x}$ for $\frac{dy}{dx}$

$$\frac{d^2y}{dx^2} = \frac{-3y + 3x\cdot\frac{y - 2x}{2y - x}}{(2y - x)^2}$$

$$= \frac{-3y\frac{2y - x}{2y - x} + 3x\cdot\frac{y - 2x}{2y - x}}{(2y - x)^2}$$

$$= \frac{-6y^2 + 3xy + 3xy - 6x^2}{(2y - x)^3}$$

$$= \frac{-6y^2 + 6xy - 6x^2}{(2y - x)^3}$$

$$= \frac{-6\left(y^2 - xy + x^2\right)}{(2y - x)^3}$$

56. $x^2 - y^2 = 5$

Differentiate implicitly to find $\frac{dy}{dx}$.

$$\frac{d}{dx}\left[x^2 - y^2\right] = \frac{d}{dx}[5]$$

$$2x - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x}{y}$$

Differentiate $\frac{dy}{dx}$ implicitly to find $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{x}{y}\right]$$

$$= \frac{y\cdot 1 - x\cdot\frac{dy}{dx}}{y^2}$$

$$= \frac{y - x\left[\frac{x}{y}\right]}{y^2} \quad \text{Substituting for } \frac{dy}{dx}$$

$$= \frac{\frac{y^2}{y} - \frac{x^2}{y}}{y^2}$$

$$= \frac{y^2 - x^2}{y^3}$$

57. $x^3 - y^3 = 8$

Differentiate implicitly to find $\frac{dy}{dx}$.

$$\frac{d}{dx}\left[x^3 - y^3\right] = \frac{d}{dx}[8]$$

$$3x^2 - 3y^2\frac{dy}{dx} = 0$$

$$-3y^2\frac{dy}{dx} = -3x^2$$

$$\frac{dy}{dx} = \frac{-3x^2}{-3y^2}$$

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

The solution is continued on the next page.

Differentiate $\frac{dy}{dx}$ implicitly to find $\frac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{x^2}{y^2}\right]$$

$$= \frac{y^2 \cdot (2x) - x^2 \cdot 2y\frac{dy}{dx}}{\left(y^2\right)^2}$$

$$= \frac{2xy^2 - 2x^2y\left[\frac{x^2}{y^2}\right]}{y^4} \quad \text{Substituting for } \frac{dy}{dx}$$

$$= \frac{2xy^2 - \frac{2x^4}{y}}{y^4}$$

Simplifying the derivative, we have:

$$\frac{d^2y}{dx^2} = \frac{\frac{2xy^2}{1} \cdot \frac{y}{y} - \frac{2x^4}{y}}{y^4}$$

$$= \frac{\frac{2xy^3 - 2x^4}{y}}{y^4}$$

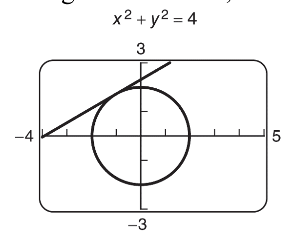
$$= \frac{2xy^3 - 2x^4}{y^5}$$

$$= \frac{2x\left(y^3 - x^3\right)}{y^5}$$

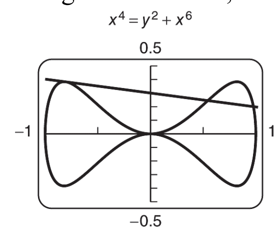
58. Given an equation in x and y where y is a function of x but where it is difficult or impossible to express y in terms of x, implicit differentiation allows us to find the derivative of y with respect to x.

59. One dictionary defines "implicit" as "capable of being understood from something else though unexpressed" or "involved in the nature or essence of something though not revealed, expressed, or developed." When a function y of x is defined implicitly it is written as an equation in x and y where y is not expressed in terms of x but where it is understood or implied that y is indeed a function of x.

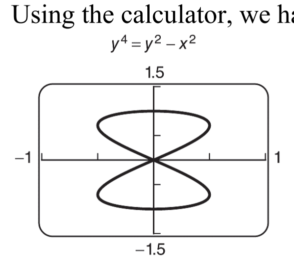
60. Using the calculator, we have:



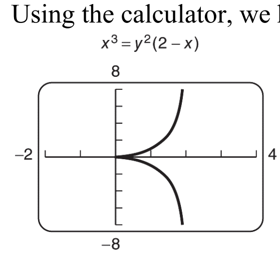
61. Using the calculator, we have:



62. Using the calculator, we have:



63. Using the calculator, we have:



64. Using the calculator, we have: