INSTRUCTOR'S SOLUTIONS MANUAL
SINGLE VARIABLE

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CALCULUS
SECOND EDITION

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Contents

Chapter 1

Functions

1.1 Review of Functions

1.1.1 A function is a rule which assigns each domain element to a unique range element. The independent variable is associated with the domain, while the dependent variable is associated with the range.

1.1.2 The independent variable belongs to the domain, while the dependent variable belongs to the range.

1.1.3 The vertical line test is used to determine whether a given graph represents a function. (Specifically, it tests whether the variable associated with the vertical axis is a function of the variable associated with the horizontal axis.) If every vertical line which intersects the graph does so in exactly one point, then the given graph represents a function. If any vertical line $x = a$ intersects the curve in more than one point, then there is more than one range value for the domain value $x = a$, so the given curve does not represent a function.

1.1.4 $f(2) = \frac{1}{2^3+1} = \frac{1}{9}$. $f(y^2) = \frac{1}{(y^2)^3+1} = \frac{1}{y^6+1}$.

1.1.5 Item i. is true while item ii. isn't necessarily true. In the definition of function, item i. is stipulated. However, item ii. need not be true – for example, the function $f(x) = x^2$ has two different domain values associated with the one range value 4, because $f(2) = f(-2) = 4$.

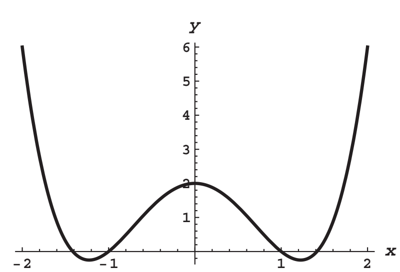
1.1.6 $(f \circ g)(x) = f(g(x)) = f(x^3 - 2) = \sqrt{x^3 - 2}$
$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = x^{3/2} - 2$.
$(f \circ f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$.
$(g \circ g)(x) = g(g(x)) = g(x^3 - 2) = (x^3 - 2)^3 - 2 = x^9 - 6x^6 + 12x^3 - 10$

1.1.7 $f(g(2)) = f(-2) = f(2) = 2$. The fact that $f(-2) = f(2)$ follows from the fact that f is an even function.

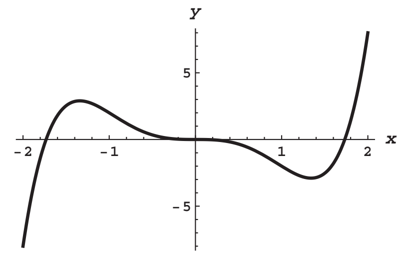
$g(f(-2)) = g(f(2)) = g(2) = -2$.

1.1.8 The domain of $f \circ g$ is the subset of the domain of g whose range is in the domain of f. Thus, we need to look for elements x in the domain of g so that $g(x)$ is in the domain of f.

1.1.9 When f is an even function, we have $f(-x) = f(x)$ for all x in the domain of f, which ensures that the graph of the function is symmetric about the y-axis.



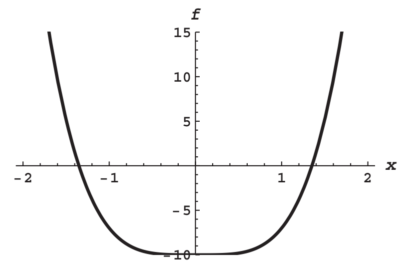
1.1.10 When f is an odd function, we have $f(-x) = -f(x)$ for all x in the domain of f, which ensures that the graph of the function is symmetric about the origin.



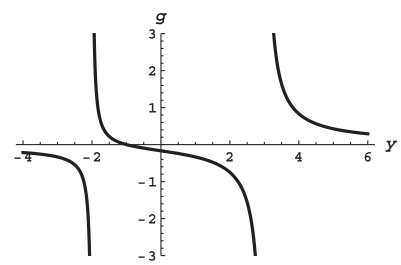
1.1.11 Graph A does not represent a function, while graph B does. Note that graph A fails the vertical line test, while graph B passes it.

1.1.12 Graph A does not represent a function, while graph B does. Note that graph A fails the vertical line test, while graph B passes it.

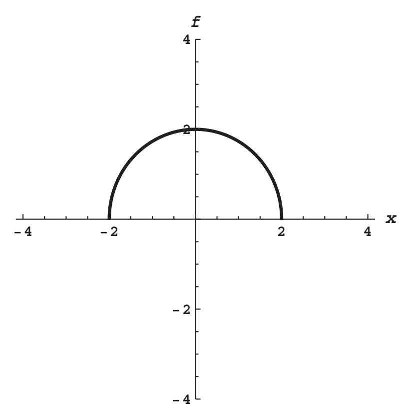
1.1.13 The domain of this function is the set of a real numbers. The range is $[-10, \infty)$.



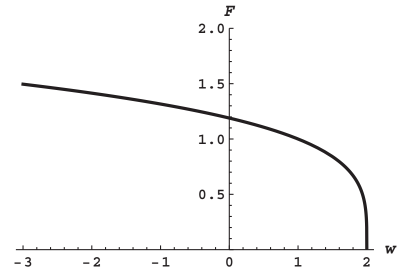
1.1.14 The domain of this function is $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$. The range is the set of all real numbers.



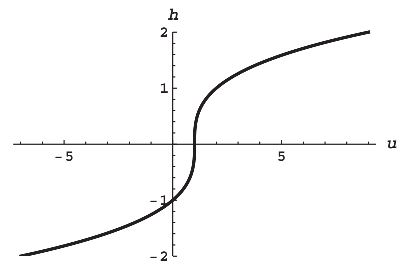
1.1.15 The domain of this function is $[-2, 2]$. The range is $[0, 2]$.



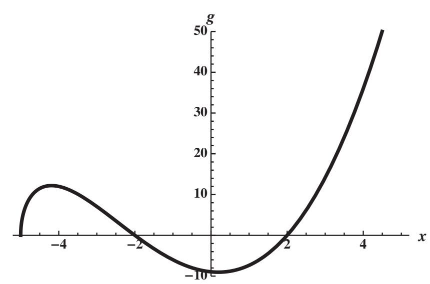
1.1.16 The domain of this function is $(-\infty, 2]$. The range is $[0, \infty)$.



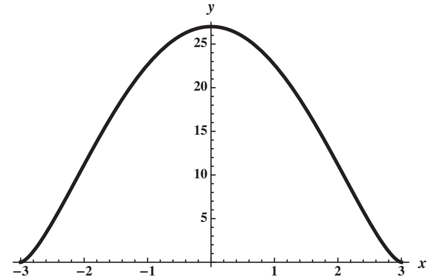
1.1.17 The domain and the range for this function are both the set of all real numbers.



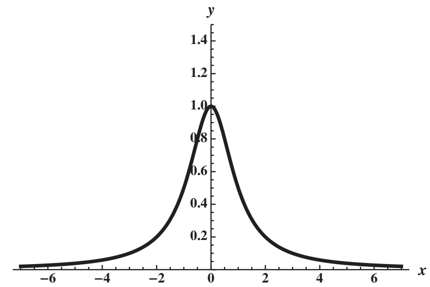
1.1.18 The domain of this function is $[-5, \infty)$. The range is approximately $[-9.03, \infty)$.



1.1.19 The domain of this function is $[-3, 3]$. The range is $[0, 27]$.



1.1.20 The domain of this function is $(-\infty, \infty)]$. The range is $(0, 1]$.



1.1.21 The independent variable t is elapsed time and the dependent variable d is distance above the ground. The domain in context is $[0, 8]$

1.1.22 The independent variable t is elapsed time and the dependent variable d is distance above the water. The domain in context is $[0, 2]$

1.1.23 The independent variable h is the height of the water in the tank and the dependent variable V is the volume of water in the tank. The domain in context is $[0, 50]$

1.1.24 The independent variable r is the radius of the balloon and the dependent variable V is the volume of the balloon. The domain in context is $[0, \sqrt[3]{3/(4\pi)}]$

1.1.25 $f(10) = 96$

1.1.26 $f(p^2) = (p^2)^2 - 4 = p^4 - 4$

1.1.27 $g(1/z) = (1/z)^3 = \frac{1}{z^3}$

1.1.28 $F(y^4) = \frac{1}{y^4-3}$

1.1.29 $F(g(y)) = F(y^3) = \frac{1}{y^3-3}$

1.1.30 $f(g(w)) = f(w^3) = (w^3)^2 - 4 = w^6 - 4$

1.1.31 $g(f(u)) = g(u^2 - 4) = (u^2 - 4)^3$

1.1.32 $\frac{f(2+h)-f(2)}{h} = \frac{(2+h)^2-4-0}{h} = \frac{4+4h+h^2-4}{h} = \frac{4h+h^2}{h} = 4 + h$

1.1.33 $F(F(x)) = F\left(\frac{1}{x-3}\right) = \frac{1}{\frac{1}{x-3}-3} = \frac{1}{\frac{1}{x-3}-\frac{3(x-3)}{x-3}} = \frac{1}{\frac{10-3x}{x-3}} = \frac{x-3}{10-3x}$

1.1.34 $g(F(f(x))) = g(F(x^2 - 4)) = g\left(\frac{1}{x^2-4-3}\right) = \left(\frac{1}{x^2-7}\right)^3$

1.1.35 $f(\sqrt{x+4}) = (\sqrt{x+4})^2 - 4 = x + 4 - 4 = x.$

1.1.36 $F((3x+1)/x) = \frac{1}{\frac{3x+1}{x}-3} = \frac{1}{\frac{3x+1-3x}{x}} = \frac{x}{3x+1-3x} = x.$

1.1.37 $g(x) = x^3 - 5$ and $f(x) = x^{10}$. The domain of h is the set of all real numbers.

1.1.38 $g(x) = x^6 + x^2 + 1$ and $f(x) = \frac{2}{x^2}$. The domain of h is the set of all real numbers.

1.1.39 $g(x) = x^4 + 2$ and $f(x) = \sqrt{x}$. The domain of h is the set of all real numbers.

1.1.40 $g(x) = x^3 - 1$ and $f(x) = \frac{1}{\sqrt{x}}$. The domain of h is the set of all real numbers for which $x^3 - 1 > 0$, which corresponds to the set $(1, \infty)$.

1.1.41 $(f \circ g)(x) = f(g(x)) = f(x^2 - 4) = |x^2 - 4|$. The domain of this function is the set of all real numbers.

1.1.42 $(g \circ f)(x) = g(f(x)) = g(|x|) = |x|^2 - 4 = x^2 - 4$. The domain of this function is the set of all real numbers.

1.1.43 $(f\circ G)(x) = f(G(x)) = f\left(\frac{1}{x-2}\right) = \left|\frac{1}{x-2}\right|$. The domain of this function is the set of all real numbers except for the number 2.

1.1.44 $(f\circ g\circ G)(x) = f(g(G(x))) = f\left(g\left(\frac{1}{x-2}\right)\right) = f\left(\left(\frac{1}{x-2}\right)^2 - 4\right) = \left|\left(\frac{1}{x-2}\right)^2 - 4\right|$. The domain of this function is the set of all real numbers except for the number 2.

1.1.45 $(G\circ g\circ f)(x) = G(g(f(x))) = G(g(|x|)) = G(x^2-4) = \frac{1}{x^2-4-2} = \frac{1}{x^2-6}$. The domain of this function is the set of all real numbers except for the numbers $\pm\sqrt{6}$.

1.1.46 $(F\circ g\circ g)(x) = F(g(g(x))) = F(g(x^2-4)) = F((x^2-4)^2-4) = \sqrt{(x^2-4)^2-4} = \sqrt{x^4-8x^2+12}$. The domain of this function consists of the numbers x so that $x^4-8x^2+12 \geq 0$. Because $x^4-8x^2+12 = (x^2-6)\cdot(x^2-2)$, we see that this expression is zero for $x=\pm\sqrt{6}$ and $x=\pm\sqrt{2}$, By looking between these points, we see that the expression is greater than or equal to zero for the set $(-\infty,-\sqrt{6}]\cup[-\sqrt{2},\sqrt{2}]\cup[\sqrt{2},\infty)$.

1.1.47 $(g\circ g)(x) = g(g(x)) = g(x^2-4) = (x^2-4)^2-4 = x^4-8x^2+16-4 = x^4-8x^2+12$. The domain is the set of all real numbers.

1.1.48 $(G\circ G)(x) = G(G(x)) = G(1/(x-2)) = \frac{1}{\frac{1}{x-2}-2} = \frac{1}{\frac{1-2(x-2)}{x-2}} = \frac{x-2}{1-2x+4} = \frac{x-2}{5-2x}$. Then $G\circ G$ is defined except where the denominator vanishes, so its domain is the set of all real numbers except for $x=\frac{5}{2}$.

1.1.49 Because $(x^2+3)-3 = x^2$, we may choose $f(x) = x-3$.

1.1.50 Because the reciprocal of x^2+3 is $\frac{1}{x^2+3}$, we may choose $f(x)=\frac{1}{x}$.

1.1.51 Because $(x^2+3)^2 = x^4+6x^2+9$, we may choose $f(x)=x^2$.

1.1.52 Because $(x^2+3)^2 = x^4+6x^2+9$, and the given expression is 11 more than this, we may choose $f(x)=x^2+11$.

1.1.53 Because $(x^2)^2+3 = x^4+3$, this expression results from squaring x^2 and adding 3 to it. Thus we may choose $f(x)=x^2$.

1.1.54 Because $x^{2/3}+3 = (\sqrt[3]{x})^2+3$, we may choose $f(x)=\sqrt[3]{x}$.

1.1.55

a. $(f\circ g)(2) = f(g(2)) = f(2) = 4$.

b. $g(f(2)) = g(4) = 1$.

c. $f(g(4)) = f(1) = 3$.

d. $g(f(5)) = g(6) = 3$.

e. $f(f(8)) = f(8) = 8$.

f. $g(f(g(5))) = g(f(2)) = g(4) = 1$.

1.1.56

a. $h(g(0)) = h(0) = -1$.

b. $g(f(4)) = g(-1) = -1$.

c. $h(h(0)) = h(-1) = 0$.

d. $g(h(f(4))) = g(h(-1)) = g(0) = 0$.

e. $f(f(f(1))) = f(f(0)) = f(1) = 0$.

f. $h(h(h(0))) = h(h(-1)) = h(0) = -1$.

g. $f(h(g(2))) = f(h(3)) = f(0) = 1$.

h. $g(f(h(4))) = g(f(4)) = g(-1) = -1$.

i. $g(g(g(1))) = g(g(2)) = g(3) = 4$.

j. $f(f(h(3))) = f(f(0)) = f(1) = 0$.

1.1.57 $\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2-x^2}{h} = \frac{(x^2+2hx+h^2)-x^2}{h} = \frac{h(2x+h)}{h} = 2x+h$.

1.1.58 $\frac{f(x+h)-f(x)}{h} = \frac{4(x+h)-3-(4x-3)}{h} = \frac{4x+4h-3-4x+3}{h} = \frac{4h}{h} = 4$.

1.1.59 $\frac{f(x+h)-f(x)}{h} = \frac{\frac{2}{x+h}-\frac{2}{x}}{h} = \frac{\frac{2x-2(x+h)}{x(x+h)}}{h} = \frac{2x-2x-2h}{hx(x+h)} = -\frac{2h}{hx(x+h)} = -\frac{2}{x(x+h)}.$

1.1.60 $\frac{f(x+h)-f(x)}{h} = \frac{2(x+h)^2-3(x+h)+1-(2x^2-3x+1)}{h} = \frac{2x^2+4xh+2h^2-3x-3h+1-2x^2+3x-1}{h} = \frac{4xh+2h^2-3h}{h} = \frac{h(4x+2h-3)}{h} = 4x+2h-3.$

1.1.61 $\frac{f(x+h)-f(x)}{h} = \frac{\frac{x+h}{x+h+1}-\frac{x}{x+1}}{h} = \frac{\frac{(x+h)(x+1)-x(x+h+1)}{(x+1)(x+h+1)}}{h} = \frac{x^2+x+hx+h-x^2-xh-x}{h(x+1)(x+h+1)} = \frac{h}{h(x+1)(x+h+1)} = \frac{1}{(x+1)(x+h+1)}$

1.1.62 $\frac{f(x)-f(a)}{x-a} = \frac{x^4-a^4}{x-a} = \frac{(x^2-a^2)(x^2+a^2)}{x-a} = \frac{(x-a)(x+a)(x^2+a^2)}{x-a} = (x+a)(x^2+a^2).$

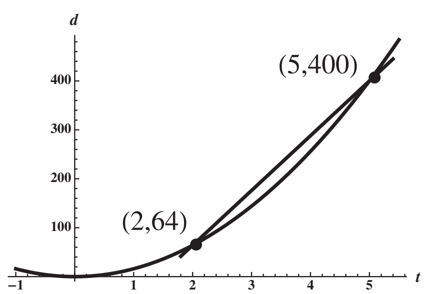
1.1.63 $\frac{f(x)-f(a)}{x-a} = \frac{x^3-2x-(a^3-2a)}{x-a} = \frac{(x^3-a^3)-2(x-a)}{x-a} = \frac{(x-a)(x^2+ax+a^2)-2(x-a)}{x-a} = \frac{(x-a)(x^2+ax+a^2-2)}{x-a} = x^2+ax+a^2-2.$

1.1.64 $\frac{f(x)-f(a)}{x-a} = \frac{4-4x-x^2-(4-4a-a^2)}{x-a} = \frac{-4(x-a)-(x^2-a^2)}{x-a} = \frac{-4(x-a)-(x-a)(x+a)}{x-a} = \frac{(x-a)(-4-(x+a))}{x-a} = -4-x-a.$

1.1.65 $\frac{f(x)-f(a)}{x-a} = \frac{\frac{-4}{x^2}-\frac{-4}{a^2}}{x-a} = \frac{\frac{-4a^2+4x^2}{a^2x^2}}{x-a} = \frac{4(x^2-a^2)}{(x-a)a^2x^2} = \frac{4(x-a)(x+a)}{(x-a)a^2x^2} = \frac{4(x+a)}{a^2x^2}.$

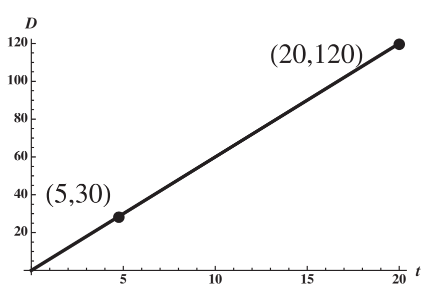
1.1.66 $\frac{f(x)-f(a)}{x-a} = \frac{\frac{1}{x}-x^2-(\frac{1}{a}-a^2)}{x-a} = \frac{\frac{1}{x}-\frac{1}{a}}{x-a} - \frac{x^2-a^2}{x-a} = \frac{\frac{a-x}{ax}}{x-a} - \frac{(x-a)(x+a)}{x-a} = -\frac{1}{ax}-(x+a).$

1.1.67

a. 

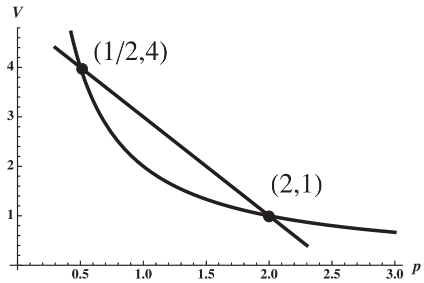
b. The slope of the secant line is given by $\frac{400-64}{5-2} = \frac{336}{3} = 112$ feet per second. The object falls at an average rate of 112 feet per second over the interval $2 \le t \le 5$.

1.1.68

a. 

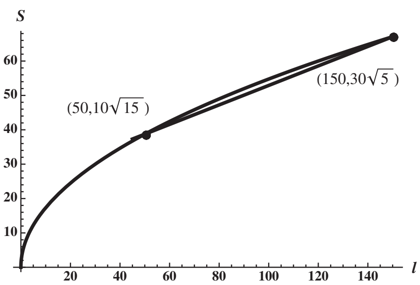
b. The slope of the secant line is given by $\frac{120-30}{20-5} = \frac{90}{15} = 6$ degrees per second. The second hand moves at an average rate of 6 degrees per second over the interval $5 \le t \le 20$.

1.1.69

a. 

b. The slope of the secant line is given by $\frac{1-4}{2-(1/2)} = -\frac{3}{3/2} = -2$ cubic cm per atmosphere. The volume decreases at an average rate of 2 cubic cm per atmosphere over the interval $0.5 \leq p \leq 2$.

1.1.70

a. 

b. The slope of the secant line is given by $\frac{30\sqrt{5}-10\sqrt{15}}{150-50} \approx .2835$ mph per foot. The speed of the car changes with an average rate of about .2835 mph per foot over the interval $50 \leq l \leq 150$.

1.1.71 This function is symmetric about the y-axis, because $f(-x) = (-x)^4 + 5(-x)^2 - 12 = x^4 + 5x^2 - 12 = f(x)$.

1.1.72 This function is symmetric about the origin, because $f(-x) = 3(-x)^5 + 2(-x)^3 - (-x) = -3x^5 - 2x^3 + x = -(3x^5 + 2x^3 - x) = f(x)$.

1.1.73 This function has none of the indicated symmetries. For example, note that $f(-2) = -26$, while $f(2) = 22$, so f is not symmetric about either the origin or about the y-axis, and is not symmetric about the x-axis because it is a function.

1.1.74 This function is symmetric about the y-axis. Note that $f(-x) = 2|-x| = 2|x| = f(x)$.

1.1.75 This curve (which is not a function) is symmetric about the x-axis, the y-axis, and the origin. Note that replacing either x by $-x$ or y by $-y$ (or both) yields the same equation. This is due to the fact that $(-x)^{2/3} = ((-x)^2)^{1/3} = (x^2)^{1/3} = x^{2/3}$, and a similar fact holds for the term involving y.

1.1.76 This function is symmetric about the origin. Writing the function as $y = f(x) = x^{3/5}$, we see that $f(-x) = (-x)^{3/5} = -(x)^{3/5} = -f(x)$.

1.1.77 This function is symmetric about the origin. Note that $f(-x) = (-x)|(-x)| = -x|x| = -f(x)$.

1.1.78 This curve (which is not a function) is symmetric about the x-axis, the y-axis, and the origin. Note that replacing either x by $-x$ or y by $-y$ (or both) yields the same equation. This is due to the fact that $|-x| = |x|$ and $|-y| = |y|$.

1.1.79 Function A is symmetric about the y-axis, so is even. Function B is symmetric about the origin, so is odd. Function C is also symmetric about the y-axis, so is even.

1.1.80 Function A is symmetric about the y-axis, so is even. Function B is symmetric about the origin, so is odd. Function C is also symmetric about the origin, so is odd.

1.1.81

a. True. A real number z corresponds to the domain element $z/2+19$, because $f(z/2+19)=2(z/2+19)-38=z+38-38=z$.

b. False. The definition of function does not require that each range element comes from a unique domain element, rather that each domain element is paired with a unique range element.

c. True. $f(1/x)=\frac{1}{1/x}=x$, and $\frac{1}{f(x)}=\frac{1}{1/x}=x$.

d. False. For example, suppose that f is the straight line through the origin with slope 1, so that $f(x)=x$. Then $f(f(x))=f(x)=x$, while $(f(x))^2=x^2$.

e. False. For example, let $f(x)=x+2$ and $g(x)=2x-1$. Then $f(g(x))=f(2x-1)=2x-1+2=2x+1$, while $g(f(x))=g(x+2)=2(x+2)-1=2x+3$.

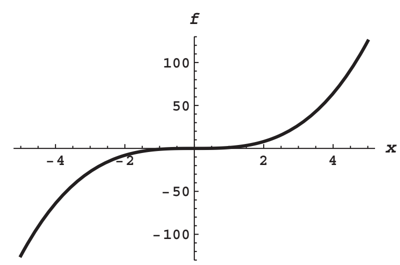
f. True. This is the definition of $f\circ g$.

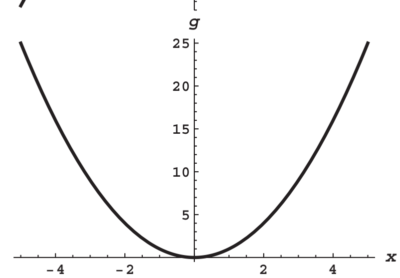
g. True. If f is even, then $f(-z)=f(z)$ for all z, so this is true in particular for $z=ax$. So if $g(x)=cf(ax)$, then $g(-x)=cf(-ax)=cf(ax)=g(x)$, so g is even.

h. False. For example, $f(x)=x$ is an odd function, but $h(x)=x+1$ isn't, because $h(2)=3$, while $h(-2)=-1$ which isn't $-h(2)$.

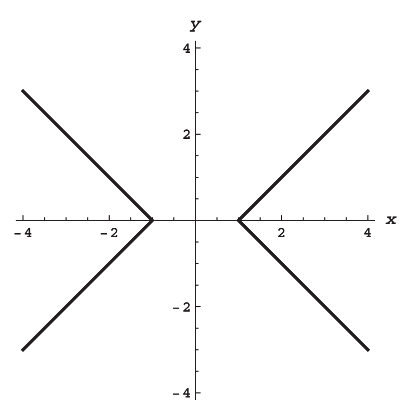
i. True. If $f(-x)=-f(x)=f(x)$, then in particular $-f(x)=f(x)$, so $0=2f(x)$, so $f(x)=0$ for all x.

1.1.82 If n is odd, then $n=2k+1$ for some integer k, and $(x)^n=(x)^{2k+1}=x(x)^{2k}$, which is less than 0 when $x<0$ and greater than 0 when $x>0$. For any number P (positive or negative) the number $\sqrt[n]{P}$ is a real number when n is odd, and $f(\sqrt[n]{P})=P$. So the range of f in this case is the set of all real numbers.
If n is even, then $n=2k$ for some integer k, and $x^n=(x^2)^k$. Thus $g(-x)=g(x)=(x^2)^k\geq 0$ for all x. Also, for any nonnegative number M, we have $g(\sqrt[n]{M})=M$, so the range of g in this case is the set of all nonnegative numbers.





1.1.83 We will make heavy use of the fact that $|x|$ is x if $x>0$, and is $-x$ if $x<0$. In the first quadrant where x and y are both positive, this equation becomes $x-y=1$ which is a straight line with slope 1 and y-intercept -1. In the second quadrant where x is negative and y is positive, this equation becomes $-x-y=1$, which is a straight line with slope -1 and y-intercept -1. In the third quadrant where both x and y are negative, we obtain the equation $-x-(-y)=1$, or $y=x+1$, and in the fourth quadrant, we obtain $x+y=1$. Graphing these lines and restricting them to the appropriate quadrants yields the following curve:



1.1.84

a. No. For example $f(x) = x^2 + 3$ is an even function, but $f(0)$ is not 0.

b. Yes. because $f(-x) = -f(x)$, and because $-0 = 0$, we must have $f(-0) = f(0) = -f(0)$, so $f(0) = -f(0)$, and the only number which is its own additive inverse is 0, so $f(0) = 0$.

1.1.85 Because the composition of f with itself has first degree, f has first degree as well, so let $f(x) = ax+b$. Then $(f \circ f)(x) = f(ax+b) = a(ax+b)+b = a^2x + (ab+b)$. Equating coefficients, we see that $a^2 = 9$ and $ab + b = -8$. If $a = 3$, we get that $b = -2$, while if $a = -3$ we have $b = 4$. So the two possible answers are $f(x) = 3x - 2$ and $f(x) = -3x + 4$.

1.1.86 Since the square of a linear function is a quadratic, we let $f(x) = ax+b$. Then $f(x)^2 = a^2x^2+2abx+b^2$. Equating coefficients yields that $a = \pm 3$ and $b = \pm 2$. However, a quick check shows that the middle term is correct only when one of these is positive and one is negative. So the two possible such functions f are $f(x) = 3x - 2$ and $f(x) = -3x + 2$.

1.1.87 Let $f(x) = ax^2 + bx + c$. Then $(f \circ f)(x) = f(ax^2 + bx + c) = a(ax^2 + bx + c)^2 + b(ax^2 + bx + c) + c$. Expanding this expression yields $a^3x^4 + 2a^2bx^3 + 2a^2cx^2 + ab^2x^2 + 2abcx + ac^2 + abx^2 + b^2x + bc + c$, which simplifies to $a^3x^4 + 2a^2bx^3 + (2a^2c + ab^2 + ab)x^2 + (2abc + b^2)x + (ac^2 + bc + c)$. Equating coefficients yields $a^3 = 1$, so $a = 1$. Then $2a^2b = 0$, so $b = 0$. It then follows that $c = -6$, so the original function was $f(x) = x^2 - 6$.

1.1.88 Because the square of a quadratic is a quartic, we let $f(x) = ax^2 + bx + c$. Then the square of f is $c^2 + 2bcx + b^2x^2 + 2acx^2 + 2abx^3 + a^2x^4$. By equating coefficients, we see that $a^2 = 1$ and so $a = \pm 1$. Because the coefficient on x^3 must be 0, we have that $b = 0$. And the constant term reveals that $c = \pm 6$. A quick check shows that the only possible solutions are thus $f(x) = x^2 - 6$ and $f(x) = -x^2 + 6$.

1.1.89 $\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \frac{1}{\sqrt{x+h}+\sqrt{x}}$.

$\frac{f(x)-f(a)}{x-a} = \frac{\sqrt{x}-\sqrt{a}}{x-a} = \frac{\sqrt{x}-\sqrt{a}}{x-a} \cdot \frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} = \frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})} = \frac{1}{\sqrt{x}+\sqrt{a}}$.

1.1.90 $\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{1-2(x+h)}-\sqrt{1-2x}}{h} = \frac{\sqrt{1-2(x+h)}-\sqrt{1-2x}}{h} \cdot \frac{\sqrt{1-2(x+h)}+\sqrt{1-2x}}{\sqrt{1-2(x+h)}+\sqrt{1-2x}} = \frac{1-2(x+h)-(1-2x)}{h(\sqrt{1-2(x+h)}+\sqrt{1-2x})} = -\frac{2}{\sqrt{1-2(x+h)}+\sqrt{1-2x}}$.

$\frac{f(x)-f(a)}{x-a} = \frac{\sqrt{1-2x}-\sqrt{1-2a}}{x-a} = \frac{\sqrt{1-2x}-\sqrt{1-2a}}{x-a} \cdot \frac{\sqrt{1-2x}+\sqrt{1-2a}}{\sqrt{1-2x}+\sqrt{1-2a}} = \frac{(1-2x)-(1-2a)}{(x-a)(\sqrt{1-2x}+\sqrt{1-2a})} = \frac{(-2)(x-a)}{(x-a)(\sqrt{1-2x}+\sqrt{1-2a})} = -\frac{2}{(\sqrt{1-2x}+\sqrt{1-2a})}$.

1.1.91 $\frac{f(x+h)-f(x)}{h} = \frac{\frac{-3}{\sqrt{x+h}} - \frac{-3}{\sqrt{x}}}{h} = \frac{-3(\sqrt{x}-\sqrt{x+h})}{h\sqrt{x}\sqrt{x+h}} = \frac{-3(\sqrt{x}-\sqrt{x+h})}{h\sqrt{x}\sqrt{x+h}} \cdot \frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}} = \frac{-3(x-(x+h))}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} = \frac{3}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}$.

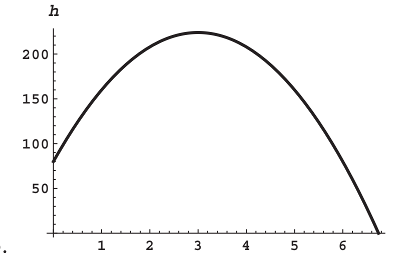
$\frac{f(x)-f(a)}{x-a} = \frac{\frac{-3}{\sqrt{x}} - \frac{-3}{\sqrt{a}}}{x-a} = \frac{-3\left(\frac{\sqrt{a}-\sqrt{x}}{\sqrt{a}\sqrt{x}}\right)}{x-a} = \frac{(-3)(\sqrt{a}-\sqrt{x})}{(x-a)\sqrt{a}\sqrt{x}} \cdot \frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}+\sqrt{x}} = \frac{(3)(x-a)}{(x-a)(\sqrt{a}\sqrt{x})(\sqrt{a}+\sqrt{x})} = \frac{3}{\sqrt{ax}(\sqrt{a}+\sqrt{x})}$.

1.1.92 $\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h} = \frac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h} \cdot \frac{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}} = \frac{(x+h)^2+1-(x^2+1)}{h(\sqrt{(x+h)^2+1}+\sqrt{x^2+1})} = \frac{x^2+2hx+h^2-x^2}{h(\sqrt{(x+h)^2+1}+\sqrt{x^2+1})} = \frac{2x+h}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}$.

$\frac{f(x)-f(a)}{x-a} = \frac{\sqrt{x^2+1}-\sqrt{a^2+1}}{x-a} = \frac{\sqrt{x^2+1}-\sqrt{a^2+1}}{x-a} \cdot \frac{\sqrt{x^2+1}+\sqrt{a^2+1}}{\sqrt{x^2+1}+\sqrt{a^2+1}} = \frac{x^2+1-(a^2+1)}{(x-a)(\sqrt{x^2+1}+\sqrt{a^2+1})} = \frac{(x-a)(x+a)}{(x-a)(\sqrt{x^2+1}+\sqrt{a^2+1})} = \frac{x+a}{\sqrt{x^2+1}+\sqrt{a^2+1}}$.

1.1.93

a. The formula for the height of the rocket is valid from $t = 0$ until the rocket hits the ground, which is the positive solution to $-16t^2 + 96t + 80 = 0$, which the quadratic formula reveals is $t = 3 + \sqrt{14}$. Thus, the domain is $[0, 3 + \sqrt{14}]$.

b. 

The maximum appears to occur at $t = 3$. The height at that time would be 224.

1.1.94

a. $d(0) = (10 - (2.2) \cdot 0)^2 = 100$.

b. The tank is first empty when $d(t) = 0$, which is when $10 - (2.2)t = 0$, or $t = 50/11$.

c. An appropriate domain would $[0, 50/11]$.

1.1.95 This would not necessarily have either kind of symmetry. For example, $f(x) = x^2$ is an even function and $g(x) = x^3$ is odd, but the sum of these two is neither even nor odd.

1.1.96 This would be an odd function, so it would be symmetric about the origin. Suppose f is even and g is odd. Then $(f \cdot g)(-x) = f(-x)g(-x) = f(x) \cdot (-g(x)) = -(f \cdot g)(x)$.

1.1.97 This would be an odd function, so it would be symmetric about the origin. Suppose f is even and g is odd. Then $\frac{f}{g}(-x) = \frac{f(-x)}{g(-x)} = \frac{f(x)}{-g(x)} = -\frac{f}{g}(x)$.

1.1.98 This would be an even function, so it would be symmetric about the y-axis. Suppose f is even and g is odd. Then $f(g(-x)) = f(-g(x)) = f(g(x))$.

1.1.99 This would be an even function, so it would be symmetric about the y-axis. Suppose f is even and g is even. Then $f(g(-x)) = f(g(x))$, because $g(-x) = g(x)$.

1.1.100 This would be an odd function, so it would be symmetric about the origin. Suppose f is odd and g is odd. Then $f(g(-x)) = f(-g(x)) = -f(g(x))$.

1.1.101 This would be an even function, so it would be symmetric about the y-axis. Suppose f is even and g is odd. Then $g(f(-x)) = g(f(x))$, because $f(-x) = f(x)$.

1.1.102

a. $f(g(-1)) = f(-g(1)) = f(3) = 3$

b. $g(f(-4)) = g(f(4)) = g(-4) = -g(4) = 2$

c. $f(g(-3)) = f(-g(3)) = f(4) = -4$

d. $f(g(-2)) = f(-g(2)) = f(1) = 2$

e. $g(g(-1)) = g(-g(1)) = g(3) = -4$

f. $f(g(0) - 1) = f(-1) = f(1) = 2$

g. $f(g(g(-2))) = f(g(-g(2))) = f(g(1)) = f(-3) = 3$

h. $g(f(f(-4))) = g(f(-4)) = g(-4) = 2$

i. $g(g(g(-1))) = g(g(-g(1))) = g(g(3)) = g(-4) = 2$

1.1.103

a. $f(g(-2)) = f(-g(2)) = f(-2) = 4$

b. $g(f(-2)) = g(f(2)) = g(4) = 1$

c. $f(g(-4)) = f(-g(4)) = f(-1) = 3$

d. $g(f(5) - 8) = g(-2) = -g(2) = -2$

e. $g(g(-7)) = g(-g(7)) = g(-4) = -1$

f. $f(1 - f(8)) = f(-7) = 7$

1.2 Representing Functions

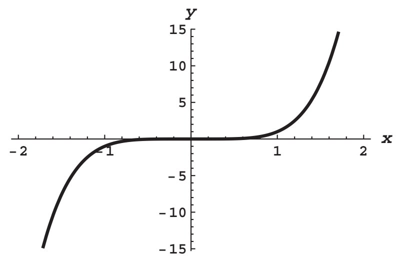
1.2.1 Functions can be defined and represented by a formula, through a graph, via a table, and by using words.

1.2.2 The domain of every polynomial is the set of all real numbers.

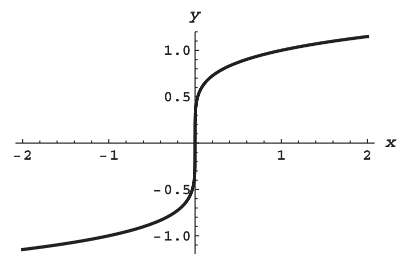
1.2.3 The domain of a rational function $\frac{p(x)}{q(x)}$ is the set of all real numbers for which $q(x) \neq 0$.

1.2.4 A piecewise linear function is one which is linear over intervals in the domain.

1.2.5



1.2.6



1.2.7 Compared to the graph of $f(x)$, the graph of $f(x+2)$ will be shifted 2 units to the left.

1.2.8 Compared to the graph of $f(x)$, the graph of $-3f(x)$ will be scaled vertically by a factor of 3 and flipped about the x axis.

1.2.9 Compared to the graph of $f(x)$, the graph of $f(3x)$ will be scaled horizontally by a factor of 3.

1.2.10 To produce the graph of $y = 4(x+3)^2 + 6$ from the graph of x^2, one must

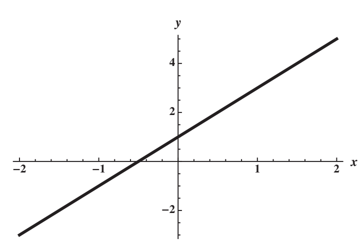
1. shift the graph horizontally by 3 units to left
2. scale the graph vertically by a factor of 4
3. shift the graph vertically up 6 units.

1.2.11 The slope of the line shown is $m = \frac{-3-(-1)}{3-0} = -2/3$. The y-intercept is $b = -1$. Thus the function is given by $f(x) = (-2/3)x - 1$.

1.2.12 The slope of the line shown is $m = \frac{1-(5)}{5-0} = -4/5$. The y-intercept is $b = 5$. Thus the function is given by $f(x) = (-4/5)x + 5$.

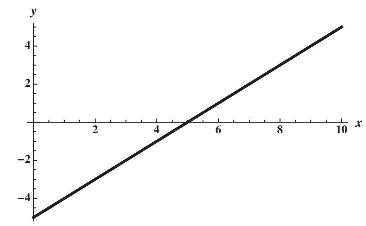
1.2.13

The slope is given by $\frac{5-3}{2-1} = 2$, so the equation of the line is $y - 3 = 2(x-1)$, which can be written as $y = 2x - 2 + 3$, or $y = 2x + 1$.

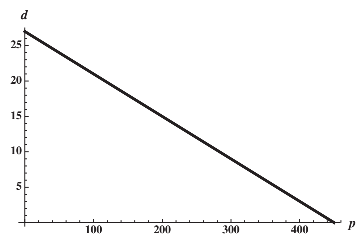


1.2.14

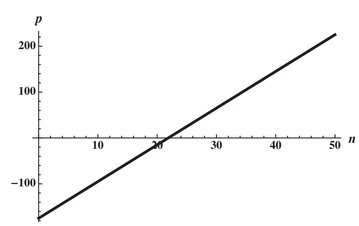
The slope is given by $\frac{0-(-3)}{5-2} = 1$, so the equation of the line is $y - 0 = 1(x - 5)$, or $y = x - 5$.



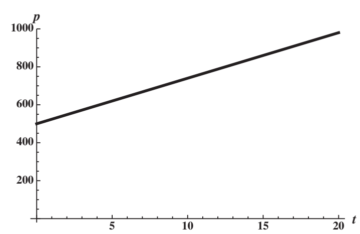
1.2.15 Using price as the independent variable p and the average number of units sold per day as the dependent variable d, we have the ordered pairs $(250, 12)$ and $(200, 15)$. The slope of the line determined by these points is $m = \frac{15-12}{200-250} = -\frac{3}{50}$. Thus the demand function has the form $d(p) = (-3/50)p + b$ for some constant b. Using the point $(200, 15)$, we find that $15 = (-3/50) \cdot 200 + b$, so $b = 27$. Thus the demand function is $d = (-3/50)p + 27$. While the domain of this linear function is the set of all real numbers, the formula is only likely to be valid for some subset of the interval $(0, 450)$, because outside of that interval either $p \leq 0$ or $d \leq 0$.



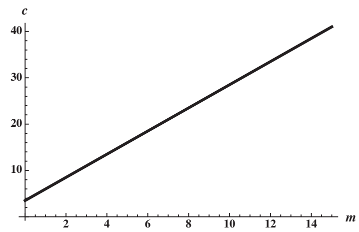
1.2.16 The profit is given by $p = f(n) = 8n - 175$. The break-even point is when $p = 0$, which occurs when $n = 175/8 = 21.875$, so they need to sell at least 22 tickets to not have a negative profit.



1.2.17 The slope is given by the rate of growth, which is 24. When $t = 0$ (years past 2015), the population is 500, so the point $(0, 500)$ satisfies our linear function. Thus the population is given by $p(t) = 24t + 500$. In 2030, we have $t = 15$, so the population will be approximately $p(15) = 360 + 500 = 860$.



1.2.18 The cost per mile is the slope of the desired line, and the intercept is the fixed cost of 3.5. Thus, the cost per mile is given by $c(m) = 2.5m + 3.5$. When $m = 9$, we have $c(9) = (2.5)(9) + 3.5 = 22.5 + 3.5 = 26$ dollars.



1.2.19 For $x < 0$, the graph is a line with slope 1 and y- intercept 3, while for $x > 0$, it is a line with slope $-1/2$ and y-intercept 3. Note that both of these lines contain the point $(0, 3)$. The function shown can thus be written

$$f(x) = \begin{cases} x + 3 & \text{if } x < 0; \\ -\frac{1}{2}x + 3 & \text{if } x \geq 0. \end{cases}$$

1.2.20 For $x < 3$, the graph is a line with slope 1 and y- intercept 1, while for $x > 3$, it is a line with slope $-1/3$. The portion to the right thus is represented by $y = (-1/3)x + b$, but because it contains the point $(6, 1)$, we must have $1 = (-1/3)(6) + b$ so $b = 3$. The function shown can thus be written

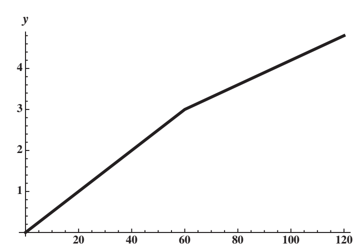
$$f(x) = \begin{cases} x + 1 & \text{if } x < 3; \\ -\frac{1}{3}x + 3 & \text{if } x \geq 3. \end{cases}$$

Note that at $x = 3$ the value of the function is 2, as indicated by our formula.

1.2.21

The cost is given by

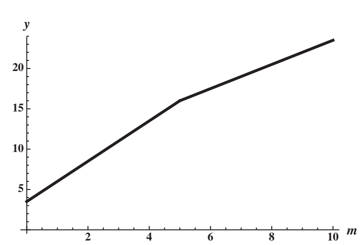
$$c(t) = \begin{cases} 0.05t & \text{for } 0 \leq t \leq 60 \\ 1.2 + 0.03t & \text{for } 60 < t \leq 120 \end{cases}.$$



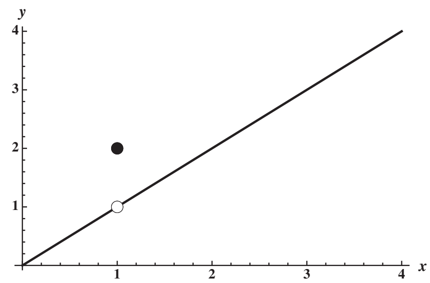
1.2.22

The cost is given by

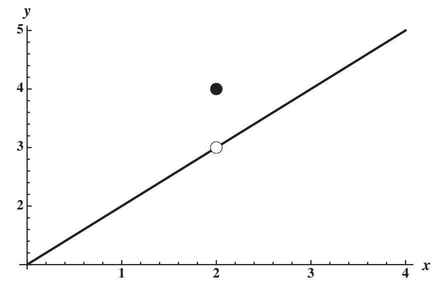
$$c(m) = \begin{cases} 3.5 + 2.5m & \text{for } 0 \leq m \leq 5 \\ 8.5 + 1.5m & \text{for } m > 5 \end{cases}.$$



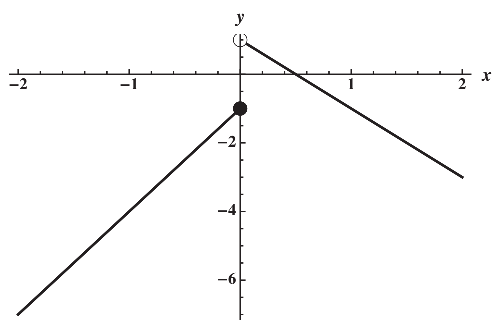
1.2.23



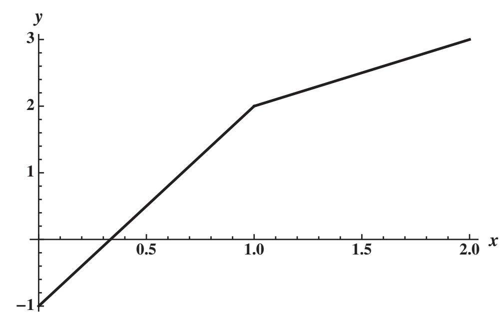
1.2.24



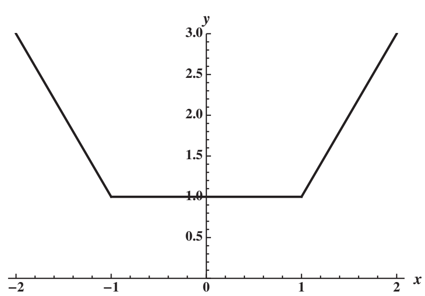
1.2.25



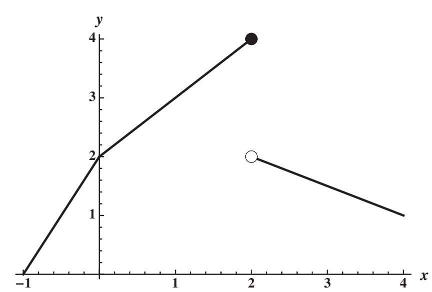
1.2.26



1.2.27

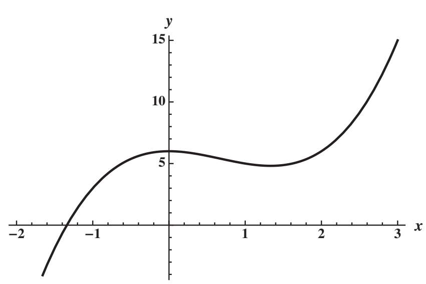


1.2.28



1.2.29

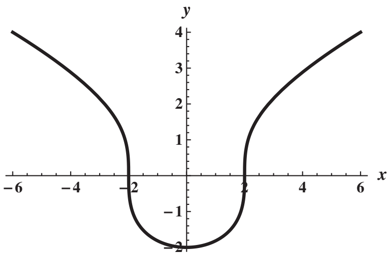
a.



b. The function is a polynomial, so its domain is the set of all real numbers.

c. It has one peak near its y-intercept of $(0, 6)$ and one valley between $x = 1$ and $x = 2$. Its x-intercept is near $x = -4/3$.

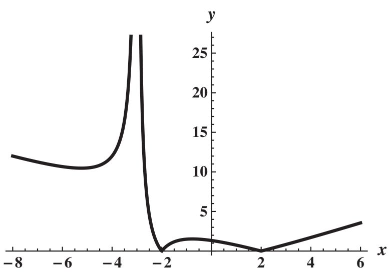
1.2.30

a. 

b. The function's domain is the set of all real numbers.

c. It has a valley at the y-intercept of $(0, -2)$, and is very steep at $x = -2$ and $x = 2$ which are the x-intercepts. It is symmetric about the y-axis.

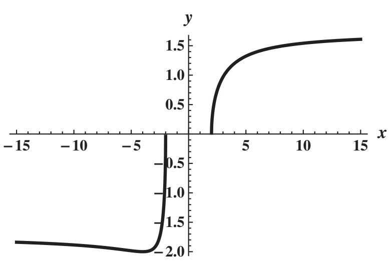
1.2.31

a. 

b. The domain of the function is the set of all real numbers except -3.

c. There is a valley near $x = -5.2$ and a peak near $x = -0.8$. The x-intercepts are at -2 and 2, where the curve does not appear to be smooth. There is a vertical asymptote at $x = -3$. The function is never below the x-axis. The y-intercept is $(0, 4/3)$.

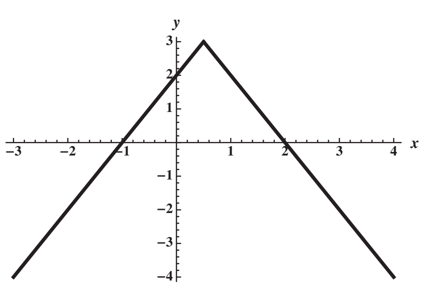
1.2.32

a. 

b. The domain of the function is $(-\infty, -2] \cup [2, \infty)$

c. x-intercepts are at -2 and 2. Because 0 isn't in the domain, there is no y-intercept. The function has a valley at $x = -4$.

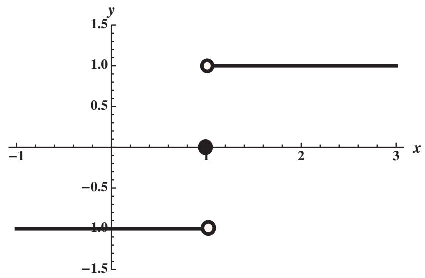
1.2.33

a. 

b. The domain of the function is $(-\infty, \infty)$

c. The function has a maximum of 3 at $x = 1/2$, and a y-intercept of 2.

1.2.34

a. 

b. The domain of the function is $(-\infty, \infty)$

c. The function contains a jump at $x = 1$. The maximum value of the function is 1 and the minimum value is -1.

1.2.35 The slope of this line is constantly 2, so the slope function is $s(x) = 2$.

1.2.36 The function can be written as $|x| = \begin{cases} -x & \text{if } x \le 0 \\ x & \text{if } x > 0 \end{cases}$.

The slope function is $s(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$.

1.2.37 The slope function is given by $s(x) = \begin{cases} 1 & \text{if } x < 0; \\ -1/2 & \text{if } x > 0. \end{cases}$

1.2.38 The slope function is given by $s(x) = \begin{cases} 1 & \text{if } x < 3; \\ -1/3 & \text{if } x > 3. \end{cases}$

1.2.39

a. Because the area under consideration is that of a rectangle with base 2 and height 6, $A(2) = 12$.

b. Because the area under consideration is that of a rectangle with base 6 and height 6, $A(6) = 36$.

c. Because the area under consideration is that of a rectangle with base x and height 6, $A(x) = 6x$.

1.2.40

a. Because the area under consideration is that of a triangle with base 2 and height 1, $A(2) = 1$.

b. Because the area under consideration is that of a triangle with base 6 and height 3, the $A(6) = 9$.

c. Because $A(x)$ represents the area of a triangle with base x and height $(1/2)x$, the formula for $A(x)$ is $\frac{1}{2} \cdot x \cdot \frac{x}{2} = \frac{x^2}{4}$.

1.2.41

a. Because the area under consideration is that of a trapezoid with base 2 and heights 8 and 4, we have $A(2) = 2 \cdot \frac{8+4}{2} = 12$.

b. Note that $A(3)$ represents the area of a trapezoid with base 3 and heights 8 and 2, so $A(3) = 3 \cdot \frac{8+2}{2} = 15$. So $A(6) = 15 + (A(6) - A(3))$, and $A(6) - A(3)$ represents the area of a triangle with base 3 and height 2. Thus $A(6) = 15 + 6 = 21$.

c. For x between 0 and 3, $A(x)$ represents the area of a trapezoid with base x, and heights 8 and $8-2x$. Thus the area is $x \cdot \frac{8+8-2x}{2} = 8x - x^2$. For $x > 3$, $A(x) = A(3) + A(x) - A(3) = 15 + 2(x-3) = 2x + 9$. Thus

$$A(x) = \begin{cases} 8x - x^2 & \text{if } 0 \le x \le 3; \\ 2x + 9 & \text{if } x > 3. \end{cases}$$

1.2.42

a. Because the area under consideration is that of trapezoid with base 2 and heights 3 and 1, we have $A(2) = 2 \cdot \frac{3+1}{2} = 4$.

b. Note that $A(6) = A(2) + (A(6) - A(2))$, and that $A(6) - A(2)$ represents a trapezoid with base $6 - 2 = 4$ and heights 1 and 5. The area is thus $4 + \left(4 \cdot \frac{1+5}{2}\right) = 4 + 12 = 16$.

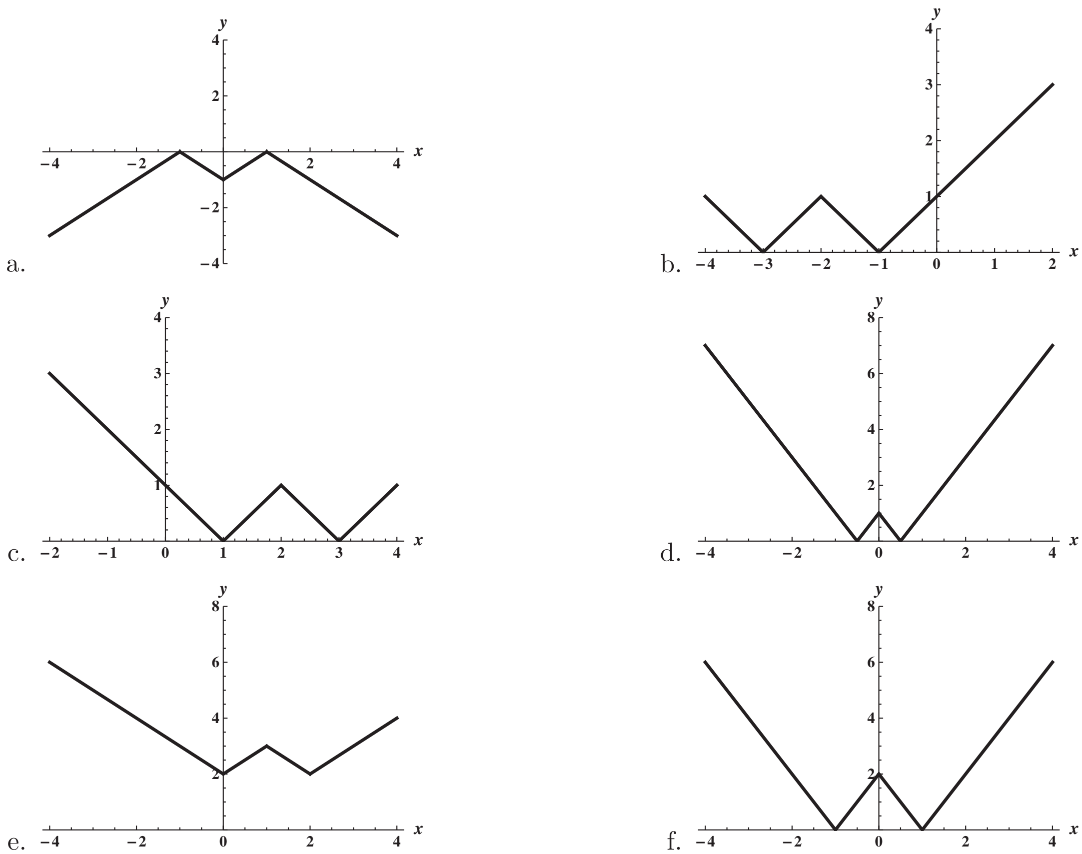
c. For x between 0 and 2, $A(x)$ represents the area of a trapezoid with base x, and heights 3 and $3-x$. Thus the area is $x \cdot \frac{3+3-x}{2} = 3x - \frac{x^2}{2}$. For $x > 2$, $A(x) = A(2) + A(x) - A(2) = 4 + (A(x) - A(2))$. Note that $A(x) - A(2)$ represents the area of a trapezoid with base $x - 2$ and heights 1 and $x - 1$. Thus $A(x) = 4 + (x-2) \cdot \frac{1+x-1}{2} = 4 + (x-2)\left(\frac{x}{2}\right) = \frac{x^2}{2} - x + 4$. Thus

$$A(x) = \begin{cases} 3x - \frac{x^2}{2} & \text{if } 0 \le x \le 2; \\ \frac{x^2}{2} - x + 4 & \text{if } x > 2. \end{cases}$$

1.2.43 $f(x) = |x-2| + 3$, because the graph of f is obtained from that of $|x|$ by shifting 2 units to the right and 3 units up.

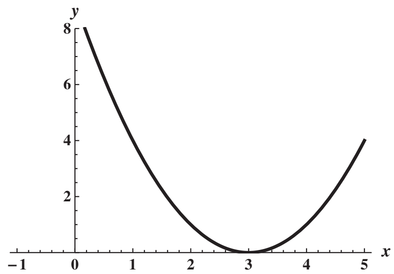
$g(x) = -|x+2| - 1$, because the graph of g is obtained from the graph of $|x|$ by shifting 2 units to the left, then reflecting about the x-axis, and then shifting 1 unit down.

1.2.44

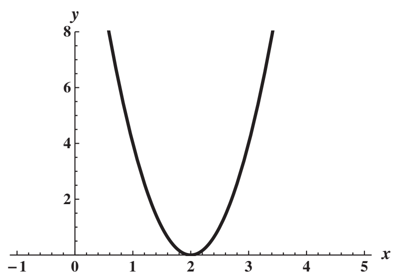


1.2.45

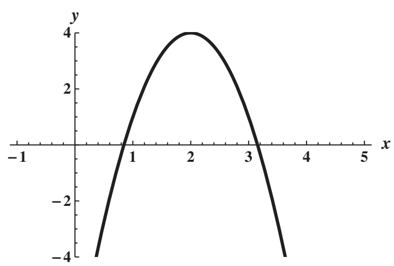
a. Shift 3 units to the right.



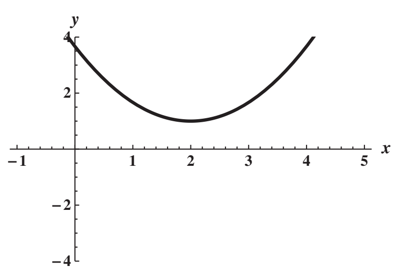
b. Horizontal scaling by a factor of 2, then shift 2 units to the right.



c. Shift to the right 2 units, vertical scaling by a factor of 3 and flip, shift up 4 units.

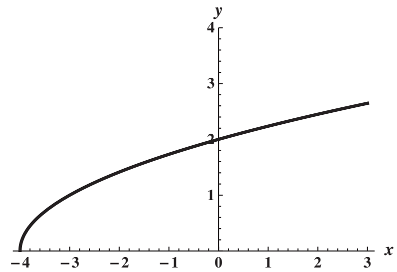


d. Horizontal scaling by a factor of $\frac{1}{3}$, horizontal shift right 2 units, vertical scaling by a factor of 6, vertical shift up 1 unit.

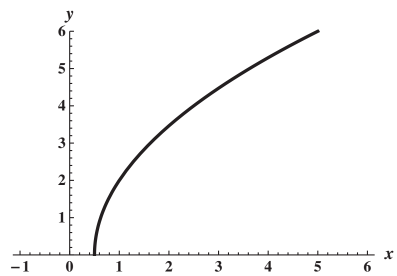


1.2.46

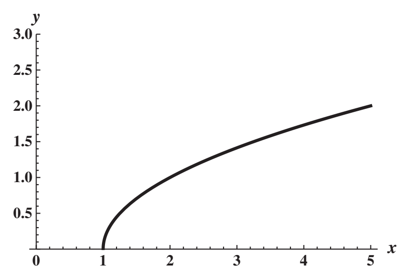
a. Shift 4 units to the left.



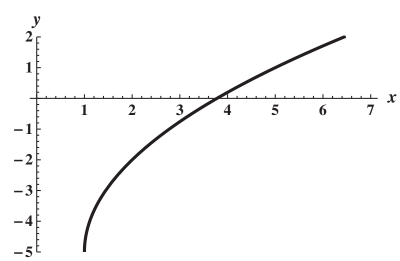
b. Horizontal scaling by a factor of 2, shift $\frac{1}{2}$ unit to the right, vertical scaling by a factor of 2.



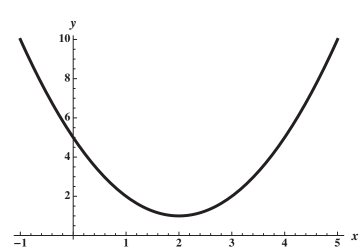
c. Shift 1 unit to the right.



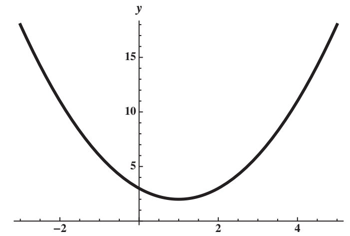
d. Shift 1 unit to the right, vertical scaling by a factor of 3, vertical shift down 5 units.



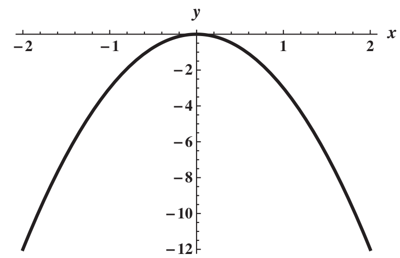
1.2.47 The graph is obtained by shifting the graph of x^2 two units to the right and one unit up.



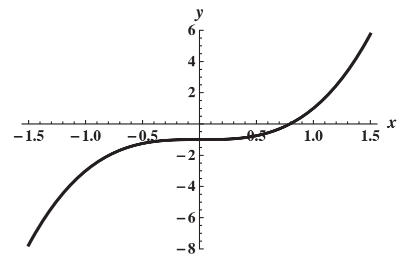
1.2.48 Write x^2-2x+3 as $(x^2-2x+1)+2=(x-1)^2+2$. The graph is obtained by shifting the graph of x^2 one unit to the right and two units up.



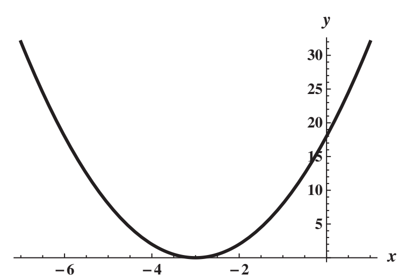
1.2.49 This function is $-3f(x)$ where $f(x)=x^2$. Vertically scale the graph of f by a factor of 3 and then flip.



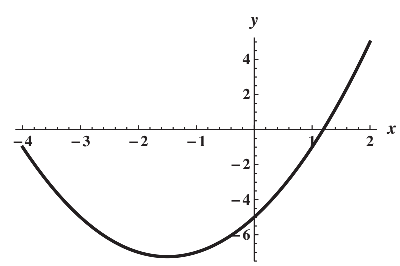
1.2.50 This function is $2f(x)-1$ where $f(x)=x^3$. Vertically scale the graph of f by a factor of 2 and then vertically shift down 1 unit.



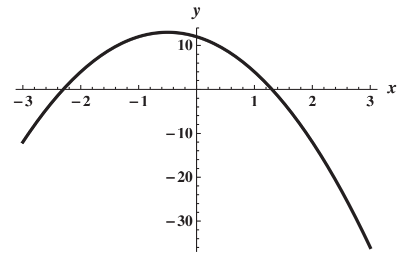
1.2.51 This function is $2f(x+3)$ where $f(x)=x^2$. Vertically scale the graph of f by a factor of 2 and then shift left 3 units.



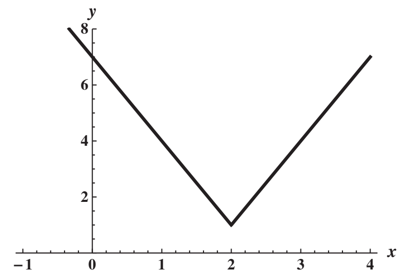
1.2.52 By completing the square, we have that $p(x) = (x^2+3x+(9/4))-(29/4)=(x+(3/2))^2-(29/4)$. So it is $f(x+(3/2))-(29/4)$ where $f(x)=x^2$. Shift the graph of f 3/2 units to the left and then down 29/4 units.



1.2.53 By completing the square, we have that $h(x) = -4(x^2 + x - 3) = -4\left(x^2 + x + \frac{1}{4} - \frac{1}{4} - 3\right) = -4(x + (1/2))^2 + 13$. So it is $-4f(x + (1/2)) + 13$ where $f(x) = x^2$. Vertically scale the graph of f by a factor of 4, then reflect about the x-axis, then shift left 1/2 unit, and then up 13 units.



1.2.54 Because $|3x-6|+1 = 3|x-2|+1$, this is $3f(x-2)+1$ where $f(x) = |x|$. Shift the graph of f 2 units to the right, vertically scale by a factor of 3, and then shift 1 unit up.



1.2.55

a. True. A polynomial $p(x)$ can be written as the ratio of polynomials $\frac{p(x)}{1}$, so it is a rational function. However, a rational function like $\frac{1}{x}$ is not a polynomial.

b. False. For example, if $f(x) = 2x$, then $(f \circ f)(x) = f(f(x)) = f(2x) = 4x$ is linear, not quadratic.

c. True. In fact, if f is degree m and g is degree n, then the degree of the composition of f and g is $m \cdot n$, regardless of the order they are composed.

d. False. The graph would be shifted two units to the left.

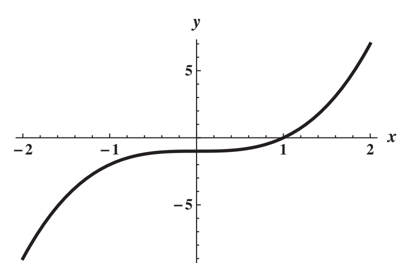
1.2.56 The points of intersection are found by solving $x^2 + 2 = x + 4$. This yields the quadratic equation $x^2 - x - 2 = 0$ or $(x - 2)(x + 1) = 0$. So the x-values of the points of intersection are 2 and -1. The actual points of intersection are $(2, 6)$ and $(-1, 3)$.

1.2.57 The points of intersection are found by solving $x^2 = -x^2 + 8x$. This yields the quadratic equation $2x^2 - 8x = 0$ or $(2x)(x - 4) = 0$. So the x-values of the points of intersection are 0 and 4. The actual points of intersection are $(0, 0)$ and $(4, 16)$.

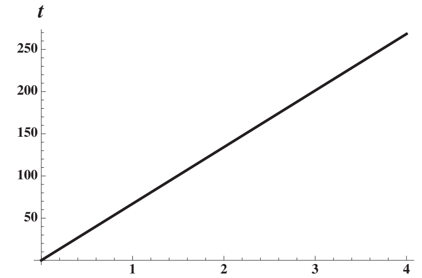
1.2.58 $y = x + 1$, because the y value is always 1 more than the x value.

1.2.59 $y = \sqrt{x} - 1$, because the y value is always 1 less than the square root of the x value.

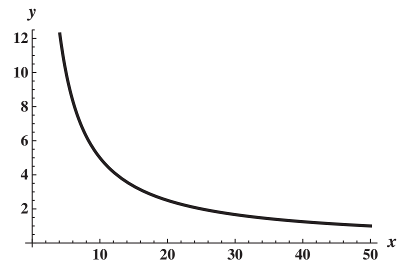
1.2.60 $y = x^3 - 1$. The domain is $(-\infty, \infty)$.



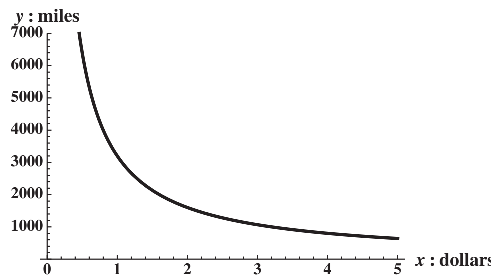
1.2.61 The car moving north has gone $30t$ miles after t hours and the car moving east has gone $60t$ miles. Using the Pythagorean theorem, we have $s(t) = \sqrt{(30t)^2 + (60t)^2} = \sqrt{900t^2 + 3600t^2} = \sqrt{4500t^2} = 30\sqrt{5}t$ miles. The context domain could be $[0, 4]$.



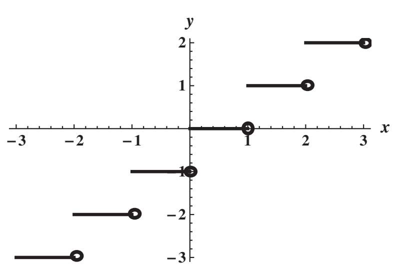
1.2.62 $y = \frac{50}{x}$. Theoretically the domain is $(0, \infty)$, but the world record for the "hour ride" is just short of 50 miles.



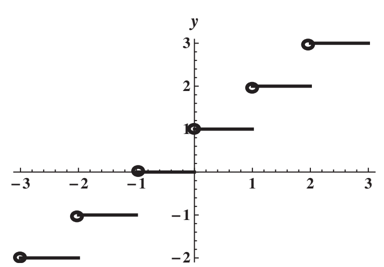
1.2.63 $y = \frac{3200}{x}$. Note that $\frac{x \text{ dollars per gallon}}{32 \text{ miles per gallon}} \cdot y \, \text{miles}$ would represent the numbers of dollars, so this must be 100. So we have $\frac{xy}{32} = 100$, or $y = \frac{3200}{x}$. We certainly have $x > 0$, and a reasonable upper bound to imagine for x is \$5 (let's hope), so the context domain is $(0, 5]$.



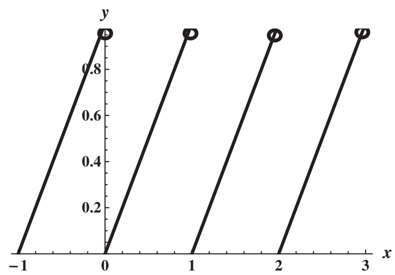
1.2.64



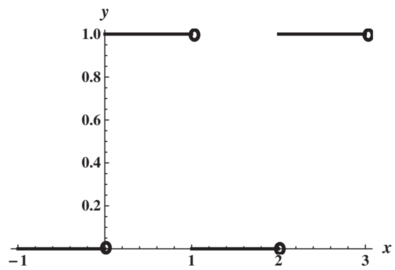
1.2.65



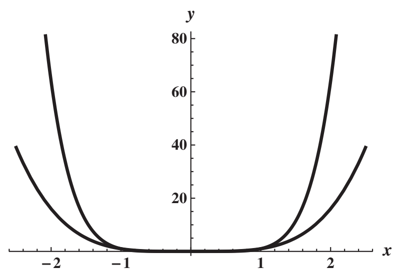
1.2.66



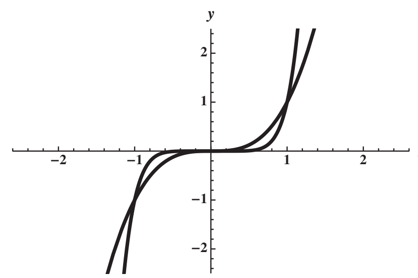
1.2.67



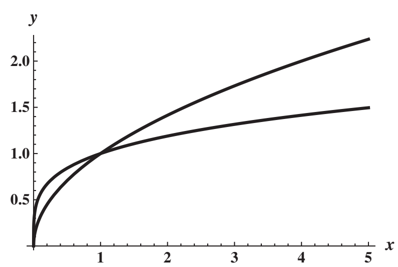
1.2.68



1.2.69



1.2.70



1.2.71

a. The zeros of f are the points where the graph crosses the x-axis, so these are points A, D, F, and I.

b. The only high point, or peak, of f occurs at point E, because it appears that the graph has larger and larger y values as x increases past point I and decreases past point A.

c. The only low points, or valleys, of f are at points B and H, again assuming that the graph of f continues its apparent behavior for larger values of x.

d. Past point H, the graph is rising, and is rising faster and faster as x increases. It is also rising between points B and E, but not as quickly as it is past point H. So the marked point at which it is rising most rapidly is I.

e. Before point B, the graph is falling, and falls more and more rapidly as x becomes more and more negative. It is also falling between points E and H, but not as rapidly as it is before point B. So the marked point at which it is falling most rapidly is A.

1.2.72

a. The zeros of g appear to be at $x = 0$, $x = 1$, $x = 1.6$, and $x \approx 3.15$.

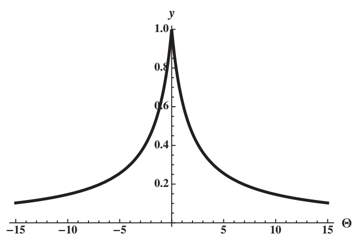
b. The two peaks of g appear to be at $x \approx 0.5$ and $x \approx 2.6$, with corresponding points $\approx (0.5, 0.4)$ and $\approx (2.6, 3.4)$.

c. The only valley of g is at $\approx (1.3, -0.2)$.

d. Moving right from $x \approx 1.3$, the graph is rising more and more rapidly until about $x = 2$, at which point it starts rising less rapidly (because, by $x \approx 2.6$, it is not rising at all). So the coordinates of the point at which it is rising most rapidly are approximately $(2.1, g(2)) \approx (2.1, 2)$. Note that while the curve is also rising between $x = 0$ and $x \approx 0.5$, it is not rising as rapidly as it is near $x = 2$.

e. To the right of $x \approx 2.6$, the curve is falling, and falling more and more rapidly as x increases. So the point at which it is falling most rapidly in the interval $[0, 3]$ is at $x = 3$, which has the approximate coordinates $(3, 1.4)$. Note that while the curve is also falling between $x \approx 0.5$ and $x \approx 1.3$, it is not falling as rapidly as it is near $x = 3$.

1.2.73

a. 

b. This appears to have a maximum when $\theta = 0$. Our vision is sharpest when we look straight ahead.

c. For $|\theta| \le .19°$. We have an extremely narrow range where our eyesight is sharp.

1.2.74

a. $f(.75) = \frac{.75^2}{1-2(.75)(.25)} = .9$. There is a 90% chance that the server will win from deuce if they win 75% of their service points.

b. $f(.25) = \frac{.25^2}{1-2(.25)(.75)} = .1$. There is a 10% chance that the server will win from deuce if they win 25% of their service points.

1.2.75

a. Using the points $(1986, 1875)$ and $(2000, 6471)$ we see that the slope is about 328.3. At $t = 0$, the value of p is 1875. Therefore a line which reasonably approximates the data is $p(t) = 328.3t + 1875$.

b. Using this line, we have that $p(9) = 4830$.

1.2.76

a. We know that the points $(32, 0)$ and $(212, 100)$ are on our line. The slope of our line is thus $\frac{100-0}{212-32} = \frac{100}{180} = \frac{5}{9}$. The function $f(F)$ thus has the form $C = (5/9)F + b$, and using the point $(32, 0)$ we see that $0 = (5/9)32 + b$, so $b = -(160/9)$. Thus $C = (5/9)F - (160/9)$

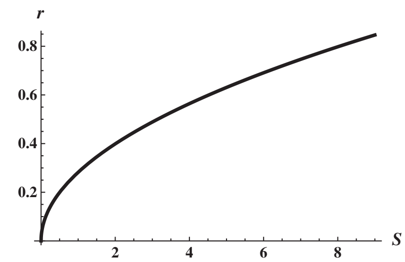
b. Solving the system of equations $C = (5/9)F - (160/9)$ and $C = F$, we have that $F = (5/9)F - (160/9)$, so $(4/9)F = -160/9$, so $F = -40$ when $C = -40$.

1.2.77

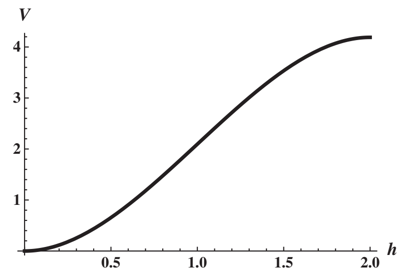
a. Because you are paying \$350 per month, the amount paid after m months is $y = 350m + 1200$.

b. After 4 years (48 months) you have paid $350 \cdot 48 + 1200 = 18000$ dollars. If you then buy the car for \$10,000, you will have paid a total of \$28,000 for the car instead of \$25,000. So you should buy the car instead of leasing it.

1.2.78 Because $S = 4\pi r^2$, we have that $r^2 = \frac{S}{4\pi}$, so $|r| = \frac{\sqrt{S}}{2\sqrt{\pi}}$, but because r is positive, we can write $r = \frac{\sqrt{S}}{2\sqrt{\pi}}$.

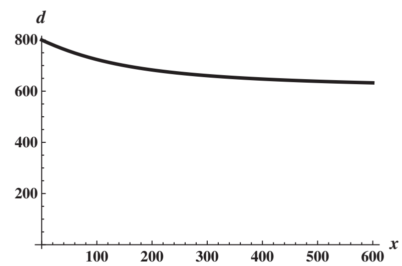


1.2.79 The function makes sense for $0 \leq h \leq 2$.

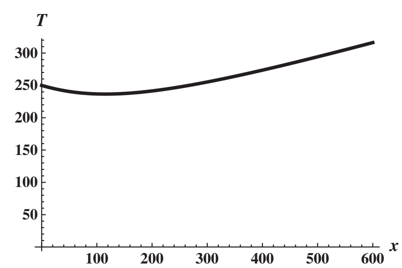


1.2.80

a. Note that the island, the point P on shore, and the point down shore x units from P form a right triangle. By the Pythagorean theorem, the length of the hypotenuse is $\sqrt{40000 + x^2}$. So Kelly must row this distance and then jog $600-x$ meters to get home. So her total distance $d(x) = \sqrt{40000 + x^2} + (600 - x)$.



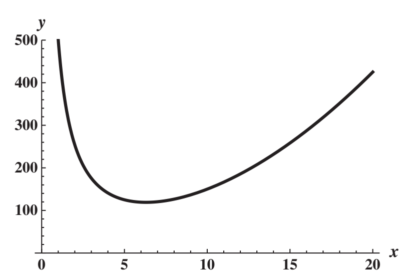
b. Because distance is rate times time, we have that time is distance divided by rate. Thus $T(x) = \frac{\sqrt{40000+x^2}}{2} + \frac{600-x}{4}$.



c. By inspection, it looks as though she should head to a point about 115 meters down shore from P. This would lead to a time of about 236.6 seconds.

1.2.81

a. The volume of the box is x^2h, but because the box has volume 125 cubic feet, we have that $x^2h = 125$, so $h = \frac{125}{x^2}$. The surface area of the box is given by x^2 (the area of the base) plus $4 \cdot hx$, because each side has area hx. Thus $S = x^2 + 4hx = x^2 + \frac{4 \cdot 125 \cdot x}{x^2} = x^2 + \frac{500}{x}$.



b. By inspection, it looks like the value of x which minimizes the surface area is about 6.3.

1.2.82 Let $f(x) = a_nx^n +$ smaller degree terms and let $g(x) = b_mx^m +$ some smaller degree terms.

a. The largest degree term in $f \cdot f$ is $a_n x^n \cdot a_n x^n = a_n^2 x^{n+n}$, so the degree of this polynomial is $n+n = 2n$.

b. The largest degree term in $f \circ f$ is $a_n \cdot (a_n x^n)^n$, so the degree is n^2.

c. The largest degree term in $f \cdot g$ is $a_n b_m x^{m+n}$, so the degree of the product is $m + n$.

d. The largest degree term in $f \circ g$ is $a_n \cdot (b_m x^m)^n$, so the degree is mn.

1.2.83 Suppose that the parabola f crosses the x-axis at a and b, with $a < b$. Then a and b are roots of the polynomial, so $(x - a)$ and $(x - b)$ are factors. Thus the polynomial must be $f(x) = c(x - a)(x - b)$ for some non-zero real number c. So $f(x) = cx^2 - c(a + b)x + abc$. Because the vertex always occurs at the x value which is $\frac{-\text{coefficient on x}}{2\cdot\text{coefficient on } x^2}$ we have that the vertex occurs at $\frac{c(a+b)}{2c} = \frac{a+b}{2}$, which is halfway between a and b.

1.2.84

a. We complete the square to rewrite the function f. Write $f(x) = ax^2 + bx + c$ as $f(x) = a(x^2 + \frac{b}{a}x + \frac{c}{a})$. Completing the square yields

$$a\left(\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a}\right) + \left(\frac{c}{a} - \frac{b^2}{4a}\right)\right) = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4}\right).$$

Thus the graph of f is obtained from the graph of x^2 by shifting $\frac{b}{2a}$ units to the left (and then doing some scaling and vertical shifting) – moving the vertex from 0 to $-\frac{b}{2a}$. The vertex is therefore $\left(\frac{-b}{2a}, c - \frac{b^2}{4}\right)$.

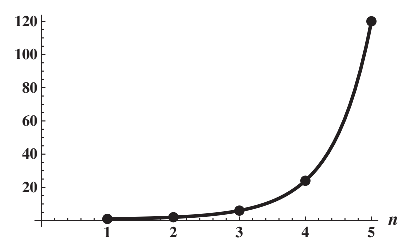
b. We know that the graph of f touches the x-axis twice if the equation $ax^2 + bx + c = 0$ has two real solutions. By the quadratic formula, we know that this occurs exactly when the discriminant $b^2 - 4ac$ is positive. So the condition we seek is for $b^2 - 4ac > 0$, or $b^2 > 4ac$.

1.2.85

a.

n	1	2	3	4	5
$n!$	1	2	6	24	120

b.



c. Using trial and error and a calculator yields that 10! is more than a million, but 9! isn't.

1.2.86

a.

n	1	2	3	4	5	6	7	8	9	10
$S(n)$	1	3	6	10	15	21	28	36	45	55

b. The domain of this function consists of the positive integers. The range is a subset of the set of positive integers.

c. Using trial and error and a calculator yields that $S(n) > 1000$ for the first time for $n = 45$.

1.2.87

a.

n	1	2	3	4	5	6	7	8	9	10
$T(n)$	1	5	14	30	55	91	140	204	285	385

b. The domain of this function consists of the positive integers.

c. Using trial and error and a calculator yields that $T(n) > 1000$ for the first time for $n = 14$.

1.3 Trigonometric Functions

1.3.1 Let O be the length of the side opposite the angle x, let A be length of the side adjacent to the angle x, and let H be the length of the hypotenuse. Then $\sin x = \frac{O}{H}$, $\cos x = \frac{A}{H}$, $\tan x = \frac{O}{A}$, $\csc x = \frac{H}{O}$, $\sec x = \frac{H}{A}$, and $\cot x = \frac{A}{O}$.

1.3.2 We consider the angle formed by the positive x axis and the ray from the origin through the point $P(x, y)$. A positive angle is one for which the rotation from the positive x axis to the other ray is counter-clockwise. We then define the six trigonometric functions as follows: let $r = \sqrt{x^2 + y^2}$. Then $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, $\tan\theta = \frac{y}{x}$, $\csc\theta = \frac{r}{y}$, $\sec\theta = \frac{r}{x}$, and $\cot\theta = \frac{x}{y}$.

1.3.3 The radian measure of an angle θ is the length of the arc s on the unit circle associated with θ.

1.3.4 The period of a function is the smallest positive real number k so that $f(x + k) = f(x)$ for all x in the domain of the function. The sine, cosine, secant, and cosecant function all have period 2π. The tangent and cotangent functions have period π.

1.3.5 $\sin^2 x + \cos^2 x = 1$, $1 + \cot^2 x = \csc^2 x$, and $\tan^2 x + 1 = \sec^2 x$.

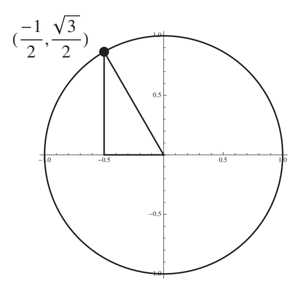
1.3.6 $\csc x = \frac{1}{\sin x}$, $\sec x = \frac{1}{\cos x}$, $\tan x = \frac{\sin x}{\cos x}$, and $\cot x = \frac{\cos x}{\sin x}$.

1.3.7 The tangent function is undefined where $\cos x = 0$, which is at all real numbers of the form $\frac{\pi}{2} + k\pi, k$ an integer. This is the set of odd multiples of $\pi/2$.

1.3.8 $\sec x$ is defined wherever $\cos x \neq 0$, which is $\{x\colon x \neq \frac{\pi}{2} + k\pi, k \text{ an integer}\}$. This is the set of odd multiples of $\pi/2$.

1.3.9

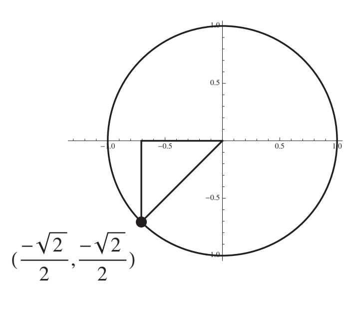
The point on the unit circle associated with $2\pi/3$ is $(-1/2, \sqrt{3}/2)$, so $\cos(2\pi/3) = -1/2$.



1.3.10 The point on the unit circle associated with $2\pi/3$ is $(-1/2, \sqrt{3}/2)$, so $\sin(2\pi/3) = \sqrt{3}/2$. See the picture from the previous problem.

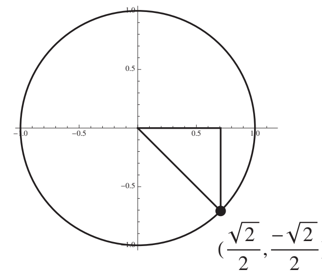
1.3.11

The point on the unit circle associated with $-3\pi/4$ is $(-\sqrt{2}/2, -\sqrt{2}/2)$, so $\tan(-3\pi/4) = 1$.



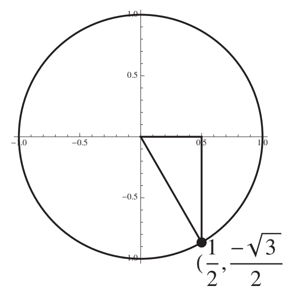
1.3.12

The point on the unit circle associated with $15\pi/4$ is $(\sqrt{2}/2, -\sqrt{2}/2)$, so $\tan(15\pi/4) = -1$.



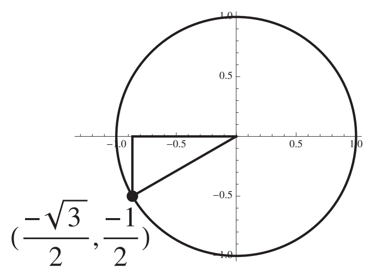
1.3.13

The point on the unit circle associated with $-13\pi/3$ is $(1/2, -\sqrt{3}/2)$, so $\cot(-13\pi/3) = -1/\sqrt{3} = -\sqrt{3}/3$.



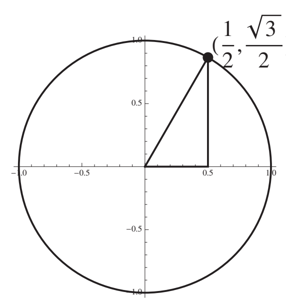
1.3.14

The point on the unit circle associated with $7\pi/6$ is $(-\sqrt{3}/2, -1/2)$, so $\sec(7\pi/6) = -2/\sqrt{3} = -2\sqrt{3}/3$.



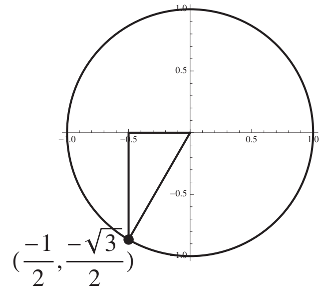
1.3.15

The point on the unit circle associated with $-17\pi/3$ is $(1/2, \sqrt{3}/2)$, so $\cot(-17\pi/3) = 1/\sqrt{3} = \sqrt{3}/3$.



1.3.16

The point on the unit circle associated with $16\pi/3$ is $(-1/2, -\sqrt{3}/2)$, so $\sin(16\pi/3) = -\sqrt{3}/2$.



1.3.17 Because the point on the unit circle associated with $\theta = 0$ is the point $(1, 0)$, we have $\cos 0 = 1$.

1.3.18 Because $-\pi/2$ corresponds to a quarter circle clockwise revolution, the point on the unit circle associated with $-\pi/2$ is the point $(0, -1)$. Thus $\sin(-\pi/2) = -1$.

1.3.19 Because $-\pi$ corresponds to a half circle clockwise revolution, the point on the unit circle associated with $-\pi$ is the point $(-1, 0)$. Thus $\cos(-\pi) = -1$.

1.3.20 Because 3π corresponds to one and a half counterclockwise revolutions, the point on the unit circle associated with 3π is $(-1, 0)$, so $\tan 3\pi = \frac{0}{-1} = 0$.

1.3.21 Because $5\pi/2$ corresponds to one and a quarter counterclockwise revolutions, the point on the unit circle associated with $5\pi/2$ is the same as the point associated with $\pi/2$, which is $(0, 1)$. Thus $\sec 5\pi/2$ is undefined.

1.3.22 Because π corresponds to one half circle counterclockwise revolution, the point on the unit circle associated with π is $(-1, 0)$. Thus $\cot \pi$ is undefined.

1.3.23 From our definitions of the trigonometric functions via a point $P(x, y)$ on a circle of radius $r = \sqrt{x^2 + y^2}$, we have $\sec\theta = \frac{r}{x} = \frac{1}{x/r} = \frac{1}{\cos\theta}$.

1.3.24 From our definitions of the trigonometric functions via a point $P(x, y)$ on a circle of radius $r = \sqrt{x^2 + y^2}$, we have $\tan\theta = \frac{y}{x} = \frac{y/r}{x/r} = \frac{\sin\theta}{\cos\theta}$.

1.3.25 We have already established that $\sin^2\theta + \cos^2\theta = 1$. Dividing both sides by $\cos^2\theta$ gives $\tan^2\theta + 1 = \sec^2\theta$.

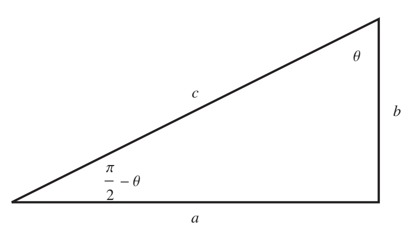
1.3.26 We have already established that $\sin^2\theta + \cos^2\theta = 1$. We can write this as $\frac{\sin\theta}{(1/\sin\theta)} + \frac{\cos\theta}{(1/\cos\theta)} = 1$, or $\frac{\sin\theta}{\csc\theta} + \frac{\cos\theta}{\sec\theta} = 1$.

1.3.27

Using the triangle pictured, we see that

$$\sec(\pi/2 - \theta) = \frac{c}{a} = \csc\theta.$$

This also follows from the sum identity $\cos(a+b) = \cos a \cos b - \sin a \sin b$ as follows: $\sec(\pi/2 - \theta) = \frac{1}{\cos(\pi/2 + (-\theta))} = \frac{1}{\cos(\pi/2)\cos(-\theta) - \sin(\pi/2)\sin(-\theta)} = \frac{1}{0 - (-\sin(\theta))} = \csc(\theta)$.



1.3.28 Using the trig identity for the cosine of a sum (mentioned in the previous solution) we have:

$$\sec(x + \pi) = \frac{1}{\cos(x + \pi)} = \frac{1}{\cos(x)\cos(\pi) - \sin(x)\sin(\pi)} = \frac{1}{\cos(x)\cdot(-1) - \sin(x)\cdot 0} = \frac{1}{-\cos(x)} = -\sec x.$$

1.3.29 Using the fact that $\frac{\pi}{12} = \frac{\pi/6}{2}$ and the half-angle identity for cosine:

$$\cos^2(\pi/12) = \frac{1+\cos(\pi/6)}{2} = \frac{1+\sqrt{3}/2}{2} = \frac{2+\sqrt{3}}{4}.$$

Thus, $\cos(\pi/12) = \sqrt{\frac{2+\sqrt{3}}{4}}$.

1.3.30 Using the fact that $\frac{3\pi}{8} = \frac{3\pi/4}{2}$ and the half-angle identities for sine and cosine, we have:

$$\cos^2(3\pi/8) = \frac{1+\cos(3\pi/4)}{2} = \frac{1+(-\sqrt{2}/2)}{2} = \frac{2-\sqrt{2}}{4},$$

and using the fact that $3\pi/8$ is in the first quadrant (and thus has positive value for cosine) we deduce that $\cos(3\pi/8) = \sqrt{2-\sqrt{2}}/2$. A similar calculation using the sine function results in $\sin(3\pi/8) = \sqrt{2+\sqrt{2}}/2$. Thus $\tan(3\pi/8) = \sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}}$, which simplifies as

$$\sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}} \cdot \frac{2+\sqrt{2}}{2+\sqrt{2}}} = \sqrt{\frac{(2+\sqrt{2})^2}{2}} = \frac{2+\sqrt{2}}{\sqrt{2}} = 1+\sqrt{2}.$$

1.3.31 First note that $\tan x = 1$ when $\sin x = \cos x$. Using our knowledge of the values of the standard angles between 0 and 2π, we recognize that the sine function and the cosine function are equal at $\pi/4$. Then, because we recall that the period of the tangent function is π, we know that $\tan(\pi/4 + k\pi) = \tan(\pi/4) = 1$ for every integer value of k. Thus the solution set is $\{\pi/4 + k\pi, \text{where } k \text{ is an integer}\}$.

1.3.32 Given that $2\theta\cos(\theta) + \theta = 0$, we have $\theta(2\cos(\theta)+1) = 0$. Which means that either $\theta = 0$, or $2\cos(\theta) + 1 = 0$. The latter leads to the equation $\cos\theta = -1/2$, which occurs at $\theta = 2\pi/3$ and $\theta = 4\pi/3$. Using the fact that the cosine function has period 2π the entire solution set is thus

$$\{0\} \cup \{2\pi/3 + 2k\pi, \text{where } k \text{ is an integer}\} \cup \{4\pi/3 + 2l\pi, \text{where } l \text{ is an integer}\}.$$

1.3.33 Given that $\sin^2\theta = \frac{1}{4}$, we have $|\sin\theta| = \frac{1}{2}$, so $\sin\theta = \frac{1}{2}$ or $\sin\theta = -\frac{1}{2}$. It follows that $\theta = \pi/6, 5\pi/6, 7\pi/6, 11\pi/6$.

1.3.34 Given that $\cos^2\theta = \frac{1}{2}$, we have $|\cos\theta| = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. Thus $\cos\theta = \frac{\sqrt{2}}{2}$ or $\cos\theta = -\frac{\sqrt{2}}{2}$. We have $\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.

1.3.35 The equation $\sqrt{2}\sin(x) - 1 = 0$ can be written as $\sin x = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. Standard solutions to this equation occur at $x = \pi/4$ and $x = 3\pi/4$. Because the sine function has period 2π the set of all solutions can be written as:

$$\{\pi/4 + 2k\pi, \text{where } k \text{ is an integer}\} \cup \{3\pi/4 + 2l\pi, \text{where } l \text{ is an integer}\}.$$

1.3.36 Let $u = 3x$. Note that because $0 \le x < 2\pi$, we have $0 \le u < 6\pi$. Because $\sin u = \sqrt{2}/2$ for $u = \pi/4$, $3\pi/4$, $9\pi/4$, $11\pi/4$, $17\pi/4$, and $19\pi/4$, we must have that $\sin 3x = \sqrt{2}/2$ for $3x = \pi/4$, $3\pi/4$, $9\pi/4$, $11\pi/4$, $17\pi/4$, and $19\pi/4$, which translates into

$$x = \pi/12, \pi/4, 3\pi/4, 11\pi/12, 17\pi/12, \text{and } 19\pi/12.$$

1.3.37 As in the previous problem, let $u = 3x$. Then we are interested in the solutions to $\cos u = \sin u$, for $0 \le u < 6\pi$.

This would occur for $u = 3x = \pi/4, 5\pi/4, 9\pi/4, 13\pi/4, 17\pi/4, \text{and } 21\pi/4$. Thus there are solutions for the original equation at

$$x = \pi/12, 5\pi/12, 3\pi/4, 13\pi/12, 17\pi/12, \text{and } 7\pi/4.$$

1.3.38 $\sin^2(\theta) - 1 = 0$ wherever $\sin^2(\theta) = 1$, which is wherever $\sin(\theta) = \pm 1$. This occurs for $\theta = \pi/2 + k\pi$, where k is an integer.

1.3.39 If $\sin\theta\cos\theta = 0$, then either $\sin\theta = 0$ or $\cos\theta = 0$. This occurs for $\theta = 0, \pi/2, \pi, 3\pi/2$.

1.3.40 If $\tan^2 2\theta = 1$, then $\sin^2 2\theta = \cos^2 2\theta$, so we have either $\sin 2\theta = \cos 2\theta$ or $\sin 2\theta = -\cos 2\theta$. This occurs for $2\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$ for $0 \le 2\theta \le 2\pi$, so the corresponding values for θ are $\pi/8$, $3\pi/8$, $5\pi/8$, $7\pi/8$, $0 \le \theta \le \pi$.

1.3.41

a. False. For example, $\sin(\pi/2 + \pi/2) = \sin(\pi) = 0 \ne \sin(\pi/2) + \sin(\pi/2) = 1 + 1 = 2$.

b. False. That equation has zero solutions, because the range of the cosine function is $[-1, 1]$.

c. False. It has infinitely many solutions of the form $\pi/6 + 2k\pi$, where k is an integer (among others.)

d. False. It has period $\frac{2\pi}{\pi/12} = 24$.

e. True. The others have a range of either $[-1, 1]$ or $(-\infty, -1] \cup [1, \infty)$.

1.3.42 If $\sin\theta = -4/5$, then the Pythagorean identity gives $|\cos\theta| = 3/5$. But if $\pi < \theta < 3\pi/2$, then the cosine of θ is negative, so $\cos\theta = -3/5$. Thus $\tan\theta = 4/3$, $\cot\theta = 3/4$, $\sec\theta = -5/3$, and $\csc\theta = -5/4$.

1.3.43 If $\cos\theta = 5/13$, then the Pythagorean identity gives $|\sin\theta| = 12/13$. But if $0 < \theta < \pi/2$, then the sine of θ is positive, so $\sin\theta = 12/13$. Thus $\tan\theta = 12/5$, $\cot\theta = 5/12$, $\sec\theta = 13/5$, and $\csc\theta = 13/12$.

1.3.44 If $\sec\theta = 5/3$, then $\cos\theta = 3/5$, and the Pythagorean identity gives $|\sin\theta| = 4/5$. But if $3\pi/2 < \theta < 2\pi$, then the sine of θ is negative, so $\sin\theta = -4/5$. Thus $\tan\theta = -4/3$, $\cot\theta = -3/4$, and $\csc\theta = -5/4$.

1.3.45 If $\csc\theta = 13/12$, then $\sin\theta = 12/13$, and the Pythagorean identity gives $|\cos\theta| = 5/13$. But if $0 < \theta < \pi/2$, then the cosine of θ is positive, so $\cos\theta = 5/13$. Thus $\tan\theta = 12/5$, $\cot\theta = 5/12$, and $\sec\theta = 13/5$.

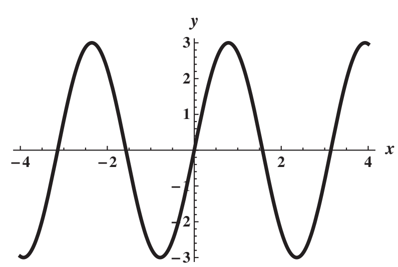
1.3.46 The amplitude is 2, and the period is $\frac{2\pi}{2} = \pi$.

1.3.47 The amplitude is 3, and the period is $\frac{2\pi}{1/3} = 6\pi$.

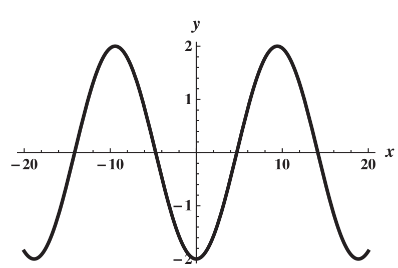
1.3.48 The amplitude is 2.5, and the period is $\frac{2\pi}{1/2} = 4\pi$.

1.3.49 The amplitude is 3.6, and the period is $\frac{2\pi}{\pi/24} = 48$.

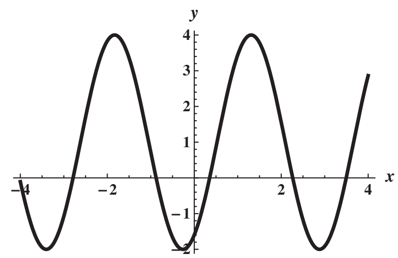
1.3.50 Scale the graph of $y = \sin x$ horizontally by a factor of 2 (steepening it) and vertically by a factor of 3.



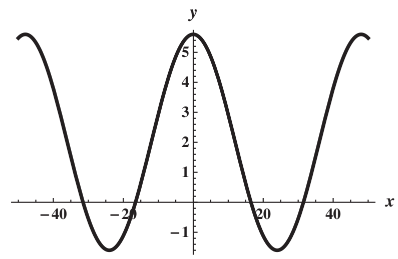
1.3.51 Stretch the graph of $y = \cos x$ horizontally by a factor of 3 and vertically by a factor of 2, and reflect across the x-axis.



1.3.52 Write $p(x) = 3\sin\left((2\left((x - \frac{\pi}{6}\right)\right) + 1$. Shift the graph of $y = \sin x$ $\frac{\pi}{6}$ units to the right, then scale horizontally by a factor of 2 (steepening it), then stretch vertically by a factor of 3 and then shift vertically by 1 unit.

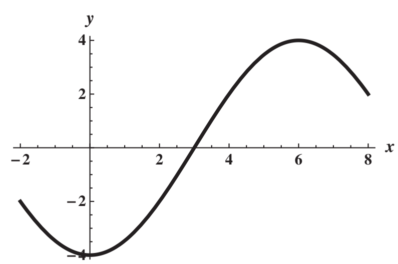


1.3.53 Stretch the graph of $y = \cos x$ horizontally by a factor of $\frac{24}{\pi}$, then stretch it vertically by a factor of 3.6 and then shift it up 2 units.



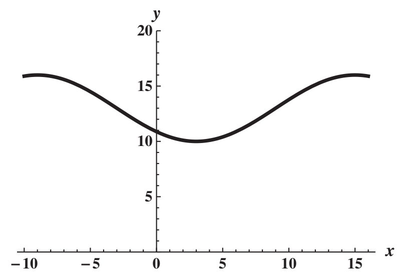
1.3.54 It is helpful to imagine first shifting the function horizontally so that the x intercept is where it should be, then stretching the function horizontally to obtain the correct period, and then stretching the function vertically to obtain the correct amplitude. Because the old x-intercept is at $x = 0$ and the new one should be at $x = 3$ (halfway between where the maximum and the minimum occur), we need to shift the function 3 units to the right. Then to get the right period, we need to multiply (before applying the sine function) by $\pi/6$ so that the new period is $\frac{2\pi}{\pi/6} = 12$. Finally, to get the right amplitude and to get the max and min at the right spots, we need to multiply on the outside by 4. Thus, the desired function is:

$$f(x) = 4\sin((\pi/6)(x - 3)) = 4\sin((\pi/6)x - \pi/2).$$



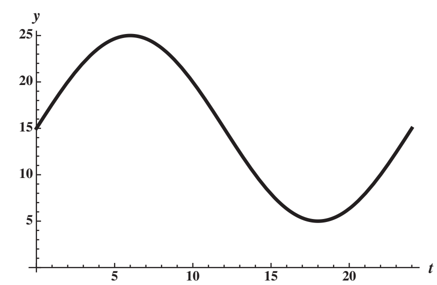
1.3.55 It is helpful to imagine first shifting the function horizontally so that the x intercept is where it should be, then stretching the function horizontally to obtain the correct period, and then stretching the function vertically to obtain the correct amplitude, and then shifting the whole graph up. Because the old x-intercept is at $x = 0$ and the new one should be at $x = 9$ (halfway between where the maximum and the minimum occur), we need to shift the function 9 units to the right. Then to get the right period, we need to multiply (before applying the sine function) by $\pi/12$ so that the new period is $\frac{2\pi}{\pi/12} = 24$. Finally, to get the right amplitude and to get the max and min at the right spots, we need to multiply on the outside by 3, and then shift the whole thing up 13 units. Thus, the desired function is:

$$f(x) = 3\sin((\pi/12)(x - 9)) + 13 = 3\sin((\pi/12)x - 3\pi/4) + 13.$$



1.3.56 It is helpful to imagine first shifting the function horizontally so that the t intercept is where it should be, then stretching the function horizontally to obtain the correct period, and then stretching the function vertically to obtain the correct amplitude, and then shifting the whole graph up. Because the old t-intercept is at $t = 0$ and the new one should be at $t = 12$ (halfway between where the maximum and the minimum occur), we need to shift the function 12 units to the right. Then to get the right period, we need to multiply (before applying the sine function) by $\pi/12$ so that the new period is $\frac{2\pi}{\pi/12} = 24$. Finally, to get the right amplitude and to get the max and min at the right spots, we need to multiply on the outside by -10, and then shift the whole thing up 15 units. Thus, the desired function is:

$$f(t) = -10\sin((\pi/12)(t - 12)) + 15.$$



1.3.57 Let C be the circumference of the earth. Then the first rope has radius $r_1 = \frac{C}{2\pi}$. The circle generated by the longer rope has circumference $C + 38$, so its radius is $r_2 = \frac{C+38}{2\pi} = \frac{C}{2\pi} + \frac{38}{2\pi} \approx r_1 + 6$, so the radius of the bigger circle is about 6 feet more than the smaller circle.

1.3.58

a. The period of this function is $\frac{2\pi}{2\pi/365} = 365$.

b. Because the maximum for the regular sine function is 1, and this function is scaled vertically by a factor of 2.8 and shifted 12 units up, the maximum for this function is $(2.8)(1) + 12 = 14.8$. Similarly, the minimum is $(2.8)(-1) + 12 = 9.2$. Because of the horizontal shift, the point at $t = 81$ is the midpoint between where the max and min occur. Thus the max occurs at $81 + (365/4) \approx 172$ and the min occurs approximately $(365/2)$ days later at about $t = 355$.

c. The solstices occur halfway between these points, at 81 and $81 + (365/2) \approx 264$.

1.3.59 We are seeking a function with amplitude 10 and period 1.5, and value 10 at time 0, so it should have the form $10\cos(kt)$, where $\frac{2\pi}{k} = 1.5$. Solving for k yields $k = \frac{4\pi}{3}$, so the desired function is $d(t) = 10\cos(4\pi t/3)$.

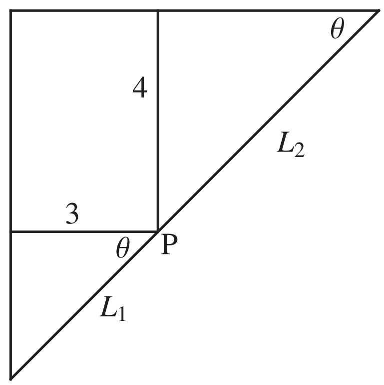
1.3.60

a. Because $\tan\theta = \frac{50}{d}$, we have $d = \frac{50}{\tan\theta}$.

b. Because $\sin\theta = \frac{50}{L}$, we have $L = \frac{50}{\sin\theta}$.

1.3.61 Let L be the line segment connecting the tops of the ladders and let M be the horizontal line segment between the walls h feet above the ground. Now note that the triangle formed by the ladders and L is equilateral, because the angle between the ladders is 60 degrees, and the other two angles must be equal and add to 120, so they are 60 degrees as well. Now we can see that the triangle formed by L, M and the right wall is similar to the triangle formed by the left ladder, the left wall, and the ground, because they are both right triangles with one angle of 75 degrees and one of 15 degrees. Thus $M = h$ is the distance between the walls.

1.3.62 Let the corner point P divide the pole into two pieces, L_1 (which spans the 3-ft hallway) and L_2 (which spans the 4-ft hallway.) Then $L = L_1 + L_2$. Now $L_2 = \frac{4}{\sin\theta}$, and $\frac{3}{L_1} = \cos\theta$ (see diagram.) Thus $L = L_1 + L_2 = \frac{3}{\cos\theta} + \frac{4}{\sin\theta}$. When $L = 10$, $\theta \approx .9273$.



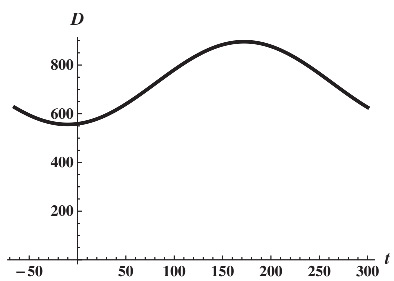
1.3.63 To find $s(t)$ note that we are seeking a periodic function with period 365, and with amplitude 87.5 (which is half of the number of minutes between 7:25 and 4:30). We need to shift the function 4 days plus one fourth of 365, which is about 95 days so that the max and min occur at $t = 4$ days and at half a year later. Also, to get the right value for the maximum and minimum, we need to multiply by negative one and add 117.5 (which represents 30 minutes plus half the amplitude, because $s = 0$ corresponds to 4:00 AM.) Thus we have

$$s(t) = 117.5 - 87.5\sin\left(\frac{\pi}{182.5}(t-95)\right).$$

A similar analysis leads to the formula

$$S(t) = 844.5 + 87.5\sin\left(\frac{\pi}{182.5}(t-67)\right).$$

The graph pictured shows $D(t) = S(t) - s(t)$, the length of day function, which has its max at the summer solstice which is about the 172nd day of the year, and its min at the winter solstice.



1.3.64 Let θ_1 be the viewing angle to the bottom of the television. Then $\tan\theta_1 = \left(\frac{3}{10}\right)$. Now $\tan(\theta + \theta_1) = \frac{10}{10} = 1$, so $\theta + \theta_1 = \frac{\pi}{4}$, so $\theta = \frac{\pi}{4} - \theta_1 \approx 0.494$.

1.3.65 The area of the entire circle is πr^2. The ratio $\frac{\theta}{2\pi}$ represents the proportion of the area swept out by a central angle θ. Thus the area of a sector of a circle is this same proportion of the entire area, so it is $\frac{\theta}{2\pi} \cdot \pi r^2 = \frac{r^2\theta}{2}$.

1.3.66 Using the given diagram, drop a perpendicular from the point $(b\cos\theta, b\sin\theta)$ to the x axis, and consider the right triangle thus formed whose hypotenuse has length c. By the Pythagorean theorem, $(b\sin\theta)^2 + (a - b\cos\theta)^2 = c^2$. Expanding the binomial gives $b^2\sin^2\theta + a^2 - 2ab\cos\theta + b^2\cos^2\theta = c^2$. Now because $b^2\sin^2\theta + b^2\cos^2\theta = b^2$, this reduces to $a^2 + b^2 - 2ab\cos\theta = c^2$.

1.3.67 Note that $\sin A = \frac{h}{c}$ and $\sin C = \frac{h}{a}$, so $h = c\sin A = a\sin C$. Thus

$$\frac{\sin A}{a} = \frac{\sin C}{c}.$$

Now drop a perpendicular from the vertex A to the line determined by $\overline{BC}$, and let h_2 be the length of this perpendicular. Then $\sin C = \frac{h_2}{b}$ and $\sin B = \frac{h_2}{C}$, so $h_2 = b\sin C = c\sin B$. Thus

$$\frac{\sin C}{c} = \frac{\sin B}{b}.$$

Putting the two displayed equations together gives

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Chapter One Review

1

a. True. For example, $f(x) = x^2$ is such a function.

b. False. For example, $\cos(\pi/2 + \pi/2) = \cos(\pi) = -1 \neq \cos(\pi/2) + \cos(\pi/2) = 0 + 0 = 0$.

c. False. Consider $f(1 + 1) = f(2) = 2m + b \neq f(1) + f(1) = (m + b) + (m + b) = 2m + 2b$. (At least these aren't equal when $b \neq 0$.)

d. True. $f(f(x)) = f(1 - x) = 1 - (1 - x) = x$.

e. False. This set is the union of the disjoint intervals $(-\infty, -7)$ and $(1, \infty)$.

2

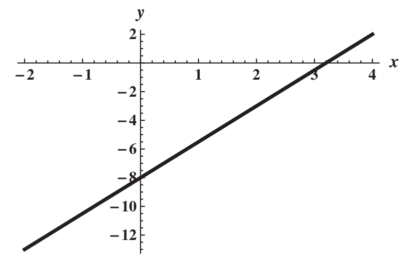
a. Because the quantity under the radical must be non-zero, the domain of f is $[0, \infty)$. The range is also $[0, \infty)$.

b. The domain is $(-\infty, 2) \cup (2, \infty)$. The range is $(-\infty, 0) \cup (0, \infty)$. (Note that if 0 were in the range then $\frac{1}{y-2} = 0$ for some value of y, but this expression has no real solutions.)

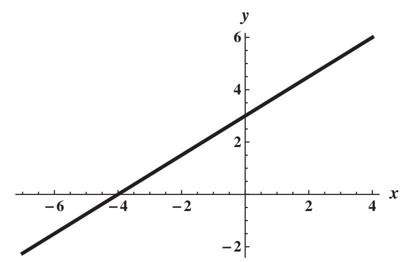
c. Because h can be written $h(z) = \sqrt{(z-3)(z+1)}$, we see that the domain is $(-\infty, -1] \cup [3, \infty)$. The range is $[0, \infty)$. (Note that as z gets large, $h(z)$ gets large as well.)

3

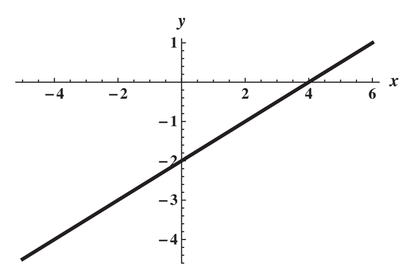
a. This line has slope $\frac{2-(-3)}{4-2} = 5/2$. Therefore the equation of the line is $y - 2 = \frac{5}{2}(x-4)$, so $y = \frac{5}{2}x - 8$.



b. This line has the form $y = (3/4)x + b$, and because $(-4, 0)$ is on the line, $0 = (3/4)(-4) + b$, so $b = 3$. Thus the equation of the line is given by $y = (3/4)x + 3$.

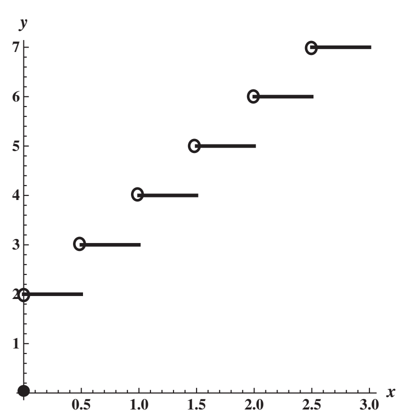


c. This line has slope $\frac{0-(-2)}{4-0} = \frac{1}{2}$, and the y-intercept is given to be -2, so the equation of this line is $y = (1/2)x - 2$.



4

The function is a piecewise step function which jumps up by one every half-hour step.

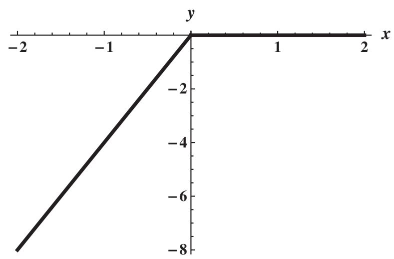


5

Because $|x| = \begin{cases} -x & \text{if } x < 0; \\ x & \text{if } x \geq 0, \end{cases}$

we have

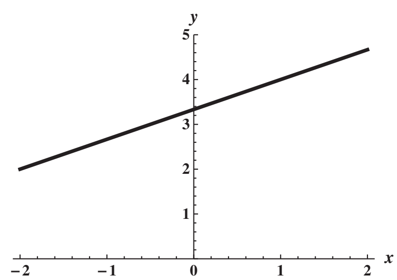
$$2(x - |x|) = \begin{cases} 2(x - (-x)) = 4x & \text{if } x < 0; \\ 2(x - x) = 0 & \text{if } x \geq 0. \end{cases}$$



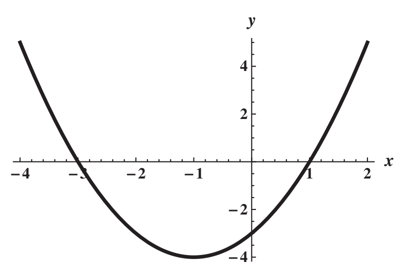
6 Because the trip is 500 miles in a car that gets 35 miles per gallon, $\frac{500}{35} = \frac{100}{7}$ represents the number of gallons required for the trip. If we multiply this times the number of dollars per gallon we will get the cost. Thus $C = f(p) = \frac{100}{7}p$ dollars.

7

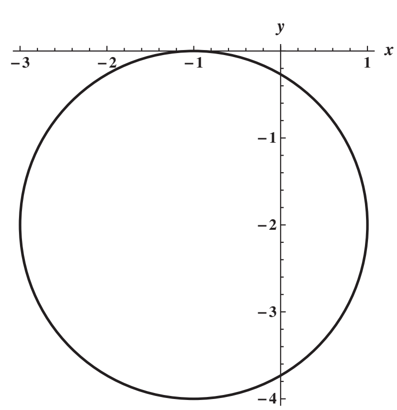
a. This is a straight line with slope 2/3 and y-intercept 10/3.



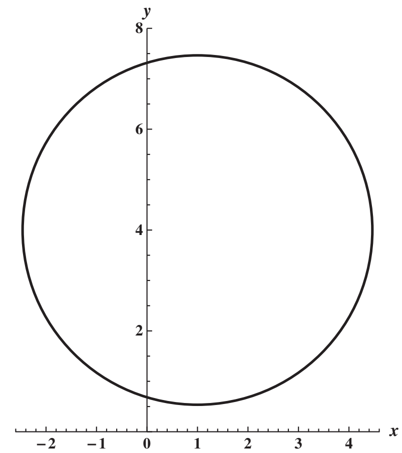
b. Completing the square gives $y = (x^2+2x+1)-4$, or $y = (x+1)^2-4$, so this is the standard parabola shifted one unit to the left and down 4 units.



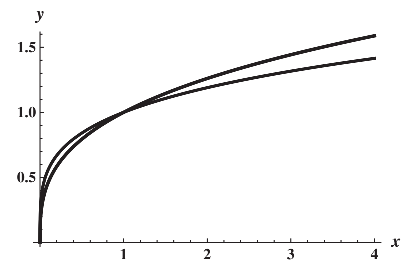
c. Completing the square, we have $x^2+2x+1+y^2+4y+4 = -1+1+4$, so we have $(x+1)^2+(y+2)^2 = 4$, a circle of radius 2 centered at $(-1, -2)$.



d. Completing the square, we have $x^2 - 2x + 1 + y^2 - 8y + 16 = -5 + 1 + 16$, or $(x-1)^2 + (y-4)^2 = 12$, which is a circle of radius $\sqrt{12}$ centered at $(1, 4)$.



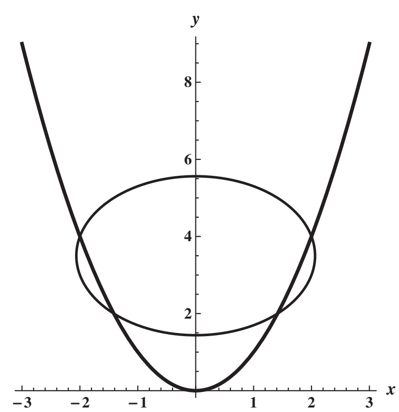
8 To solve $x^{1/3} = x^{1/4}$ we raise each side to the 12th power, yielding $x^4 = x^3$. This gives $x^4 - x^3 = 0$, or $x^3(x-1) = 0$, so the only solutions are $x = 0$ and $x = 1$ (which can be easily verified as solutions.) Between 0 and 1, $x^{1/4} > x^{1/3}$, but for $x > 1$, $x^{1/3} > x^{1/4}$.



9 The domain of $x^{1/7}$ is the set of all real numbers, as is its range. The domain of $x^{1/4}$ is the set of non-negative real numbers, as is its range.

10

Completing the square in the second equation, we have $x^2 + y^2 - 7y + \frac{49}{4} = -8 + \frac{49}{4}$, which can be written as $x^2 + (y - (7/2))^2 = \frac{17}{4}$. Thus we have a circle of radius $\sqrt{17}/2$ centered at $(0, 7/2)$, along with the standard parabola. These intersect when $y = 7y - y^2 - 8$, which occurs for $y^2 - 6y + 8 = 0$, so for $y = 2$ and $y = 4$, with corresponding x values of ± 2 and $\pm\sqrt{2}$.



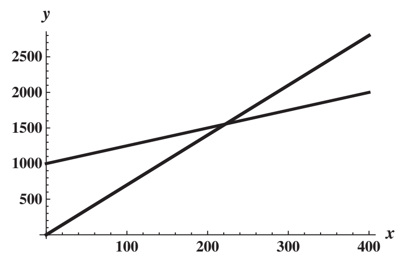
11 We are looking for the line between the points $(0, 212)$ and $(6000, 200)$. The slope is $\frac{212-200}{0-6000} = -\frac{12}{6000} = -\frac{1}{500}$. Because the intercept is given, we deduce that the line is $B = f(a) = -\frac{1}{500}a + 212$.

12

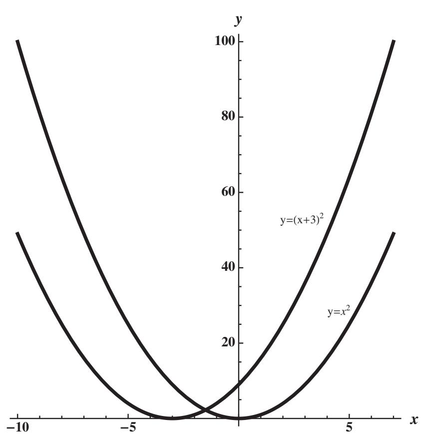
a. The cost of producing x books is $C(x) = 1000 + 2.5x$.

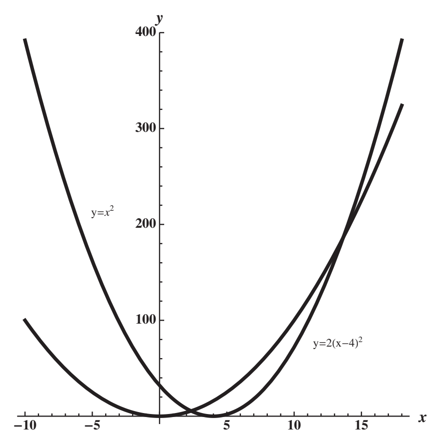
b. The revenue generated by selling x books is $R(x) = 7x$.

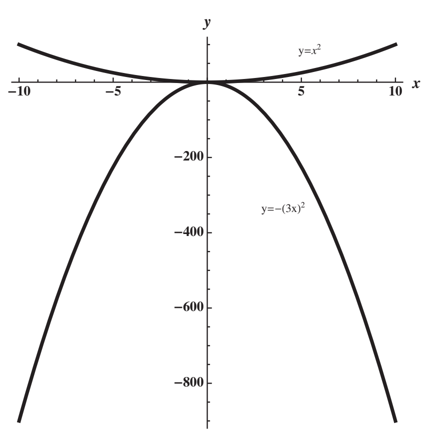
c. The break-even point is where $R(x) = C(x)$. This is where $7x = 1000 + 2.5x$, or $4.5x = 1000$. So $x = \frac{1000}{4.5} \approx 222$.

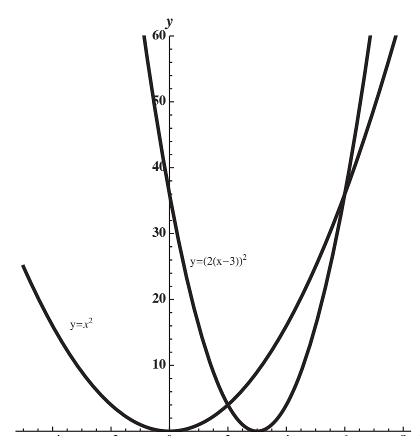


13

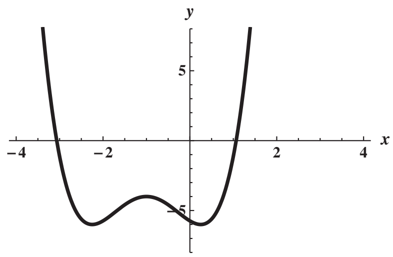
a. 

b. 

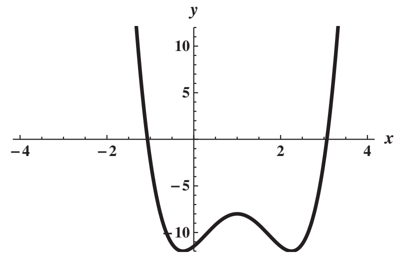
c. 

d. 

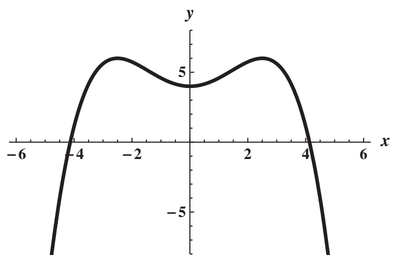
14



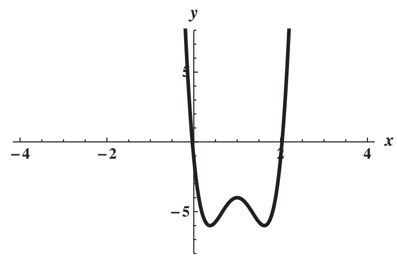
a.



b.



c.



d.

15

a. $h(g(\pi/2)) = h(1) = 1$

b. $h(f(x)) = h(x^3) = x^{3/2}$.

c. $f(g(h(x))) = f(g(\sqrt{x})) = f(\sin(\sqrt{x})) = (\sin(\sqrt{x}))^3$.

d The domain of $g(f(x))$ is $\mathbb{R}$, because the domain of both functions is the set of all real numbers.

e. The range of $f(g(x))$ is $[-1, 1]$. This is because the range of g is $[-1, 1]$, and on the restricted domain $[-1, 1]$, the range of f is also $[-1, 1]$.

16

a. If $g(x) = x^2 + 1$ and $f(x) = \sin x$, then $f(g(x)) = f(x^2 + 1) = \sin(x^2 + 1)$.

b. If $g(x) = x^2 - 4$ and $f(x) = x^{-3}$ then $f(g(x)) = f(x^2 - 4) = (x^2 - 4)^{-3}$.

17 $\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2-2(x+h)-(x^2-2x)}{h} = \frac{x^2+2hx+h^2-2x-2h-x^2+2x}{h} = \frac{2hx+h^2-2h}{h} = 2x + h - 2$.

$\frac{f(x)-f(a)}{x-a} = \frac{x^2-2x-(a^2-2a)}{x-a} = \frac{(x^2-a^2)-2(x-a)}{x-a} = \frac{(x-a)(x+a)-2(x-a)}{x-a} = x + a - 2$.

18 $\frac{f(x+h)-f(x)}{h} = \frac{4-5(x+h)-(4-5x)}{h} = \frac{4-5x-5h-4+5x}{h} = -\frac{5h}{h} = -5$.

$\frac{f(x)-f(a)}{x-a} = \frac{4-5x-(4-5a)}{x-a} = -\frac{5(x-a)}{x-a} = -5$.

19 $\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^3+2-(x^3+2)}{h} = \frac{x^3+3x^2h+3xh^2+h^3+2-x^3-2}{h} = \frac{h(3x^2+3xh+h^2)}{h} = 3x^2 + 3xh + h^2$.

$\frac{f(x)-f(a)}{x-a} = \frac{x^3+2-(a^3+2)}{x-a} = \frac{x^3-a^3}{x-a} = \frac{(x-a)(x^2+ax+a^2)}{x-a} = x^2 + ax + a^2$.

20 $\frac{f(x+h)-f(x)}{h} = \frac{\frac{7}{x+h+3}-\frac{7}{x+3}}{h} = \frac{\frac{7x+21-(7x+7h+21)}{(x+3)(x+h+3)}}{h} = -\frac{7h}{(h)(x+3)(x+h+3)} = -\frac{7}{(x+3)(x+h+3)}$.

$\frac{f(x)-f(a)}{x-a} = \frac{\frac{7}{x+3}-\frac{7}{a+3}}{x-a} = \frac{\frac{7a+21-(7x+21)}{(x+3)(a+3)}}{x-a} = -\frac{7(x-a)}{(x-a)(x+3)(a+3)} = -\frac{7}{(x+3)(a+3)}$.

21

a. Because $f(-x) = \cos -3x = \cos 3x = f(x)$, this is an even function, and is symmetric about the y-axis.

b. Because $f(-x) = 3(-x)^4 - 3(-x)^2 + 1 = 3x^4 - 3x^2 + 1 = f(x)$, this is an even function, and is symmetric about the y-axis.

c. Because replacing x by $-x$ and/or replacing y by $-y$ gives the same equation, this represents a curve which is symmetric about the y-axis and about the origin and about the x-axis.

22

a. $\frac{1+\cos\theta}{\sin\theta} = \frac{1+\cos\theta}{\sin\theta} \cdot \frac{1-\cos\theta}{1-\cos\theta} = \frac{1-\cos^2\theta}{(\sin\theta)(1-\cos\theta)} = \frac{\sin^2\theta}{(\sin\theta)(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}$.

b. $\frac{\sec\theta-1}{\tan\theta} = \frac{\sec\theta-1}{\tan\theta} \cdot \frac{\sec\theta+1}{\sec\theta+1} = \frac{\sec^2\theta-1}{(\tan\theta)(\sec\theta+1)} = \frac{\tan^2\theta}{(\tan\theta)(\sec\theta+1)} = \frac{\tan\theta}{\sec\theta+1}$.

23

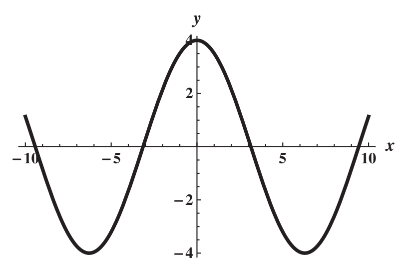
a. A 135 degree angle measures $135 \cdot (\pi/180)$ radians, which is $3\pi/4$ radians.

b. A $4\pi/5$ radian angle measues $4\pi/5 \cdot (180/\pi)$ degrees, which is 144 degrees.

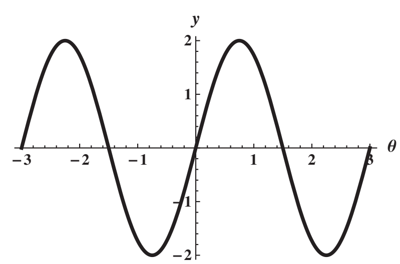
c. Because the length of the arc is the measure of the subtended angle (in radians) times the radius, this arc would be $4\pi/3 \cdot 10 = \frac{40\pi}{3}$ units long.

24

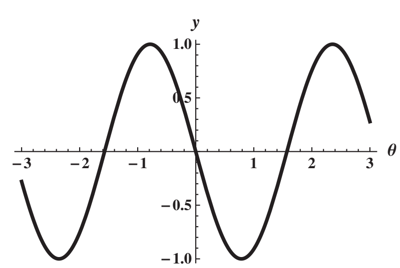
a. This function has period $\frac{2\pi}{1/2} = 4\pi$ and amplitude 4.



b. This function has period $\frac{2\pi}{2\pi/3} = 3$ and amplitude 2.



c. This function has period $\frac{2\pi}{2} = \pi$ and amplitude 1. Compared to the ordinary cosine function it is compressed horizontally, flipped about the x-axis, and shifted $\pi/4$ units to the right.



25

a. We need to scale the ordinary cosine function so that its period is 6, and then shift it 3 units to the right, and multiply it by 2. So the function we seek is $y = 2\cos((\pi/3)(t-3)) = -2\cos(\pi t/3)$.

b. We need to scale the ordinary cosine function so that its period is 24, and then shift it to the right 6 units. We then need to change the amplitude to be half the difference between the maximum and minimum, which would be 5. Then finally we need to shift the whole thing up by 15 units. The function we seek is thus $y = 15 + 5\cos((\pi/12)(t-6)) = 15 + 5\sin(\pi t/12)$.

26 The pictured function has a period of π, an amplitude of 2, and a maximum of 3 and a minimum of -1. It can be described by $y = 1 + 2\cos(2(x - \pi/2))$.

27

a. $-\sin x$ is pictured in F.

b. $\cos 2x$ is pictured in E.

c. $\tan(x/2)$ is pictured in D.

d. $-\sec x$ is pictured in B.

e. $\cot 2x$ is pictured in C.

f. $\sin^2 x$ is pictured in A.

28 If $\sec x = 2$, then $\cos x = \frac{1}{2}$. This occurs for $x = -\pi/3$ and $x = \pi/3$, so the intersection points are $(-\pi/3, 2)$ and $(\pi/3, 2)$.

29 $\sin x = -\frac{1}{2}$ for $x = 7\pi/6$ and for $x = 11\pi/6$, so the intersection points are $(7\pi/6, -1/2)$ and $(11\pi/6, -1/2)$.

30 Let N be the north pole, and C the center of the given circle, and consider the angle CNP. This angle measures $\frac{\pi-\varphi}{2}$. (Note that the triangle CNP is isosceles.) Now consider the triangle NOX where O is the origin and X is the point $(x, 0)$. Using triangle NOX, we have $\tan\left(\frac{\pi-\varphi}{2}\right) = \frac{x}{2R}$, so $x = 2R\tan\left(\frac{\pi-\varphi}{2}\right)$.

Chapter 2

Limits

2.1 The Idea of Limits

2.1.1 The average velocity of the object between time $t = a$ and $t = b$ is the change in position divided by the elapsed time: $v_{\text{av}} = \frac{s(b)-s(a)}{b-a}$.

2.1.2 In order to compute the instantaneous velocity of the object at time $t = a$, we compute the average velocity over smaller and smaller time intervals of the form $[a, t]$, using the formula: $v_{\text{av}} = \frac{s(t)-s(a)}{t-a}$. We let t approach a. If the quantity $\frac{s(t)-s(a)}{t-a}$ approaches a limit as $t \to a$, then that limit is called the instantaneous velocity of the object at time $t = a$.

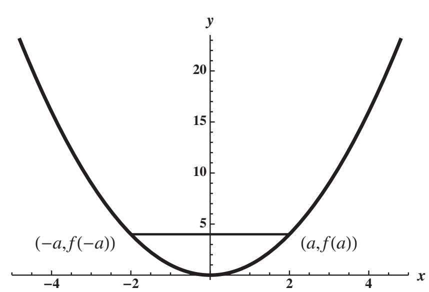
2.1.3 The slope of the secant line between points $(a, f(a))$ and $(b, f(b))$ is the ratio of the differences $f(b) - f(a)$ and $b - a$. Thus $m_{\text{sec}} = \frac{f(b)-f(a)}{b-a}$.

2.1.4 In order to compute the slope of the tangent line to the graph of $y = f(t)$ at $(a, f(a))$, we compute the slope of the secant line over smaller and smaller time intervals of the form $[a, t]$. Thus we consider $\frac{f(t)-f(a)}{t-a}$ and let $t \to a$. If this quantity approaches a limit, then that limit is the slope of the tangent line to the curve $y = f(t)$ at $t = a$.

2.1.5 Both problems involve the same mathematics, namely finding the limit as $t \to a$ of a quotient of differences of the form $\frac{g(t)-g(a)}{t-a}$ for some function g.

2.1.6

Because $f(x) = x^2$ is an even function, $f(-a) = f(a)$ for all a. Thus the slope of the secant line between the points $(a, f(a))$ and $(-a, f(-a))$ is $m_{\text{sec}} = \frac{f(-a)-f(a)}{-a-a} = \frac{0}{-2a} = 0$. The slope of the tangent line at $x = 0$ is also zero.



2.1.7 The average velocity is $\frac{s(3)-s(2)}{3-2} = 156 - 136 = 20$.

2.1.8 The average velocity is $\frac{s(4)-s(1)}{4-1} = \frac{144-84}{3} = \frac{60}{3} = 20$.

2.1.9

a. Over $[1,4]$, we have $v_{\text{av}} = \frac{s(4)-s(1)}{4-1} = \frac{256-112}{3} = 48$.

b. Over $[1,3]$, we have $v_{\text{av}} = \frac{s(3)-s(1)}{3-1} = \frac{240-112}{2} = 64$.

c. Over $[1,2]$, we have $v_{\text{av}} = \frac{s(2)-s(1)}{2-1} = \frac{192-112}{1} = 80$.

d. Over $[1,1+h]$, we have $v_{\text{av}} = \frac{s(1+h)-s(1)}{1+h-1} = \frac{-16(1+h)^2+128(1+h)-(112)}{h} = \frac{-16h^2-32h+128h}{h} = \frac{h(-16h+96)}{h} = 96-16h = 16(6-h)$.

2.1.10

a. Over $[0,3]$, we have $v_{\text{av}} = \frac{s(3)-s(0)}{3-0} = \frac{65.9-20}{3} = 15.3$.

b. Over $[0,2]$, we have $v_{\text{av}} = \frac{s(2)-s(0)}{2-0} = \frac{60.4-20}{2} = 20.2$.

c. Over $[0,1]$, we have $v_{\text{av}} = \frac{s(1)-s(0)}{1-0} = \frac{45.1-20}{1} = 25.1$.

d. Over $[0,h]$, we have $v_{\text{av}} = \frac{s(h)-s(0)}{h-0} = \frac{-4.9h^2+30h+20-20}{h} = \frac{(h)(-4.9h+30)}{h} = -4.9h+30$.

2.1.11

a. $\frac{s(2)-s(0)}{2-0} = \frac{72-0}{2} = 36$.

b. $\frac{s(1.5)-s(0)}{1.5-0} = \frac{66-0}{1.5} = 44$.

c. $\frac{s(1)-s(0)}{1-0} = \frac{52-0}{1} = 52$.

d. $\frac{s(0.5)-s(0)}{0.5-0} = \frac{30-0}{0.5} = 60$.

2.1.12

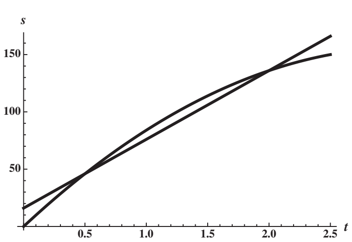
a. $\frac{s(2.5)-s(0.5)}{2.5-0.5} = \frac{150-46}{2} = 52$.

b. $\frac{s(2)-s(0.5)}{2-0.5} = \frac{136-46}{1.5} = 60$.

c. $\frac{s(1.5)-s(0.5)}{1.5-0.5} = \frac{114-46}{1} = 68$.

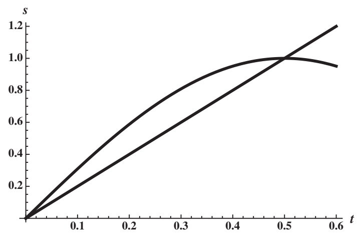
d. $\frac{s(1)-s(0.5)}{1-0.5} = \frac{84-46}{0.5} = 76$.

2.1.13



The slope of the secant line is given by $\frac{s(2)-s(0.5)}{2-0.5} = \frac{136-46}{1.5} = 60$. This represents the average velocity of the object over the time interval $[0.5, 2]$.

2.1.14



The slope of the secant line is given by $\frac{s(0.5)-s(0)}{0.5-0} = \frac{1}{0.5} = 2$. This represents the average velocity of the object over the time interval $[0, 0.5]$.

2.1.15

Time Interval	$[1, 2]$	$[1, 1.5]$	$[1, 1.1]$	$[1, 1.01]$	$[1, 1.001]$
Average Velocity	80	88	94.4	95.84	95.984

The instantaneous velocity appears to be 96 ft/s.

2.1.16

Time Interval	$[2, 3]$	$[2, 2.25]$	$[2, 2.1]$	$[2, 2.01]$	$[2, 2.001]$
Average Velocity	5.5	9.175	9.91	10.351	10.395

The instantaneous velocity appears to be 10.4 m/s.

2.1.17 $\frac{s(1.01)-s(1)}{0.01} = 47.84$, while $\frac{s(1.001)-s(1)}{0.001} = 47.984$ and $\frac{s(1.0001)-s(1)}{0.0001} = 47.9984$. It appears that the instantaneous velocity at $t = 1$ is approximately 48.

2.1.18 $\frac{s(2.01)-s(2)}{0.01} = -4.16$, while $\frac{s(2.001)-s(2)}{0.001} = -4.016$ and $\frac{s(2.0001)-s(2)}{0.0001} = -4.0016$. It appears that the instantaneous velocity at $t = 2$ is approximately -4.

2.1.19

Time Interval	$[2, 3]$	$[2.9, 3]$	$[2.99, 3]$	$[2.999, 3]$	$[2.9999, 3]$	$[2.99999, 3]$
Average Velocity	20	5.6	4.16	4.016	4.002	4.0002

The instantaneous velocity appears to be 4 ft/s.

2.1.20

Time Interval	$[\pi/2, \pi]$	$[\pi/2, \pi/2 + 0.1]$	$[\pi/2, \pi/2 + .01]$	$[\pi/2, \pi/2 + .001]$	$[\pi/2, \pi/2 + .0001]$
Average Velocity	-1.90986	-0.149875	-0.0149999	-0.0015	-0.00015

The instantaneous velocity appears to be 0 ft/s.

2.1.21

Time Interval	$[3, 3.1]$	$[3, 3.01]$	$[3, 3.001]$	$[3, 3.0001]$
Average Velocity	-17.6	-16.16	-16.016	-16.002

The instantaneous velocity appears to be -16 ft/s.

2.1.22

Time Interval	$[\pi/2, \pi/2 + 0.1]$	$[\pi/2, \pi/2 + 0.01]$	$[\pi/2, \pi/2 + 0.001]$	$[\pi/2, \pi/2 + 0.0001]$
Average Velocity	-19.9667	-19.9997	-20.0000	-20.0000

The instantaneous velocity appears to be -20 ft/s.

2.1.23

Time Interval	$[0, 0.1]$	$[0, 0.01]$	$[0, 0.001]$	$[0, 0.0001]$
Average Velocity	79.468	79.995	80.000	80.0000

The instantaneous velocity appears to be 80 ft/s.

2.1.24

Time Interval	$[0, 1]$	$[0, 0.1]$	$[0, 0.01]$	$[0, 0.001]$
Average Velocity	-10	-18.1818	-19.802	-19.98

The instantaneous velocity appears to be -20 ft/s.

2.1.25

x **Interval**	$[2, 2.1]$	$[2, 2.01]$	$[2, 2.001]$	$[2, 2.0001]$
Slope of Secant Line	8.2	8.02	8.002	8.0002

The slope of the tangent line appears to be 8.

2.1.26

x **Interval**	$[\pi/2, \pi/2 + 0.1]$	$[\pi/2, \pi/2 + 0.01]$	$[\pi/2, \pi/2 + 0.001]$	$[\pi/2, \pi/2 + 0.0001]$
Slope of Secant Line	-2.995	-2.99995	-3.0000	-3.0000

The slope of the tangent line appears to be -3.

2.1.27

x **Interval**	$[-1, -0.9]$	$[-1, -0.99]$	$[-1, -0.999]$	$[-1, -0.9999]$
Slope of the Secant Line	0.524862	0.5025	0.50025	0.500025

The slope of the tangent line appears to be 0.5.

2.1.28

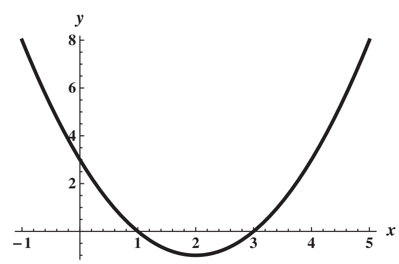
x **Interval**	$[1, 1.1]$	$[1, 1.01]$	$[1, 1.001]$	$[1, 1.0001]$
Slope of the Secant Line	2.31	2.0301	2.003	2.0003

The slope of the tangent line appears to be 2.

2.1.29

a. Note that the graph is a parabola with vertex $(2, -1)$.

b. At $(2, -1)$ the function has tangent line with slope 0.



c.

x **Interval**	$[2, 2.1]$	$[2, 2.01]$	$[2, 2.001]$	$[2, 2.0001]$
Slope of the Secant Line	.1	.01	.001	.0001

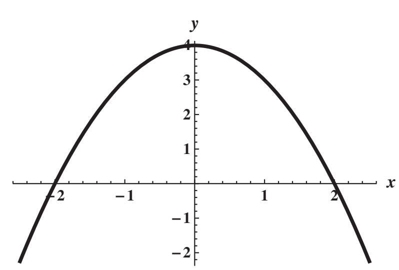
The slope of the tangent line at $(2, -1)$ appears to be 0.

2.1.30

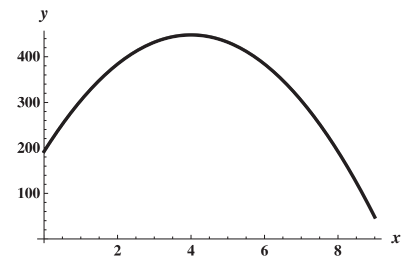
a. Note that the graph is a parabola with vertex $(0, 4)$.

b. At $(0, 4)$ the function has a tangent line with slope 0.

c. This is true for this function – because the function is symmetric about the y-axis and we are taking pairs of points symmetrically about the y axis. Thus $f(0 + h) = 4 - (0 + h)^2 = 4 - (-h)^2 = f(0 - h)$. So the slope of any such secant line is $\frac{4-h^2-(4-h^2)}{h-(-h)} = \frac{0}{2h} = 0$.



2.1.31



a. Note that the graph is a parabola with vertex $(4, 448)$.

b. At $(4, 448)$ the function has tangent line with slope 0, so $a = 4$.

c.

x **Interval**	$[4, 4.1]$	$[4, 4.01]$	$[4, 4.001]$	$[4, 4.0001]$
Slope of the Secant Line	-1.6	-0.16	-0.016	-0.0016

The slopes of the secant lines appear to be approaching zero.

d. On the interval $[0, 4)$ the instantaneous velocity of the projectile is positive.

e. On the interval $(4, 9]$ the instantaneous velocity of the projectile is negative.

2.1.32

a. The rock strikes the water when $s(t) = 96$. This occurs when $16t^2 = 96$, or $t^2 = 6$, whose only positive solution is $t = \sqrt{6} \approx 2.45$ seconds.

b.

t **Interval**	$[\sqrt{6} - 0.1, \sqrt{6}]$	$[\sqrt{6} - 0.01, \sqrt{6}]$	$[\sqrt{6} - 0.001, \sqrt{6}]$	$[\sqrt{6} - 0.0001, \sqrt{6}]$
Average Velocity	76.7837	78.2237	78.3677	78.3821

When the rock strikes the water, its instantaneous velocity is about 78.38 ft/s.

2.1.33 For line AD, we have

$$m_{AD} = \frac{y_D - y_A}{x_D - x_A} = \frac{f(\pi) - f(\pi/2)}{\pi - (\pi/2)} = \frac{1}{\pi/2} \approx .63662.$$

For line AC, we have

$$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{f(\pi/2 + 0.5) - f(\pi/2)}{(\pi/2 + 0.5) - (\pi/2)} = -\frac{\cos(\pi/2 + 0.5)}{0.5} \approx 0.958851.$$

For line AB, we have

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{f(\pi/2 + 0.05) - f(\pi/2)}{(\pi/2 + 0.05) - (\pi/2)} = -\frac{\cos(\pi/2 + 0.05)}{0.05} \approx 0.999583.$$

Computing one more slope of a secant line:

$$m_{\text{sec}} = \frac{f(\pi/2 + 0.01) - f(\pi/2)}{(\pi/2 + 0.01) - (\pi/2)} = -\frac{\cos(\pi/2 + 0.01)}{0.01} \approx 0.999983.$$

Conjecture: The slope of the tangent line to the graph of f at $x = \pi/2$ is 1.

2.2 Definitions of Limits

2.2.1 Suppose the function f is defined for all x near a except possibly at a. If $f(x)$ is arbitrarily close to a number L whenever x is sufficiently close to (but not equal to) a, then we write $\lim_{x\to a} f(x) = L$.

2.2.2 False. For example, consider the function $f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 4 & \text{if } x = 0. \end{cases}$

Then $\lim_{x\to 0} f(x) = 0$, but $f(0) = 4$.

2.2.3 Suppose the function f is defined for all x near a but greater than a. If $f(x)$ is arbitrarily close to L for x sufficiently close to (but strictly greater than) a, then we write $\lim_{x\to a^+} f(x) = L$.

2.2.4 Suppose the function f is defined for all x near a but less than a. If $f(x)$ is arbitrarily close to L for x sufficiently close to (but strictly less than) a, then we write $\lim_{x\to a^-} f(x) = L$.

2.2.5 It must be true that $L = M$.

2.2.6 Because graphing utilities generally just plot a sampling of points and "connect the dots," they can sometimes mislead the user investigating the subtleties of limits.

2.2.7

a. $h(2) = 5$.

b. $\lim_{x\to 2} h(x) = 3$.

c. $h(4)$ does not exist.

d. $\lim_{x\to 4} f(x) = 1$.

e. $\lim_{x\to 5} h(x) = 2$.

2.2.8

a. $g(0) = 0$.

b. $\lim_{x\to 0} g(x) = 1$.

c. $g(1) = 2$.

d. $\lim_{x\to 1} g(x) = 2$.

2.2.9

a. $f(1) = -1$.

b. $\lim_{x\to 1} f(x) = 1$.

c. $f(0) = 2$.

d. $\lim_{x\to 0} f(x) = 2$.

2.2.10

a. $f(2) = 2$.

b. $\lim_{x\to 2} f(x) = 4$.

c. $\lim_{x\to 4} f(x) = 4$.

d. $\lim_{x\to 5} f(x) = 2$.

2.2.11

a.

x	1.9	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.1
$f(x) = \frac{x^2-4}{x-2}$	3.9	3.99	3.999	3.9999	undefined	4.0001	4.001	4.01	4.1

b. $\lim_{x\to 2} f(x) = 4$.

2.2.12

a.

x	.9	.99	.999	.9999	1	1.0001	1.001	1.01	1.1
$f(x) = \frac{x^3-1}{x-1}$	2.71	2.9701	2.997	2.9997	undefined	3.0003	3.003	3.0301	3.31

b. $\lim_{x\to 1} \dfrac{x^3-1}{x-1} = 3$

2.2.13

a.

t	8.9	8.99	8.999	9	9.001	9.01	9.1
$g(t) = \frac{t-9}{\sqrt{t}-3}$	5.98329	5.99833	5.99983	undefined	6.00017	6.00167	6.01662

b. $\lim_{t\to 9} \frac{t-9}{\sqrt{t}-3} = 6.$

2.2.14

a.

x	.01	.001	.0001	.00001
$f(x) = (1+x)^{1/x}$	2.70481	2.71692	2.71815	2.71827

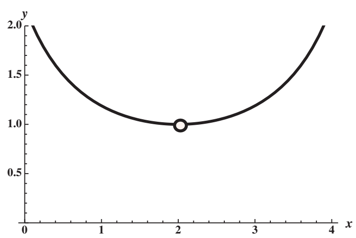
x	-0.01	-0.001	-0.0001	-0.00001
$f(x) = (1+x)^{1/x}$	2.732	2.71964	2.71842	2.71830

b. $\lim_{x\to 0}(1+x)^{1/x} \approx 2.718.$

c. $\lim_{x\to 0}(1+x)^{1/x} = e.$

2.2.15

a.



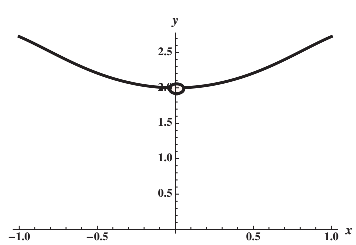
b.

x	1.8	1.9	1.99	2.01	2.1	2.2
$f(x)$	1.0067	1.00167	1.00002	1.00002	1.00167	1.0067

From both the graph and the table, the limit appears to be 1.

2.2.16

a.



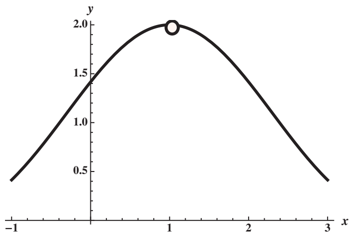
b.

x	-0.2	-0.1	-0.01	0.01	0.1	0.2
$f(x)$	2.03336	2.00834	2.00008	2.00008	2.00834	2.03336

From both the graph and the table, the limit appears to be 2.

2.2.17

a.



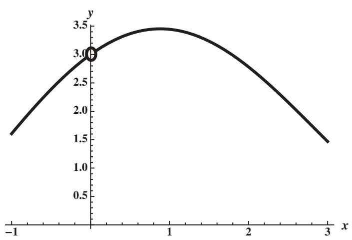
b.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	1.993342	1.999933	1.999999	1.999999	1.999933	1.993342

From both the graph and the table, the limit appears to be 2.

2.2.18

a.



b.

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$	2.8951	2.99	2.999	3.001	3.0099	3.0949

From both the graph and the table, the limit appears to be 3.

2.2.19

x	4.9	4.99	4.999	4.9999	5	5.0001	5.001	5.01	5.1
$f(x) = \frac{x^2-25}{x-5}$	9.9	9.99	9.999	9.9999	undefined	10.0001	10.001	10.01	10.1

$\lim_{x\to 5^+} \dfrac{x^2-25}{x-5} = 10$, $\lim_{x\to 5^-} \dfrac{x^2-25}{x-5} = 10$, and thus $\lim_{x\to 5} \dfrac{x^2-25}{x-5} = 10$.

2.2.20

x	99.9	99.99	99.999	99.9999	100	100.0001	100.001	100.01	100.1
$f(x) = \frac{x-100}{\sqrt{x}-10}$	19.995	19.9995	19.99995	≈ 20	undefined	≈ 20	20.0005	20.00005	20.005

$\lim_{x\to 100^+} \dfrac{x-100}{\sqrt{x}-10} = 20$, $\lim_{x\to 100^-} \dfrac{x-100}{\sqrt{x}-10} = 20$, and thus $\lim_{x\to 100} \dfrac{x-100}{\sqrt{x}-10} = 20$.

2.2.21

a. $f(1) = 0$.

b. $\lim_{x\to 1^-} f(x) = 1$.

c. $\lim_{x\to 1^+} f(x) = 0$.

d. $\lim_{x\to 1} f(x)$ does not exist, since the two one-sided limits aren't equal.

2.2.22

a. $g(2) = 3$.

b. $\lim_{x\to 2^-} g(x) = 2$.

c. $\lim_{x\to 2^+} g(x) = 3$.

d. $\lim_{x\to 2} g(x)$ does not exist.

e. $g(3) = 2$.

f. $\lim_{x\to 3^-} g(x) = 3$.

g. $\lim_{x\to 3^+} g(x) = 2$.

h. $g(4) = 3$.

i. $\lim_{x\to 4} g(x) = 3$.

2.2.23

a. $f(1) = 3$.

b. $\lim_{x\to 1^-} f(x) = 2$.

c. $\lim_{x\to 1^+} f(x) = 2$.

d. $\lim_{x\to 1} f(x) = 2$.

e. $f(3) = 2$.

f. $\lim_{x\to 3^-} f(x) = 4$.

g. $\lim_{x\to 3^+} f(x) = 1$.

h. $\lim_{x\to 3} f(x)$ does not exist.

i. $f(2) = 3$.

j. $\lim_{x\to 2^-} f(x) = 3$.

k. $\lim_{x\to 2^+} f(x) = 3$.

l. $\lim_{x\to 2} f(x) = 3$.

2.2.24

a. $g(-1) = 3$.

b. $\lim_{x\to -1^-} g(x) = 2$.

c. $\lim_{x\to -1^+} g(x) = 2$.

d. $\lim_{x\to -1} g(x) = 2$.

e. $g(1) = 2$.

f. $\lim_{x\to 1} g(x)$ does not exist.

g. $\lim_{x\to 3} g(x) = 4$.

h. $g(5) = 5$.

i. $\lim_{x\to 5^-} g(x) = 5$.

2.2.25

a.

x	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$	$\frac{2}{9\pi}$	$\frac{2}{11\pi}$
$f(x) = \sin(1/x)$	1	-1	1	-1	1	-1

If $x_n = \frac{2}{(2n+1)\pi}$, then $f(x_n) = (-1)^n$ where n is a non-negative integer.

b. As $x \to 0$, $1/x \to \infty$. So the values of $f(x)$ oscillate dramatically between -1 and 1.

c. $\lim_{x\to 0} \sin(1/x)$ does not exist.

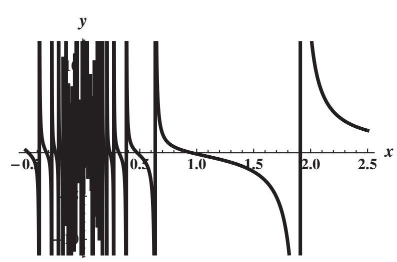
2.2.26

a.

x	$\frac{12}{\pi}$	$\frac{12}{3\pi}$	$\frac{12}{5\pi}$	$\frac{12}{7\pi}$	$\frac{12}{9\pi}$	$\frac{12}{11\pi}$
$f(x) = \tan(3/x)$	1	-1	1	-1	1	-1

We have alternating 1's and -1's.

b. $\tan 3x$ alternates between 1 and -1 infinitely many times on $(0, h)$ for any $h > 0$.



c. $\lim_{x\to 0} \tan(3/x)$ does not exist.

2.2.27

a. False. In fact $\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3}(x+3) = 6$.

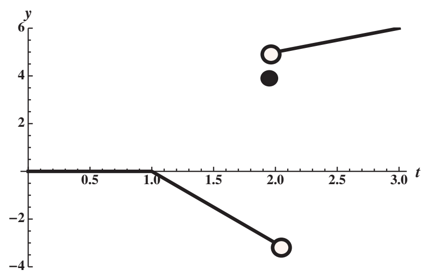
b. False. For example, if $f(x) = \begin{cases} x^2 & \text{if } x \neq 0; \\ 5 & \text{if } x = 0 \end{cases}$ and if $a = 0$ then $f(a) = 5$ but $\lim_{x \to a} f(x) = 0$.

c. False. For example, the limit in part a of this problem exists, even though the corresponding function is undefined at $a = 3$.

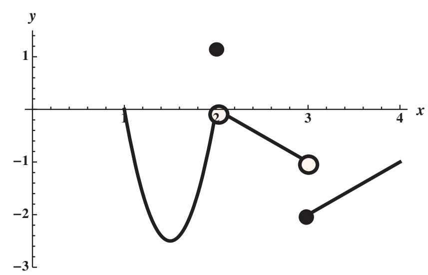
d. False. It is true that the limit of $\sqrt{x}$ as x approaches zero from the right is zero, but because the domain of $\sqrt{x}$ does not include any numbers to the left of zero, the two-sided limit doesn't exist.

e. True. Note that $\lim_{x \to \pi/2} \cos x = 0$ and $\lim_{x \to \pi/2} \sin x = 1$, so $\lim_{x \to \pi/2} \dfrac{\cos x}{\sin x} = \dfrac{0}{1} = 0$.

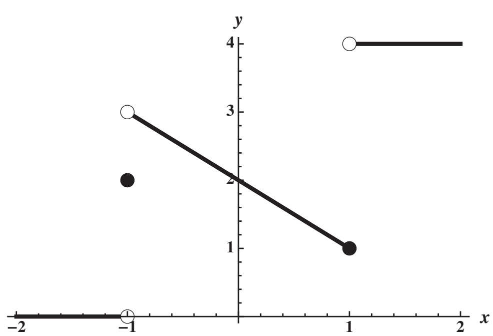
2.2.28



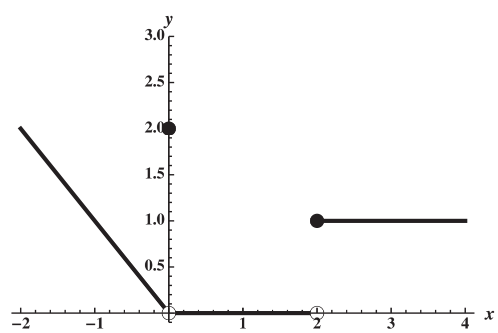
2.2.29



2.2.30



2.2.31



2.2.32

h	0.01	0.001	0.0001	−0.01	−0.001	−0.0001
$\sin(h)/h$	0.99998	0.9999998	0.999999998	0.99998	0.9999998	0.999999998

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1$$

2.2.33

h	0.01	0.001	0.0001	−0.01	−0.001	−0.0001
$\tan(3h)/h$	3.0009	3.000009	3.00000009	3.0009	3.000009	3.00000009

$$\lim_{h \to 0} \frac{\tan(3h)}{h} = 3$$

2.2.34

h	0.01	0.001	0.0001	−0.01	−0.001	−0.0001
$(\sqrt{(h+4)} - 2)/h$	0.24984	0.249984	0.25000	0.250156	0.250016	0.25000

$$\lim_{h\to 0} \frac{\sqrt{h+4}-2}{h} = \frac{1}{4}$$

2.2.35

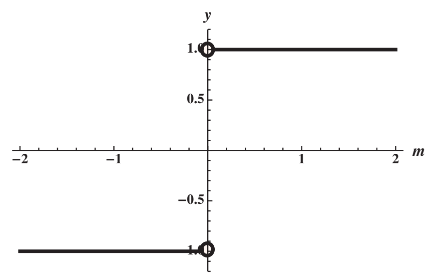
h	0.01	0.001	0.0001	−0.01	−0.001	−0.0001
$(1-\cos(h))/h$	0.0049999	0.0005	0.00005	−0.0049999	−0.0005	−0.00005

$$\lim_{h\to 0} \frac{1-\cos(h)}{h} = 0$$

2.2.36

a. Note that $f(x) = \frac{|x|}{x}$ is undefined at 0, and $\lim_{x\to 0^-} f(x) = -1$ and $\lim_{x\to 0^+} f(x) = 1$.

b. $\lim_{x\to 0} f(x)$ does not exist, since the two one-side limits aren't equal.



2.2.37

a. $\lim_{x\to -1^-} \lfloor x \rfloor = -2$, $\lim_{x\to -1^+} \lfloor x \rfloor = -1$, $\lim_{x\to 2^-} \lfloor x \rfloor = 1$, $\lim_{x\to 2^+} \lfloor x \rfloor = 2$.

b. $\lim_{x\to 2.3^-} \lfloor x \rfloor = 2$, $\lim_{x\to 2.3^+} \lfloor x \rfloor = 2$, $\lim_{x\to 2.3} \lfloor x \rfloor = 2$.

c. In general, for an integer a, $\lim_{x\to a^-} \lfloor x \rfloor = a-1$ and $\lim_{x\to a^+} \lfloor x \rfloor = a$.

d. In general, if a is not an integer, $\lim_{x\to a^-} \lfloor x \rfloor = \lim_{x\to a^+} \lfloor x \rfloor = \lfloor a \rfloor$.

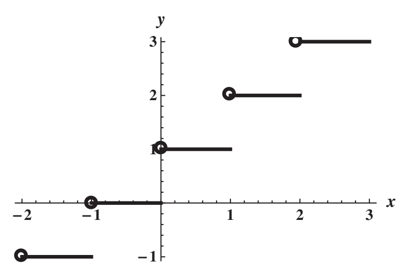
e. $\lim_{x\to a} \lfloor x \rfloor$ exists and is equal to $\lfloor a \rfloor$ for non-integers a.

2.2.38

a. Note that the graph is piecewise constant.

b. $\lim_{x\to 2^-} \lceil x \rceil = 2$, $\lim_{x\to 1^+} \lceil x \rceil = 2$, $\lim_{x\to 1.5} \lceil x \rceil = 2$.

c. $\lim_{x\to a} \lceil x \rceil$ exists and is equal to $\lceil a \rceil$ for non-integers a.



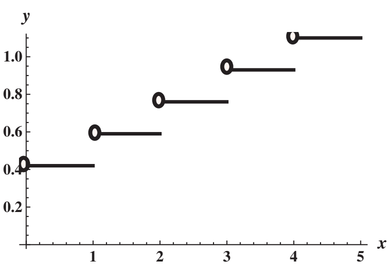
2.2.39 By zooming in closely, you should be able to convince yourself that the answer is 0.

2.2.40 By zooming in closely, you should be able to convince yourself that the answer is 2.

2.2.41 By zooming in closely, you should be able to convince yourself that the answer is 16.

2.2.42 By zooming in closely, you should be able to convince yourself that the answer is 0.

2.2.43



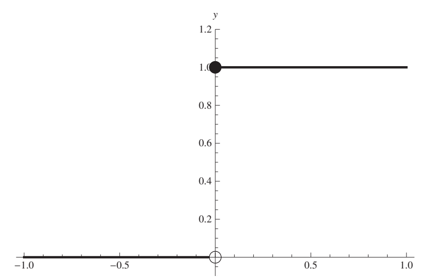
a. Note that the function is piecewise constant.

b. $\lim_{w\to 3.3} f(w) = .95$.

c. $\lim_{w\to 1^+} f(w) = .61$ corresponds to the fact that for any piece of mail that weighs slightly over 1 ounce, the postage will cost 61 cents. $\lim_{w\to 1^-} f(w) = .44$ corresponds to the fact that for any piece of mail that weighs slightly less than 1 ounce, the postage will cost 44 cents.

d. $\lim_{w\to 4} f(w)$ does not exist because the two corresponding one-side limits don't exist. (The limit from the left is .95, while the limit from the right is 1.12.)

2.2.44



a. Note that H is piecewise constant.

b. $\lim_{x\to 0^-} H(x) = 0$, $\lim_{x\to 0^+} H(x) = 1$, and so $\lim_{x\to 0} H(x)$ does not exist.

2.2.45

a. Because of the symmetry about the y axis, we must have $\lim_{x\to -2^+} f(x) = 8$.

b. Because of the symmetry about the y axis, we must have $\lim_{x\to -2^-} f(x) = 5$.

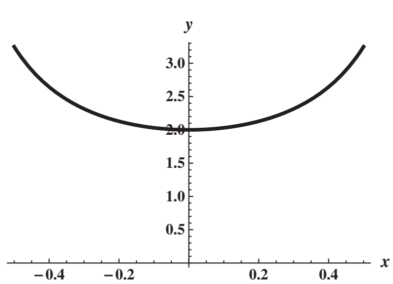
2.2.46

a. Because of the symmetry about the origin, we must have $\lim_{x\to -2^+} g(x) = -8$.

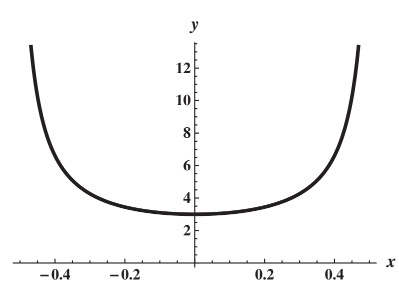
b. Because of the symmetry about the origin, we must have $\lim_{x\to -2^-} g(x) = -5$.

2.2.47

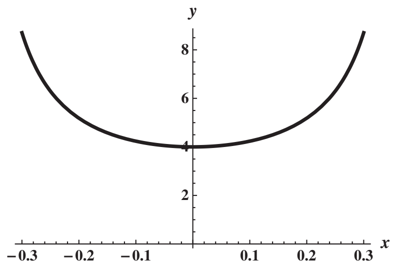
a.



$\lim_{x\to 0} \frac{\tan 2x}{\sin x} = 2$.



$\lim_{x\to 0} \frac{\tan 3x}{\sin x} = 3$.

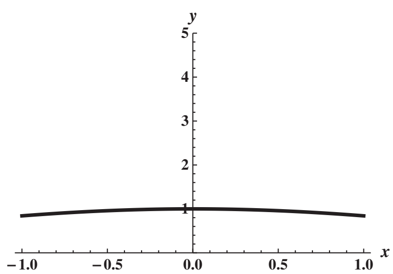


$\lim_{x\to 0} \frac{\tan 4x}{\sin x} = 4.$

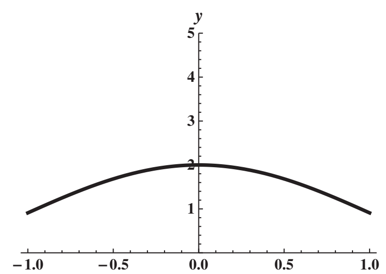
b. It appears that $\displaystyle\lim_{x\to 0} \frac{\tan(px)}{\sin x} = p.$

2.2.48

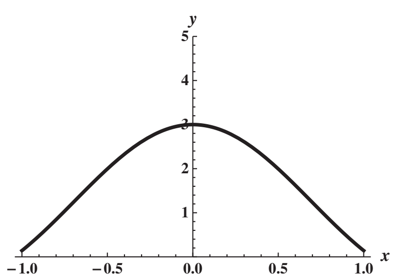
a.



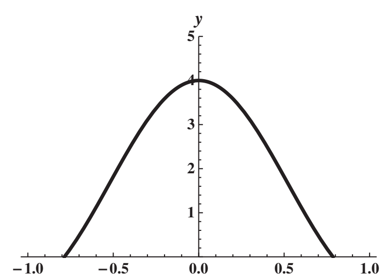
$\lim_{x\to 0} \frac{\sin x}{x} = 1.$



$\lim_{x\to 0} \frac{\sin 2x}{x} = 2.$



$\lim_{x\to 0} \frac{\sin 3x}{x} = 3.$

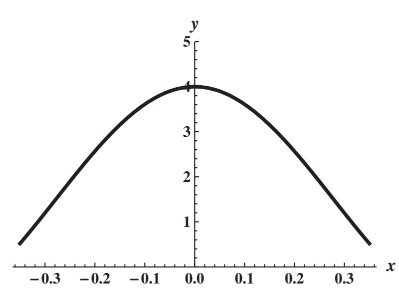


$\lim_{x\to 0} \frac{\sin 4x}{x} = 4.$

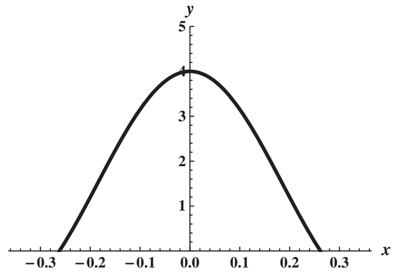
b. It appears that $\displaystyle\lim_{x\to 0} \frac{\sin(px)}{x} = p.$

2.2.49

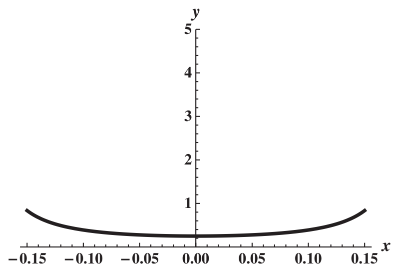
For $p = 8$ and $q = 2$, it appears that the limit is 4.



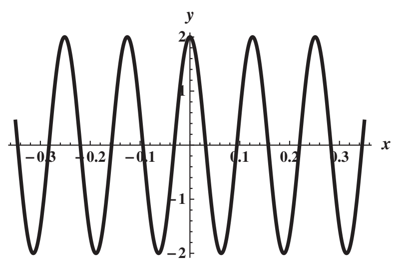
For $p = 12$ and $q = 3$, it appears that the limit is 4.



For $p = 4$ and $q = 16$, it appears that the limit is $1/4$.



For $p = 100$ and $q = 50$, it appears that the limit is 2.



Conjecture: $\lim_{x \to 0} \dfrac{\sin px}{\sin qx} = \dfrac{p}{q}$.

2.3 Techniques for Computing Limits

2.3.1 If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then $\lim_{x\to a} f(x) = \lim_{x\to a}(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)$
$= a_n(\lim_{x\to a} x)^n + a_{n-1}(\lim_{x\to a} x)^{n-1} + \cdots + a_1 \lim_{x\to a} x + \lim_{x\to a} a_0$
$= a_n a^n + a_{n-1}a^{n-1} + \cdots + a_1 a + a_0 = f(a)$.

2.3.2 If $f(x)$ is a polynomial, then $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$.

2.3.3 For a rational function $r(x)$, we have $\lim_{x\to a} r(x) = r(a)$ exactly for those numbers a which are in the domain of r.

2.3.4 If $f(x) = g(x)$ for $x \neq 3$, and $\lim_{x\to 3} g(x) = 4$, then $\lim_{x\to 3} f(x) = 4$ as well.

2.3.5 Because $\frac{x^2-7x+12}{x-3} = \frac{(x-3)(x-4)}{x-3} = x - 4$ (for $x \neq 3$), we can see that the graphs of these two functions are the same except that one is undefined at $x = 3$ and the other is a straight line that is defined everywhere. Thus the function $\frac{x^2-7x+12}{x-3}$ is a straight line except that it has a "hole" at $(3, -1)$. The two functions have the same limit as $x \to 3$, namely $\lim_{x\to 3} \dfrac{x^2 - 7x + 12}{x - 3} = \lim_{x\to 3}(x - 4) = -1$.

2.3.6 $\lim\limits_{x\to 2} f(x)^{2/3} = \left(\lim\limits_{x\to 2} f(x)\right)^{2/3} = (-8)^{2/3} = (-2)^2 = 4.$

2.3.7 If p and q are polynomials then $\lim\limits_{x\to 0} \dfrac{p(x)}{q(x)} = \dfrac{\lim\limits_{x\to 0} p(x)}{\lim\limits_{x\to 0} q(x)} = \dfrac{p(0)}{q(0)}$. Because this quantity is given to be equal to 10, we have $\frac{p(0)}{2} = 10$, so $p(0) = 20$.

2.3.8 By a direct application of the squeeze theorem, $\lim\limits_{x\to 2} g(x) = 5$.

2.3.9 $\lim\limits_{x\to 5} \sqrt{x^2-9} = \sqrt{\lim\limits_{x\to 5}(x^2-9)} = \sqrt{16} = 4.$

2.3.10 $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} 4 = 4$, and $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+}(x+2) = 5$.

2.3.11 $\lim\limits_{x\to 4}(3x-7) = 3\lim\limits_{x\to 4} x - 7 = 3\cdot 4 - 7 = 5.$

2.3.12 $\lim\limits_{x\to 1}(-2x+5) = -2\lim\limits_{x\to 1} x + 5 = -2\cdot 1 + 5 = 3.$

2.3.13 $\lim\limits_{x\to -9}(5x) = 5\lim\limits_{x\to -9} x = 5\cdot -9 = -45.$

2.3.14 $\lim\limits_{x\to 2}(-3x) = -3\lim\limits_{x\to 2} x = -3\cdot 2 = -6.$

2.3.15 $\lim\limits_{x\to 6} 4 = 4.$

2.3.16 $\lim\limits_{x\to -5} \pi = \pi.$

2.3.17 $\lim\limits_{x\to 1} 4f(x) = 4\lim\limits_{x\to 1} f(x) = 4\cdot 8 = 32$. This follows from the Constant Multiple Law.

2.3.18 $\lim\limits_{x\to 1} \dfrac{f(x)}{h(x)} = \dfrac{\lim\limits_{x\to 1} f(x)}{\lim\limits_{x\to 1} h(x)} = \dfrac{8}{2} = 4$. This follows from the Quotient Law.

2.3.19 $\lim\limits_{x\to 1}(f(x) - g(x)) = \lim\limits_{x\to 1} f(x) - \lim\limits_{x\to 1} g(x) = 8 - 3 = 5$. This follows from the Difference Law.

2.3.20 $\lim\limits_{x\to 1} f(x)h(x) = \lim\limits_{x\to 1} f(x)\cdot \lim\limits_{x\to 1} h(x) = 8\cdot 2 = 16$. This follows from the Product Law.

2.3.21 $\lim\limits_{x\to 1} \dfrac{f(x)g(x)}{h(x)} = \dfrac{\lim\limits_{x\to 1}(f(x)g(x))}{\lim\limits_{x\to 1} h(x)} = \dfrac{\lim\limits_{x\to 1} f(x)\cdot \lim\limits_{x\to 1} g(x)}{\lim\limits_{x\to 1} h(x)} = \dfrac{8\cdot 3}{2} = 12$. This follows from the Quotient and Product Laws.

2.3.22 $\lim\limits_{x\to 1} \dfrac{f(x)}{g(x)-h(x)} = \dfrac{\lim\limits_{x\to 1} f(x)}{\lim\limits_{x\to 1}[g(x)-h(x)]} = \dfrac{\lim\limits_{x\to 1} f(x)}{\lim\limits_{x\to 1} g(x) - \lim\limits_{x\to 1} h(x)} = \dfrac{8}{3-2} = 8$. This follows from the Quotient and Difference Laws.

2.3.23 $\lim\limits_{x\to 1}(h(x))^5 = \left(\lim\limits_{x\to 1} h(x)\right)^5 = (2)^5 = 32$. This follows from the Power Law.

2.3.24 $\lim\limits_{x\to 1} \sqrt[3]{f(x)g(x)+3} = \sqrt[3]{\lim\limits_{x\to 1}(f(x)g(x)+3)} = \sqrt[3]{\lim\limits_{x\to 1} f(x)\cdot \lim\limits_{x\to 1} g(x) + \lim\limits_{x\to 1} 3} = \sqrt[3]{8\cdot 3+3} = \sqrt[3]{27} = 3$. This follows form the Root, Product, Sum and Constant Laws.

2.3.25 $\lim\limits_{x\to 1}(2x^3-3x^2+4x+5) = \lim\limits_{x\to 1} 2x^3 - \lim\limits_{x\to 1} 3x^2 + \lim\limits_{x\to 1} 4x + \lim\limits_{x\to 1} 5 = 2(\lim\limits_{x\to 1} x)^3 - 3(\lim\limits_{x\to 1} x)^2 + 4(\lim\limits_{x\to 1} x) + 5 = 2(1)^3 - 3(1)^2 + 4\cdot 1 + 5 = 8.$

2.3.26 $\lim_{t\to-2}(t^2+5t+7) = \lim_{t\to-2} t^2 + \lim_{t\to-2} 5t + \lim_{t\to-2} 7 = \left(\lim_{t\to-2} t\right)^2 + 5\lim_{t\to-2} t + 7 = (-2)^2 + 5\cdot(-2) + 7 = 1.$

2.3.27 $\lim_{x\to1} \frac{5x^2+6x+1}{8x-4} = \frac{\lim_{x\to1}(5x^2+6x+1)}{\lim_{x\to1}(8x-4)} = \frac{5(\lim_{x\to1} x)^2 + 6\lim_{x\to1} x + \lim_{x\to1} 1}{8\lim_{x\to1} x - \lim_{x\to1} 4} = \frac{5(1)^2+6\cdot1+1}{8\cdot1-4} = 3.$

2.3.28 $\lim_{t\to3} \sqrt[3]{t^2-10} = \sqrt[3]{\lim_{t\to3}(t^2-10)} = \sqrt[3]{\lim_{t\to3} t^2 - \lim_{t\to3} 10} = \sqrt[3]{\left(\lim_{t\to3} t\right)^2 - 10} = \sqrt[3]{(3)^2-10} = -1.$

2.3.29 $\lim_{b\to2} \frac{3b}{\sqrt{4b+1}-1} = \frac{\lim_{b\to2} 3b}{\lim_{b\to2}(\sqrt{4b+1}-1)} = \frac{3\lim_{b\to2} b}{\lim_{b\to2}\sqrt{4b+1} - \lim_{b\to2} 1} = \frac{3\cdot2}{\sqrt{\lim_{b\to2}(4b+1)}-1} = \frac{6}{3-1} = 3.$

2.3.30 $\lim_{x\to2}(x^2-x)^5 = \left(\lim_{x\to2}(x^2-x)\right)^5 = \left(\lim_{x\to2} x^2 - \lim_{x\to2} x\right)^5 = (4-2)^5 = 32.$

2.3.31 $\lim_{x\to3} \frac{-5x}{\sqrt{4x-3}} = \frac{\lim_{x\to3} -5x}{\lim_{x\to3}\sqrt{4x-3}} = \frac{-5\lim_{x\to3} x}{\sqrt{\lim_{x\to3}(4x-3)}} = \frac{-5\cdot3}{\sqrt{4\lim_{x\to3} x - \lim_{x\to3} 3}} = \frac{-15}{\sqrt{4\cdot3-3}} = -5.$

2.3.32 $\lim_{h\to0} \frac{3}{\sqrt{16+3h}+4} = \frac{\lim_{h\to0} 3}{\lim_{h\to0}(\sqrt{16+3h}+4)} = \frac{3}{\sqrt{\lim_{h\to0}(16+3h)} + \lim_{h\to0} 4} = \frac{3}{\sqrt{\lim_{h\to0} 16 + \lim_{h\to0} 3h}+4} = \frac{3}{\sqrt{16+3\cdot0}+4} = \frac{3}{4+4} = \frac{3}{8}.$

2.3.33

a. $\lim_{x\to-1^-} f(x) = \lim_{x\to-1^-}(x^2+1) = (-1)^2+1 = 2.$

b. $\lim_{x\to-1^+} f(x) = \lim_{x\to-1^+}\sqrt{x+1} = \sqrt{-1+1} = 0.$

c. $\lim_{x\to-1} f(x)$ does not exist.

2.3.34

a. $\lim_{x\to-5^-} f(x) = \lim_{x\to-5^-} 0 = 0.$

b. $\lim_{x\to-5^+} f(x) = \lim_{x\to-5^+}\sqrt{25-x^2} = \sqrt{25-25} = 0.$

c. $\lim_{x\to-5} f(x) = 0.$

d. $\lim_{x\to5^-} f(x) = \lim_{x\to5^-}\sqrt{25-x^2} = \sqrt{25-25} = 0.$

e. $\lim_{x\to5^+} f(x) = \lim_{x\to5^+} 3x = 15.$

f. $\lim_{x\to5} f(x)$ does not exist.

2.3.35

a. $\lim_{x\to2^+}\sqrt{x-2} = \sqrt{2-2} = 0.$

b. The domain of $f(x) = \sqrt{x-2}$ is $[2,\infty)$. Thus, any question about this function that involves numbers less than 2 doesn't make any sense, because those numbers aren't in the domain of f.

2.3.36

a. Note that the domain of $f(x) = \sqrt{\frac{x-3}{2-x}}$ is $(2,3]$. $\lim_{x\to3^-}\sqrt{\frac{x-3}{2-x}} = 0.$

b. Because the numbers to the right of 3 aren't in the domain of this function, the limit as $x\to3^+$ of this function doesn't make any sense.

2.3.37 Using the definition of $|x|$ given, we have $\lim\limits_{x\to 0^-}|x| = \lim\limits_{x\to 0^-}(-x) = -0 = 0$. Also, $\lim\limits_{x\to 0^+}|x| = \lim\limits_{x\to 0^+} x = 0$. Because the two one-sided limits are both 0, we also have $\lim\limits_{x\to 0}|x| = 0$.

2.3.38

If $a > 0$, then for x near a, $|x| = x$. So in this case, $\lim\limits_{x\to a}|x| = \lim\limits_{x\to a} x = a = |a|$.

If $a < 0$, then for x near a, $|x| = -x$. So in this case, $\lim\limits_{x\to a}|x| = \lim\limits_{x\to a}(-x) = -a = |a|$, (because $a < 0$).

If $a = 0$, we have already seen in a previous problem that $\lim\limits_{x\to 0}|x| = 0 = |0|$.

Thus in all cases, $\lim\limits_{x\to a}|x| = |a|$.

2.3.39 $\lim\limits_{x\to 1}\dfrac{x^2-1}{x-1} = \lim\limits_{x\to 1}\dfrac{(x+1)(x-1)}{x-1} = \lim\limits_{x\to 1}(x+1) = 2.$

2.3.40 $\lim\limits_{x\to 3}\dfrac{x^2-2x-3}{x-3} = \lim\limits_{x\to 3}\dfrac{(x-3)(x+1)}{x-3} = \lim\limits_{x\to 3}(x+1) = 4.$

2.3.41 $\lim\limits_{x\to 4}\dfrac{x^2-16}{4-x} = \lim\limits_{x\to 4}\dfrac{(x+4)(x-4)}{-(x-4)} = \lim\limits_{x\to 4}[-(x+4)] = -8.$

2.3.42 $\lim\limits_{t\to 2}\dfrac{3t^2-7t+2}{2-t} = \lim\limits_{t\to 2}\dfrac{(t-2)(3t-1)}{-(t-2)} = \lim\limits_{t\to 2}[-(3t-1)] = -5.$

2.3.43 $\lim\limits_{x\to b}\dfrac{(x-b)^{50}-x+b}{x-b} = \lim\limits_{x\to b}\dfrac{(x-b)^{50}-(x-b)}{x-b} = \lim\limits_{x\to b}\dfrac{(x-b)((x-b)^{49}-1)}{x-b} = \lim\limits_{x\to b}[(x-b)^{49}-1] = -1.$

2.3.44 $\lim\limits_{x\to -b}\dfrac{(x+b)^7+(x+b)^{10}}{4(x+b)} = \lim\limits_{x\to -b}\dfrac{(x+b)((x+b)^6+(x+b)^9)}{4(x+b)} = \lim\limits_{x\to -b}\dfrac{(x+b)^6+(x+b)^9}{4} = \dfrac{0}{4} = 0.$

2.3.45 $\lim\limits_{x\to -1}\dfrac{(2x-1)^2-9}{x+1} = \lim\limits_{x\to -1}\dfrac{(2x-1-3)(2x-1+3)}{x+1} = \lim\limits_{x\to -1}\dfrac{2(x-2)2(x+1)}{x+1} = \lim\limits_{x\to -1}4(x-2) = 4\cdot(-3) = -12.$

2.3.46 $\lim\limits_{h\to 0}\dfrac{\frac{1}{5+h}-\frac{1}{5}}{h} = \lim\limits_{h\to 0}\dfrac{\left(\frac{1}{5+h}-\frac{1}{5}\right)\cdot 5\cdot(5+h)}{h\cdot 5\cdot(5+h)} = \lim\limits_{h\to 0}\dfrac{5-(5+h)}{5h(5+h)} = \lim\limits_{h\to 0}\dfrac{-h}{5h(5+h)} = \lim\limits_{h\to 0}\dfrac{-1}{5(5+h)} = \dfrac{-1}{25}.$

2.3.47 $\lim\limits_{x\to 9}\dfrac{\sqrt{x}-3}{x-9} = \lim\limits_{x\to 9}\dfrac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)} = \lim\limits_{x\to 9}\dfrac{x-9}{(x-9)(\sqrt{x}+3)} = \lim\limits_{x\to 9}\dfrac{1}{\sqrt{x}+3} = \dfrac{1}{6}.$

2.3.48 Expanding gives

$$\lim_{t\to 3}\left(4t-\frac{2}{t-3}\right)(6+t-t^2) = \lim_{t\to 3}\left(4t(6+t-t^2)-\frac{2(6+t-t^2)}{t-3}\right) = \lim_{t\to 3}\left(4t(6+t-t^2)-\frac{2(3-t)(2+t)}{t-3}\right).$$

Now because $t-3 = -(3-t)$, we have

$$\lim_{t\to 3}\left(4t(6+t-t^2)+2(2+t)\right) = 12(6+3-9)+2(2+3) = 10.$$

2.3.49 $\lim\limits_{x\to a}\dfrac{x-a}{\sqrt{x}-\sqrt{a}} = \lim\limits_{x\to a}\dfrac{x-a}{\sqrt{x}-\sqrt{a}}\cdot\dfrac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} = \lim\limits_{x\to a}\dfrac{(x-a)(\sqrt{x}+\sqrt{a})}{x-a} = \lim\limits_{x\to a}(\sqrt{x}+\sqrt{a}) = 2\sqrt{a}.$

2.3.50 $\lim\limits_{x\to a}\dfrac{x^2-a^2}{\sqrt{x}-\sqrt{a}} = \lim\limits_{x\to a}\dfrac{x^2-a^2}{\sqrt{x}-\sqrt{a}}\cdot\dfrac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} = \lim\limits_{x\to a}\dfrac{(x-a)(x+a)(\sqrt{x}+\sqrt{a})}{x-a} = (a+a)(\sqrt{a}+\sqrt{a}) = 4a^{3/2}.$

2.3.51 $\lim_{h\to 0}\frac{\sqrt{16+h}-4}{h} = \lim_{h\to 0}\frac{(\sqrt{16+h}-4)(\sqrt{16+h}+4)}{h(\sqrt{16+h}+4)} = \lim_{h\to 0}\frac{(16+h)-16}{h(\sqrt{16+h}+4)} = \lim_{h\to 0}\frac{h}{h(\sqrt{16+h}+4)}$ $= \lim_{h\to 0}\frac{1}{(\sqrt{16+h}+4)} = \frac{1}{8}$.

2.3.52 Note that $x^3 - a^3 = (x-a)(x^2+ax+a^2)$, and thus as long as $x \neq a$, we have

$$\frac{x^3-a^3}{x-a} = x^2+ax+a^2.$$

Thus,

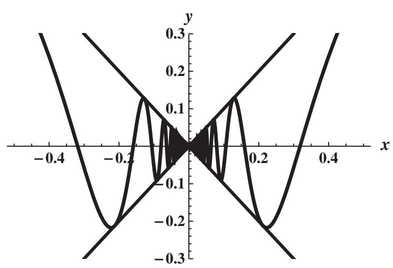
$$\lim_{x\to a}\frac{x^3-a^3}{x-a} = \lim_{x\to a}(x^2+ax+a^2) = a^2+a^2+a^2 = 3a^2.$$

2.3.53

a. The statement we are trying to prove can be stated in cases as follows: For $x > 0$, $-x \le x\sin(1/x) \le x$, and for $x < 0$, $x \le x\sin(1/x) \le -x$.

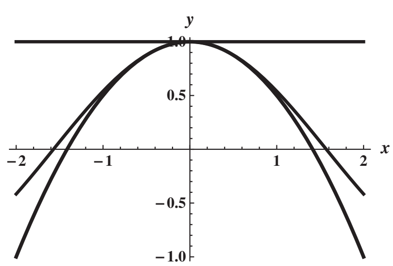
Now for all $x \neq 0$, note that $-1 \le \sin(1/x) \le 1$ (because the range of the sine function is $[-1,1]$). We will consider the two cases $x > 0$ and $x < 0$ separately, but in each case, we will multiply this inequality through by x, switching the inequalities for the $x < 0$ case.

For $x > 0$ we have $-x \le x\sin(1/x) \le x$, and for $x < 0$ we have $-x \ge x\sin(1/x) \ge x$, which are exactly the statements we are trying to prove.

b. 

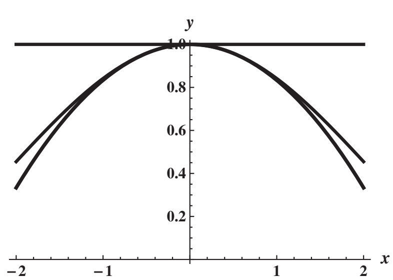
c. Because $\lim_{x\to 0} -|x| = \lim_{x\to 0}|x| = 0$, and because $-|x| \le x\sin(1/x) \le |x|$, the Squeeze Theorem assures us that $\lim_{x\to 0}[x\sin(1/x)] = 0$ as well.

2.3.54

a. 

b. Note that $\lim_{x\to 0}\left[1-\frac{x^2}{2}\right] = 1 = \lim_{x\to 0} 1$. So because $1-\frac{x^2}{2} \le \cos x \le 1$, the squeeze theorem assures us that $\lim_{x\to 0}\cos x = 1$ as well.

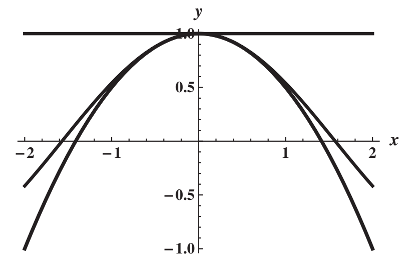
2.3.55

a. 

b. Note that $\lim_{x\to 0}\left[1-\frac{x^2}{6}\right] = 1 = \lim_{x\to 0} 1$. So because $1-\frac{x^2}{6} \le \frac{\sin x}{x} \le 1$, the squeeze theorem assures us that $\lim_{x\to 0}\frac{\sin x}{x} = 1$ as well.

2.3.56

a. The graphs of $x^2 \sec x^2$ and x^2+x^4 are indistinguishable if both are graphed near zero, so it is hard to draw a graph that shows the relationship between the two. However, the inequality in the problem statement holds if and only if (after dividing the inequality by x^2), $0 \le \sec x^2 \le x^2+1$. The graphs of these functions near zero look like this (note that the scale on the x-axis has been greatly expanded to show more clearly the relationship between the graphs)



b. $\lim_{x\to 0} x^4 + x^2 = 0 = \lim_{x\to 0} 0$. Because $x^2\sec(x^2)$ lies between these two near zero, we also have $\lim_{x\to 0} x^2 \sec(x^2) = 0$.

2.3.57

a. False. For example, if $f(x) = \begin{cases} x & \text{if } x \ne 1; \\ 4 & \text{if } x = 1, \end{cases}$ then $\lim_{x\to 1} f(x) = 1$ but $f(1) = 4$.

b. False. For example, if $f(x) = \begin{cases} x+1 & \text{if } x \le 1; \\ x-6 & \text{if } x > 1, \end{cases}$ then $\lim_{x\to 1^-} f(x) = 2$ but $\lim_{x\to 1^+} f(x) = -5$.

c. False. For example, if $f(x) = \begin{cases} x & \text{if } x \ne 1; \\ 4 & \text{if } x = 1, \end{cases}$ and $g(x) = 1$, then f and g both have limit 1 as $x \to 1$, but $f(1) = 4 \ne g(1)$.

d. False. For example $\lim_{x\to 2} \frac{x^2-4}{x-2}$ exists and is equal to 4.

e. False. For example, it would be possible for the domain of f to be $[1,\infty)$, so that the one-sided limit exists but the two-sided limit doesn't even make sense. This would be true, for example, if $f(x) = x-1$.

2.3.58 $\lim_{h\to 0} \frac{100}{(10h-1)^{11}+2} = \frac{100}{(-1)^{11}+2} = \frac{100}{1} = 100.$

2.3.59 $\lim_{x\to 2} (5x-6)^{3/2} = (5\cdot 2-6)^{3/2} = 4^{3/2} = 2^3 = 8.$

2.3.60 $\lim_{x\to 3} \frac{\frac{1}{x^2+2x} - \frac{1}{15}}{x-3} = \lim_{x\to 3} \frac{\frac{15-(x^2+2x)}{15(x^2+2x)}}{x-3} = \lim_{x\to 3} \frac{15-(x^2+2x)}{15(x^2+2x)(x-3)} = \lim_{x\to 3} \frac{15-2x-x^2}{15(x^2+2x)(x-3)} = \lim_{x\to 3} \frac{(3-x)(5+x)}{15(x^2+2x)(x-3)} = \lim_{x\to 3} \frac{-(5+x)}{15(x^2+2x)} = \frac{-8}{225}.$

2.3.61 $\lim_{x\to 1} \frac{\sqrt{10x-9}-1}{x-1} = \lim_{x\to 1} \frac{(\sqrt{10x-9}-1)(\sqrt{10x-9}+1)}{(x-1)(\sqrt{10x-9}+1)} = \lim_{x\to 1} \frac{(10x-9)-1}{(x-1)(\sqrt{10x-9}+1)} = \lim_{x\to 1} \frac{10(x-1)}{(x-1)(\sqrt{10x-9}+1)} = \lim_{x\to 1} \frac{10}{(\sqrt{10x-9}+1)} = \frac{10}{2} = 5.$

2.3.62 $\lim_{x\to 2} \left(\frac{1}{x-2} - \frac{2}{x^2-2x}\right) = \lim_{x\to 2} \left(\frac{x}{x(x-2)} - \frac{2}{x(x-2)}\right) = \lim_{x\to 2} \left(\frac{x-2}{x(x-2)}\right) = \lim_{x\to 2} \frac{1}{x} = \frac{1}{2}.$

2.3.63 $\lim_{h\to 0} \frac{(5+h)^2-25}{h} = \lim_{h\to 0} \frac{25+10h+h^2-25}{h} = \lim_{h\to 0} \frac{h(10+h)}{h} = \lim_{h\to 0} (10+h) = 10.$

2.3.64 $\lim_{x\to c}\frac{x^2-2cx+c^2}{x-c}=\lim_{x\to c}\frac{(x-c)^2}{x-c}=\lim_{x\to c}x-c=c-c=0.$

2.3.65 We have

$$\lim_{w\to -k}\frac{w^2+5kw+4k^2}{w^2+kw}=\lim_{w\to -k}\frac{(w+4k)(w+k)}{(w)(w+k)}=\lim_{w\to -k}\frac{w+4k}{w}=\frac{-k+4k}{-k}=-3.$$

If $k=0$, we have $\lim_{w\to -k}\frac{w^2+5kw+4k^2}{w^2+kw}=\lim_{w\to 0}\frac{w^2}{w^2}=1.$

2.3.66 In order for $\lim_{x\to 2}f(x)$ to exist, we need the two one-sided limits to exist and be equal. We have $\lim_{x\to 2^-}f(x)=\lim_{x\to 2^-}(3x+b)=6+b$, and $\lim_{x\to 2^+}f(x)=\lim_{x\to 2^+}(x-2)=0$. So we need $6+b=0$, so we require that $b=-6$. Then $\lim_{x\to 2}f(x)=0$.

2.3.67 In order for $\lim_{x\to -1}g(x)$ to exist, we need the two one-sided limits to exist and be equal. We have $\lim_{x\to -1^-}g(x)=\lim_{x\to -1^-}(x^2-5x)=6$, and $\lim_{x\to -1^+}g(x)=\lim_{x\to -1^+}(ax^3-7)=-a-7$. So we need $-a-7=6$, so we require that $a=-13$. Then $\lim_{x\to -1}f(x)=6$.

2.3.68 $\lim_{x\to 2}\frac{x^5-32}{x-2}=\lim_{x\to 2}\frac{(x-2)(x^4+2x^3+4x^2+8x+16)}{x-2}=\lim_{x\to 2}(x^4+2x^3+4x^2+8x+16)=16+16+16+16+16=80.$

2.3.69 $\lim_{x\to 1}\frac{x^6-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x^5+x^4+x^3+x^2+x+1)}{x-1}=\lim_{x\to 1}(x^5+x^4+x^3+x^2+x+1)=6.$

2.3.70 $\lim_{x\to -1}\frac{x^7+1}{x+1}=\lim_{x\to -1}\frac{(x+1)(x^6-x^5+x^4-x^3+x^2-x+1)}{x+1}=\lim_{x\to -1}(x^6-x^5+x^4-x^3+x^2-x+1)=7.$

2.3.71 $\lim_{x\to a}\frac{x^5-a^5}{x-a}=\lim_{x\to a}\frac{(x-a)(x^4+ax^3+a^2x^2+a^3x+a^4)}{x-a}=\lim_{x\to a}(x^4+ax^3+a^2x^2+a^3x+a^4)=5a^4.$

2.3.72 $\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{x\to a}\frac{(x-a)(x^{n-1}+ax^{n-2}+\cdots+a^{n-2}x+a^{n-1})}{x-a}=\lim_{x\to a}(x^{n-1}+ax^{n-2}+\cdots+a^{n-2}x+a^{n-1})=na^{n-1}.$

2.3.73 $\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}=\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{(\sqrt[3]{x}-1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}=\lim_{x\to 1}\frac{1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}=\frac{1}{3}.$

2.3.74 $\lim_{x\to 16}\frac{\sqrt[4]{x}-2}{x-16}=\lim_{x\to 16}\frac{\sqrt[4]{x}-2}{(\sqrt[4]{x}-2)(\sqrt[4]{x^3}+2\sqrt[4]{x^2}+4\sqrt[4]{x}+8)}=\lim_{x\to 16}\frac{1}{\sqrt[4]{x^3}+2\sqrt[4]{x^2}+4\sqrt[4]{x}+8}=\frac{1}{32}.$

2.3.75 $\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}=\lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)}=\lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{x-1}=\lim_{x\to 1}(\sqrt{x}+1)=2.$

2.3.76 $\lim_{x\to 1}\frac{x-1}{\sqrt{4x+5}-3}=\lim_{x\to 1}\frac{(x-1)(\sqrt{4x+5}+3)}{(\sqrt{4x+5}-3)(\sqrt{4x+5}+3)}=\lim_{x\to 1}\frac{(x-1)(\sqrt{4x+5}+3)}{4x+5-9}=\lim_{x\to 1}\frac{(x-1)(\sqrt{4x+5}+3)}{4(x-1)}=\lim_{x\to 1}\frac{(\sqrt{4x+5}+3)}{4}=\frac{6}{4}=\frac{3}{2}.$

2.3.77 $\lim_{x\to 4}\frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}}=\lim_{x\to 4}\frac{3(x-4)(\sqrt{x+5})(3+\sqrt{x+5})}{(3-\sqrt{x+5})(3+\sqrt{x+5})}=\lim_{x\to 4}\frac{3(x-4)(\sqrt{x+5})(3+\sqrt{x+5})}{9-(x+5)}=\lim_{x\to 4}\frac{3(x-4)(\sqrt{x+5})(3+\sqrt{x+5})}{-(x-4)}=\lim_{x\to 4}[-3(\sqrt{x+5})(3+\sqrt{x+5})]=(-3)(3)(3+3)=-54.$

2.3.78 Assume $c \neq 0$. $\lim_{x\to 0} \dfrac{x}{\sqrt{cx+1}-1} = \lim_{x\to 0} \dfrac{x(\sqrt{cx+1}+1)}{(\sqrt{cx+1}-1)(\sqrt{cx+1}+1)} = \lim_{x\to 0} \dfrac{x(\sqrt{cx+1}+1)}{(cx+1)-1} = \lim_{x\to 0} \dfrac{x(\sqrt{cx+1}+1)}{cx} = \lim_{x\to 0} \dfrac{(\sqrt{cx+1}+1)}{c} = \dfrac{2}{c}$.

2.3.79 Let $f(x) = x-1$ and $g(x) = \frac{5}{x-1}$. Then $\lim_{x\to 1} f(x) = 0$, $\lim_{x\to 1} f(x)g(x) = \lim_{x\to 1} \dfrac{5(x-1)}{x-1} = \lim_{x\to 1} 5 = 5$.

2.3.80 Let $f(x) = x^2-1$. Then $\lim_{x\to 1} \dfrac{f(x)}{x-1} = \lim_{x\to 1} \dfrac{x^2-1}{x-1} = \lim_{x\to 1}(x+1) = 2$.

2.3.81 Let $p(x) = x^2+2x-8$. Then $\lim_{x\to 2} \dfrac{p(x)}{x-2} = \lim_{x\to 2} \dfrac{(x-2)(x+4)}{x-2} = \lim_{x\to 2}(x+4) = 6$.

The constants are unique. We know that 2 must be a root of p (otherwise the given limit couldn't exist), so it must have the form $p(x) = (x-2)q(x)$, and q must be a degree 1 polynomial with leading coefficient 1 (otherwise p wouldn't have leading coefficient 1.) So we have $p(x) = (x-2)(x+d)$, but because $\lim_{x\to 2} \dfrac{p(x)}{x-2} = \lim_{x\to 2}(x+d) = 2+d = 6$, we are forced to realize that $d = 4$. Therefore, we have deduced that the only possibility for p is $p(x) = (x-2)(x+4) = x^2+2x-8$.

2.3.82

a. $L(c/2) = L_0\sqrt{1-\frac{(c/2)^2}{c^2}} = L_0\sqrt{1-(1/4)} = \sqrt{3}L_0/2$.

b. $L(3c/4) = L_0\sqrt{1-(1/c^2)(3c/4)^2} = L_0\sqrt{1-(9/16)} = \sqrt{7}L_0/4$.

c. It appears that that the observed length L of the ship decreases as the ship speed increases.

d. $\lim_{x\to c^-} L_0\sqrt{1-(\nu^2/c^2)} = L_0 \cdot 0 = 0$. As the speed of the ship approaches the speed of light, the observed length of the ship shrinks to 0.

2.3.83 $\lim_{S\to 0^+} r(S) = \lim_{S\to 0^+} (1/2)\left(\sqrt{100+\dfrac{2S}{\pi}}-10\right) = 0$.

The radius of the circular cylinder approaches zero as the surface area approaches zero.

2.3.84 $\lim_{t\to 200^-} d(t) = \lim_{t\to 200^-} (3-0.015t)^2 = (3-(0.015)(200))^2 = (3-3)^2 = 0$. As time approaches 200 seconds, the depth of the water in the tank is approaching 0.

2.3.85 $\lim_{x\to 10} E(x) = \lim_{x\to 10} \dfrac{4.35}{x\sqrt{x^2+0.01}} = \dfrac{4.35}{10\sqrt{100.01}} \approx .0435$ N/C.

2.3.86 Because $\lim_{x\to 1} f(x) = 4$, we know that f is near 4 when x is near 1 (but not equal to 1). It follows that $\lim_{x\to -1} f(x^2) = 4$ as well, because when x is near but not equal to -1, x^2 is near 1 but not equal to 1. Thus $f(x^2)$ is near 4 when x is near -1.

2.3.87

a. As $x \to 0^+$, $(1-x) \to 1^-$. So $\lim_{x\to 0^+} g(x) = \lim_{(1-x)\to 1^-} f(1-x) = \lim_{z\to 1^-} f(z) = 6$. (Where $z = 1-x$.)

b. As $x \to 0^-$, $(1-x) \to 1^+$. So $\lim_{x\to 0^-} g(x) = \lim_{(1-x)\to 1^+} f(1-x) = \lim_{z\to 1^+} f(z) = 4$. (Where $z = 1-x$.)

2.3.88

a. Suppose $0 < \theta < \pi/2$. Note that $\sin\theta > 0$, so $|\sin\theta| = \sin\theta$. Also, $\sin\theta = \frac{|AC|}{1}$, so $|AC| = |\sin\theta|$.

Now suppose that $-\pi/2 < \theta < 0$. Then $\sin\theta$ is negative, so $|\sin\theta| = -\sin\theta$. We have $\sin\theta = \frac{-|AC|}{1}$, so $|AC| = -\sin\theta = |\sin\theta|$.

b. Suppose $0 < \theta < \pi/2$. Because AB is the hypotenuse of triangle ABC, we know that $|AB| > |AC|$. We have $|\sin\theta| = |AC| < |AB| <$ the length of arc $AB = \theta = |\theta|$.

If $-\pi/2 < \theta < 0$, we can make a similar argument. We have

$$|\sin\theta| = |AC| < |AB| < \text{the length of arc } AB = -\theta = |\theta|.$$

c. If $0 < \theta < \pi/2$, we have $\sin\theta = |\sin\theta| < |\theta|$, and because $\sin\theta$ is positive, we have $-|\theta| \le 0 < \sin\theta$. Putting these together gives $-|\theta| < \sin\theta < |\theta|$.

If $-\pi/2 < \theta < 0$, then $|\sin\theta| = -\sin\theta$. From the previous part, we have $|\sin\theta| = -\sin\theta < |\theta|$. Therefore, $-|\theta| < \sin\theta$. Now because $\sin\theta$ is negative on this interval, we have $\sin\theta < 0 \le |\theta|$. Putting these together gives $-|\theta| < \sin\theta < |\theta|$.

d. If $0 < \theta < \pi/2$, we have

$$0 \le 1 - \cos\theta = |OB| - |OC| = |BC| < |AB| < \text{the length of arc } AB = \theta = |\theta|.$$

For $-\pi/2 < \theta < 0$, we have

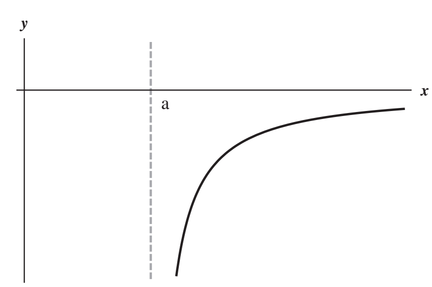
$$0 \le 1 - \cos\theta = |OB| - |OC| = |BC| < |AB| < \text{the length of arc } AB = -\theta = |\theta|.$$

2.3.89 $\lim_{x\to a} p(x) = \lim_{x\to a}(a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0) = \lim_{x\to a}(a_nx^n) + \lim_{x\to a}(a_{n-1}x^{n-1}) + \cdots + \lim_{x\to a}(a_1x) + \lim_{x\to a} a_0 = a_n \lim_{x\to a} x^n + a_{n-1}\lim_{x\to a} x^{n-1} + \cdots + a_1 \lim_{x\to a} x + a_0 = a_n(\lim_{x\to a} x)^n + a_{n-1}(\lim_{x\to a} x)^{n-1} + \cdots + a_1(\lim_{x\to a} x) + a_0 = a_na^n + a_{n-1}a^{n-1} + \cdots + a_1a + a_0 = p(a)$.

2.4 Infinite Limits

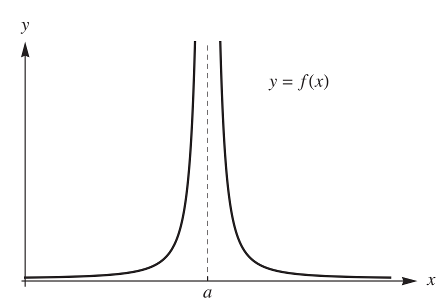
2.4.1

$\lim_{x\to a^+} f(x) = -\infty$ means that when x is very close to (but a little bigger than) a, the corresponding values for $f(x)$ are negative numbers whose absolute value is very large.



2.4.2

$\lim_{x\to a} f(x) = \infty$ means that when x is close to (but not equal to) a, the corresponding values for $f(x)$ are very large positive numbers.



2.4.3 A vertical asymptote for a function f is a vertical line $x = a$ so that one or more of the following are true: $\lim_{x\to a^-} f(x) = \pm\infty$, $\lim_{x\to a^+} f(x) = \pm\infty$.

2.4.4 No. For example, if $f(x) = x^2 - 4$ and $g(x) = x - 2$ and $a = 2$, we would have $\lim_{x\to 2} \frac{f(x)}{g(x)} = 4$, even though $g(2) = 0$.

2.4.5 Because the numerator is approaching a non-zero constant while the denominator is approaching zero, the quotient of these numbers is getting big – at least the absolute value of the quotient is getting big. The quotient is actually always negative, because a number near 100 divided by a negative number is always negative. Thus $\lim_{x\to 2} \frac{f(x)}{g(x)} = -\infty$.

2.4.6 Using the same sort of reasoning as in the last problem – as $x \to 3$ the numerator is fixed at 1, but the denominator is getting small, so the quotient is getting big. It remains to investigate the sign of the quotient. As $x \to 3^-$, the quantity $x - 3$ is negative, so the quotient of the positive number 1 and this small negative number is negative. On the other hand, as $x \to 3^+$, the quantity $x - 3$ is positive, so the quotient of 1 and this number is positive. Thus: $\lim_{x\to 3^-} \frac{1}{x-3} = -\infty$, and $\lim_{x\to 3^+} \frac{1}{x-3} = \infty$.

2.4.7

x	$\frac{x+1}{(x-1)^2}$	x	$\frac{x+1}{(x-1)^2}$
1.1	210	.9	190
1.01	20,100	.99	19,900
1.001	2,001,000	.999	1,999,000
1.0001	200,010,000	.9999	199,990,000

From the data given, it appears that $\lim_{x\to 1} f(x) = \infty$.

2.4.8 $\lim_{x\to 3} f(x) = \infty$, and $\lim_{x\to -1} f(x) = -\infty$.

2.4.9

a. $\lim_{x\to 1^-} f(x) = \infty$.
b. $\lim_{x\to 1^+} f(x) = \infty$.
c. $\lim_{x\to 1} f(x) = \infty$.
d. $\lim_{x\to 2^-} f(x) = \infty$.
e. $\lim_{x\to 2^+} f(x) = -\infty$.
f. $\lim_{x\to 2} f(x)$ does not exist.

2.4.10

a. $\lim_{x\to 2^-} g(x) = \infty$.
b. $\lim_{x\to 2^+} g(x) = -\infty$.
c. $\lim_{x\to 2} g(x)$ does not exist.
d. $\lim_{x\to 4^-} g(x) = -\infty$.
e. $\lim_{x\to 4^+} g(x) = -\infty$.
f. $\lim_{x\to 4} g(x) = -\infty$.

2.4.11

a. $\lim_{x\to -2^-} h(x) = -\infty$.
b. $\lim_{x\to -2^+} h(x) = -\infty$.
c. $\lim_{x\to -2} h(x) = -\infty$.
d. $\lim_{x\to 3^-} h(x) = \infty$.
e. $\lim_{x\to 3^+} h(x) = -\infty$.
f. $\lim_{x\to 3} h(x)$ does not exist.

2.4.12

a. $\lim_{x\to -2^-} p(x) = -\infty$.
b. $\lim_{x\to -2^+} p(x) = -\infty$.
c. $\lim_{x\to -2} p(x) = -\infty$.
d. $\lim_{x\to 3^-} p(x) = -\infty$.
e. $\lim_{x\to 3^+} p(x) = -\infty$.
f. $\lim_{x\to 3} p(x) = -\infty$.

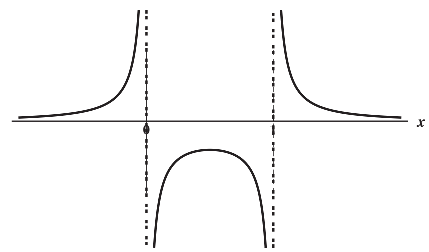
2.4.13

a. $\lim_{x\to 0^-} \frac{1}{x^2 - x} = \infty.$

b. $\lim_{x\to 0^+} \frac{1}{x^2 - x} = -\infty.$

c. $\lim_{x\to 1^-} \frac{1}{x^2 - x} = -\infty.$

d. $\lim_{x\to 1^+} \frac{1}{x^2 - x} = \infty.$



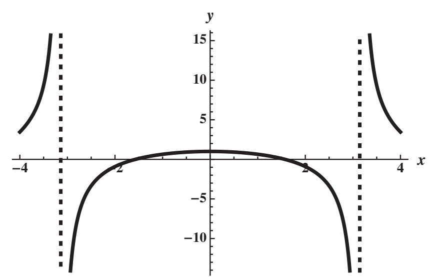
2.4.14

a. $\lim_{x\to \pi^+} x \cot x = \infty.$

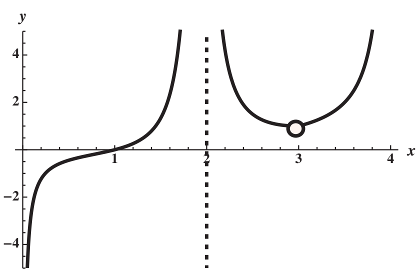
b. $\lim_{x\to \pi^-} x \cot x = -\infty.$

c. $\lim_{x\to -\pi^+} x \cot x = -\infty.$

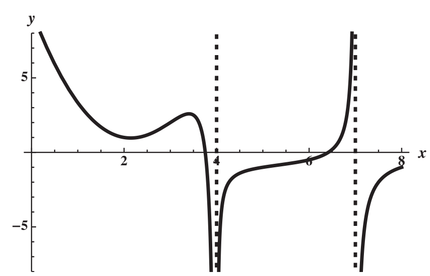
d. $\lim_{x\to -\pi^-} x \cot x = \infty.$



2.4.15



2.4.16



2.4.17

a. $\lim_{x\to 2^+} \frac{1}{x-2} = \infty.$

b. $\lim_{x\to 2^-} \frac{1}{x-2} = -\infty.$

c. $\lim_{x\to 2} \frac{1}{x-2}$ does not exist.

2.4.18

a. $\lim_{x\to 3^+} \frac{2}{(x-3)^3} = \infty.$

b. $\lim_{x\to 3^-} \frac{2}{(x-3)^3} = -\infty.$

c. $\lim_{x\to 3} \frac{2}{(x-3)^3}$ does not exist.

2.4.19

a. $\lim_{x\to 4^+} \frac{x-5}{(x-4)^2} = -\infty.$

b. $\lim_{x\to 4^-} \frac{x-5}{(x-4)^2} = -\infty.$

c. $\lim_{x\to 4} \frac{x-5}{(x-4)^2} = -\infty.$

2.4.20

a. $\lim_{x\to 1^+} \frac{x-2}{(x-1)^3} = -\infty.$

b. $\lim_{x\to 1^-} \frac{x-2}{(x-1)^3} = \infty.$

c. $\lim_{x\to 1} \frac{x-2}{(x-1)^3}$ does not exist.

2.4.21

a. $\lim_{x\to 3^+} \frac{(x-1)(x-2)}{(x-3)} = \infty$. b. $\lim_{x\to 3^-} \frac{(x-1)(x-2)}{(x-3)} = -\infty$. c. $\lim_{x\to 3} \frac{(x-1)(x-2)}{(x-3)}$ does not exist.

2.4.22

a. $\lim_{x\to -2^+} \frac{(x-4)}{x(x+2)} = \infty$. b. $\lim_{x\to -2^-} \frac{(x-4)}{x(x+2)} = -\infty$. c. $\lim_{x\to -2} \frac{(x-4)}{x(x+2)}$ does not exist.

2.4.23

a. $\lim_{x\to 2^+} \frac{x^2-4x+3}{(x-2)^2} = -\infty$. b. $\lim_{x\to 2^-} \frac{x^2-4x+3}{(x-2)^2} = -\infty$. c. $\lim_{x\to 2} \frac{x^2-4x+3}{(x-2)^2} = -\infty$.

2.4.24

a. $\lim_{x\to -2^+} \frac{x^3-5x^2+6x}{x^4-4x^2} = \lim_{x\to -2^+} \frac{x(x-2)(x-3)}{x^2(x-2)(x+2)} = \lim_{x\to -2^+} \frac{x-3}{x(x+2)} = \infty$.

b. $\lim_{x\to -2^-} \frac{x^3-5x^2+6x}{x^4-4x^2} = \lim_{x\to -2^-} \frac{x(x-2)(x-3)}{x^2(x-2)(x+2)} = \lim_{x\to -2^-} \frac{x-3}{x(x+2)} = -\infty$.

c. Because the two one-sided limits differ, $\lim_{x\to -2} \frac{x^3-5x^2+6x}{x^4-4x^2}$ does not exist.

d. $\lim_{x\to 2} \frac{x^3-5x^2+6x}{x^4-4x^2} = \lim_{x\to 2} \frac{x-3}{x(x+2)} = \frac{-1}{8}$.

2.4.25 $\lim_{x\to 0} \frac{x^3-5x^2}{x^2} = \lim_{x\to 0} \frac{x^2(x-5)}{x^2} = \lim_{x\to 0}(x-5) = -5$.

2.4.26 $\lim_{t\to 5} \frac{4t^2-100}{t-5} = \lim_{t\to 5} \frac{4(t-5)(t+5)}{t-5} = \lim_{t\to 5}[4(t+5)] = 40$.

2.4.27 $\lim_{x\to 1^+} \frac{x^2-5x+6}{x-1} = \lim_{x\to 1^+} \frac{(x-2)(x-3)}{x-1} = \infty$. (Note that as $x \to 1^+$, the numerator is near 2, while the denominator is near zero, but is positive. So the quotient is positive and large.)

2.4.28 $\lim_{z\to 4} \frac{z-5}{(z^2-10z+24)^2} = \lim_{z\to 4} \frac{z-5}{(z-4)^2(z-6)^2} = -\infty$. (Note that as $z \to 4$, the numerator is near -1 while the denominator is near zero but is positive. So the quotient is negative with large absolute value.)

2.4.29

a. $\lim_{x\to 5} \frac{x-5}{x^2-25} = \lim_{x\to 5} \frac{1}{x+5} = \frac{1}{10}$, so there isn't a vertical asymptote at $x = 5$.

b. $\lim_{x\to -5^-} \frac{x-5}{x^2-25} = \lim_{x\to -5^-} \frac{1}{x+5} = -\infty$, so there is a vertical asymptote at $x = -5$.

c. $\lim_{x\to -5^+} \frac{x-5}{x^2-25} = \lim_{x\to -5^+} \frac{1}{x+5} = \infty$. This also implies that $x = -5$ is a vertical asymptote, as we already noted in part b.

2.4.30

a. $\lim_{x\to 7^-} \frac{x+7}{x^4-49x^2} = \lim_{x\to 7^-} \frac{x+7}{x^2(x+7)(x-7)} = \lim_{x\to 7^-} \frac{1}{x^2(x-7)} = -\infty$, so there is a vertical asymptote at $x = 7$.

b. $\lim_{x\to 7^+} \dfrac{x+7}{x^4-49x^2} = \lim_{x\to 7^+} \dfrac{x+7}{x^2(x+7)(x-7)} = \lim_{x\to 7^+} \dfrac{1}{x^2(x-7)} = \infty$. This also implies that there is a vertical asymptote at $x = 7$, as we already noted in part a.

c. $\lim_{x\to -7} \dfrac{x+7}{x^4-49x^2} = \lim_{x\to -7} \dfrac{x+7}{x^2(x+7)(x-7)} = \lim_{x\to -7} \dfrac{1}{x^2(x-7)} = \dfrac{1}{-686}$. So there is not a vertical asymptote at $x = 7$.

d. $\lim_{x\to 0} \dfrac{x+7}{x^4-49x^2} = \lim_{x\to 0} \dfrac{x+7}{x^2(x+7)(x-7)} = \lim_{x\to 0} \dfrac{1}{x^2(x-7)} = -\infty$. So there is a vertical asymptote at $x = 0$.

2.4.31 $f(x) = \frac{x^2-9x+14}{x^2-5x+6} = \frac{(x-2)(x-7)}{(x-2)(x-3)}$. Note that $x = 3$ is a vertical asymptote, while $x = 2$ appears to be a candidate but isn't one. We have $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \dfrac{x-7}{x-3} = -\infty$ and $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \dfrac{x-7}{x-3} = \infty$, and thus $\lim_{x\to 3} f(x)$ doesn't exist. Note that $\lim_{x\to 2} f(x) = 5$.

2.4.32 $f(x) = \frac{\cos x}{x(x+2)}$ has vertical asymptotes at $x = 0$ and at $x = -2$. Note that $\cos x$ is near 1 when x is near 0 , and $\cos x$ is near $-.4$ when x is near -2. Thus, $\lim_{x\to 0^+} f(x) = +\infty$, $\lim_{x\to 0^-} f(x) = -\infty$, $\lim_{x\to -2^+} f(x) = \infty$, and $\lim_{x\to -2^-} f(x) = -\infty$.

2.4.33 $f(x) = \frac{x+1}{x^3-4x^2+4x} = \frac{x+1}{x(x-2)^2}$. There are vertical asymptotes at $x = 0$ and $x = 2$. We have $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \dfrac{x+1}{x(x-2)^2} = -\infty$, while $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \dfrac{x+1}{x(x-2)^2} = \infty$, and thus $\lim_{x\to 0} f(x)$ doesn't exist.

Also we have $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \dfrac{x+1}{x(x-2)^2} = \infty$, while $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \dfrac{x+1}{x(x-2)^2} = \infty$, and thus $\lim_{x\to 2} f(x) = \infty$ as well.

2.4.34 $g(x) = \frac{x^3-10x^2+16x}{x^2-8x} = \frac{x(x-2)(x-8)}{x(x-8)}$. This function has no vertical asymptotes.

2.4.35 $\lim_{\theta\to 0^+} \csc\theta = \lim_{\theta\to 0^+} \dfrac{1}{\sin\theta} = \infty$.

2.4.36 $\lim_{x\to 0^-} \csc x = \lim_{x\to 0^-} \dfrac{1}{\sin x} = -\infty$.

2.4.37 $\lim_{x\to 0^+} -10\cot x = \lim_{x\to 0^+} \dfrac{-10\cos x}{\sin x} = -\infty$. (Note that as $x \to 0^+$, the numerator is near -10 and the denominator is near zero, but is positive. Thus the quotient is a negative number whose absolute value is large.)

2.4.38 $\lim_{\theta\to(\pi/2)^+} \dfrac{1}{3}\tan\theta = \lim_{\theta\to(\pi/2)^+} \dfrac{\sin\theta}{3\cos\theta} = -\infty$. (Note that as $\theta \to (\pi/2)^+$, the numerator is near 1 and the denominator is near 0, but is negative. Thus the quotient is a negative number whose absolute value is large.)

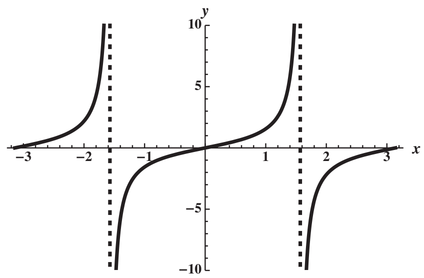
2.4.39

a. $\lim_{x\to(\pi/2)^+} \tan x = -\infty$.

b. $\lim_{x\to(\pi/2)^-} \tan x = \infty$.

c. $\lim_{x\to(-\pi/2)^+} \tan x = -\infty$.

d. $\lim_{x\to(-\pi/2)^-} \tan x = \infty$.



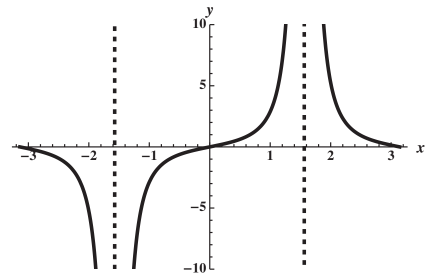
2.4.40

a. $\lim_{x\to(\pi/2)^+} \sec x \tan x = \infty$.

b. $\lim_{x\to(\pi/2)^-} \sec x \tan x = \infty$.

c. $\lim_{x\to(-\pi/2)^+} \sec x \tan x = -\infty$.

d. $\lim_{x\to(-\pi/2)^-} \sec x \tan x = -\infty$.



2.4.41

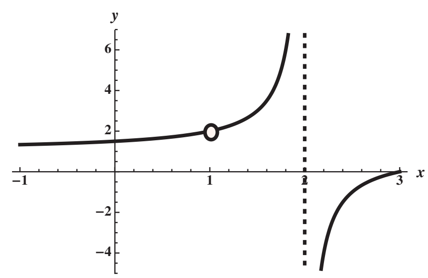
a. False. $\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = \lim_{x\to1} f(x) = \lim_{x\to1} \dfrac{(x-1)(x-6)}{(x-1)(x+1)} = \dfrac{-5}{2}$.

b. True. For example, $\lim_{x\to-1^+} f(x) = \lim_{x\to-1^+} \dfrac{(x-1)(x-6)}{(x-1)(x+1)} = -\infty$.

c. False. For example $g(x) = \frac{1}{x-1}$ has $\lim_{x\to1^+} g(x) = \infty$, but $\lim_{x\to1^-} g(x) = -\infty$.

2.4.42

One such function is $f(x) = \frac{x^2-4x+3}{x^2-3x+2} = \frac{(x-1)(x-3)}{(x-1)(x-2)}$.



2.4.43 One example is $f(x) = \frac{1}{x-6}$.

2.4.44

function	a	b	c	d	e	f
graph	D	C	F	B	A	E

2.4.45 $f(x) = \frac{x^2-3x+2}{x^{10}-x^9} = \frac{(x-2)(x-1)}{x^9(x-1)}$. f has a vertical asymptote at $x = 0$, because $\lim_{x\to0^+} f(x) = -\infty$ (and $\lim_{x\to0^-} f(x) = \infty$.) Note that $\lim_{x\to1} f(x) = -1$, so there isn't a vertical asymptote at $x = 1$.

2.4.46 $g(x) = \cot\left(x - \dfrac{\pi}{2}\right)$ has vertical asymptotes at $x = \pm\dfrac{\pi}{2}$, because $\lim_{x\to\pi/2^-} \cot\left(x - \dfrac{\pi}{2}\right) = -\infty$, and $\lim_{x\to-\pi/2^+} \cot\left(x - \dfrac{\pi}{2}\right) = \infty$.

2.4.47 $h(x) = \dfrac{\cos x}{(x+1)^3}$ has a vertical asymptote at $x = -1$, because $\lim_{x\to-1^+} \dfrac{\cos x}{(x+1)^3} = \infty$, and because $\lim_{x\to-1^-} h(x) = -\infty$.

2.4.48 $p(x) = \sec(\pi x/2) = \frac{1}{\cos(\pi x/2)}$ has a vertical asymptote on $(-2, 2)$ at $x = \pm1$.

2.4.49 $g(\theta) = \tan(\pi\theta/10) = \frac{\sin(\pi\theta/10)}{\cos(\pi\theta/10)}$ has a vertical asymptote at each $\theta = 10n + 5$ where n is an integer. This is due to the fact that $\cos(\pi\theta/10) = 0$ when $\pi\theta/10 = \pi/2 + n\pi$ where n is an integer, which is the same as $\{\theta\colon \theta = 10n + 5, n \text{ an integer}\}$. Note that at all of these numbers which make the denominator zero, the numerator isn't zero.

2.4.50 $q(s) = \frac{\pi}{s-\sin s}$ has a vertical asymptote at $s = 0$. Note that this is the only number where $\sin s = s$.

2.4.51 $f(x) = \frac{1}{\sqrt{x}\sec x} = \frac{\cos x}{\sqrt{x}}$ has a vertical asymptote at $x = 0$.

2.4.52 Note that $g(x)$ is defined only when $x(x^2-1) > 0$, i.e. for $x \in (-1,0)\cup(1,\infty)$. Then $\lim_{x\to-1^+} g(x) = \infty$, $\lim_{x\to0^-} g(x) = \infty$, $\lim_{x\to1^+} g(x) = \infty$, so that $g(x)$ has vertical asymptotes for $x = \pm1, 0$.

2.4.53

a. Note that the numerator of the given expression factors as $(x-3)(x-4)$. So if $a = 3$ or if $a = 4$ the limit would be a finite number. In fact, $\lim_{x\to3} \dfrac{(x-3)(x-4)}{x-3} = -1$ and $\lim_{x\to4} \dfrac{(x-3)(x-4)}{x-4} = 1$.

b. For any number other than 3 or 4, the limit would be either $\pm\infty$. Because $x - a$ is always positive as $x \to a^+$, the limit would be $+\infty$ exactly when the numerator is positive, which is for a in the set $(-\infty,3)\cup(4,\infty)$.

c. The limit would be $-\infty$ for a in the set $(3,4)$.

2.4.54

a. The slope of the secant line is given by $\frac{f(h)-f(0)}{h} = \frac{h^{1/3}}{h} = h^{-2/3}$.

b. $\lim_{h\to0} \dfrac{1}{\sqrt[3]{h^2}} = \infty$. This tells us that the slope of the tangent line is infinite – which means that the tangent line at $(0,0)$ is vertical.

2.4.55

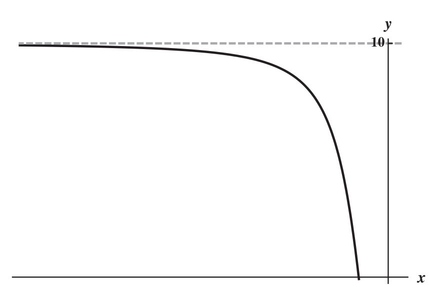
a. The slope of the secant line is $\frac{f(h)-f(0)}{h} = \frac{h^{2/3}}{h} = h^{-1/3}$.

b. $\lim_{h\to0^+} \dfrac{1}{h^{1/3}} = \infty$, and $\lim_{h\to0^-} \dfrac{1}{h^{1/3}} = -\infty$. The tangent line is infinitely steep at the origin (i.e., it is a vertical line.)

2.5 Limits at Infinity

2.5.1

As $x < 0$ becomes large in absolute value, the corresponding values of f level off near 10.



2.5.2 A horizontal asymptote is a horizontal line $y = L$ so that either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$ (or both.)

2.5.3 If $f(x) \to 100,000$ as $x \to \infty$ and $g(x) \to \infty$ as $x \to \infty$, then the ratio $\frac{f(x)}{g(x)} \to 0$ as $x \to \infty$. (Because *eventually* the values of f are small compared to the values of g.)

2.5.4 As $x \to \pm\infty$, $\sin x$ remains between -1 and 1, so that $\lim_{x\to\infty} g(x) = 0$ and $\lim_{x\to-\infty} g(x) = 0$.

2.5.5 $\lim\limits_{x\to\infty}(-2x^3) = -\infty$, and $\lim\limits_{x\to-\infty}(-2x^3) = \infty$.

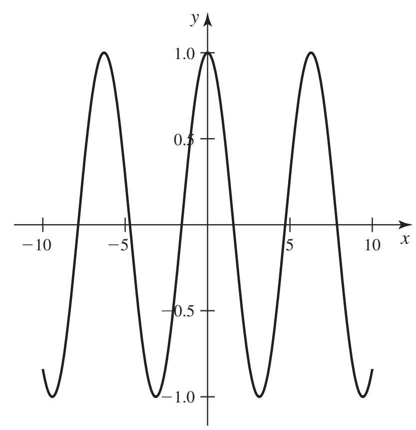
2.5.6 The line $y = 0$ may be a horizontal asymptote, the line $y = a$ where $a \neq 0$ may be a horizontal asymptote, and the limit at $\pm\infty$ may not exist either with or without a slant asymptote.

2.5.7

$$\lim_{x\to\infty} \frac{1-x}{2x} = -\frac{1}{2}, \quad \lim_{x\to\infty} \frac{1-x}{x^2} = 0, \quad \lim_{x\to\infty} \frac{1-x^2}{2x} = -\infty$$

2.5.8

As $x \to \infty$ (or as $x \to -\infty$), $\cos x$ oscillates between -1 and 1. Thus neither $\lim\limits_{x\to\infty} \cos x$ nor $\lim\limits_{x\to-\infty} \cos x$ exists.



2.5.9 $\lim\limits_{x\to\infty}(3 + 10/x^2) = 3 + \lim\limits_{x\to\infty}(10/x^2) = 3 + 0 = 3$.

2.5.10 $\lim\limits_{x\to\infty}(5 + 1/x + 10/x^2) = 5 + \lim\limits_{x\to\infty}(1/x) + \lim\limits_{x\to\infty}(10/x^2) = 5 + 0 + 0 = 5$.

2.5.11 $\lim\limits_{\theta\to\infty} \dfrac{\cos\theta}{\theta^2} = 0$. Note that $-1 \le \cos\theta \le 1$, so $-\frac{1}{\theta^2} \le \frac{\cos\theta}{\theta^2} \le \frac{1}{\theta^2}$. The result now follows from the squeeze theorem.

2.5.12 $\lim\limits_{x\to\infty} \dfrac{3+2x+4x^2}{x^2} = \lim\limits_{x\to\infty} \dfrac{3}{x^2} + \lim\limits_{x\to\infty} \dfrac{2x}{x^2} + \lim\limits_{x\to\infty} \dfrac{4x^2}{x^2} = 0 + \lim\limits_{x\to\infty} \dfrac{2}{x} + \lim\limits_{x\to\infty} 4 = 0 + 0 + 4 = 4$.

2.5.13 $\lim\limits_{x\to\infty} \dfrac{\cos x^5}{\sqrt{x}} = 0$. Note that $-1 \le \cos x^5 \le 1$, so $\frac{-1}{\sqrt{x}} \le \frac{\cos x^5}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$. Because $\lim\limits_{x\to\infty} \dfrac{1}{\sqrt{x}} = \lim\limits_{x\to\infty} \dfrac{-1}{\sqrt{x}} = 0$, we have $\lim\limits_{x\to\infty} \dfrac{\cos x^5}{\sqrt{x}} = 0$ by the squeeze theorem.

2.5.14 $\lim\limits_{x\to-\infty} \left(5 + \dfrac{100}{x} + \dfrac{\sin^4(x^3)}{x^2}\right) = 5 + 0 + 0 = 5$. For this last limit, note that $0 \le \sin^4(x^3) \le 1$, so $0 \le \frac{\sin^4(x^3)}{x^2} \le \frac{1}{x^2}$. The result now follows from the squeeze theorem.

2.5.15 $\lim\limits_{x\to\infty} x^{12} = \infty$. Note that x^{12} is positive when $x > 0$.

2.5.16 $\lim\limits_{x\to-\infty} 3x^{11} = -\infty$. Note that x^{11} is negative when $x < 0$.

2.5.17 $\lim\limits_{x\to\infty} x^{-6} = \lim\limits_{x\to\infty} \dfrac{1}{x^6} = 0.$

2.5.18 $\lim\limits_{x\to-\infty} x^{-11} = \lim\limits_{x\to-\infty} \dfrac{1}{x^{11}} = 0.$

2.5.19 $\lim\limits_{x\to\infty} (3x^{12} - 9x^7) = \infty.$

2.5.20 $\lim\limits_{x\to-\infty} (3x^7 + x^2) = -\infty.$

2.5.21 $\lim\limits_{x\to-\infty} (-3x^{16} + 2) = -\infty.$

2.5.22 $\lim\limits_{x\to-\infty} 2x^{-8} = \lim\limits_{x\to-\infty} \dfrac{2}{x^8} = 0.$

2.5.23 $\lim\limits_{x\to\infty} (-12x^{-5}) = \lim\limits_{x\to\infty} -\dfrac{12}{x^5} = 0.$

2.5.24 $\lim\limits_{x\to-\infty} (2x^{-8} + 4x^3) = 0 + \lim\limits_{x\to-\infty} 4x^3 = -\infty.$

2.5.25 $\lim\limits_{x\to\infty} \dfrac{4x}{20x+1} = \lim\limits_{x\to\infty} \dfrac{4x}{20x+1} \cdot \dfrac{1/x}{1/x} = \lim\limits_{x\to\infty} \dfrac{4}{20+1/x} = \dfrac{4}{20} = \dfrac{1}{5}$. Thus, the line $y = \frac{1}{5}$ is a horizontal asymptote.

$\lim\limits_{x\to-\infty} \dfrac{4x}{20x+1} = \lim\limits_{x\to-\infty} \dfrac{4x}{20x+1} \cdot \dfrac{1/x}{1/x} = \lim\limits_{x\to-\infty} \dfrac{4}{20+1/x} = \dfrac{4}{20} = \dfrac{1}{5}$. This shows that the curve is also asymptotic to the asymptote in the negative direction.

2.5.26 $\lim\limits_{x\to\infty} \dfrac{3x^2-7}{x^2+5x} = \lim\limits_{x\to\infty} \dfrac{3x^2-7}{x^2+5x} \cdot \dfrac{1/x^2}{1/x^2} = \lim\limits_{x\to\infty} \dfrac{3-(7/x^2)}{1+(5/x)} = \dfrac{3-0}{1+0} = 3$. Thus, the line $y = 3$ is a horizontal asymptote.

$\lim\limits_{x\to-\infty} \dfrac{3x^2-7}{x^2+5x} = \lim\limits_{x\to-\infty} \dfrac{3x^2-7}{x^2+5x} \cdot \dfrac{1/x^2}{1/x^2} = \lim\limits_{x\to-\infty} \dfrac{3-(7/x^2)}{1+(5/x)} = \dfrac{3-0}{1+0} = 3$. Thus, the curve is also asymptotic to the asymptote in the negative direction.

2.5.27 $\lim\limits_{x\to\infty} \dfrac{(6x^2-9x+8)}{(3x^2+2)} \cdot \dfrac{1/x^2}{1/x^2} = \lim\limits_{x\to\infty} \dfrac{6-9/x+8/x^2}{3+2/x^2} = \dfrac{6-0+0}{3+0} = 2$. Similarly $\lim\limits_{x\to-\infty} f(x) = 2$. The line $y = 2$ is a horizontal asymptote.

2.5.28 $\lim\limits_{x\to\infty} \dfrac{(4x^2-7)}{(8x^2+5x+2)} \cdot \dfrac{1/x^2}{1/x^2} = \lim\limits_{x\to\infty} \dfrac{4-7/x^2}{8+5/x+2/x^2} = \dfrac{4-0}{8+0+0} = \dfrac{1}{2}$. Similarly $\lim\limits_{x\to-\infty} f(x) = \dfrac{1}{2}$. The line $y = \frac{1}{2}$ is a horizontal asymptote.

2.5.29 $\lim\limits_{x\to\infty} \dfrac{3x^3-7}{x^4+5x^2} = \lim\limits_{x\to\infty} \dfrac{3x^3-7}{x^4+5x^2} \cdot \dfrac{3/x^4}{1/x^4} = \lim\limits_{x\to\infty} \dfrac{1/x-(7/x^4)}{1+(5/x^2)} = \dfrac{0-0}{1+0} = 0$. Thus, the line $y = 0$ (the x-axis) is a horizontal asymptote.

$\lim\limits_{x\to-\infty} \dfrac{3x^3-7}{x^4+5x^2} = \lim\limits_{x\to-\infty} \dfrac{3x^3-7}{x^4+5x^2} \cdot \dfrac{3/x^4}{1/x^4} = \lim\limits_{x\to-\infty} \dfrac{1/x-(7/x^4)}{1+(5/x^2)} = \dfrac{0-0}{1+0} = 0$. Thus, the curve is asymptotic to the x-axis in the negative direction as well.

2.5.30 $\lim\limits_{x\to\infty} \dfrac{x^4+7}{x^5+x^2-x} = \lim\limits_{x\to\infty} \dfrac{x^4+7}{x^5+x^2-x} \cdot \dfrac{1/x^5}{1/x^5} = \lim\limits_{x\to\infty} \dfrac{(1/x)+(7/x^5)}{1+(1/x^3)-(1/x^4)} = \dfrac{0+0}{1+0-0} = 0$. Thus, the line $y = 0$ (the x-axis) is a horizontal asymptote.

$\lim\limits_{x\to-\infty} \dfrac{x^4+7}{x^5+x^2-x} = \lim\limits_{x\to-\infty} \dfrac{x^4+7}{x^5+x^2-x} \cdot \dfrac{1/x^5}{1/x^5} = \lim\limits_{x\to-\infty} \dfrac{(1/x)+(7/x^5)}{1+(1/x^3)-(1/x^4)} = \dfrac{0+0}{1+0-0} = 0$. Thus, the curve is asymptotic to the x-axis in the negative direction as well.

2.5.31 $\lim_{x\to\infty} \frac{(2x+1)}{(3x^4-2)} \cdot \frac{1/x^4}{1/x^4} = \lim_{x\to\infty} \frac{2/x^3+1/x^4}{3-2/x^4} = \frac{0+0}{3-0} = 0$. Similarly $\lim_{x\to-\infty} f(x) = 0$. The line $y=0$ is a horizontal asymptote.

2.5.32 $\lim_{x\to\infty} \frac{(12x^8-3)}{(3x^8-2x^7)} \cdot \frac{1/x^8}{1/x^8} = \lim_{x\to\infty} \frac{12-3/x^8}{3-2/x} = \frac{12-0}{3-0} = 4$. Similarly $\lim_{x\to-\infty} f(x) = 4$. The line $y=4$ is a horizontal asymptote.

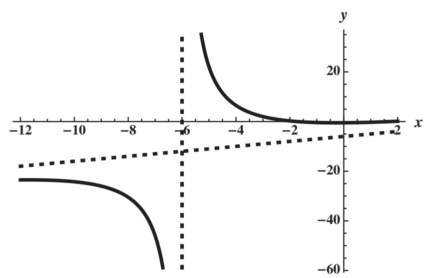
2.5.33 $\lim_{x\to\infty} \frac{(40x^5+x^2)}{(16x^4-2x)} \cdot \frac{1/x^4}{1/x^4} = \lim_{x\to\infty} \frac{40x+1/x^2}{16-2/x^3} = \infty$. Similarly $\lim_{x\to-\infty} f(x) = -\infty$. There are no horizontal asymptotes.

2.5.34 $\lim_{x\to\infty} \frac{(-x^3+1)}{(2x+8)} \cdot \frac{1/x}{1/x} = \lim_{x\to\infty} \frac{-x^2+1/x}{2+8/x} = -\infty$. Similarly $\lim_{x\to-\infty} f(x) = -\infty$. There are no horizontal asymptotes.

2.5.35

a. $f(x) = \frac{x^2-3}{x+6} = x-6+\frac{33}{x+6}$. The oblique asymptote of f is $y = x-6$.

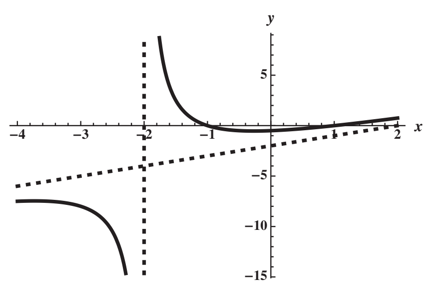
b. Because $\lim_{x\to-6^+} f(x) = \infty$, there is a vertical asymptote at $x=-6$. Note also that $\lim_{x\to-6^-} f(x) = -\infty$.

c. 

2.5.36

a. $f(x) = \frac{x^2-1}{x+2} = x-2+\frac{3}{x+2}$. The oblique asymptote of f is $y = x-2$.

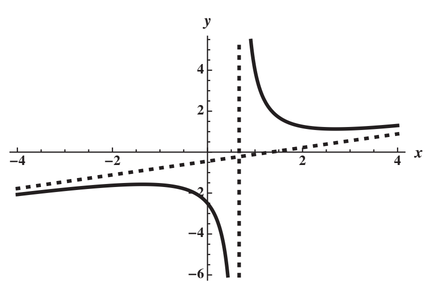
b. Because $\lim_{x\to-2^+} f(x) = \infty$, there is a vertical asymptote at $x=-2$. Note also that $\lim_{x\to-2^-} f(x) = -\infty$.

c. 

2.5.37

a. $f(x) = \frac{x^2-2x+5}{3x-2} = (1/3)x - 4/9 + \frac{37}{9(3x-2)}$. The oblique asymptote of f is $y = (1/3)x - 4/9$.

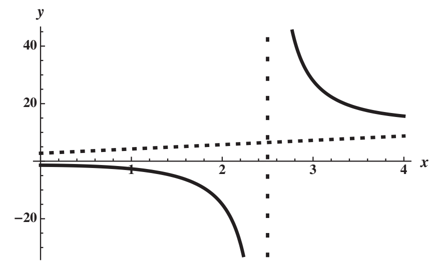
b. Because $\lim_{x\to(2/3)^+} f(x) = \infty$, there is a vertical asymptote at $x = 2/3$. Note also that $\lim_{x\to(2/3)^-} f(x) = -\infty$.

c. 

2.5.38

a. $f(x) = \frac{3x^2-2x+7}{2x-5} = (3/2)x + 11/4 + \frac{83}{4(2x-5)}$. The oblique asymptote of f is $y = (3/2)x + 11/4$.

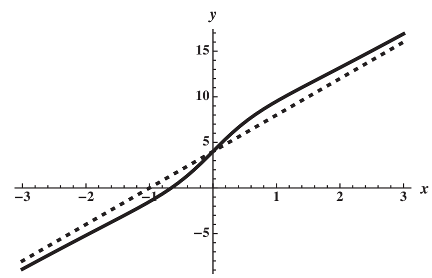
b. Because $\lim_{x\to(5/2)^+} f(x) = \infty$, there is a vertical asymptote at $x = 5/2$. Note also that $\lim_{x\to(5/2)^-} f(x) = -\infty$.

c. 

2.5.39

a. $f(x) = \frac{4x^3+4x^2+7x+4}{1+x^2} = 4x + 4 + \frac{3x}{1+x^2}$. The oblique asymptote of f is $y = 4x + 4$.

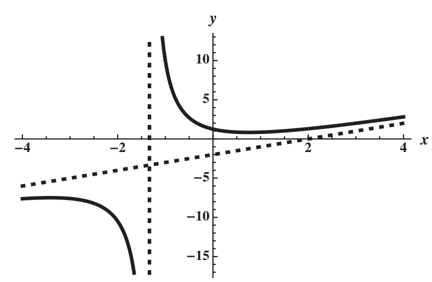
b. There are no vertical asymptotes.

c. 

2.5.40

a. $f(x) = \frac{3x^2-2x+5}{3x+4} = x - 2 + \frac{13}{3x+4}$. The oblique asymptote of f is $y = x - 2$.

b. Because $\lim_{x\to(-4/3)^+} f(x) = \infty$, there is a vertical asymptote at $x = -4/3$. Note also that $\lim_{x\to(-4/3)^-} f(x) = -\infty$.

c. 

2.5.41 First note that $\sqrt{x^6} = x^3$ if $x > 0$, but $\sqrt{x^6} = -x^3$ if $x < 0$. We have $\lim_{x\to\infty} \frac{4x^3+1}{(2x^3+\sqrt{16x^6+1})} \cdot \frac{1/x^3}{1/x^3} = \lim_{x\to\infty} \frac{4+1/x^3}{2+\sqrt{16+1/x^6}} = \frac{4+0}{2+\sqrt{16+0}} = \frac{2}{3}$.

However, $\lim_{x\to-\infty} \frac{4x^3+1}{(2x^3+\sqrt{16x^6+1})} \cdot \frac{1/x^3}{1/x^3} = \lim_{x\to-\infty} \frac{4+1/x^3}{2-\sqrt{16+1/x^6}} = \frac{4+0}{2-\sqrt{16+0}} = \frac{4}{-2} = -2$.

So $y = \frac{2}{3}$ is a horizontal asymptote (as $x \to \infty$) and $y = -2$ is a horizontal asymptote (as $x \to -\infty$).

2.5.42 First note that $\sqrt{x^2} = x$ for $x > 0$, while $\sqrt{x^2} = -x$ for $x < 0$. Then $\lim_{x\to\infty} f(x)$ can be written as

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{2x+1} \cdot \frac{1/\sqrt{x^2}}{1/x} = \lim_{x\to\infty} \frac{\sqrt{1+1/x^2}}{2+1/x} = \frac{1}{2}.$$

However, $\lim_{x\to-\infty} f(x)$ can be written as

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{2x+1} \cdot \frac{1/\sqrt{x^2}}{-1/x} = \lim_{x\to-\infty} \frac{\sqrt{1+1/x^2}}{-2-1/x} = -\frac{1}{2}.$$

2.5.43 First note that $\sqrt[3]{x^6} = x^2$ and $\sqrt{x^4} = x^2$ for all x (even when $x < 0$.) We have $\lim_{x\to\infty} \frac{\sqrt[3]{x^6+8}}{(4x^2+\sqrt{3x^4+1})} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{\sqrt[3]{1+8/x^6}}{4+\sqrt{3+1/x^4}} = \frac{1}{4+\sqrt{3+0}} = \frac{1}{4+\sqrt{3}}$.

The calculation as $x \to -\infty$ is similar. So $y = \frac{1}{4+\sqrt{3}}$ is a horizontal asymptote.

2.5.44 First note that $\sqrt{x^2} = x$ for $x > 0$ and $\sqrt{x^2} = -x$ for $x < 0$.

We have

$$\begin{aligned}\lim_{x\to\infty} 4x(3x - \sqrt{9x^2+1}) &= \lim_{x\to\infty} \frac{4x(3x-\sqrt{9x^2+1})(3x+\sqrt{9x^2+1})}{3x+\sqrt{9x^2+1}} \\ &= \lim_{x\to\infty} \frac{(4x)(-1)}{(3x+\sqrt{9x^2+1})} \cdot \frac{1/x}{1/x} \\ &= \lim_{x\to\infty} -\frac{4}{3+\sqrt{9+1/x^2}} = -\frac{4}{6} = -\frac{2}{3}.\end{aligned}$$

Moreover, as $x \to -\infty$ we have

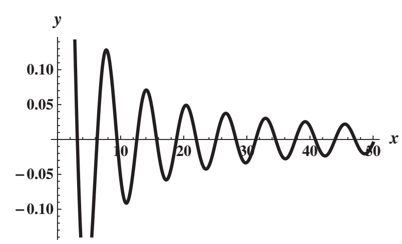
$$\begin{aligned}\lim_{x\to-\infty} 4x(3x - \sqrt{9x^2+1}) &= \lim_{x\to-\infty} \frac{4x(3x-\sqrt{9x^2+1})(3x+\sqrt{9x^2+1})}{3x+\sqrt{9x^2+1}} \\ &= \lim_{x\to-\infty} \frac{(4x)(-1)}{(3x+\sqrt{9x^2+1})} \cdot \frac{1/x}{1/x} \\ &= \lim_{x\to-\infty} -\frac{4}{3-\sqrt{9+1/x^2}} = \infty.\end{aligned}$$

Note that this last equality is due to the fact that the numerator is the constant -4 and the denominator is approaching zero (from the left) so the quotient is positive and is getting large.

So $y = -\frac{2}{3}$ is the only horizontal asymptote.

2.5.45

a. False. For example, the function $y = \frac{\sin x}{x}$ on the domain $[1, \infty)$ has a horizontal asymptote of $y = 0$, and it crosses the x-axis infinitely many times.



b. False. If f is a rational function, and if $\lim_{x\to\infty} f(x) = L \neq 0$, then the degree of the polynomial in the numerator must equal the degree of the polynomial in the denominator. In this case, both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x) = \frac{a_n}{b_n}$ where a_n is the leading coefficient of the polynomial in the numerator and b_n is the leading coefficient of the polynomial in the denominator. In the case where $\lim_{x\to\infty} f(x) = 0$, then the degree of the numerator is strictly less than the degree of the denominator. This case holds for $\lim_{x\to-\infty} f(x) = 0$ as well.

c. True. There are only two directions which might lead to horizontal asymptotes: there could be one as $x \to \infty$ and there could be one as $x \to -\infty$, and those are the only possibilities.

2.5.46

a. $\lim_{x\to\infty} \frac{x^2-4x+3}{x-1} = \infty$, and $\lim_{x\to-\infty} \frac{x^2-4x+3}{x-1} = -\infty$. There are no horizontal asymptotes.

b. It appears that $x = 1$ is a candidate to be a vertical asymptote, but note that $f(x) = \frac{x^2-4x+3}{x-1} = \frac{(x-1)(x-3)}{x-1}$. Thus $\lim_{x\to 1} f(x) = \lim_{x\to 1}(x-3) = -2$. So f has no vertical asymptotes.

2.5.47

a. $\lim_{x\to\infty} \frac{2x^3+10x^2+12x}{x^3+2x^2} \cdot \frac{(1/x^3)}{(1/x^3)} = \lim_{x\to\infty} \frac{2+10/x+12/x^2}{1+2/x} = 2$. Similarly, $\lim_{x\to-\infty} f(x) = 2$. Thus, $y = 2$ is a horizontal asymptote.

b. Note that $f(x) = \frac{2x(x+2)(x+3)}{x^2(x+2)}$. So $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{2(x+3)}{x} = \infty$, and similarly, $\lim_{x\to 0^-} f(x) = -\infty$. There is a vertical asymptote at $x = 0$. Note that there is no asymptote at $x = -2$ because $\lim_{x\to -2} f(x) = -1$.

2.5.48

a. We have $\lim_{x\to\infty} \frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4} \cdot \frac{(1/x^2)}{(1/x^2)} = \lim_{x\to\infty} \frac{\sqrt{16+64/x^2}+1}{2-4/x^2} = \frac{5}{2}$. Similarly, $\lim_{x\to-\infty} f(x) = \frac{5}{2}$. So $y = \frac{5}{2}$ is a horizontal asymptote.

b. $\lim_{x\to\sqrt{2}^+} f(x) = \lim_{x\to-\sqrt{2}^-} f(x) = \infty$, and $\lim_{x\to\sqrt{2}^-} f(x) = \lim_{x\to-\sqrt{2}^+} f(x) = -\infty$ so there are vertical asymptotes at $x = \pm\sqrt{2}$.

2.5.49

a. We have $\lim_{x\to\infty} \frac{3x^4+3x^3-36x^2}{x^4-25x^2+144} \cdot \frac{(1/x^4)}{(1/x^4)} = \lim_{x\to\infty} \frac{3+3/x-36/x^2}{1-25/x^2+144/x^4} = 3$. Similarly, $\lim_{x\to-\infty} f(x) = 3$. So $y = 3$ is a horizontal asymptote.

b. Note that $f(x) = \frac{3x^2(x+4)(x-3)}{(x+4)(x-4)(x+3)(x-3)}$. Thus, $\lim_{x\to-3^+} f(x) = -\infty$ and $\lim_{x\to-3^-} f(x) = \infty$. Also, $\lim_{x\to 4^-} f(x) = -\infty$ and $\lim_{x\to 4^+} f(x) = \infty$. Thus there are vertical asymptotes at $x = -3$ and $x = 4$.

2.5.50

a. First note that

$$f(x) = 16x^2(4x^2 - \sqrt{16x^4+1}) \cdot \frac{4x^2+\sqrt{16x^4+1}}{4x^2+\sqrt{16x^4+1}} = -\frac{16x^2}{4x^2+\sqrt{16x^4+1}}.$$

We have $\lim_{x\to\infty} -\frac{16x^2}{4x^2+\sqrt{16x^4+1}} \cdot \frac{(1/x^2)}{(1/x^2)} = \lim_{x\to\infty} -\frac{16}{4+\sqrt{16+1/x^4}} = -2$. Similarly, the limit as $x \to -\infty$ of $f(x)$ is -2 as well. so $y = -2$ is a horizontal asymptote.

b. f has no vertical asymptotes.

2.5.51

a. $\lim_{x\to\infty} \frac{x^2-9}{x^2-3x} \cdot \frac{(1/x^2)}{(1/x^2)} = \lim_{x\to\infty} \frac{1-9/x^2}{1-3/x} = 1$. A similar result holds as $x \to -\infty$. So $y = 1$ is a horizontal asymptote.

b. Because $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x+3}{x} = \infty$ and $\lim_{x\to 0^-} f(x) = -\infty$, there is a vertical asymptote at $x = 0$.

2.5.52

a. $\lim_{x\to\infty} \frac{x-1}{x^{2/3}-1} \cdot \frac{1/x^{2/3}}{1/x^{2/3}} = \lim_{x\to\infty} \frac{x^{1/3} - 1/x^{2/3}}{1 - 1/x^{2/3}} = \infty$. Similarly, $\lim_{x\to -\infty} f(x) = -\infty$. So there are no horizontal asymptotes.

b. There is a vertical asymptote at $x = -1$. The easiest way to see this is to factor the denominator as the difference of squares, and the numerator as the difference of cubes. We have

$$f(x) = \frac{x-1}{x^{2/3}-1} = \frac{(x^{1/3}-1)(x^{2/3}+x^{1/3}+1)}{(x^{1/3}+1)(x^{1/3}-1)}.$$

Thus,

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \frac{x^{2/3}+x^{1/3}+1}{x^{1/3}+1} = \infty.$$

Similarly, $\lim_{x\to -1^-} f(x) = -\infty$.

2.5.53

a. First note that $f(x) = \frac{\sqrt{x^2+2x+6}-3}{x-1} \cdot \frac{\sqrt{x^2+2x+6}+3}{\sqrt{x^2+2x+6}+3} = \frac{x^2+2x+6-9}{(x-1)(\sqrt{x^2+2x+6}+3)} = \frac{(x-1)(x+3)}{(x-1)(\sqrt{x^2+2x+6}+3)}$.

Thus

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x+3}{\sqrt{x^2+2x+6}+3} \cdot \frac{1/x}{1/x} = \lim_{x\to\infty} \frac{1+3/x}{\sqrt{1+2/x+6/x^2}+3/x} = 1.$$

Using the fact that $\sqrt{x^2} = -x$ for $x < 0$, we have $\lim_{x\to -\infty} f(x) = -1$. Thus the lines $y = 1$ and $y = -1$ are horizontal asymptotes.

b. f has no vertical asymptotes.

2.5.54

a. Note that when x is large $|1-x^2| = x^2 - 1$. We have $\lim_{x\to\infty} \frac{|1-x^2|}{x^2+x} = \lim_{x\to\infty} \frac{x^2-1}{x^2+x} = 1$. Likewise $\lim_{x\to -\infty} \frac{|1-x^2|}{x^2+x} = \lim_{x\to -\infty} \frac{x^2-1}{x^2+x} = 1$. So there is a horizontal asymptote at $y = 1$.

b. Note that when x is near 0, we have $|1-x^2| = 1-x^2 = (1-x)(1+x)$. So $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1-x}{x} = \infty$. Similarly, $\lim_{x\to 0^-} f(x) = -\infty$. There is a vertical asymptote at $x = 0$.

2.5.55

a. Note that when $x > 1$, we have $|x| = x$ and $|x-1| = x-1$. Thus

$$f(x) = (\sqrt{x} - \sqrt{x-1}) \cdot \frac{\sqrt{x}+\sqrt{x-1}}{\sqrt{x}+\sqrt{x-1}} = \frac{1}{\sqrt{x}+\sqrt{x-1}}.$$

Thus $\lim_{x\to\infty} f(x) = 0$.

When $x < 0$, we have $|x| = -x$ and $|x-1| = 1-x$. Thus

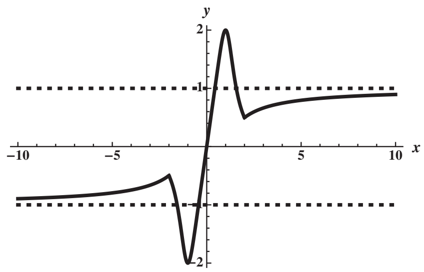
$$f(x) = (\sqrt{-x} - \sqrt{1-x}) \cdot \frac{\sqrt{-x}+\sqrt{1-x}}{\sqrt{-x}+\sqrt{1-x}} = -\frac{1}{\sqrt{-x}+\sqrt{1-x}}.$$

Thus, $\lim_{x\to -\infty} f(x) = 0$. There is a horizontal asymptote at $y = 0$.

b. f has no vertical asymptotes.

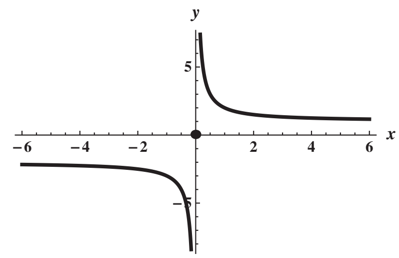
2.5.56

One possible such graph is:



2.5.57

One possible such graph is:



2.5.58

$$\lim_{x\to\pm\infty} \frac{2x}{\sqrt{x^2-x-2}} = \lim_{x\to\pm\infty} \frac{2}{\sqrt{1-1/x-2/x^2}} = 2,$$

so that $f(x)$ has a horizontal asymptote at $y = 2$. It has vertical asymptotes where the denominator is zero, which happens when $x^2 - x - 2 = (x-2)(x+1) = 0$. Thus the vertical asymptotes are at $x = 2$ and $x = -1$.

2.5.59 $\lim_{x\to 0^+} \frac{\cos x + 2\sqrt{x}}{\sqrt{x}} = \infty$. $\lim_{x\to\infty} \frac{\cos x + 2\sqrt{x}}{\sqrt{x}} = \lim_{x\to\infty} \left(2 + \frac{\cos x}{\sqrt{x}}\right) = 2$. There is a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 2$.

2.5.60 $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \frac{2500}{t+1} = 0$. The steady state exists. The steady state value is 0.

2.5.61 $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \frac{3500t}{t+1} = 3500$. The steady state exists. The steady state value is 3500.

2.5.62 $\lim_{t\to\infty} \frac{1500t^2}{2t^2+3} = \lim_{t\to\infty} \frac{1500}{2+3/t^2} = 750$, so that the steady-state population value is 750.

2.5.63 $\lim_{t\to\infty} a(t) = \lim_{t\to\infty} 2\left(\frac{t+\sin t}{t}\right) = \lim_{t\to\infty} 2\left(1 + \frac{\sin t}{t}\right) = 2$. The steady state exists. The steady state value is 2.

2.5.64 $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \frac{4}{n} = 0$.

2.5.65 $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty} [1 - (1/n)] = 1$.

2.5.66 $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \frac{n^2}{n+1} = \lim_{n\to\infty} \frac{n}{1+1/n} = \infty$, so the limit does not exist.

2.5.67 $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \frac{n+1}{n^2} = \lim_{n\to\infty} [1/n + 1/n^2] = 0$.

2.5.68

a. Suppose $m = n$.

$$\begin{aligned}\lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty} \frac{p(x)}{q(x)} \cdot \frac{1/x^n}{1/x^n} \\ &= \lim_{x\to\pm\infty} \frac{a_n + a_{n-1}/x + \cdots + a_1/x^{n-1} + a_0/x^n}{b_n + b_{n-1}/x + \cdots + b_1/x^{n-1} + b_0/x^n} \\ &= \frac{a_n}{b_n}.\end{aligned}$$

b. Suppose $m < n$.

$$\begin{aligned}\lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty} \frac{p(x)}{q(x)} \cdot \frac{1/x^n}{1/x^n} \\ &= \lim_{x\to\pm\infty} \frac{a_n/x^{n-m} + a_{n-1}/x^{n-m+1} + \cdots + a_1/x^{n-1} + a_0/x^n}{b_n + b_{n-1}/x + \cdots + b_1/x^{n-1} + b_0/x^n} \\ &= \frac{0}{b_n} = 0.\end{aligned}$$

2.5.69

a. No. If $m = n$, there will be a horizontal asymptote, and if $m = n + 1$, there will be a slant asymptote.

b. Yes. For example, $f(x) = \frac{x^4}{\sqrt{x^6+1}}$ has a slant asymptote of $y = x$ as $x \to \infty$ and a slant asymptote of $y = -x$ as $x \to -\infty$.

2.6 Continuity

2.6.1

a. $a(t)$ is a continuous function during the time period from when she jumps from the plane and when she touches down on the ground, because her position is changing continuously with time.

b. $n(t)$ is not a continuous function of time. The function "jumps" at the times when a quarter must be added.

c. $T(t)$ is a continuous function, because temperature varies continuously with time.

d. $p(t)$ is not continuous – it jumps by whole numbers when a player scores a point.

2.6.2 In order for f to be continuous at $x = a$, the following conditions must hold:

- f must be defined at a (i.e. a must be in the domain of f),
- $\lim_{x\to a} f(x)$ must exist, and
- $\lim_{x\to a} f(x)$ must equal $f(a)$.

2.6.3 A function f is continuous on an interval I if it is continuous at all points in the interior of I, and it must be continuous from the right at the left endpoint (if the left endpoint is included in I) and it must be continuous from the left at the right endpoint (if the right endpoint is included in I.)

2.6.4 The words "hole" and "break" are not mathematically precise, so a strict mathematical definition can not be based on them.

2.6.5

a. A function f is continuous from the left at $x = a$ if a is in the domain of f, and $\lim_{x \to a^-} f(x) = f(a)$.

b. A function f is continuous from the right at $x = a$ if a is in the domain of f, and $\lim_{x \to a^+} f(x) = f(a)$.

2.6.6 A rational function is discontinuous at each point not in its domain.

2.6.7 The domain of $f(x) = \sqrt{1 - x^2}$ is the set of real numbers x so that $1 - x^2 \geq 0$, i.e. $[-1, 1]$. The function is continuous on its domain.

2.6.8 The Intermediate Value Theorem says that if f is continuous on $[a, b]$ and if L is strictly between $f(a)$ and $f(b)$, then there must be a domain value c (with $a < c < b$) where $f(c) = L$. This means that a continuous function assumes all the intermediate values between the values at the endpoints of an interval.

2.6.9 f is discontinuous at $x = 1$, at $x = 2$, and at $x = 3$. At $x = 1$, $f(1)$ does not exist (so the first condition is violated). At $x = 2$, $f(2)$ exists and $\lim_{x \to 2} f(x)$ exists, but $\lim_{x \to 2} f(x) \neq f(2)$ (so condition 3 is violated). At $x = 3$, $\lim_{x \to 3} f(x)$ does not exist (so condition 2 is violated).

2.6.10 f is discontinuous at $x = 1$, at $x = 2$, and at $x = 3$. At $x = 1$, $\lim_{x \to 1} f(x) \neq f(1)$ (so condition 3 is violated). At $x = 2$, $\lim_{x \to 2} f(x)$ does not exist (so condition 2 is violated). At $x = 3$, $f(3)$ does not exist (so condition 1 is violated).

2.6.11 f is discontinuous at $x = 1$, at $x = 2$, and at $x = 3$. At $x = 1$, $\lim_{x \to 1} f(x)$ does not exist, and $f(1)$ does not exist (so conditions 1 and 2 are violated). At $x = 2$, $\lim_{x \to 2} f(x)$ does not exist (so condition 2 is violated). At $x = 3$, $f(3)$ does not exist (so condition 1 is violated).

2.6.12 f is discontinuous at $x = 2$, at $x = 3$, and at $x = 4$. At $x = 2$, $\lim_{x \to 2} f(x)$ does not exist (so condition 2 is violated). At $x = 3$, $f(3)$ does not exist and $\lim_{x \to 3} f(x)$ does not exist (so conditions 1 and 2 are violated). At $x = 4$, $\lim_{x \to 4} f(x) \neq f(4)$ (so condition 3 is violated).

2.6.13 The function is defined at 5, in fact $f(5) = \frac{50+15+1}{25+25} = \frac{66}{50} = \frac{33}{25}$. Also, $\lim_{x \to 5} f(x) = \lim_{x \to 5} \dfrac{2x^2 + 3x + 1}{x^2 + 5x} = \dfrac{33}{25} = f(5)$. The function is continuous at $a = 5$.

2.6.14 The number -5 is not in the domain of f, because the denominator is equal to 0 when $x = -5$. Thus, the function is not continuous at -5.

2.6.15 f is discontinuous at 1, because 1 is not in the domain of f.

2.6.16 g is discontinuous at 3 because 3 is not in the domain of g.

2.6.17 f is discontinuous at 1, because $\lim_{x \to 1} f(x) \neq f(1)$. In fact, $f(1) = 3$, but $\lim_{x \to 1} f(x) = 2$.

2.6.18 f is continuous at 3, because $\lim_{x \to 3} f(x) = f(3)$. In fact, $f(3) = 2$ and $\lim_{x \to 3} f(x) = \lim_{x \to 3} \dfrac{(x-3)(x-1)}{x-3} = \lim_{x \to 3} (x - 1) = 2$.

2.6.19 f is discontinuous at 4, because 4 is not in the domain of f.

2.6.20 f is discontinuous at -1 because $\lim_{x\to-1} f(x) = \lim_{x\to-1} \frac{x(x+1)}{x+1} = \lim_{x\to-1} x = -1 \neq f(-1) = 2$.

2.6.21 Because f is a polynomial, it is continuous on all of $\mathbb{R}$.

2.6.22 Because g is a rational function, it is continuous on its domain, which is all of $\mathbb{R}$. (Because x^2+x+1 has no real roots.)

2.6.23 Because f is a rational function, it is continuous on its domain. Its domain is $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$.

2.6.24 Because s is a rational function, it is continuous on its domain. Its domain is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$.

2.6.25 Because f is a rational function, it is continuous on its domain. Its domain is $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$.

2.6.26 Because f is a rational function, it is continuous on its domain. Its domain is $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$.

2.6.27 Because $f(x) = \left(x^8 - 3x^6 - 1\right)^{40}$ is a polynomial, it is continuous everywhere, including at 0. Thus $\lim_{x\to 0} f(x) = f(0) = (-1)^{40} = 1$.

2.6.28 Because $f(x) = \left(\frac{3}{2x^5-4x^2-50}\right)^4$ is a rational function, it is continuous at all points in its domain, including at $x=2$. So $\lim_{x\to 2} f(x) = f(2) = \frac{81}{16}$.

2.6.29 Because $f(x) = \left(\frac{x+5}{x+2}\right)^4$ is a rational function, it is continuous at all points in its domain, including at $x=1$. Thus $\lim_{x\to 1} f(x) = f(1) = 16$.

2.6.30 $\lim_{x\to\infty} \left(\frac{2x+1}{x}\right)^3 = \lim_{x\to\infty} (2+(1/x))^3 = 2^3 = 8$.

2.6.31 Because $x^3-2x^2-8x = x(x^2-2x-8) = x(x-4)(x+2)$, we have (as long as $x \neq 4$)

$$\sqrt{\frac{x^3-2x^2-8x}{x-4}} = \sqrt{x(x+2)}.$$

Thus, $\lim_{x\to 4} \sqrt{\frac{x^3-2x^2-8x}{x-4}} = \lim_{x\to 4} \sqrt{x(x+2)} = \sqrt{24}$, using Theorem 2.11 and the fact that the square root is a continuous function.

2.6.32 Note that $t-4 = (\sqrt{t}-2)(\sqrt{t}+2)$, so for $t \neq 4$, we have

$$\frac{t-4}{\sqrt{t}-2} = \sqrt{t}+2.$$

Thus, $\lim_{t\to 4} \frac{t-4}{\sqrt{t}-2} = \lim_{t\to 4}(\sqrt{t}+2) = 4$. Then using Theorem 2.11 and the fact that the tangent function is continuous at 4, we have $\lim_{t\to 4} \tan\left(\frac{t-4}{\sqrt{t}-2}\right) = \tan\left(\lim_{t\to 4} \frac{t-4}{\sqrt{t}-2}\right) = \tan 4$.

2.6.33 Recall that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Now noting that the function $f(x) = \cos 2x$ is continuous at 1, we have by Theorem 2.11 that $\lim_{x\to 0} \cos\left(\frac{2\sin x}{x}\right) = \cos\left(2\left(\lim_{x\to 0} \frac{\sin x}{x}\right)\right) = \cos(2\cdot 1) = \cos 2$.

2.6.34 First note that

$$\lim_{x\to 0}\frac{x}{\sqrt{16x+1}-1}=\lim_{x\to 0}\frac{x}{(\sqrt{16x+1}-1)}\cdot\frac{(\sqrt{16x+1}+1)}{(\sqrt{16x+1}+1)}=\lim_{x\to 0}\frac{x(\sqrt{16x+1}+1)}{16x}=\frac{2}{16}=\frac{1}{8}.$$

Then because $f(x)=x^{1/3}$ is continuous at $1/8$, we have $\lim_{x\to 0}\left(\frac{x}{\sqrt{16x+1}-1}\right)^{1/3}=\left(\frac{1}{8}\right)^{1/3}=\frac{1}{2}$, by Theorem 2.11.

2.6.35 f is continuous on $[0,1)$, on $(1,2)$, on $(2,3]$, and on $(3,4]$.

2.6.36 f is continuous on $[0,1)$, on $(1,2]$, on $(2,3)$, and on $(3,4]$.

2.6.37 f is continuous on $[0,1)$, on $(1,2)$, on $[2,3)$, and on $(3,5]$.

2.6.38 f is continuous on $[0,2]$, on $(2,3)$, on $(3,4)$, and on $(4,5]$.

2.6.39

a. f is defined at 1. We have $f(1)=1^2+(3)(1)=4$. To see whether or not $\lim_{x\to 1}f(x)$ exists, we investigate the two one-sided limits. $\lim_{x\to 1^-}f(x)=\lim_{x\to 1^-}2x=2$, and $\lim_{x\to 1^+}f(x)=\lim_{x\to 1^+}(x^2+3x)=4$, so $\lim_{x\to 1}f(x)$ does not exist. Thus f is discontinuous at $x=1$.

b. f is continuous from the right, because $\lim_{x\to 1^+}f(x)=4=f(1)$.

c. f is continuous on $(-\infty,1)$ and on $[1,\infty)$.

2.6.40

a. f is defined at 0, in fact $f(0)=1$. However, $\lim_{x\to 0^-}f(x)=\lim_{x\to 0^-}(x^3+4x+1)=1$, while $\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}2x^3=0$. So $\lim_{x\to 0}f(x)$ does not exist.

b. f is continuous from the left at 0, because $\lim_{x\to 0^-}f(x)=f(0)=1$.

c. f is continuous on $(-\infty,0]$ and on $(0,\infty)$.

2.6.41 f is continuous on $(-\infty,-\sqrt{8}]$ and on $[\sqrt{8},\infty)$.

2.6.42 g is continuous on $(-\infty,-1]$ and on $[1,\infty)$.

2.6.43 Because f is the composition of two functions which are continuous everywhere, it is continuous everywhere.

2.6.44 f is continuous on $(-\infty,-1]$ and on $[1,\infty)$.

2.6.45 Because f is the composition of two functions which are continuous everywhere, it is continuous everywhere.

2.6.46 f is continuous on $[1,\infty)$.

2.6.47 $\lim_{x\to 2}\sqrt{\frac{4x+10}{2x-2}}=\sqrt{\frac{18}{2}}=3$.

2.6.48 $\lim_{x\to -1}\left(x^2-4+\sqrt[3]{x^2-9}\right)=(-1)^2-4+\sqrt[3]{(-1)^2-9}=-3+\sqrt[3]{-8}=-3+-2=-5$.

2.6.49 $\lim_{x\to 3}\sqrt{x^2+7}=\sqrt{9+7}=4$.

2.6.50 $\lim_{t\to 2} \dfrac{t^2+5}{1+\sqrt{t^2+5}} = \dfrac{9}{1+\sqrt{9}} = \dfrac{9}{4}$.

2.6.51 $f(x) = \csc x$ isn't defined at $x = k\pi$ where k is an integer, so it isn't continuous at those points. So it is continuous on intervals of the form $(k\pi, (k+1)\pi)$ where k is an integer. $\lim_{x\to\pi/4} \csc x = \sqrt{2}$. $\lim_{x\to 2\pi^-} \csc x = -\infty$.

2.6.52 $f(x) = \sqrt{\sin x}$ is continuous wherever it is defined. It is defined when $\sin x \ge 0$, i.e. for $2n\pi \le x \le (2n+1)\pi$ where n is an integer. Then $\lim_{x\to\pi/2} \sqrt{\sin x} = \sqrt{\sin \pi/2} = 1$, and $\lim_{x\to 0} \sqrt{\sin x} = \sqrt{\sin 0} = 0$.

2.6.53 f isn't defined for any number of the form $\pi/2 + k\pi$ where k is an integer, so it isn't continuous there. It is continuous on intervals of the form $(\pi/2 + k\pi, \pi/2 + (k+1)\pi)$, where k is an integer.

$\lim_{x\to\pi/2^-} f(x) = \infty$. $\lim_{x\to 4\pi/3} f(x) = \dfrac{1-\sqrt{3}/2}{-1/2} = \sqrt{3} - 2$.

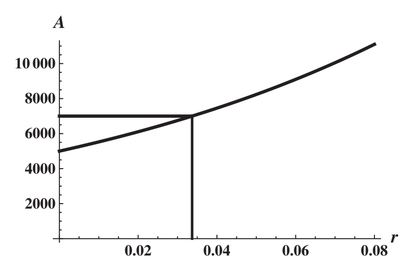
2.6.54 $f(x) = \dfrac{1}{2\cos x - 1}$ is continuous where it is defined. It is defined when $2\cos x - 1 \ne 0$, i.e. for all real numbers x except for $x = \pm\pi/3 + 2n\pi$ where n is an integer. Then

$$\lim_{x\to\pi/6} \frac{1}{2\cos x - 1} = \frac{1}{2(\sqrt{3}/2) - 1} = \frac{1}{\sqrt{3} - 1}.$$

2.6.55

a. Because A is a continuous function of r on $[0, .08]$, and because $A(0) = 5000$ and $A(.08) \approx 11098.2$, (and 7000 is an intermediate value between these two numbers) the Intermediate Value Theorem guarantees a value of r between 0 and .08 where $A(r) = 7000$.

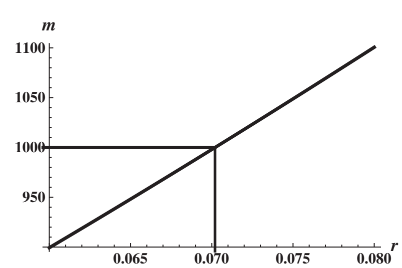
b. Solving $5000(1 + (r/12))^{120} = 7000$ for r, we see that $(1 + (r/12))^{120} = 7/5$, so $1 + r/12 = \sqrt[120]{7/5}$, so $r = 12(\sqrt[120]{7/5} - 1) \approx 0.034$.



2.6.56

a. Because m is a continuous function of r on $[.06, .08]$, and because $m(.06) \approx 899.33$ and $m(.08) \approx 1100.65$, (and 1000 is an intermediate value between these two numbers) the Intermediate Value Theorem guarantees a value of r between .06 and .08 where $m(r) = 1000$.

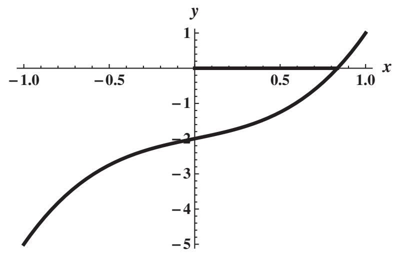
b. Using a computer algebra system, we see that the required interest rate is about 0.0702.



2.6.57

a. Note that $f(x) = 2x^3 + x - 2$ is continuous everywhere, so in particular it is continuous on $[-1, 1]$. Note that $f(-1) = -5 < 0$ and $f(1) = 1 > 0$. Because 0 is an intermediate value between $f(-1)$ and $f(1)$, the Intermediate Value Theorem guarantees a number c between -1 and 1 where $f(c) = 0$.

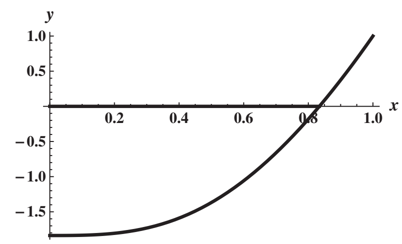
b. Using a graphing calculator and a computer algebra system, we see that the root of f is about 0.835.

c. 

2.6.58

a. Note that $f(x) = \sqrt{x^4 + 25x^3 + 10} - 5$ is continuous on its domain, so in particular it is continuous on $[0, 1]$. Note that $f(0) = \sqrt{10} - 5 < 0$ and $f(1) = 6 - 5 = 1 > 0$. Because 0 is an intermediate value between $f(0)$ and $f(1)$, the Intermediate Value Theorem guarantees a number c between 0 and 1 where $f(c) = 0$.

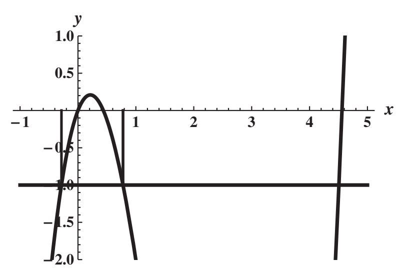
b. Using a graphing calculator and a computer algebra system, we see the root of $f(x)$ is at about .834.

c. 

2.6.59

a. Note that $f(x) = x^3 - 5x^2 + 2x$ is continuous everywhere, so in particular it is continuous on $[-1, 5]$. Note that $f(-1) = -8 < -1$ and $f(5) = 10 > -1$. Because -1 is an intermediate value between $f(-1)$ and $f(5)$, the Intermediate Value Theorem guarantees a number c between -1 and 5 where $f(c) = -1$.

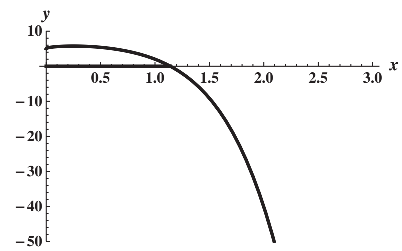
b. Using a graphing calculator and a computer algebra system, we see that there are actually three different values of c between -1 and 5 for which $f(c) = -1$. They are $c \approx -0.285$, $c \approx 0.778$, and $c \approx 4.507$.

c. 

2.6.60

a. Note that $f(x) = -x^5 - 4x^2 + 2\sqrt{x} + 5$ is continuous on its domain, so in particular it is continuous on $[0, 3]$. Note that $f(0) = 5 > 0$ and $f(3) \approx -270.5 < 0$. Because 0 is an intermediate value between $f(0)$ and $f(3)$, the Intermediate Value Theorem guarantees a number c between 0 and 3 where $f(c) = 0$.

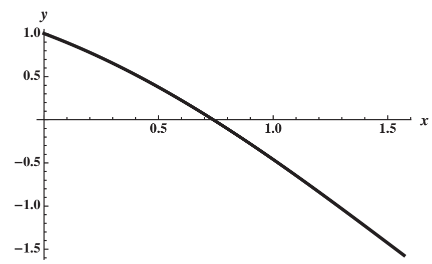
b. Using a graphing calculator and a computer algebra system, we see that the value of c guaranteed by the theorem is about 1.141.

c. 

2.6.61

a. Note that $f(x) = \cos x - x$ is continuous on its domain, so in particular it is continuous on $[0, \pi/2]$. Note that $f(0) = 1 > 0$ and $f(\pi/2) = -\pi/2 < 0$. Because 0 is an intermediate value between $f(0)$ and $f(\pi/2)$, the Intermediate Value Theorem guarantees a number c between 0 and $\pi/2$ where $f(c) = 0$.

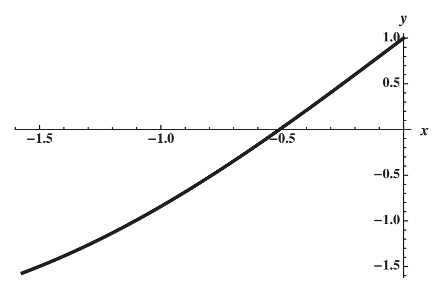
b. Using a graphing calculator and a computer algebra system, we see that the value of c guaranteed by the theorem is about 0.739085.

c. 

2.6.62

a. Note that $f(x) = 1 + x + \sin x$ is continuous on its domain, so in particular it is continuous on $[-\pi/2, 0]$. Note that $f(-\pi/2) = -\pi/2 < 0$ and $f(0) = 1 > 0$. Because 0 is an intermediate value between $f(-\pi/2)$ and $f(0)$, the Intermediate Value Theorem guarantees a number c between $-\pi/2$ and 0 where $f(c) = 0$.

b. Using a graphing calculator and a computer algebra system, we see that the value of c guaranteed by the theorem is about -0.510973.

c. 

2.6.63

a. True. If f is right continuous at a, then $f(a)$ exists and the limit from the right at a exists and is equal to $f(a)$. Because it is left continuous, the limit from the left exists — so we now know that the limit as $x \to a$ of $f(x)$ exists, because the two one-sided limits are both equal to $f(a)$.

b. True. If $\lim_{x\to a} f(x) = f(a)$, then $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to a^-} f(x) = f(a)$.

c. False. The statement would be true if f were continuous. However, if f isn't continuous, then the statement doesn't hold. For example, suppose that $f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1; \\ 1 & \text{if } 1 \le x \le 2, \end{cases}$ Note that $f(0) = 0$ and $f(2) = 1$, but there is no number c between 0 and 2 where $f(c) = 1/2$.

d. False. Consider $f(x) = x^2$ and $a = -1$ and $b = 1$. Then f is continuous on $[a, b]$, but $\frac{f(1)+f(-1)}{2} = 1$, and there is no c on (a, b) with $f(c) = 1$.

2.6.64 Let $f(x) = |x|$.

For values of a other than 0, it is clear that $\lim_{x\to a} |x| = |a|$ because f is defined to be either the polynomial x (for values greater than 0) or the polynomial $-x$ (for values less than 0.) For the value of $a = 0$, we have $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0 = f(0)$. Also, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(-x) = -0 = 0$. Thus $\lim_{x\to 0} f(x) = f(0)$, so f is continuous at 0.

2.6.65 Because $f(x) = x^3 + 3x - 18$ is a polynomial, it is continuous on $(-\infty, \infty)$, and because the absolute value function is continuous everywhere, $|f(x)|$ is continuous everywhere.

2.6.66 Let $f(x) = \frac{x+4}{x^2-4}$. Then f is continuous on $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$. So $g(x) = |f(x)|$ is also continuous on this set.

2.6.67 Let $f(x) = \frac{1}{\sqrt{x}-4}$. Then f is continuous on $[0, 16) \cup (16, \infty)$. So $h(x) = |f(x)|$ is continuous on this set as well.

2.6.68 Because $x^2 + 2x + 5$ is a polynomial, it is continuous everywhere, as is $|x^2 + 2x + 5|$. So $h(x) = |x^2 + 2x + 5| + \sqrt{x}$ is continuous on its domain, namely $[0, \infty)$.

2.6.69 $\displaystyle\lim_{x\to\pi} \frac{\cos^2 x + 3\cos x + 2}{\cos x + 1} = \lim_{x\to\pi} \frac{(\cos x + 1)(\cos x + 2)}{\cos x + 1} = \lim_{x\to\pi}(\cos x + 2) = 1.$

2.6.70 $\displaystyle\lim_{x\to 3\pi/2} \frac{\sin^2 x + 6\sin x + 5}{\sin^2 x - 1} = \lim_{x\to 3\pi/2} \frac{(\sin x + 5)(\sin x + 1)}{(\sin x - 1)(\sin x + 1)} = \lim_{x\to 3\pi/2} \frac{\sin x + 5}{\sin x - 1} = \frac{4}{-2} = -2.$

2.6.71 $\displaystyle\lim_{x\to\pi/2} \frac{\sin x - 1}{\sqrt{\sin x} - 1} = \lim_{x\to\pi/2}(\sqrt{\sin x} + 1) = 2.$

2.6.72 $\displaystyle\lim_{\theta\to 0} \frac{\frac{1}{2+\sin\theta} - \frac{1}{2}}{\sin\theta} \cdot \frac{(2)(2+\sin\theta)}{(2)(2+\sin\theta)} = \lim_{\theta\to 0} \frac{2 - (2+\sin\theta)}{(\sin\theta)(2)(2+\sin\theta)} = \lim_{\theta\to 0} -\frac{1}{2(2+\sin\theta)} = -\frac{1}{4}.$

2.6.73 $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{\sin^2 x} = \lim_{x\to 0} \frac{\cos x - 1}{1 - \cos^2 x} = \lim_{x\to 0} \frac{\cos x - 1}{(1 - \cos x)(1 + \cos x)} = \lim_{x\to 0} -\frac{1}{1 + \cos x} = -\frac{1}{2}.$

2.6.74 $\displaystyle\lim_{x\to 0^+} \frac{1 - \cos^2 x}{\sin x} = \lim_{x\to 0^+} \frac{\sin^2 x}{\sin x} = \lim_{x\to 0^+} \sin x = 0.$

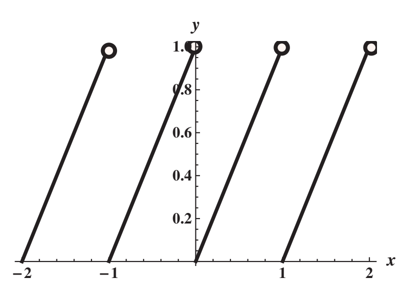
2.6.75 Because $-1 \le \sin t \le 1$ for all t, we have $\frac{-1}{t^2} \le \frac{\sin t}{t^2} \le \frac{1}{t^2}$, so by the Squeeze Theorem, $\displaystyle\lim_{t\to\infty} \frac{\sin t}{t^2} = 0$.

2.6.76 Recall that $\cos 0 = 1$, so as $x \to 0^+$, the numerator of $\frac{\cos x}{x}$ is approaching 1, while the denominator is approaching 0 (but is positive,) so the ratio is increasing without bound. Thus $\displaystyle\lim_{x\to 0^+} \frac{\cos x}{x} = \infty$.

2.6.77

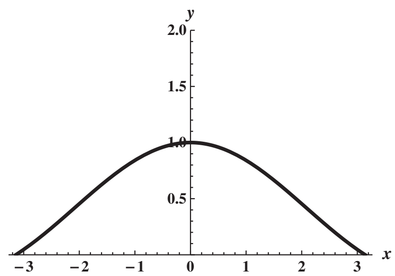
The graph shown isn't drawn correctly at the integers. At an integer a, the value of the function is 0, whereas the graph shown appears to take on all the values from 0 to 1.

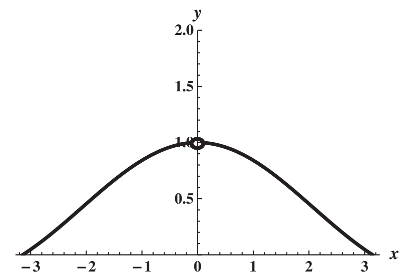
Note that in the correct graph, $\lim_{x\to a^-} f(x) = 1$ and $\lim_{x\to a^+} f(x) = 0$ for every integer a.



2.6.78

The graph as drawn on most graphing calculators appears to be continuous at $x = 0$, but it isn't, of course (because the function isn't defined at $x = 0$). A better drawing would show the "hole" in the graph at $(0, 1)$.

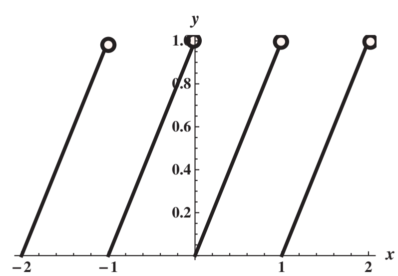
a. 

b. 

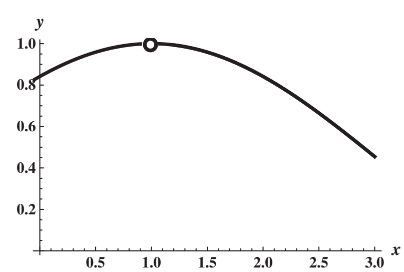
c. It appears that $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

2.6.79 With slight modifications, we can use the examples from the previous two problems.

a. The function $y = x - \lfloor x \rfloor$ is defined at $x = 1$ but isn't continuous there.



b. The function $y = \frac{\sin(x-1)}{x-1}$ has a limit at $x = 1$, but isn't defined there, so isn't continuous there.



2.6.80 In order for this function to be continuous at $x = -1$, we require $\lim_{x \to -1} f(x) = f(-1) = a$. So the value of a must be equal to the value of $\lim_{x \to -1} \frac{x^2 + 3x + 2}{x + 1} = \lim_{x \to -1} \frac{(x+2)(x+1)}{x+1} = \lim_{x \to -1} (x + 2) = 1$. Thus we must have $a = 1$.

2.6.81

a. In order for g to be continuous from the left at $x = 1$, we must have $\lim_{x \to 1^-} g(x) = g(1) = a$. We have $\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} (x^2 + x) = 2$. So we must have $a = 2$.

b. In order for g to be continuous from the right at $x = 1$, we must have $\lim_{x \to 1^+} g(x) = g(1) = a$. We have $\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} (3x + 5) = 8$. So we must have $a = 8$.

c. Because the limit from the left and the limit from the right at $x = 1$ don't agree, there is no value of a which will make the function continuous at $x = 1$.

2.6.82 Let $f(x) = x^3 + 10x^2 - 100x + 50$. Note that $f(-20) < 0$, $f(-5) > 0$, $f(5) < 0$, and $f(10) > 0$. Because the given polynomial is continuous everywhere, the Intermediate Value Theorem guarantees us a root on $(-20, -5)$, at least one on $(-5, 5)$, and at least one on $(5, 10)$. Because there can be at most 3 roots and there are at least 3 roots, there must be exactly 3 roots. The roots are $x_1 \approx -16.32$, $x_2 \approx 0.53$ and $x_3 \approx 5.79$.

2.6.83 Let $f(x) = 70x^3 - 87x^2 + 32x - 3$. Note that $f(0) < 0$, $f(.2) > 0$, $f(.55) < 0$, and $f(1) > 0$. Because the given polynomial is continuous everywhere, the Intermediate Value Theorem guarantees us a root on $(0, .2)$, at least one on $(.2, .55)$, and at least one on $(.55, 1)$. Because there can be at most 3 roots and there are at least 3 roots, there must be exactly 3 roots. The roots are $x_1 = 1/7$, $x_2 = 1/2$ and $x_3 = 3/5$.

2.6.84 The function is continuous on $(0, 15]$, on $(15, 30]$, on $(30, 45]$, and on $(45, 60]$.

2.6.85

a. Note that $A(.01) \approx 2615.55$ and $A(.1) \approx 3984.36$. By the Intermediate Value Theorem, there must be a number r_0 between .01 and .1 so that $A(r_0) = 3500$.

b. The desired value is $r_0 \approx 0.0728$ or 7.28%.

2.6.86

a. We have $f(0) = 0$, $f(2) = 3$, $g(0) = 3$ and $g(2) = 0$.

b. $h(t) = f(t) - g(t)$, $h(0) = -3$ and $h(2) = 3$.

c. By the Intermediate Value Theorem, because h is a continuous function and 0 is an intermediate value between -3 and 3, there must be a time c between 0 and 2 where $h(c) = 0$. At this point $f(c) = g(c)$, and at that time, the distance from the car is the same on both days, so the hiker is passing over the exact same point at that time.

2.6.87 We can argue essentially like the previous problem, or we can imagine an identical twin to the original monk, who takes an identical version of the original monk's journey up the winding path while the monk is taking the return journey down. Because they must pass somewhere on the path, that point is the one we are looking for.

2.6.88

a. Because $|-1| = 1$, $|g(x)| = 1$, for all x.

b. The function g isn't continuous at $x = 0$, because $\lim_{x\to 0^+} g(x) = 1 \neq -1 = \lim_{x\to 0^-} g(x)$.

c. This constant function is continuous everywhere, in particular at $x = 0$.

d. This example shows that in general, the continuity of $|g|$ does not imply the continuity of g.

2.6.89 The discontinuity is not removable, because $\lim_{x\to a} f(x)$ does not exist. The discontinuity pictured is a jump discontinuity.

2.6.90 The discontinuity is not removable, because $\lim_{x\to a} f(x)$ does not exist. The discontinuity pictured is an infinite discontinuity.

2.6.91 Note that $\lim_{x\to 2} \frac{x^2 - 7x + 10}{x - 2} = \lim_{x\to 2} \frac{(x-2)(x-5)}{x-2} = \lim_{x\to 2}(x - 5) = -3$. Because this limit exists, the discontinuity is removable.

2.6.92 Note that $\lim_{x\to 1}\dfrac{x^2-1}{1-x} = \lim_{x\to 1}\dfrac{(x-1)(x+1)}{1-x} = \lim_{x\to 1}[-(x+1)] = -2$. Because this limit exists, the discontinuity is removable.

2.6.93

a. Note that $-1 \le \sin(1/x) \le 1$ for all $x \ne 0$, so $-x \le x\sin(1/x) \le x$ (for $x > 0$. For $x < 0$ we would have $x \le x\sin(1/x) \le -x$.) Because both $x \to 0$ and $-x \to 0$ as $x \to 0$, the Squeeze Theorem tells us that $\lim_{x\to 0} x\sin(1/x) = 0$ as well. Because this limit exists, the discontinuity is removable.

b. Note that as $x \to 0^+$, $1/x \to \infty$, and thus $\lim_{x\to 0^+} \sin(1/x)$ does not exist. So the discontinuity is not removable.

2.6.94 This is a jump discontinuity, because $\lim_{x\to 2^+} f(x) = 1$ and $\lim_{x\to 2^-} f(x) = -1$.

2.6.95 Note that $h(x) = \frac{x^3-4x^2+4x}{x(x-1)} = \frac{x(x-2)^2}{x(x-1)}$. Thus $\lim_{x\to 0} h(x) = -4$, and the discontinuity at $x = 0$ is removable. However, $\lim_{x\to 1} h(x)$ does not exist, and the discontinuity at $x = 1$ is not removable (it is infinite.)

2.6.96 Because g is continuous at a, as $x \to a$, $g(x) \to g(a)$. Because f is continuous at $g(a)$, as $z \to g(a)$, $f(z) \to f(g(a))$. Let $z = g(x)$, and suppose $x \to a$. Then $g(x) = z \to g(a)$, so $f(z) = f(g(x)) \to f(g(a))$, as desired.

2.6.97

a. Consider $g(x) = x + 1$ and $f(x) = \frac{|x-1|}{x-1}$. Note that both g and f are continuous at $x = 0$. However $f(g(x)) = f(x+1) = \frac{|x|}{x}$ is not continuous at 0.

b. The previous theorem says that the composition of f and g is continuous at a if g is continuous at a and f is continuous at $g(a)$. It does not say that if g and f are both continuous at a that the composition is continuous at a.

2.6.98 The Intermediate Value Theorem requires that our function be continuous on the given interval. In this example, the function f is not continuous on $[-2, 2]$ because it isn't continuous at 0.

2.6.99

a. Using the hint, we have

$$\sin x = \sin(a + (x-a)) = \sin a\cos(x-a) + \sin(x-a)\cos a.$$

Note that as $x \to a$, we have that $\cos(x-a) \to 1$ and $\sin(x-a) \to 0$.

So,

$$\lim_{x\to a}\sin x = \lim_{x\to a}\sin(a+(x-a)) = \lim_{x\to a}(\sin a\cos(x-a) + \sin(x-a)\cos a) = (\sin a)\cdot 1 + 0\cdot\cos a = \sin a.$$

b. Using the hint, we have

$$\cos x = \cos(a + (x-a)) = \cos a\cos(x-a) - \sin a\sin(x-a).$$

So,

$$\lim_{x\to a}\cos x = \lim_{x\to a}\cos(a+(x-a)) = \lim_{x\to a}((\cos a)\cos(x-a) - (\sin a)\sin(x-a)) = (\cos a)\cdot 1 - (\sin a)\cdot 0 = \cos a.$$

2.7 Precise Definitions of Limits

2.7.1 Note that all the numbers in the interval $(1, 3)$ are within 1 unit of the number 2. So $|x - 2| < 1$ is true for all numbers in that interval. In fact, $\{x: 0 < |x - 2| < 1\}$ is exactly the set $(1, 3)$ with $x \neq 2$.

2.7.2 Note that all the numbers in the interval $(2, 6)$ are within 2 units of the number 4. So $|f(x) - 4| < \epsilon$ for $\epsilon = 2$.

2.7.3

$(3, 8)$ has center 5.5, so it is not symmetric about the number 5.

$(1, 9)$ and $(4, 6)$ and $(4.5, 5.5)$ are symmetric about the number 5.

2.7.4 No. At $x = a$, we would have $|x - a| = 0$, not $|x - a| > 0$, so a is not included in the given set.

2.7.5 $\lim_{x\to a} f(x) = L$ if for any arbitrarily small positive number ϵ, there exists a number δ, so that $f(x)$ is within ϵ units of L for any number x within δ units of a (but not including a itself).

2.7.6 The set of all x for which $|f(x) - L| < \epsilon$ is the set of numbers so that the value of the function f at those numbers is within ϵ units of L.

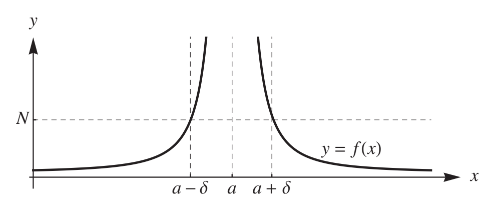
2.7.7 We are given that $|f(x) - 5| < .1$ for values of x in the interval $(0, 5)$, so we need to ensure that the set of x values we are allowing fall in this interval.

Note that the number 0 is two units away from the number 2 and the number 5 is three units away from the number 2. In order to be sure that we are talking about numbers in the interval $(0, 5)$ when we write $|x - 2| < \delta$, we would need to have $\delta = 2$ (or a number less than 2). In fact, the set of numbers for which $|x - 2| < 2$ is the interval $(0, 4)$ which is a subset of $(0, 5)$.

If we were to allow δ to be any number greater than 2, then the set of all x so that $|x - 2| < \delta$ would include numbers less than 0, and those numbers aren't on the interval $(0, 5)$.

2.7.8

$\lim_{x\to a} f(x) = \infty$, if for any $N > 0$, there exists $\delta > 0$ so that if $0 < |x - a| < \delta$ then $f(x) > N$.



2.7.9

a. In order for f to be within 2 units of 5, it appears that we need x to be within 1 unit of 2. So $\delta = 1$.

b. In order for f to be within 1 unit of 5, it appears that we would need x to be within 1/2 unit of 2. So $\delta = .5$.

2.7.10

a. In order for f to be within 1 unit of 4, it appears that we would need x to be within 1 unit of 2. So $\delta = 1$.

b. In order for f to be within 1/2 unit of 4, it appears that we would need x to be within 1/2 unit of 2. So $\delta = 1/2$.

2.7.11

a. In order for f to be within 3 units of 6, it appears that we would need x to be within 2 units of 3. So $\delta = 2$.

b. In order for f to be within 1 unit of 6, it appears that we would need x to be within 1/2 unit of 3. So $\delta = 1/2$.

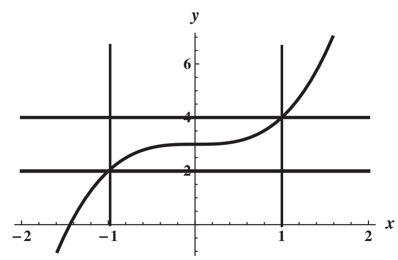
2.7.12

a. In order for f to be within 1 unit of 5, it appears that we would need x to be within 3 units of 4. So $\delta = 3$.

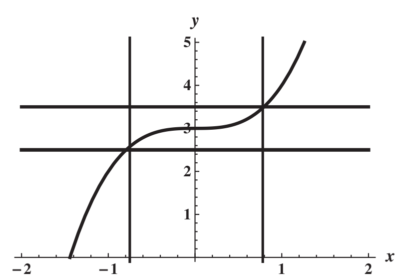
b. In order for f to be within 1/2 unit of 5, it appears that we would need x to be within 2 units of 4. So $\delta = 2$.

2.7.13

a. If $\epsilon = 1$, we need $|x^3 + 3 - 3| < 1$. So we need $|x| < \sqrt[3]{1} = 1$ in order for this to happen. Thus $\delta = 1$ will suffice, or any δ so that $0 < \delta \leq 1$.

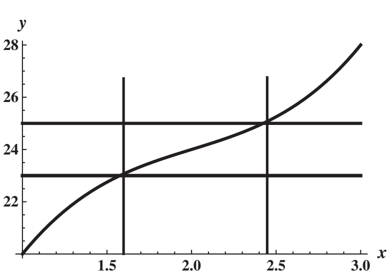


b. If $\epsilon = .5$, we need $|x^3 + 3 - 3| < .5$. So we need $|x| < \sqrt[3]{.5}$ in order for this to happen. Thus $\delta = \sqrt[3]{.5} \approx .079$ will suffice, or any δ so that $0 < \delta \leq 0.79$.

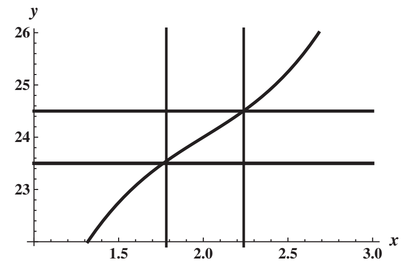


2.7.14

a. By looking at the graph, it appears that for $\epsilon = 1$, we would need δ to be about 0.4 or less. So δ should satisfy $0 < \delta \leq 0.4$.



b. By looking at the graph, it appears that for $\epsilon = 0.5$, we would need δ to be about 0.2 or less. So δ should satisfy $0 < \delta \le 0.2$.



2.7.15

a. For $\epsilon = 1$, the required value of δ would also be 1. A larger value of δ would work to the right of 2, but this is the largest one that would work to the left of 2. So we require $0 < \delta \le 1$.

b. For $\epsilon = 1/2$, the required value of δ would also be $1/2$, so we require $0 < \delta \le 1/2$.

c. It appears that for a given value of ϵ with $0 < \epsilon < 2$, it would be wise to let δ satisfy $0 < \delta \le \epsilon$.

2.7.16

a. For $\epsilon = 2$, the required value of δ would be 1 (or smaller). This is the largest value of δ that works on either side, so we are requiring $0 < \delta \le 1$.

b. For $\epsilon = 1$, the required value of δ would be $1/2$ (or smaller). This is the largest value of δ that works on the right of 4, so we are requiring $0 < \delta \le 1/2$.

c. It appears that for a given value of ϵ with $0 < \epsilon < 2$, the corresponding value of δ should satisfy $0 < \delta \le \epsilon/2$.

2.7.17

a. For $\epsilon = 2$, it appears that a value of $\delta = 1$ (or smaller) would work.

b. For $\epsilon = 1$, it appears that a value of $\delta = 1/2$ (or smaller) would work.

c. For an arbitrary ϵ, a value of $\delta = \epsilon/2$ or smaller appears to suffice.

2.7.18

a. For $\epsilon = 1/2$, it appears that a value of $\delta = 1$ (or smaller) would work.

b. For $\epsilon = 1/4$, it appears that a value of $\delta = 1/2$ (or smaller) would work.

c. For an arbitrary ϵ, a value of 2ϵ or smaller appears to suffice.

2.7.19 For any $\epsilon > 0$, let $\delta = \epsilon/8$. Then if $0 < |x-1| < \delta$, we would have $|x-1| < \epsilon/8$. Then $|8x-8| < \epsilon$, so $|(8x+5)-13| < \epsilon$. This last inequality has the form $|f(x)-L| < \epsilon$, which is what we were attempting to show. Thus, $\lim_{x\to 1}(8x+5) = 13$.

2.7.20 For any $\epsilon > 0$, let $\delta = \epsilon/2$. Then if $0 < |x-3| < \delta$, we would have $|x-3| < \epsilon/2$. Then $|2x-6| < \epsilon$, so $|-2x+6| < \epsilon$, so $|(-2x+8)-2| < \epsilon$. This last inequality has the form $|f(x)-L| < \epsilon$, which is what we were attempting to show. Thus, $\lim_{x\to 3}(-2x+8) = 2$.

2.7.21 First note that if $x \ne 4$, $f(x) = \frac{x^2-16}{x-4} = x+4$.

Now if $\epsilon > 0$ is given, let $\delta = \epsilon$. Now suppose $0 < |x-4| < \delta$. Then $x \ne 4$, so the function $f(x)$ can be described by $x+4$. Also, because $|x-4| < \delta$, we have $|x-4| < \epsilon$. Thus $|(x+4)-8| < \epsilon$. This last inequality has the form $|f(x)-L| < \epsilon$, which is what we were attempting to show. Thus, $\lim_{x\to 4}\dfrac{x^2-16}{x-4} = 8$.

2.7.22 First note that if $x \neq 3$, $f(x) = \frac{x^2-7x+12}{x-3} = \frac{(x-4)(x-3)}{x-3} = x - 4$.

Now if $\epsilon > 0$ is given, let $\delta = \epsilon$. Now suppose $0 < |x - 3| < \delta$. Then $x \neq 3$, so the function $f(x)$ can be described by $x - 4$. Also, because $|x - 3| < \delta$, we have $|x - 3| < \epsilon$. Thus $|(x - 4) - (-1)| < \epsilon$. This last inequality has the form $|f(x) - L| < \epsilon$, which is what we were attempting to show. Thus, $\lim_{x\to 3} f(x) = -1$.

2.7.23 Let $\epsilon > 0$ be given. Let $\delta = \sqrt{\epsilon}$. Then if $0 < |x - 0| < \delta$, we would have $|x| < \sqrt{\epsilon}$. But then $|x^2| < \epsilon$, which has the form $|f(x) - L| < \epsilon$. Thus, $\lim_{x\to 0} f(x) = 0$.

2.7.24 Let $\epsilon > 0$ be given. Let $\delta = \sqrt{\epsilon}$. Then if $0 < |x - 3| < \delta$, we would have $|x - 3| < \sqrt{\epsilon}$. But then $|(x - 3)^2| < \epsilon$, which has the form $|f(x) - L| < \epsilon$. Thus, $\lim_{x\to 3} f(x) = 0$.

2.7.25 Let $\epsilon > 0$ be given.

Because $\lim_{x\to a} f(x) = L$, we know that there exists a $\delta_1 > 0$ so that $|f(x)-L| < \epsilon/2$ when $0 < |x-a| < \delta_1$. Also, because $\lim_{x\to a} g(x) = M$, there exists a $\delta_2 > 0$ so that $|g(x) - M| < \epsilon/2$ when $0 < |x - a| < \delta_2$.

Now let $\delta = \min(\delta_1, \delta_2)$.

Then if $0 < |x - a| < \delta$, we would have $|f(x) - g(x) - (L - M)| = |(f(x) - L) + (M - g(x))| \leq |f(x) - L| + |M - g(x)| = |f(x) - L| + |g(x) - M| \leq \epsilon/2 + \epsilon/2 = \epsilon$. Note that the key inequality in this sentence follows from the triangle inequality.

2.7.26 First note that the theorem is trivially true if $c = 0$. So assume $c \neq 0$.

Let $\epsilon > 0$ be given. Because $\lim_{x\to a} f(x) = L$, there exists a $\delta > 0$ so that if $0 < |x - a| < \delta$, we have $|f(x) - L| < \epsilon/|c|$. But then $|c||f(x) - L| = |cf(x) - cL| < \epsilon$, as desired. Thus, $\lim_{x\to a} cf(x) = cL$.

2.7.27

a. Let $\epsilon > 0$ be given. It won't end up mattering what δ is, so let $\delta = 1$. Note that the statement $|f(x) - L| < \epsilon$ amounts to $|c - c| < \epsilon$, which is true for any positive number ϵ, without any restrictions on x. So $\lim_{x\to a} c = c$.

b. Let $\epsilon > 0$ be given. Let $\delta = \epsilon$. Note that the statement $|f(x) - L| < \epsilon$ has the form $|x - a| < \epsilon$, which follows whenever $0 < |x - a| < \delta$ (because $\delta = \epsilon$). Thus $\lim_{x\to a} x = a$.

2.7.28 First note that if $m = 0$, this follows from exercise 27a. So assume $m \neq 0$.

Let $\epsilon > 0$ be given. Let $\delta = \epsilon/|m|$. Now if $0 < |x-a| < \delta$, we would have $|x-a| < \epsilon/|m|$, so $|mx-ma| < \epsilon$. This can be written as $|(mx + b) - (ma + b)| < \epsilon$, which has the form $|f(x) - L| < \epsilon$. Thus, $\lim_{x\to a} f(x) = f(a)$, which implies that f is continuous at $x = a$ by the definition of continuity at a point. Because a is an arbitrary number, f must be continuous at all real numbers.

2.7.29 Let $N > 0$ be given. Let $\delta = 1/\sqrt{N}$. Then if $0 < |x - 4| < \delta$, we have $|x - 4| < 1/\sqrt{N}$. Taking the reciprocal of both sides, we have $\frac{1}{|x-4|} > \sqrt{N}$, and squaring both sides of this inequality yields $\frac{1}{(x-4)^2} > N$. Thus $\lim_{x\to 4} f(x) = \infty$.

2.7.30 Let $N > 0$ be given. Let $\delta = 1/\sqrt[4]{N}$. Then if $0 < |x - (-1)| < \delta$, we have $|x + 1| < 1/\sqrt[4]{N}$. Taking the reciprocal of both sides, we have $\frac{1}{|x+1|} > \sqrt[4]{N}$, and raising both sides to the 4th power yields $\frac{1}{(x+1)^4} > N$. Thus $\lim_{x\to -1} f(x) = \infty$.

2.7.31 Let $N > 1$ be given. Let $\delta = 1/\sqrt{N-1}$. Suppose that $0 < |x - 0| < \delta$. Then $|x| < 1/\sqrt{N-1}$, and taking the reciprocal of both sides, we see that $1/|x| > \sqrt{N-1}$. Then squaring both sides yields $\frac{1}{x^2} > N-1$, so $\frac{1}{x^2} + 1 > N$. Thus $\lim_{x\to 0} f(x) = \infty$.

2.7.32 Let $N > 0$ be given. Let $\delta = 1/\sqrt[4]{N+1}$. Then if $0 < |x - 0| < \delta$, we would have $|x| < 1/\sqrt[4]{N+1}$. Taking the reciprocal of both sides yields $\frac{1}{|x|} > \sqrt[4]{N+1}$, and then raising both sides to the 4th power gives $\frac{1}{x^4} > N + 1$, so $\frac{1}{x^4} - 1 > N$. Now because $-1 \leq \sin x \leq 1$, we can surmise that $\frac{1}{x^4} - \sin x > N$ as well, because $\frac{1}{x^4} - \sin x \geq \frac{1}{x^4} - 1$. Hence $\lim_{x\to 0} \left(\frac{1}{x^4} - \sin x\right) = \infty$.

2.7.33

a. False. In fact, if the statement is true for a specific value of δ_1, then it would be true for any value of $\delta < \delta_1$. This is because if $0 < |x-a| < \delta$, it would automatically follow that $0 < |x-a| < \delta_1$.

b. False. This statement is not equivalent to the definition – note that it says "for an arbitrary δ there exists an ϵ" rather than "for an arbitrary ϵ there exists a δ."

c. True. This is the definition of $\lim_{x\to a} f(x) = L$.

d. True. Both inequalities describe the set of x's which are within δ units of a.

2.7.34

a. We want it to be true that $|f(x)-2| < .25$. So we need $|x^2-2x+3-2| = |x^2-2x+1| = (x-1)^2 < .25$. Therefore we need $|x-1| < \sqrt{.25} = .5$. Thus we should let $\delta = .5$.

b. We want it to be true that $|f(x)-2| < \epsilon$. So we need $|x^2-2x+3-2| = |x^2-2x+1| = (x-1)^2 < \epsilon$. Therefore we need $|x-1| < \sqrt{\epsilon}$. Thus we should let $\delta = \sqrt{\epsilon}$.

2.7.35 Assume $|x-3| < 1$, as indicated in the hint. Then $2 < x < 4$, so $\frac{1}{4} < \frac{1}{x} < \frac{1}{2}$, and thus $\left|\frac{1}{x}\right| < \frac{1}{2}$.
Also note that the expression $\left|\frac{1}{x} - \frac{1}{3}\right|$ can be written as $\left|\frac{x-3}{3x}\right|$.
Now let $\epsilon > 0$ be given. Let $\delta = \min(6\epsilon, 1)$. Now assume that $0 < |x-3| < \delta$. Then

$$|f(x) - L| = \left|\frac{x-3}{3x}\right| < \left|\frac{x-3}{6}\right| < \frac{6\epsilon}{6} = \epsilon.$$

Thus we have established that $\left|\frac{1}{x} - \frac{1}{3}\right| < \epsilon$ whenever $0 < |x-3| < \delta$.

2.7.36 Note that for $x \neq 4$, the expression $\frac{x-4}{\sqrt{x}-2} = \frac{x-4}{\sqrt{x}-2} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \sqrt{x}+2$. Also note that if $|x-4| < 1$, then x is between 3 and 5, so $\sqrt{x} > 0$. Then it follows that $\sqrt{x}+2 > 2$, and therefore $\frac{1}{\sqrt{x}+2} < \frac{1}{2}$. We will use this fact below.

Let $\epsilon > 0$ be given. Let $\delta = \min(2\epsilon, 1)$. Suppose that $0 < |x-4| < \delta$, so $|x-4| < 2\epsilon$. We have

$$\begin{aligned} |f(x) - L| = |\sqrt{x}+2-4| = |\sqrt{x}-2| &= \left|\frac{x-4}{\sqrt{x}+2}\right| \\ &< \frac{|x-4|}{2} < \frac{2\epsilon}{2} = \epsilon. \end{aligned}$$

2.7.37 Assume $|x-(1/10)| < (1/20)$, as indicated in the hint. Then $1/20 < x < 3/20$, so $\frac{20}{3} < \frac{1}{x} < \frac{20}{1}$, and thus $\left|\frac{1}{x}\right| < 20$.
Also note that the expression $\left|\frac{1}{x} - 10\right|$ can be written as $\left|\frac{10x-1}{x}\right|$.
Let $\epsilon > 0$ be given. Let $\delta = \min(\epsilon/200, 1/20)$. Now assume that $0 < |x-(1/10)| < \delta$. Then

$$\begin{aligned} |f(x) - L| = \left|\frac{10x-1}{x}\right| &< |(10x-1)\cdot 20| \\ &\leq |x-(1/10)| \cdot 200 < \frac{\epsilon}{200} \cdot 200 = \epsilon. \end{aligned}$$

Thus we have established that $\left|\frac{1}{x} - 10\right| < \epsilon$ whenever $0 < |x-(1/10)| < \delta$.

2.7.38 Note that if $|x-5| < 1$, then $4 < x < 6$, so that $9 < x+5 < 11$, so $|x+5| < 11$. Note also that $16 < x^2 < 36$, so $\frac{1}{x^2} < \frac{1}{16}$.
Let $\epsilon > 0$ be given. Let $\delta = \min(1, \frac{400}{11}\epsilon)$. Assume that $0 < |x-5| < \delta$. Then

$$\begin{aligned} |f(x) - L| = \left|\frac{1}{x^2} - \frac{1}{25}\right| &= \frac{|x+5||x-5|}{25x^2} \\ &< \frac{11|x-5|}{25x^2} < \frac{11}{25\cdot 16}|x-5| < \frac{11}{400}\frac{400\epsilon}{11} = \epsilon. \end{aligned}$$

2.7.39 Because we are approaching a from the right, we are only considering values of x which are close to, but a little larger than a. The numbers x to the right of a which are within δ units of a satisfy $0 < x - a < \delta$.

2.7.40 Because we are approaching a from the left, we are only considering values of x which are close to, but a little smaller than a. The numbers x to the left of a which are within δ units of a satisfy $0 < a - x < \delta$.

2.7.41

a. Let $\epsilon > 0$ be given. let $\delta = \epsilon/2$. Suppose that $0 < x < \delta$. Then $0 < x < \epsilon/2$ and

$$\begin{aligned} |f(x) - L| &= |2x - 4 - (-4)| = |2x| = 2|x| \\ &= 2x < \epsilon. \end{aligned}$$

b. Let $\epsilon > 0$ be given. let $\delta = \epsilon/3$. Suppose that $0 < 0 - x < \delta$. Then $-\delta < x < 0$ and $-\epsilon/3 < x < 0$, so $\epsilon > -3x$. We have

$$\begin{aligned} |f(x) - L| &= |3x - 4 - (-4)| = |3x| = 3|x| \\ &= -3x < \epsilon. \end{aligned}$$

c. Let $\epsilon > 0$ be given. Let $\delta = \epsilon/3$. Because $\epsilon/3 < \epsilon/2$, we can argue that $|f(x) - L| < \epsilon$ whenever $0 < |x| < \delta$ exactly as in the previous two parts of this problem.

2.7.42

a. This statement holds for $\delta = 2$ (or any number less than 2).

b. This statement holds for $\delta = 2$ (or any number less than 2).

c. This statement holds for $\delta = 1$ (or any number less than 1).

d. This statement holds for $\delta = .5$ (or any number less than 0.5).

2.7.43 Let $\epsilon > 0$ be given, and let $\delta = \epsilon^2$. Suppose that $0 < x < \delta$, which means that $x < \epsilon^2$, so that $\sqrt{x} < \epsilon$. Then we have

$$|f(x) - L| = |\sqrt{x} - 0| = \sqrt{x} < \epsilon.$$

as desired.

2.7.44

a. Suppose that $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$. Let $\epsilon > 0$ be given. There exists a number δ_1 so that $|f(x) - L| < \epsilon$ whenever $0 < x - a < \delta_1$, and there exists a number δ_2 so that $|f(x) - L| < \epsilon$ whenever $0 < a - x < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$. It immediately follows that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$, as desired.

b. Suppose $\lim_{x\to a} f(x) = L$, and let $\epsilon > 0$ be given. We know that a δ exists so that $|f(x)-L| < \epsilon$ whenever $0 < |x - a| < \delta$. In particular, it must be the case that $|f(x) - L| < \epsilon$ whenever $0 < x - a < \delta$ and also that $|f(x) - L| < \epsilon$ whenever $0 < a - x < \delta$. Thus $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$.

2.7.45

a. We say that $\lim_{x\to a^+} f(x) = \infty$ if for each positive number N, there exists $\delta > 0$ such that

$$f(x) > N \quad \text{whenever} \quad a < x < a + \delta.$$

b. We say that $\lim_{x\to a^-} f(x) = -\infty$ if for each negative number N, there exists $\delta > 0$ such that

$$f(x) < N \quad \text{whenever} \quad a - \delta < x < a.$$

c. We say that $\lim_{x\to a^-} f(x) = \infty$ if for each positive number N, there exists $\delta > 0$ such that

$$f(x) > N \quad \text{whenever} \quad a - \delta < x < a.$$

2.7.46 Let $N < 0$ be given. Let $\delta = -1/N$, and suppose that $1 < x < 1 + \delta$. Then $1 < x < \frac{N-1}{N}$, so $\frac{1-N}{N} < -x < -1$, and therefore $1 + \frac{1-N}{N} < 1 - x < 0$, which can be written as $\frac{1}{N} < 1 - x < 0$. Taking reciprocals yields the inequality $N > \frac{1}{1-x}$, as desired.

2.7.47 Let $N > 0$ be given. Let $\delta = 1/N$, and suppose that $1 - \delta < x < 1$. Then $\frac{N-1}{N} < x < 1$, so $\frac{1-N}{N} > -x > -1$, and therefore $1 + \frac{1-N}{N} > 1 - x > 0$, which can be written as $\frac{1}{N} > 1 - x > 0$. Taking reciprocals yields the inequality $N < \frac{1}{1-x}$, as desired.

2.7.48 Let $M < 0$ be given. Let $\delta = \sqrt{-2/M}$. Suppose that $0 < |x - 1| < \delta$. Then $(x - 1)^2 < -2/M$, so $\frac{1}{(x-1)^2} > \frac{M}{-2}$, and $\frac{-2}{(x-1)^2} < M$, as desired.

2.7.49 Let $M < 0$ be given. Let $\delta = \sqrt[4]{-10/M}$. Suppose that $0 < |x + 2| < \delta$. Then $(x + 2)^4 < -10/M$, so $\frac{1}{(x+2)^4} > \frac{M}{-10}$, and $\frac{-10}{(x+2)^4} < M$, as desired.

2.7.50 Let $\epsilon > 0$ be given. Let $N = \frac{10}{\epsilon}$. Suppose that $x > N$. Then $x > \frac{10}{\epsilon}$ so $0 < \frac{10}{x} < \epsilon$. Thus, $|\frac{10}{x} - 0| < \epsilon$, as desired.

2.7.51 Let $\epsilon > 0$ be given. Let $N = 1/\epsilon$. Suppose that $x > N$. Then $\frac{1}{x} < \epsilon$, and so $|f(x)-L| = |2+\frac{1}{x}-2| < \epsilon$.

2.7.52 Let $M > 0$ be given. Let $N = 100M$. Suppose that $x > N$. Then $x > 100M$, so $\frac{x}{100} > M$, as desired.

2.7.53 Let $M > 0$ be given. Let $N = M - 1$. Suppose that $x > N$. Then $x > M - 1$, so $x + 1 > M$, and thus $\frac{x^2+x}{x} > M$, as desired.

2.7.54 Let $\epsilon > 0$ be given. Because $\lim_{x\to a} f(x) = L$, there exists a number δ_1 so that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta_1$. And because $\lim_{x\to a} h(x) = L$, there exists a number δ_2 so that $|h(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$, and suppose that $0 < |x - a| < \delta$. Because $f(x) \le g(x) \le h(x)$ for x near a, we also have that $f(x) - L \le g(x) - L \le h(x) - L$. Now whenever x is within δ units of a (but $x \ne a$), we also note that $-\epsilon < f(x) - L \le g(x) - L \le h(x) - L < \epsilon$. Therefore $|g(x) - L| < \epsilon$, as desired.

2.7.55 Let $\epsilon > 0$ be given. Let $N = \lfloor (1/\epsilon) \rfloor + 1$. By assumption, there exists an integer $M > 0$ so that $|f(x) - L| < 1/N$ whenever $|x - a| < 1/M$. Let $\delta = 1/M$.

Now assume $0 < |x - a| < \delta$. Then $|x - a| < 1/M$, and thus $|f(x) - L| < 1/N$. But then

$$|f(x) - L| < \frac{1}{\lfloor (1/\epsilon) \rfloor + 1} < \epsilon,$$

as desired.

2.7.56 Suppose that $\epsilon = 1$. Then no matter what δ is, there are numbers in the set $0 < |x - 2| < \delta$ so that $|f(x) - 2| > \epsilon$. For example, when x is only slightly greater than 2, the value of $|f(x) - 2|$ will be 2 or more.

2.7.57 Let $f(x) = \frac{|x|}{x}$ and suppose $\lim_{x \to 0} f(x)$ does exist and is equal to L. Let $\epsilon = 1/2$. There must be a value of δ so that when $0 < |x| < \delta$, $|f(x) - L| < 1/2$. Now consider the numbers $\delta/3$ and $-\delta/3$, both of which are within δ of 0. We have $f(\delta/3) = 1$ and $f(-\delta/3) = -1$. However, it is impossible for both $|1 - L| < 1/2$ and $|-1 - L| < 1/2$, because the former implies that $1/2 < L < 3/2$ and the latter implies that $-3/2 < L < -1/2$. Thus $\lim_{x \to 0} f(x)$ does not exist.

2.7.58 Suppose that $\lim_{x \to a} f(x)$ exists and is equal to L. Let $\epsilon = 1/2$. By the definition of limit, there must be a number δ so that $|f(x) - L| < \frac{1}{2}$ whenever $0 < |x - a| < \delta$. Now in every set of the form $(a, a + \delta)$ there are both rational and irrational numbers, so there will be value of f equal to both 0 and 1. Thus we have $|0 - L| < 1/2$, which means that L lies in the interval $(-1/2, 1/2)$, and we have $|1 - L| < 1/2$, which means that L lies in the interval $(1/2, 3/2)$. Because these both can't be true, we have a contradiction.

2.7.59 Because f is continuous at a, we know that $\lim_{x \to a} f(x)$ exists and is equal to $f(a) > 0$. Let $\epsilon = f(a)/3$. Then there is a number $\delta > 0$ so that $|f(x) - f(a)| < f(a)/3$ whenever $|x - a| < \delta$. Then whenever x lies in the interval $(a - \delta, a + \delta)$ we have $-f(a)/3 \le f(x) - f(a) \le f(a)/3$, so $2f(a)/3 \le f(x) \le 4f(a)/3$, so f is positive in this interval.

Chapter Two Review

1

a. False. Because $\lim_{x \to 1} \frac{x - 1}{x^2 - 1} = \lim_{x \to 1} \frac{1}{x + 1} = \frac{1}{2}$, f doesn't have a vertical asymptote at $x = 1$.

b. False. In general, these methods are too imprecise to produce accurate results.

c. False. For example, the function $f(x) = \begin{cases} 2x & \text{if } x < 0; \\ 1 & \text{if } x = 0; \\ 4x & \text{if } x > 0 \end{cases}$ has a limit of 0 as $x \to 0$, but $f(0) = 1$.

d. True. When we say that a limit exists, we are saying that there is a real number L that the function is approaching. If the limit of the function is ∞, it is still the case that there is no real number that the function is approaching. (There is no real number called "infinity.")

e. False. It could be the case that $\lim_{x \to a^-} f(x) = 1$ and $\lim_{x \to a^+} f(x) = 2$.

f. False.

g. False. For example, the function $f(x) = \begin{cases} 2 & \text{if } 0 < x < 1; \\ 3 & \text{if } 1 \le x < 2, \end{cases}$ is continuous on $(0, 1)$, and on $[1, 2)$, but isn't continuous on $(0, 2)$.

h. True. $\lim_{x\to a} f(x) = f(a)$ if and only if f is continuous at a.

2

a. $f(-1) = 1$

b. $\lim_{x\to -1^-} f(x) = 3$.

c. $\lim_{x\to -1^+} f(x) = 1$.

d. $\lim_{x\to -1} f(x)$ does not exist.

e. $f(1) = 5$.

f. $\lim_{x\to 1} f(x) = 5$.

g. $\lim_{x\to 2} f(x) = 4$.

h. $\lim_{x\to 3^-} f(x) = 3$.

i. $\lim_{x\to 3^+} f(x) = 5$.

j. $\lim_{x\to 3} f(x)$ does not exist.

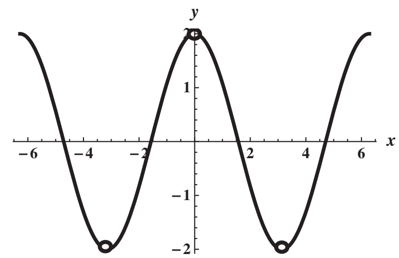
3 This function is discontinuous at $x = -1$, at $x = 1$, and at $x = 3$. At $x = -1$ it is discontinuous because $\lim_{x\to -1} f(x)$ does not exist. At $x = 1$, it is discontinuous because $\lim_{x\to 1} f(x) \neq f(1)$. At $x = 3$, it is discontinuous because $f(3)$ does not exist, and because $\lim_{x\to 3} f(x)$ does not exist.

4

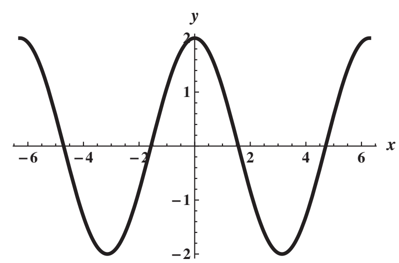
a. The graph drawn by most graphing calculators and computer algebra systems doesn't show the discontinuities where $\sin\theta = 0$.

b. It appears to be equal to 2

c. Using a trigonometric identity, $\lim_{\theta\to 0} \frac{\sin 2\theta}{\sin\theta} = \lim_{\theta\to 0} \frac{2\sin\theta\cos\theta}{\sin\theta}$. This can then be seen to be $\lim_{\theta\to 0} 2\cos\theta = 2$.



True graph, showing discontinuities where $\sin\theta = 0$.



Graph shown without discontinuities.

5

a.

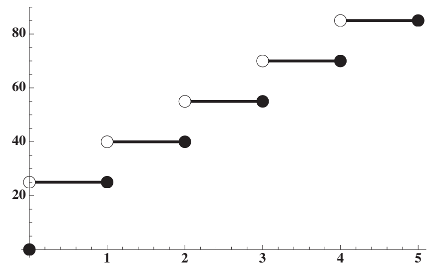
x	$.9\pi/4$	$.99\pi/4$	$.999\pi/4$	$.9999\pi/4$
$f(x)$	1.4098	1.4142	1.4142	1.4142

x	$1.1\pi/4$	$1.01\pi/4$	$1.001\pi/4$	$1.0001\pi/4$
$f(x)$	1.4098	1.4142	1.4142	1.4142

The limit appears to be approximately 1.4142.

b. $\lim_{x\to\pi/4} \frac{\cos 2x}{\cos x - \sin x} = \lim_{x\to\pi/4} \frac{\cos^2 x - \sin^2 x}{\cos x - \sin x} = \lim_{x\to\pi/4} (\cos x + \sin x) = \sqrt{2}$.

6

a. 

b. $\lim_{t\to 2.9} f(t) = 55$.

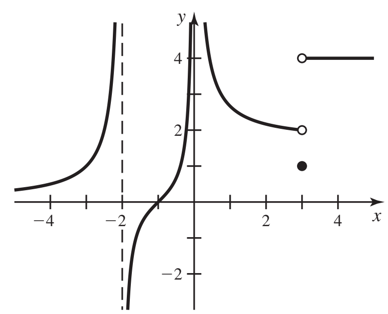
c. $\lim_{t\to 3^-} f(t) = 55$ and $\lim_{t\to 3^+} f(t) = 70$.

d. The cost of the rental jumps by \$15 exactly at $t = 3$. A rental lasting slightly less than 3 days cost \$55 and rentals lasting slightly more than 3 days cost \$70.

e. The function f is continuous everywhere except at the integers. The cost of the rental jumps by \$15 at each integer.

7

There are infinitely many different correct functions which you could draw. One of them is:



8 $\lim_{x\to 1000} 18\pi^2 = 18\pi^2$.

9 $\lim_{x\to 1} \sqrt{5x+6} = \sqrt{11}$.

10

$$\lim_{h\to 0} \frac{\sqrt{5x+5h}-\sqrt{5x}}{h} \cdot \frac{\sqrt{5x+5h}+\sqrt{5x}}{\sqrt{5x+5h}+\sqrt{5x}} = \lim_{h\to 0} \frac{(5x+5h)-5x}{h(\sqrt{5x+5h}+\sqrt{5x})} = \lim_{h\to 0} \frac{5}{\sqrt{5x+5h}+\sqrt{5x}} = \frac{5}{2\sqrt{5x}}.$$

11 $\lim_{x\to 1} \frac{x^3-7x^2+12x}{4-x} = \frac{1-7+12}{4-1} = \frac{6}{3} = 2$.

12 $\lim_{x\to 4} \frac{x^3-7x^2+12x}{4-x} = \lim_{x\to 4} \frac{x(x-3)(x-4)}{4-x} = \lim_{x\to 4} x(3-x) = -4$.

13 $\lim_{x\to 1} \frac{1-x^2}{x^2-8x+7} = \lim_{x\to 1} \frac{(1-x)(1+x)}{(x-7)(x-1)} = \lim_{x\to 1} \frac{-(x+1)}{x-7} = \frac{1}{3}$.

14 $\lim_{x\to 3} \frac{\sqrt{3x+16}-5}{x-3} \cdot \frac{\sqrt{3x+16}+5}{\sqrt{3x+16}+5} = \lim_{x\to 3} \frac{3(x-3)}{(x-3)(\sqrt{3x+16}+5)} = \lim_{x\to 3} \frac{3}{\sqrt{3x+16}+5} = \frac{3}{10}$.

15

$$\begin{aligned}\lim_{x\to 3}\frac{1}{x-3}\left(\frac{1}{\sqrt{x+1}}-\frac{1}{2}\right) &= \lim_{x\to 3}\frac{2-\sqrt{x+1}}{2(x-3)\sqrt{x+1}}\cdot\frac{(2+\sqrt{x+1})}{(2+\sqrt{x+1})}\\ &= \lim_{x\to 3}\frac{4-(x+1)}{2(x-3)(\sqrt{x+1})(2+\sqrt{x+1})}\\ &= \lim_{x\to 3}\frac{-(x-3)}{2(x-3)(\sqrt{x+1})(2+\sqrt{x+1})}\\ &= \lim_{x\to 3}-\frac{1}{2\sqrt{x+1}(2+\sqrt{x+1})}=-\frac{1}{16}.\end{aligned}$$

16 $\displaystyle\lim_{t\to 1/3}\frac{t-\frac{1}{3}}{(3t-1)^2}=\lim_{t\to 1/3}\frac{3t-1}{3(3t-1)^2}=\lim_{t\to 1/3}\frac{1}{3(3t-1)}$, which does not exist.

17 $\displaystyle\lim_{x\to 3}\frac{x^4-81}{x-3}=\lim_{x\to 3}\frac{(x-3)(x+3)(x^2+9)}{x-3}=\lim_{x\to 3}(x+3)(x^2+9)=108.$

18 Note that $\frac{p^5-1}{p-1}=p^4+p^3+p^2+p+1$. (Use long division.)
$\displaystyle\lim_{p\to 1}\frac{p^5-1}{p-1}=\lim_{p\to 1}(p^4+p^3+p^2+p+1)=5.$

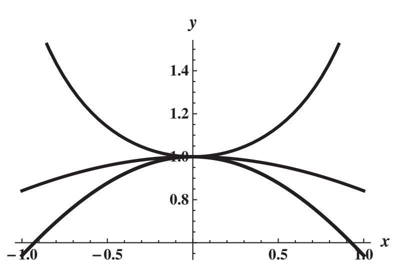
19 $\displaystyle\lim_{x\to 81}\frac{\sqrt[4]{x}-3}{x-81}=\lim_{x\to 81}\frac{\sqrt[4]{x}-3}{(\sqrt{x}+9)(\sqrt[4]{x}+3)(\sqrt[4]{x}-3)}=\lim_{x\to 81}\frac{1}{(\sqrt{x}+9)(\sqrt[4]{x}+3)}=\frac{1}{108}.$

20 $\displaystyle\lim_{\theta\to\pi/4}\frac{\sin^2\theta-\cos^2\theta}{\sin\theta-\cos\theta}=\lim_{\theta\to\pi/4}\frac{(\sin\theta-\cos\theta)(\sin\theta+\cos\theta)}{\sin\theta-\cos\theta}=\lim_{\theta\to\pi/4}(\sin\theta+\cos\theta)=\sqrt{2}.$

21 $\displaystyle\lim_{x\to\pi/2}\frac{\frac{1}{\sqrt{\sin x}}-1}{x+\pi/2}=\frac{0}{\pi}=0.$

22 The domain of $f(x)=\sqrt{\frac{x-1}{x-3}}$ is $(-\infty,1]$ and $(3,\infty)$, so $\displaystyle\lim_{x\to 1^+}f(x)$ doesn't exist.
However, we have $\displaystyle\lim_{x\to 1^-}f(x)=0$.

23

a. 

b. Because $\displaystyle\lim_{x\to 0}\cos x=\lim_{x\to 0}\frac{1}{\cos x}=1$, the squeeze theorem assures us that $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}=1$ as well.

24 Note that $\displaystyle\lim_{x\to 0}(\sin^2 x+1)=1$. Thus if $1\le g(x)\le \sin^2 x+1$, the squeeze theorem assures us that $\displaystyle\lim_{x\to 0}g(x)=1$ as well.

25 $\displaystyle\lim_{x\to 5}\frac{x-7}{x(x-5)^2}=-\infty.$

26 $\displaystyle\lim_{x\to -5^+}\frac{x-5}{x+5}=-\infty.$

27 $\lim_{x\to 3^-} \frac{x-4}{x^2-3x} = \lim_{x\to 3^-} \frac{x-4}{x(x-3)} = \infty.$

28 $\lim_{x\to 0^+} \frac{u-1}{\sin u} = -\infty.$

29 $\lim_{x\to 0^-} \frac{2}{\tan x} = -\infty.$

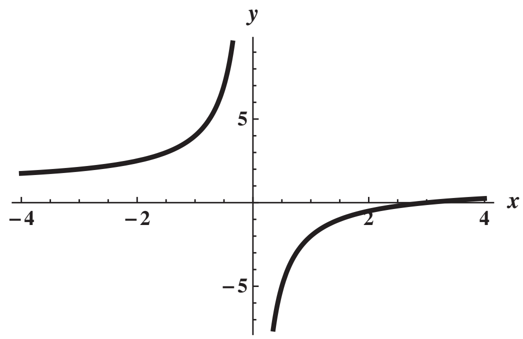
30

First note that $f(x) = \frac{x^2-5x+6}{x^2-2x} = \frac{(x-3)(x-2)}{x(x-2)}$.

a. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{(x-3)(x-2)}{x(x-2)} = \infty.$
$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{(x-3)(x-2)}{x(x-2)} = -\infty.$
$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x-3}{x} = -\frac{1}{2}.$
$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x-3}{x} = -\frac{1}{2}.$

b. By the above calculations and the definition of vertical asymptote, f has a vertical asymptote at $x = 0$.

c. Note that the actual graph has a "hole" at the point $(2, -1/2)$, because $x = 2$ isn't in the domain, but $\lim_{x\to 2} f(x) = -1/2$.



31 $\lim_{x\to\infty} \frac{2x-3}{4x+10} = \lim_{x\to\infty} \frac{2-(3/x)}{4+(10/x)} = \frac{2}{4} = \frac{1}{2}.$

32 $\lim_{x\to\infty} \frac{x^4-1}{x^5+2} = \lim_{x\to\infty} \frac{(1/x)-(1/x^5)}{1+(2/x^5)} = \frac{0-0}{1+0} = 0.$

33 $\lim_{x\to-\infty} (-3x^3+5) = \infty.$

34 $\lim_{x\to\infty} \frac{x}{\sqrt{4x^2+1}} = \lim_{x\to\infty} \frac{1}{\sqrt{4+1/x^2}} = \frac{1}{2}.$

35 $\lim_{x\to\infty} \frac{\sqrt{25x^2+8}}{x+2} = \lim_{x\to\infty} \frac{\sqrt{25+8/x^2}}{1+2/x} = 5.$

36 Because $\cos r$ oscillates between -1 and 1, this limit does not exist. For any $N > 0$, there are values of r with $|r| > N$ such that $\cos r = 0$, $\cos r = 1$, or $\cos r = -1$. Thus the value of the expression $\frac{1}{\cos r + 1}$ takes on the values 1, 1/2, and is undefined infinitely often as $r \to \infty$.

37 $\lim_{x\to\infty} \frac{4x^3+1}{1-x^3} = \lim_{x\to\infty} \frac{4+(1/x^3)}{(1/x^3)-1} = \frac{4+0}{0-1} = -4.$ A similar result holds as $x \to -\infty$. Thus, $y = -4$ is a horizontal asymptote as $x \to \infty$ and as $x \to -\infty$.

38 First note that $\sqrt{\frac{1}{x^2}} = \left|\frac{1}{x}\right| = \begin{cases} \frac{1}{x} & \text{if } x > 0; \\ -\frac{1}{x} & \text{if } x < 0. \end{cases}$.

$\lim_{x\to\infty} \frac{x+1}{\sqrt{9x^2+x}} = \lim_{x\to\infty} \frac{1+(1/x)}{\sqrt{9+\frac{1}{x}}} = \frac{1}{3}.$

On the other hand, $\lim_{x\to-\infty} \frac{x+1}{\sqrt{9x^2+x}} = \lim_{x\to\infty} \frac{1+(1/x)}{-\sqrt{9+\frac{1}{x}}} = -\frac{1}{3}$.

So $y = \frac{1}{3}$ is a horizontal asymptote as $x \to \infty$, and $y = -\frac{1}{3}$ is a horizontal asymptote as $x \to -\infty$.

39 $\lim_{x\to\pm\infty} \frac{12x^2}{\sqrt{16x^4+7}} = \lim_{x\to\pm\infty} \frac{12}{\sqrt{16+7/x^4}} = 3$.

40 $\lim_{x\to\pm\infty} \sqrt[3]{\frac{8x+1}{x-3}} = \lim_{x\to\pm\infty} \sqrt[3]{\frac{8+1/x}{1-3/x}} = \sqrt[3]{8} = 2$.

41 $\lim_{x\to\pm\infty} \frac{x^2-x}{x^2-1} = \lim_{x\to\pm\infty} \frac{x}{x+1} = \lim_{x\to\pm\infty} \frac{1}{1+1/x} = 1$, so that $f(x)$ has a horizontal asymptote of $y = 1$. There is a vertical asymptote where $x+1=0$, i.e. for $x=-1$.

42 Note that $f(x) = \frac{2x^2+6}{2x^2+3x-2} = \frac{2(x^2+3)}{(2x-1)(x+2)}$.

We have $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2+6/x^2}{2+3/x-2/x^2} = 1$. A similar result holds as $x \to -\infty$.

$\lim_{x\to 1/2^-} f(x) = -\infty$. $\lim_{x\to 1/2^+} f(x) = \infty$.

$\lim_{x\to -2^-} f(x) = \infty$. $\lim_{x\to -2^+} f(x) = -\infty$.

Thus, $y = 1$ is a horizontal asymptote as $x \to \infty$ and as $x \to -\infty$. Also, $x = \frac{1}{2}$ and $x = -2$ are vertical asymptotes.

43 $\lim_{x\to\infty} \frac{3x^2+2x-1}{4x+1} = \lim_{x\to\infty} \frac{3x^2+2x-1}{4x+1} \cdot \frac{1/x}{1/x} = \lim_{x\to\infty} \frac{3x+2-1/x}{4+1/x} = \infty$.

$\lim_{x\to-\infty} \frac{3x^2+2x-1}{4x+1} = \lim_{x\to-\infty} \frac{3x^2+2x-1}{4x+1} \cdot \frac{1/x}{1/x} = \lim_{x\to-\infty} \frac{3x+2-1/x}{4+1/x} = -\infty$.

By long division, we can write $f(x)$ as $f(x) = \frac{3x}{4} + \frac{5}{16} + \frac{-21/16}{4x+1}$, so the line $y = \frac{3x}{4} + \frac{5}{16}$ is the slant asymptote.

44 $\lim_{x\to\infty} \frac{9x^2+4}{(2x-1)^2} = \lim_{x\to\infty} \frac{9x^2+4}{4x^2-4x+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{9+4/x^2}{4-4/x+1/x^2} = \frac{9}{4}$.

$\lim_{x\to-\infty} \frac{9x^2+4}{(2x-1)^2} = \lim_{x\to-\infty} \frac{9x^2+4}{4x^2-4x+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \frac{9+4/x^2}{4-4/x+1/x^2} = \frac{9}{4}$. Because there is a horizontal asymptote, there is not a slant asymptote.

45 $\lim_{x\to\infty} \frac{1+x-2x^2-x^3}{x^2+1} = \lim_{x\to\infty} \frac{1+x-2x^2-x^3}{x^2+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{1/x^2+1/x-2-x}{1+1/x^2} = -\infty$.

$\lim_{x\to-\infty} \frac{1+x-2x^2-x^3}{x^2+1} = \lim_{x\to-\infty} \frac{1+x-2x^2-x^3}{x^2+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \frac{1/x^2+1/x-2-x}{1+1/x^2} = \infty$.

By long division, we can write $f(x)$ as $f(x) = -x-2+\frac{2x+3}{x^2+1}$, so the line $y = -x-2$ is the slant asymptote.

46 $\lim_{x\to\infty} \frac{x(x+2)^3}{3x^2-4x} = \lim_{x\to\infty} \frac{x^4+6x^3+12x^2+8x}{3x^2-4x} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{x^2+6x+12+8/x}{3-4/x} = \infty$.

$\lim_{x\to-\infty} \frac{x(x+2)^3}{3x^2-4x} = \lim_{x\to-\infty} \frac{x^4+6x^3+12x^2+8x}{3x^2-4x} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \frac{x^2+6x+12+8/x}{3-4/x} = \infty$.

Because the degree of the numerator of this rational function is two more than the degree of the denominator, there is no slant asymptote.

47 f is discontinuous at 5, because $f(5)$ does not exist, and also because $\lim_{x\to 5} f(x)$ does not exist

48 g is discontinuous at 4 because $\lim_{x\to 4} g(x) = \lim_{x\to 4} \frac{(x+4)(x-4)}{x-4} = 8 \neq g(4)$.

49 h is not continuous at 3 because $\lim_{x\to 3^-} h(x)$ does not exist, so $\lim_{x\to 3} h(x)$ does not exist.

50 g is continuous at 4 because $\lim_{x\to 4} g(x) = \lim_{x\to 4} \frac{(x+4)(x-4)}{x-4} = 8 = g(4)$.

51 The domain of f is $(-\infty, -\sqrt{5}]$ and $[\sqrt{5}, \infty)$, and f is continuous on that domain.

52 The domain of g is all x such that $x^2 - 5x + 6 = (x-3)(x-2) \geq 0$, i.e. $(-\infty, 2] \cup [3, \infty)$. g is continuous on its domain. At $x = 2$ the function is continuous from the left only; at $x = 3$ it is continuous from the right only.

53 The domain of h is $(-\infty, -5)$, $(-5, 0)$, $(0, 5)$, $(5, \infty)$, and like all rational functions, it is continuous on its domain.

54 The domain of g is the set of real numbers x such that $x \geq 0$ (so that $\sqrt{x}$ is defined). g is continuous on its domain; at $x = 0$ it is continuous from the right only.

55 In order for g to be left continuous at 1, it is necessary that $\lim_{x\to 1^-} g(x) = g(1)$, which means that $a = 3$. In order for g to be right continuous at 1, it is necessary that $\lim_{x\to 1^+} g(x) = g(1)$, which means that $a + b = 3 + b = 3$, so $b = 0$.

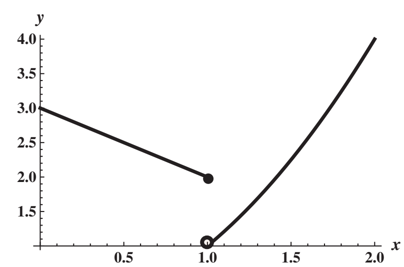
56

a. Because the domain of h is $(-\infty, -3]$ and $[3, \infty)$, there is no way that h can be left continuous at 3.

b. h is right continuous at 3, because $\lim_{x\to 3^+} h(x) = 0 = h(3)$.

57

One such possible graph is pictured to the right.



58 a. Consider the function $f(x) = x^5 + 7x + 5$. f is continuous everywhere, and $f(-1) = -3 < 0$ while $f(0) = 5 > 0$. Therefore, 0 is an intermediate value between $f(-1)$ and $f(0)$. By the IVT, there must a number c between 0 and 1 so that $f(c) = 0$.

b. Using a computer algebra system, one can find that $c \approx -0.691671$ is a root.

59

a. Any such rectangle with length x has width $y = 100/x$, so its perimeter is

$$P(x) = 2x + 2y = 2x + \frac{200}{x}.$$

b. P is continuous for $x > 0$. $P(10) = 40$ while $P(30) = 60 + \frac{200}{30} > 60$. Thus by the Intermediate Value Theorem, there must be some x with $10 \leq x \leq 30$ and $P(x) = 50$.

c. The rectangle with perimeter 50 can be determined by solving $P(x) = 50$.

$$2x + \frac{200}{x} = 50$$
$$2x^2 - 50x + 200 = 0$$
$$x = 5, 20$$

So a perimeter of 50 occurs for a 5×20 or a 20×5 rectangle.

d. If there were a rectangle with perimeter 30, it would be a solution to $P(x) = 30$, so

$$2x + \frac{200}{x} = 30, \qquad 2x^2 - 30x + 200 = 0$$

The discriminant of this quadratic is $900 - 1600 < 0$, so the quadratic has no (real) solution.

e. Plotting $P(x)$ for various viewing windows seems to show that the smallest possible perimeter is 40, for a 10×10 rectangle.

60 Let $\epsilon > 0$ be given. Let $\delta = \epsilon/5$. Now suppose that $0 < |x - 1| < \delta$. Then

$$|f(x) - L| = |(5x - 2) - 3| = |5x - 5|$$
$$= 5|x - 1| < 5 \cdot \frac{\epsilon}{5} = \epsilon.$$

61 Let $\epsilon > 0$ be given. Let $\delta = \epsilon$. Now suppose that $0 < |x - 5| < \delta$. Then

$$|f(x) - L| = \left|\frac{x^2 - 25}{x - 5} - 10\right| = \left|\frac{(x-5)(x+5)}{x - 5} - 10\right| = |x + 5 - 10|$$
$$= |x - 5| < \epsilon.$$

62

a. Assume $L > 0$. (If $L = 0$, the result follows immediately because that would imply that the function f is the constant function 0, and then $f(x)g(x)$ is also the constant function 0.) Assume that δ_1 is a number so that $|f(x)| \le L$ for $|x - a| < \delta_1$.

Let $\epsilon > 0$ be given. Because $\lim_{x \to a} g(x) = 0$, we know that there exists a number $\delta_2 > 0$ so that $|g(x)| < \epsilon/L$ whenever $0 < |x - a| < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$.

Then

$$|f(x)g(x) - 0| = |f(x)||g(x)| < L \cdot \frac{\epsilon}{L} = \epsilon,$$

whenever $0 < |x - a| < \delta$.

b. Let $f(x) = \frac{x^2}{x-2}$. Then

$$\lim_{x \to 2} f(x)(x - 2) = \lim_{x \to 2} \frac{x^2(x-2)}{x - 2} = \lim_{x \to 2} x^2 = 4 \ne 0.$$

This doesn't violate the previous result because the given function f is not bounded near $x = 2$.

c. Because $|H(x)| \le 1$ for all x, the result follows directly from part a) of this problem (using $L = 1$, $a = 0$, $f(x) = H(x)$, and $g(x) = x$).

63 Let $N > 0$ be given. Let $\delta = 1/\sqrt[4]{N}$. Suppose that $0 < |x - 2| < \delta$. Then $|x - 2| < \frac{1}{\sqrt[4]{N}}$, so $\frac{1}{|x-2|} > \sqrt[4]{N}$, and $\frac{1}{(x-2)^4} > N$, as desired.