CHAPTER 3
Applications of the Derivative

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Section 3.1 Increasing and Decreasing Functions

Skills Warm Up

1.
$$x^2 = 8x$$
$$x^2 - 8x = 0$$
$$x(x - 8) = 0$$
$$x = 0$$
$$x - 8 = 0 \Rightarrow x = 8$$

2.
$$15x = \tfrac{5}{8}x^2$$
$$15x - \tfrac{5}{8}x^2 = 0$$
$$x\left(15 - \tfrac{5}{8}x\right) = 0$$
$$x = 0$$
$$15 - \tfrac{5}{8}x = 0 \Rightarrow x = 24$$

3.
$$\frac{x^2 - 25}{x^3} = 0$$
$$\frac{1}{x} - \frac{25}{x^3} = 0$$
$$\frac{1}{x} = \frac{25}{x^3}$$
$$x^3 = 25x$$
$$x^3 - 25x = 0$$
$$x(x^2 - 25) = 0$$
$$x(x + 5)(x - 5) = 0$$
$$x = 0 \text{ Extraneous}$$
$$x + 5 = 0 \Rightarrow x = -5$$
$$x - 5 = 0 \Rightarrow x = 5$$

4.
$$\frac{2x}{\sqrt{1 - x^2}} = 0$$
$$2x = 0$$
$$x = 0$$

5. The domain of $\dfrac{x + 3}{x - 3}$ is $(-\infty, 3) \cup (3, \infty)$.

6. The domain of $\dfrac{2}{\sqrt{1 - x}}$ is $(-\infty, 1)$.

7. The domain of $\dfrac{2x + 1}{x^2 - 3x - 10}$ is $(-\infty, -2) \cup (-2, 5) \cup (5, \infty)$.

8. The domain of $\dfrac{3x}{\sqrt{9 - 3x^2}}$ is $\left(-\sqrt{3}, \sqrt{3}\right)$.

9. When $x = -2$: $-2(-2 + 1)(-2 - 1) = -6$

When $x = 0$: $-2(0 + 1)(0 - 1) = 2$

When $x = 2$: $-2(2 + 1)(2 - 1) = -6$

10. When $x = -2$: $4[2(-2) + 1][2(-2) - 1] = 60$

When $x = 0$: $4(2 \cdot 0 + 1)(2 \cdot 0 - 1) = -4$

When $x = 2$: $4(2 \cdot 2 + 1)(2 \cdot 2 - 1) = 60$

11. When $x = -2$: $\dfrac{2(-2) + 1}{(-2 - 1)^2} = -\dfrac{1}{3}$

When $x = 0$: $\dfrac{2 \cdot 0 + 1}{(0 - 1)^2} = 1$

When $x = 2$: $\dfrac{2 \cdot 2 + 1}{(2 - 1)^2} = 5$

12. When $x = -2$: $\dfrac{-2(-2 + 1)}{(-2 - 4)^2} = \dfrac{1}{18}$

When $x = 0$: $\dfrac{-2(0 + 1)}{(0 - 4)^2} = -\dfrac{1}{8}$

When $x = 2$: $\dfrac{-2(2 + 1)}{(2 - 4)^2} = -\dfrac{3}{2}$

1. f has a critical number at $x = -1$.

f is increasing on $(-\infty, -1)$.

f is decreasing on $(-1, \infty)$.

2. f has critical numbers at $x = \pm 2$.

f is increasing on $(-\infty, -2) \cup (2, \infty)$.

f is decreasing on $(-2, 2)$.

3. f has critical numbers at $x = \pm 1, 0$.

f is increasing on $(-1, 0) \cup (1, \infty)$.

f is decreasing on $(-\infty, -1) \cup (0, 1)$.

4. f has critical numbers at $x = \pm 3, 0$.

f is increasing on $(-\infty, -3) \cup (0, 3)$.

f is decreasing on $(-3, 0) \cup (3, \infty)$.

5. $f(x) = 4x^2 - 6x$

$f'(x) = 8x - 6 = 2(4x - 3)$

Set $f'(x) = 0$.

$$2(4x - 3) = 0$$
$$4x - 3 = 0$$
$$x = \frac{3}{4}$$

$f'(x)$ is never undefined.

Critical number: $x = \frac{3}{4}$

6. $f(x) = 3x^2 + 10$

$f'(x) = 6x$

Set $f'(x) = 0$.

$$6x = 0$$
$$x = 0$$

$f'(x)$ is never undefined.

Critical number: $x = 0$

7. $y = x^4 + 4x^3 + 8$

$y' = 4x^3 + 12x^2 = 4x^2(x + 3)$

Set $y' = 0$.

$$4x^2(x + 3) = 0$$
$$4x^2 = 0 \qquad x + 3 = 0$$
$$x = 0 \qquad x = -3$$

y' is never undefined.

Critical numbers: $x = 0, -3$

8. $g(x) = 2x^2 - 54x$

$g'(x) = 4x - 54 = 4(x - 16)$

Set $g'(x) = 0$.

$$4(x - 16) = 0$$
$$x - 16 = 0$$
$$x = 16$$

$g'(x)$ is undefined.

Critical number: $x = 16$

9. $f(x) = \sqrt{x^2 - 4} = (x^2 - 4)^{1/2}$

$$f'(x) = \frac{1}{2}(x^2 - 4)^{-1/2}(2x)$$
$$= \frac{x}{(x^2 - 4)^{1/2}}$$

Set $f'(x) = 0$.

$x = 0$ (**Note:** $x = 0$ is not in the domain of $f(x)$.)

$f'(x)$ is undefined at $x = \pm 2$.

Critical numbers: $x = \pm 2$

10. $y = \dfrac{x}{x^2 + 16}$

$$y'(x) = \frac{(x^2 + 16)(1) - x(2x)}{(x^2 + 16)^2}$$

$$y' = \frac{16 - x^2}{(x^2 + 16)^2}$$

Set $y' = 0$.

$$16 - x^2 = 0$$
$$x^2 = 16$$
$$x = \pm 4$$

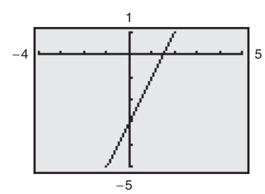
y' is never undefined.

Critical numbers: $x = \pm 4$

11. $f(x) = 2x - 3$

$f'(x) = 2$

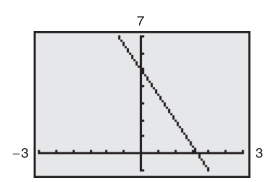
There are no critical numbers. Since the derivative is positive for all x, the function is increasing on $(-\infty, \infty)$.



12. $f(x) = 5 - 3x$

$f'(x) = -3$

There are no critical numbers. Since the derivative is negative for all x, the function is decreasing on $(-\infty, \infty)$.

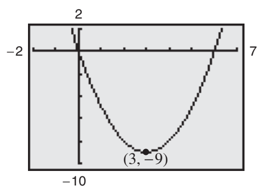


13. $y = x^2 - 6x$

$y' = 2x - 6$

Critical number: $x = 3$

Interval	$-\infty < x < 3$	$3 < x < \infty$
Sign of y'	$y' < 0$	$y' > 0$
Conclusion	Decreasing	Increasing

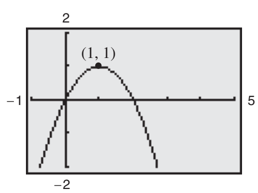


14. $y = -(x^2 - 2x) = 2x - x^2$

$y' = -2x + 2$

Critical number: $x = 1$

Interval	$-\infty < x < 1$	$1 < x < \infty$
Sign of y'	$y' > 0$	$y' < 0$
Conclusion	Increasing	Decreasing

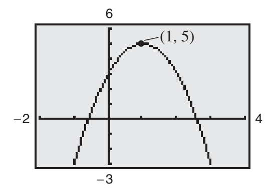


15. $f(x) = -2x^2 + 4x + 3$

$f'(x) = -4x + 4$

Critical number: $x = 1$

Interval	$-\infty < x < 1$	$1 < x < \infty$
Sign of f'	$f' > 0$	$f' < 0$
Conclusion	Increasing	Decreasing

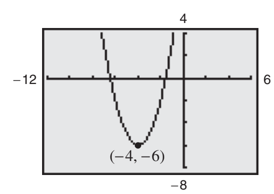


16. $f(x) = x^2 + 8x + 10$

$f'(x) = 2x + 8$

Critical number: $x = -4$

Interval	$-\infty < x < -4$	$-4 < x < \infty$
Sign of f'	$f' < 0$	$f' > 0$
Conclusion	Decreasing	Increasing

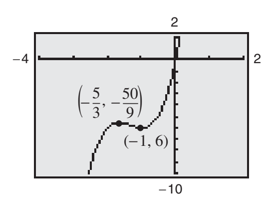


17. $y = 3x^3 + 12x^2 + 15x$

$y' = 9x^2 + 24x + 15 = 3(x + 1)(3x + 5)$

Critical numbers: $x = -1, -\frac{5}{3}$

Interval	$-\infty < x < -\frac{5}{3}$	$-\frac{5}{3} < x < -1$	$-1 < x < \infty$
Sign of y'	$y' > 0$	$y' < 0$	$y' > 0$
Conclusion	Increasing	Decreasing	Increasing

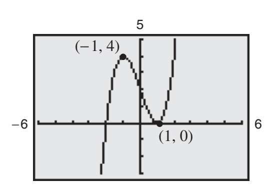


18. $f(x) = x^3 - 3x + 2 = (x - 1)^2(x + 2)$

$f'(x) = 3x^2 - 3 = 3(x + 1)(x - 1)$

Critical numbers: $x = \pm 1$

Interval	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of f'	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion	Increasing	Decreasing	Increasing

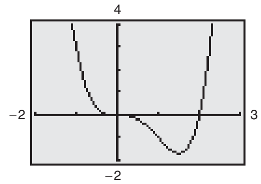


19. $f(x) = x^4 - 2x^3$

$f'(x) = 4x^3 - 6x^2 = 2x^2(2x - 3)$

Critical numbers: $x = 0, x = \frac{3}{2}$

Interval	$-\infty < x < 0$	$0 < x < \frac{3}{2}$	$\frac{3}{2} < x < \infty$
Sign of f'	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion	Decreasing	Decreasing	Increasing



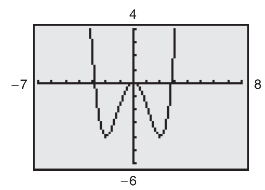
f is decreasing on $\left(-\infty, \frac{3}{2}\right)$ and increasing on $\left(\frac{3}{2}, \infty\right)$.

20. $f(x) = \frac{1}{4}x^4 - 2x^2$

$f'(x) = x^3 - 4x = x(x - 2)(x + 2)$

Critical numbers: $x = 0, 2, -2$

Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
Sign of f'	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion	Decreasing	Increasing	Decreasing	Increasing

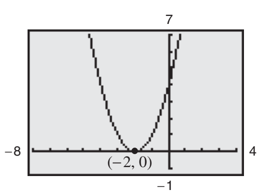


21. $g(x) = (x + 2)^2$

$g'(x) = 2(x + 2)$

Critical number: $x = -2$

Interval	$-\infty < x < -2$	$-2 < x < \infty$
Sign of g'	$g' < 0$	$g' > 0$
Conclusion	Decreasing	Increasing

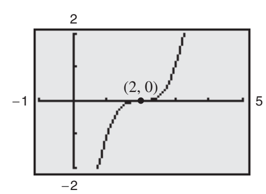


22. $y = (x - 2)^3$

$y' = 3(x - 2)^2$

Critical number: $x = 2$

Interval	$-\infty < x < 2$	$2 < x < \infty$
Sign of y'	$y' > 0$	$y' > 0$
Conclusion	Increasing	Increasing



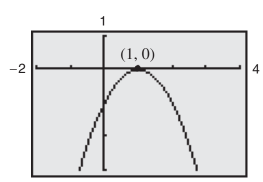
y is increasing on $(-\infty, \infty)$.

23. $g(x) = -(x - 1)^2$

$g'(x) = -2(x - 1)$

Critical number: $x = 1$

Interval	$-\infty < x < 1$	$1 < x < \infty$
Sign of g'	$g' > 0$	$g' < 0$
Conclusion	Increasing	Decreasing

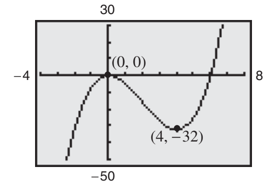


24. $y = x^3 - 6x^2$

$y' = 3x^2 - 12x = 3x(x - 4)$

Critical numbers: $x = 0$, $x = 4$

Interval	$-\infty < x < 0$	$0 < x < 4$	$4 < x < \infty$
Sign of y'	$y' > 0$	$y' < 0$	$y' > 0$
Conclusion	Increasing	Decreasing	Increasing

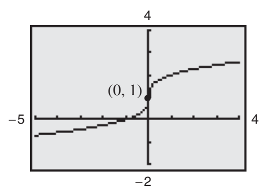


25. $y = x^{1/3} + 1$

$y' = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$

Critical number: $x = 0$

Interval	$-\infty < x < 0$	$0 < x < \infty$
Sign of y'	$y' > 0$	$y' > 0$
Conclusion	Increasing	Increasing



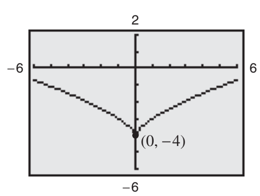
y is increasing on $(-\infty, \infty)$.

26. $y = x^{2/3} - 4$

$y' = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$

Critical number: $x = 0$

Interval	$-\infty < x < 0$	$0 < x < \infty$
Sign of y'	$y' < 0$	$y' > 0$
Conclusion	Decreasing	Increasing



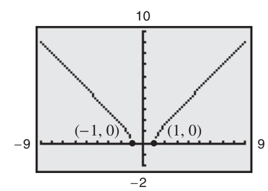
27. $f(x) = \sqrt{x^2 - 1} = (x^2 - 1)^{1/2}$

Domain: $(-\infty, -1] \cup [1, \infty)$

$f'(x) = \frac{1}{2}(x^2 - 1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 - 1}}$

Critical numbers: $x = \pm 1$ ($x = 0$ not in domain)

Interval	$-\infty < x < -1$	$1 < x < \infty$
Sign of f'	$f' < 0$	$f' > 0$
Conclusion	Decreasing	Increasing



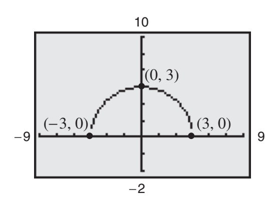
28. $f(x) = \sqrt{9 - x^2} = (9 - x^2)^{1/2}$

Domain: $-3 \le x \le 3$

$f'(x) = \frac{1}{2}(9 - x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{9 - x^2}}$

Critical numbers: $x = 0, \pm 3$

Interval	$-3 < x < 0$	$0 < x < 3$
Sign of f'	$f' > 0$	$f' < 0$
Conclusion	Increasing	Decreasing



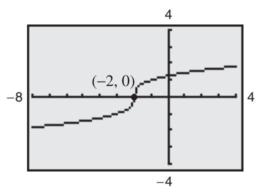
29. $g(x) = (x + 2)^{1/3}$

$$g'(x) = \frac{1}{3}(x + 2)^{-2/3}(1) = \frac{1}{3(x + 2)^{2/3}}$$

$g'(-2)$ is undefined.

Critical number: $x = -2$

Interval	$-\infty < x < -2$	$-2 < x < \infty$
Sign of g'	$g' > 0$	$g' > 0$
Conclusion	Increasing	Increasing

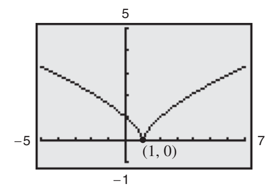


30. $g(x) = (x - 1)^{2/3}$

$$g'(x) = \frac{2}{3}(x - 1)^{-1/3} = \frac{2}{3(x - 1)^{1/3}}$$

Critical number: $x = 1$

Interval	$-\infty < x < 1$	$1 < x < \infty$
Sign of g'	$g' < 0$	$g' > 0$
Conclusion	Decreasing	Increasing



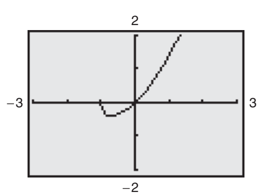
31. $f(x) = x\sqrt{x + 1} = x(x + 1)^{1/2}$

Domain: $[-1, \infty)$

$$f'(x) = x\left[\frac{1}{2}(x + 1)^{-1/2}\right] + (x + 1)^{1/2} = \frac{1}{2}(x + 1)^{-1/2}\left[x + 2(x + 1)\right] = \frac{3x + 2}{2\sqrt{x + 1}}$$

Critical numbers: $x = -1, -\dfrac{2}{3}$

Interval	$-1 < x < -\frac{2}{3}$	$-\frac{2}{3} < x < \infty$
Sign of f'	$f' < 0$	$f' > 0$
Conclusion	Decreasing	Increasing

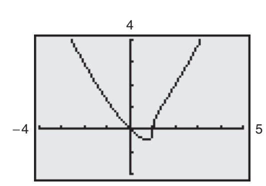


32. $h(x) = x\sqrt[3]{x - 1} = x(x - 1)^{1/3}$

$$h'(x) = \frac{1}{3}x(x - 1)^{-2/3} + (x - 1)^{1/3} = \frac{1}{3}(x - 1)^{-2/3}\left[x + 3(x - 1)\right] = \frac{4x - 3}{3(x - 1)^{2/3}}$$

Critical numbers: $x = \dfrac{3}{4}, 1$

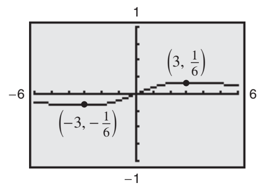
Interval	$-\infty < x < \frac{3}{4}$	$\frac{3}{4} < x < 1$	$1 < x < \infty$
Sign of h'	$h' < 0$	$h' > 0$	$h' > 0$
Conclusion	Decreasing	Increasing	Increasing



h is decreasing on $\left(-\infty, \frac{3}{4}\right)$ and increasing on $\left(\frac{3}{4}, \infty\right)$.

33. $f(x) = \dfrac{x}{x^2 + 9}$

$f'(x) = \dfrac{(x^2 + 9)(1) - x(2x)}{(x^2 + 9)^2} = \dfrac{9 - x^2}{(x^2 + 9)^2}$



$f'(x) = 0$:

$9 - x^2 = 0$

$x^2 = 9$

$x = \pm 3$

Critical numbers: $x = \pm 3$

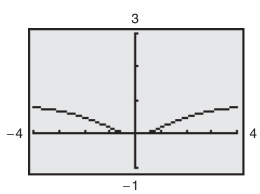
Interval	$-\infty < x < -3$	$-3 < x < 3$	$3 < x < \infty$
Sign of f'	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion	Decreasing	Increasing	Decreasing

34. $f(x) = \dfrac{x^2}{x^2 + 4}$

$f'(x) = \dfrac{(x^2 + 4)(2x) - (x^2)(2x)}{(x^2 + 4)^2} = \dfrac{8x}{(x^2 + 4)^2}$

Critical number: $x = 0$

Interval	$-\infty < x < 0$	$0 < x < \infty$
Sign of f'	$f' < 0$	$f' > 0$
Conclusion	Decreasing	Increasing

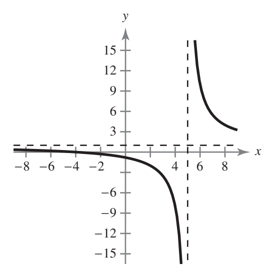


35. $f(x) = \dfrac{x + 4}{x - 5}$

$f'(x) = \dfrac{(x - 5)(1) - (x + 4)(1)}{(x - 5)^2} = -\dfrac{9}{(x - 5)^2}$

Discontinuity: $x = 5$

Interval	$-\infty < x < 5$	$5 < x < \infty$
Sign of f'	$f' < 0$	$f' < 0$
Conclusion	Decreasing	Decreasing

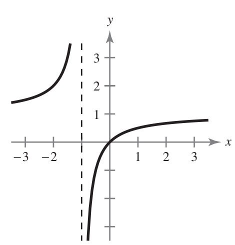


36. $f(x) = \dfrac{x}{x + 1}$

$f'(x) = \dfrac{(x + 1) - (x)}{(x + 1)^2} = \dfrac{1}{(x + 1)^2}$

Discontinuity: $x = -1$

Interval	$-\infty < x < -1$	$-1 < x < \infty$
Sign of f'	$f' > 0$	$f' > 0$
Conclusion	Increasing	Increasing

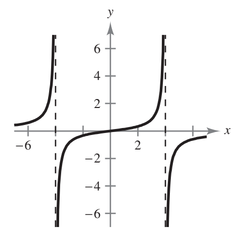


37. $f(x) = \dfrac{2x}{16 - x^2}$

$$f'(x) = \frac{(16 - x^2)2 - 2x(-2x)}{(16 - x^2)^2} = \frac{2x^2 + 32}{(16 - x^2)^2}$$

Discontinuities: $x = \pm 4$

Interval	$-\infty < x < -4$	$-4 < x < 4$	$4 < x < \infty$
Sign of f'	$f' > 0$	$f' > 0$	$f' > 0$
Conclusion	Increasing	Increasing	Increasing



38. $f(x) = \dfrac{x^2}{x^2 - 9}$

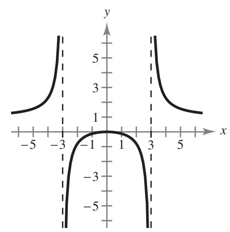
$$f'(x) = \frac{(x^2 - 9)(2x) - x^2(2x)}{(x^2 - 9)^2} = -\frac{-18x}{(x^2 - 9)^2}$$

$f'(x) = 0$: $-18x = 0$

$x = 0$

Critical number: $x = 0$

Discontinuities: $x = \pm 3$



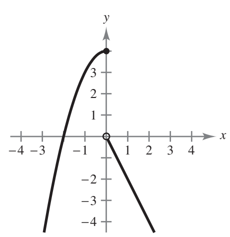
Interval	$-\infty < x < -3$	$-3 < x < 0$	$0 < x < 3$	$3 < x < \infty$
Sign of f'	$f' > 0$	$f' > 0$	$f' < 0$	$f'(x) < 0$
Conclusion	Increasing	Increasing	Decreasing	Decreasing

39. $y = \begin{cases} 4 - x^2, & x \le 0 \\ -2x, & x > 0 \end{cases}$

$y' = \begin{cases} -2x, & x < 0 \\ -2, & x > 0 \end{cases}$

$y'(0)$ is undefined.

Critical number: $x = 0$



Interval	$-\infty < x < 0$	$0 < x < \infty$
Sign of y'	$y' > 0$	$y' < 0$
Conclusion	Increasing	Decreasing

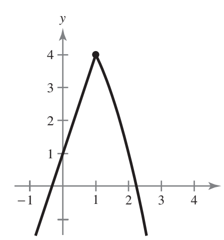
40. $y = \begin{cases} 3x + 1, & x \le 1 \\ 5 - x^2, & x > 1 \end{cases}$

$y' = \begin{cases} 3, & x < 1 \\ -2x, & x > 1 \end{cases}$

$y'(1)$ is undefined.

Critical number: $x = 1$

($x = 0$ is not a critical number.)



Interval	$-\infty < x < 1$	$1 < x < \infty$
Sign of y'	$y' > 0$	$y' < 0$
Conclusion	Increasing	Decreasing

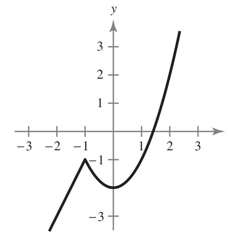
41. $y = \begin{cases} 2x + 1, & x \le -1 \\ x^2 - 2, & x > -1 \end{cases}$

$y' = \begin{cases} 2, & x < -1 \\ 2x, & x > -1 \end{cases}$

$y'(-1)$ is undefined.

Critical numbers: $x = -1, 0$

Interval	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < \infty$
Sign of y'	$y' > 0$	$y' < 0$	$y' > 0$
Conclusion	Increasing	Decreasing	Increasing



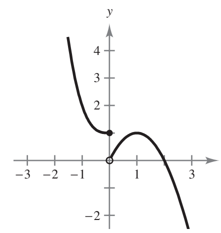
42. $y = \begin{cases} -x^3 + 1, & x \le 0 \\ -x^2 + 2x, & x > 0 \end{cases}$

$y' = \begin{cases} -3x^2, & x < 0 \\ -2x + 2, & x > 0 \end{cases}$

$y'(0)$ is undefined.

Critical numbers: $x = 0, 1$

Interval	$-\infty < x < 0$	$0 < x < 1$	$1 < x < \infty$
Sign of y'	$y' < 0$	$y' > 0$	$y' < 0$
Conclusion	Decreasing	Increasing	Decreasing



43. $S = -1.598t^2 + 45.61t + 130.2,\ 3 \le t \le 9$

$S' = -3.196t + 45.61$

Since $S' > 0$ when $t \approx 14.3$, there is no critical number in the interval $[3, 9]$.

$S' > 0$ on the interval $[3, 9]$, so S is increasing on $[3, 9]$.

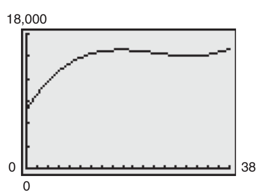
Sales for Wal-Mart are strictly increasing from 2003 to 2009.

44. As the temperature increases, the average velocity (the curve's peak) and the speed of velocities increases. For each temperature, the number of N_2 molecules increases to the peak of the curve and decreases after the peak.

45. $y = 0.692t^3 - 50.11t^2 + 1119.7t + 7894$

$y' = 2.076t^2 - 100.22t + 1119.7$

(a)



Increasing from 1970 to late 1988 and from late 2001 to 2008

Decreasing from late 1988 to late 2001

(b) Set $y' = 0$.

$2.076t^2 - 100.22t + 1119.7 = 0$

$$t = \frac{-(-100.22) \pm \sqrt{(-100.22)^2 - 4(2.076)(1119.7)}}{2(2.076)}$$

Critical numbers: $t \approx 17.6, 30.7$

Interval	$0 < t < 17.6$	$17.6 < t < 30.7$	$30.7 < t < 38$
Sign of y'	$y' > 0$	$y' < 0$	$y' > 0$
Conclusion	Increasing	Decreasing	Decreasing

Therefore, the model is increasing from 1970 to late 1988 and from late 2001 to 2008 and decreasing from late 1988 to late 2001.

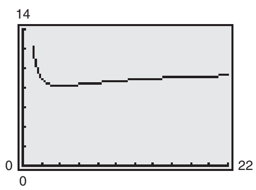
46. $C = 10\left(\frac{1}{x} + \frac{x}{x+3}\right),\ 1 \le x$

(a) $\frac{dC}{dx} = 10\left[-x^{-2} + \frac{(x+3)-(x)}{(x+3)^2}\right] = 10\left[-\frac{1}{x^2} + \frac{3}{(x+3)^2}\right] = 10\left[\frac{-(x+3)^2 + 3x^2}{x^2(x+3)^2}\right] = 10\left[\frac{2x^2 - 6x - 9}{x^2(x+3)^2}\right]$

By the Quadratic Formula, $2x^2 - 6x - 9 = 0$ when

$x = \frac{6 \pm \sqrt{108}}{4} = \frac{3 \pm 3\sqrt{3}}{2}$.

The only critical number in the domain is $x \approx 4.10$. C is decreasing on $[1, 4.10)$ and increasing on $(4.10, \infty)$.

(b) 

(c) $C = 9$ when $x = 2$ and $x = 15$.

Answers will vary.

47. $P = 2.36x - \frac{x^2}{25{,}000} - 3500,\ 0 \le x \le 50{,}000$

(a) $P' = 2.36 - \frac{1}{12{,}500}x$

Domain: $0 \le x \le 50{,}000$

Critical number: $x = 29{,}500$

Interval	$0 \le x \le 29{,}500$	$29{,}500 < x \le 50{,}000$
Sign of P'	$P' > 0$	$P' < 0$
Conclusion	Increasing	Decreasing

(b) You should charge the price that yields sales of $x = 29{,}500$ bags of popcorn. Since the function changes from increasing to decreasing at $x = 29{,}500$, the maximum profit occurs at this value.

48. (a) $P = R - C$

$$P = \frac{1}{20{,}000}(65{,}000x - x^2) - (0.6x + 7500)$$

$$= 3.25x - \frac{1}{20{,}000}x^2 - 0.6x - 7500$$

$$= -\frac{1}{20{,}000}x^2 + 2.65x - 7500$$

(b) $P' = -0.0001x + 2.65$

Domain: $0 \le x \le 50{,}000$

Critical number: $x = 26{,}500$

Interval	$0 < x < 26{,}500$	$26{,}500 < x < 50{,}000$
Sign of P'	$P' > 0$	$P' < 0$
Conclusion	Increasing	Decreasing

(c) The restaurant needs to sell 26,500 hamburgers to obtain a maximum profit. Because the function changes from increasing to decreasing at $x = 26{,}500$, the maximum profit occurs at this value

Section 3.2 Extrema and the First-Derivative Test

Skills Warm Up

1. $f(x) = 4x^4 - 2x^2 + 1$

$f'(x) = 16x^3 - 4x = 0$

$4x(4x^2 - 1) = 0$

$4x(2x + 1)(2x - 1) = 0$

$x = 0,\ x = \pm\frac{1}{2}$

2. $f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 - 10x$

$f'(x) = x^2 - 3x - 10 = 0$

$(x - 5)(x + 2) = 0$

$x = -2,\ x = 5$

3. $f(x) = 5x^{4/5} - 4x$

$f'(x) = 4x^{-1/5} - 4$

$= \frac{4}{x^{1/5}} - 4 = 0$

$x^{1/5} = 1$

$x = 1$

4. $f(x) = \frac{1}{2}x^2 - 3x^{5/3}$

$f'(x) = x - 5x^{2/3} = 0$

$x^{2/3}(x^{1/3} - 5) = 0$

$x = 0,\ x = 125$

5. $f(x) = \frac{x + 4}{x^2 + 1}$

$f'(x) = \frac{-x^2 - 8x + 1}{(x^2 + 1)^2} = 0$

$-x^2 - 8x + 1 = 0$

$$x = \frac{8 \pm \sqrt{(-8)^2 - 4(-1)(1)}}{2(-1)} = \frac{8 \pm \sqrt{68}}{-2} = \frac{8 \pm 2\sqrt{17}}{-2} = -4 \pm \sqrt{17}$$

6. $f(x) = \frac{x - 1}{x^2 + 4}$

$f'(x) = \frac{-x^2 + 2x + 4}{(x^2 + 4)^2} = 0$

$-x^2 + 2x + 4 = 0$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(-1)(4)}}{2(-1)} = \frac{-2 \pm \sqrt{20}}{-2} = \frac{-2 \pm 2\sqrt{5}}{-2} = 1 \pm \sqrt{5}$$

7. $g'(x) = -5x^4 - 8x^3 + 12x^2 + 2$

$g'(-4) = -574 < 0$

8. $g'(x) = -5x^4 - 8x^3 + 12x^2 + 2$

$g'(0) = 2 > 0$

9. $g'(x) = -5x^4 - 8x^3 + 12x^2 + 2$

$g'(1) = 1 > 0$

10. $g'(x) = -5x^4 - 8x^3 + 12x^2 + 2$

$g'(3) = -511 < 0$

11. $f(x) = 2x^2 - 11x - 6,\ (3, 6)$

$f'(x) = 4x - 11$

$f'(4) = 5 > 0$

So, f is increasing on $(3, 6)$.

12. $f(x) = x^3 + 2x^2 - 4x - 8,\ (-2, 0)$

$f'(x) = 3x^2 + 4x - 4$

$f'(-1) = -5 < 0$

So, f is decreasing on $(-2, 0)$.

1. $f(x) = -2x^2 + 4x + 3$

$f'(x) = 4 - 4x = 4(1 - x)$

Critical number: $x = 1$

Interval	$-\infty < x < 1$	$1 < x < \infty$
Sign of f'	$f' > 0$	$f' < 0$
Conclusion	Increasing	Decreasing

Relative maximum: $(1, 5)$

2. $f(x) = x^2 + 8x + 10$

$f'(x) = 2x + 8 = 2(x + 4)$

Critical number: $x = -4$

Interval	$(-\infty, -4)$	$(-4, \infty)$
Sign of f'	$f' < 0$	$f' > 0$
Conclusion	Decreasing	Increasing

Relative minimum: $(-4, -6)$

3. $f(x) = x^2 - 6x$

$f'(x) = 2x - 6 = 2(x - 3)$

Critical number: $x = 3$

Interval	$-\infty < x < 3$	$3 < x < \infty$
Sign of f'	$f' < 0$	$f' > 0$
Conclusion	Decreasing	Increasing

Relative minimum: $(3, -9)$

4. $f(x) = -4x^2 + 4x + 1$

$f'(x) = -8x + 4 = -4(2x - 1)$

Critical number: $x = \frac{1}{2}$

Interval	$-\infty < x < \frac{1}{2}$	$\frac{1}{2} < x < \infty$
Sign of f'	$f' > 0$	$f' < 0$
Conclusion	Increasing	Decreasing

Relative maximum: $\left(\frac{1}{2}, 2\right)$

5. $f(x) = x^4 - 12x^3$

$f'(x) = 4x^3 - 36x^2 = 4x^2(x - 9)$

Critical numbers: $x = 0, x = 9$

Interval	$-\infty < x < 0$	$0 < x < 9$	$9 < x < \infty$
Sign of f'	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion	Decreasing	Decreasing	Increasing

Relative minimum: $(9, -2187)$

6. $g(x) = \frac{1}{5}x^5 - x = \frac{1}{5}(x^5 - 5x)$

$g'(x) = x^4 - 1 = (x^2 + 1)(x + 1)(x - 1)$

Critical numbers: $x = \pm 1$

Interval	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of g'	$g' > 0$	$g' < 0$	$g' > 0$
Conclusion	Increasing	Decreasing	Increasing

Relative maximum: $\left(-1, \frac{4}{5}\right)$

Relative minimum: $\left(1, -\frac{4}{5}\right)$

7. $h(x) = -(x + 4)^3$

$h'(x) = -3(x + 4)^2$

Critical number: $x = -4$

Interval	$-\infty < x < -4$	$-4 < x < \infty$
Sign of h'	$h' < 0$	$h' < 0$
Conclusion	Decreasing	Decreasing

No relative extrema

8. $h(x) = 2(x - 3)^3$

$h'(x) = 6(x - 3)^2$

Critical number: $x = 3$

Interval	$-\infty < x < 3$	$3 < x < \infty$
Sign of h'	$h' > 0$	$h' > 0$
Conclusion	Increasing	Increasing

No relative extrema

9. $f(x) = x^3 - 6x^2 + 15$

$f'(x) = 3x^2 - 12x = 3x(x - 4)$

Critical numbers: $x = 0,\ x = 4$

Interval	$-\infty < x < 0$	$0 < x < 4$	$4 < x < \infty$
Sign of f'	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion	Increasing	Decreasing	Increasing

Relative maximum: $(0, 15)$

Relative minimum: $(4, -17)$

10. $f(x) = x^4 - 32x + 4$

$f'(x) = 4x^3 - 32 = 4(x^3 - 8)$

Critical number: $x = 2$

Interval	$-\infty < x < 2$	$2 < x < \infty$
Sign of f'	$f' < 0$	$f' > 0$
Conclusion	Decreasing	Increasing

Relative minimum: $(2, -44)$

11. $f(x) = 6x^{2/3} + 4x$

$f'(x) = 4x^{-1/3} + 4 = \dfrac{4}{x^{1/3}} + 4$

$f'(0)$ is undefined.

Set $f'(x) = 0$.

$$\frac{4}{x^{1/3}} + 4 = 0$$
$$4 = -4x^{1/3}$$
$$-1 = x^{1/3}$$
$$-1 = x$$

Critical numbers: $x = -1, 0$

Interval	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < \infty$
Sign of f'	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion	Increasing	Decreasing	Increasing

Relative maximum: $(-1, 2)$

Relative minimum: $(0, 0)$

12. $f(x) = 3x - 36x^{1/3}$

$f'(x) = 3 - 12x^{-2/3} = 3 - \dfrac{12}{x^{2/3}}$

$f'(0)$ is undefined.

Set $f'(x) = 0$.

$$3 - \frac{12}{x^{2/3}} = 0$$

$$3 = \frac{12}{x^{2/3}}$$

$$3x^{2/3} = 12$$

$$x^{2/3} = 4$$

$$x = (4)^{3/2} = \pm 8$$

Critical numbers: $x = \pm 8, 0$

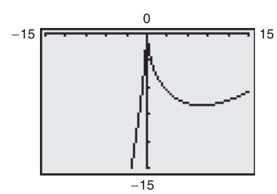
Interval	$-\infty < x < -8$	$-8 < x < 0$	$0 < x < 8$	$8 < x < \infty$
Sign of f'	$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion	Increasing	Decreasing	Decreasing	Increasing

Relative maximum: $(-8, 48)$

Relative minimum: $(8, -48)$

13. $f(x) = 2x - 6x^{2/3}$

$f'(x) = 2 - 4x^{-1/3} = 2 - \dfrac{4}{x^{1/3}} = \dfrac{2(x^{1/3} - 2)}{x^{1/3}}$



Critical numbers: $x = 0, 8$

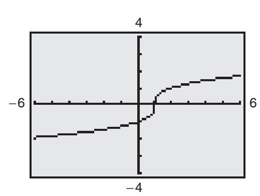
Interval	$-\infty < x < 0$	$0 < x < 8$	$8 < x < \infty$
Sign of f'	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion	Increasing	Decreasing	Increasing

Relative maximum: $(0, 0)$

Relative minimum: $(8, -8)$

14. $f(t) = (t - 1)^{1/3}$

$f'(t) = \dfrac{1}{3}(t - 1)^{-2/3} = \dfrac{1}{3(t - 1)^{2/3}}$



Critical number: $t = 1$

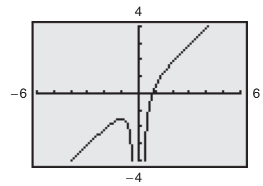
Interval	$-\infty < t < 1$	$1 < t < \infty$
Sign of f'	$f' > 0$	$f' > 0$
Conclusion	Increasing	Increasing

No relative extrema

15. $g(t) = t - \dfrac{1}{2t^2} = t - \dfrac{1}{2}t^{-2}$

$g'(t) = 1 + t^{-3}$

$= \dfrac{t^3 + 1}{t^3}$



Critical number: $t = -1$

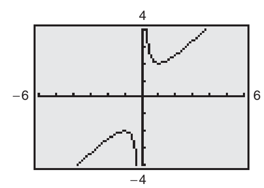
Discontinuity: $t = 0$

Interval	$-\infty < t < -1$	$-1 < t < 0$	$0 < t < \infty$
Sign of g'	$g' > 0$	$g' < 0$	$g' > 0$
Conclusion	Increasing	Decreasing	Increasing

Relative maximum: $\left(-1, -\frac{3}{2}\right)$

16. $f(x) = x + \dfrac{1}{x} = x + x^{-1}$

$f'(x) = 1 - x^{-2} = 1 - \dfrac{1}{x^2}$



Critical numbers: $x = \pm 1$

Interval	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
Sign of f'	$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion	Increasing	Decreasing	Decreasing	Increasing

Relative maximum: $(-1, -2)$

Relative minimum: $(1, 2)$

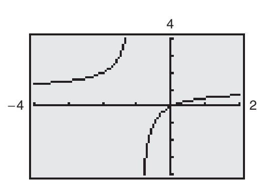
17. $f(x) = \dfrac{x}{x + 1}$

$f'(x) = \dfrac{(x + 1) - x}{(x + 1)^2} = \dfrac{1}{(x + 1)^2}$

Discontinuity: $x = -1$

Interval	$-\infty < x < -1$	$-1 < x < \infty$
Sign of f'	$f' > 0$	$f' > 0$
Conclusion	Increasing	Increasing

No relative extrema



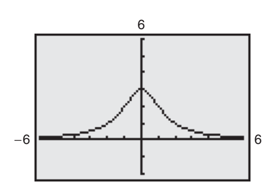
18. $h(x) = \dfrac{6}{x^2 + 2} = 6(x^2 + 2)^{-1}$

$h'(x) = -6(x^2 + 2)^{-2}(2x) = \dfrac{-12x}{(x^2 + 2)^2}$

Critical number: $x = 0$

Interval	$-\infty < x < 0$	$0 < x < \infty$
Sign of h'	$h' > 0$	$h' < 0$
Conclusion	Increasing	Decreasing

Relative maximum: $(0, 3)$



19. $f(x) = 2(3 - x), [-1, 2]$

$f'(x) = -2$

No critical numbers

x-value	Endpoint $x = -1$	Endpoint $x = 2$
$f(x)$	8	2
Conclusion	Maximum	Minimum

20. $f(x) = \frac{1}{3}(2x + 5), [0, 5]$

$f'(x) = \frac{2}{3}$

No critical numbers

x-value	Endpoint $x = 0$	Endpoint $x = 5$
$f(x)$	$\frac{5}{3}$	5
Conclusion	Minimum	Maximum

21. $f(x) = 5 - 2x^2, [0, 3]$

$f'(x) = -4x$

Critical number: $x = 0$ (endpoint)

x-value	Endpoint $x = 0$	Endpoint $x = 3$
$f(x)$	5	−13
Conclusion	Maximum	Minimum

22. $f(x) = x^2 + 2x - 4, [-1, 1]$

$f'(x) = 2x + 2$

Critical number: $x = -1$ (endpoint)

x-value	Endpoint $x = -1$	Endpoint $x = 1$
$f(x)$	−5	−1
Conclusion	Minimum	Maximum

23. $f(x) = x^3 - 3x^2, [-1, 3]$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$

Critical numbers: $x = 0, x = 2$

x-value	Endpoint $x = -1$	Critical $x = 0$	Critical $x = 2$	Endpoint $x = 3$
$f(x)$	−4	0	−4	0
Conclusion	Minimum	Maximum	Minimum	Maximum

24. $f(x) = x^3 - 12x, [0, 4]$

$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 0$

Critical numbers: $x = \pm 2$

x-value	Endpoint $x = 0$	Critical $x = 2$	Endpoint $x = 4$
$f(x)$	0	−16	16
Conclusion		Minimum	Maximum

25. $h(s) = \dfrac{1}{3 - s} = (3 - s)^{-1}, [0, 2]$

$h'(s) = -(3 - s)^{-2}(-1) = \dfrac{1}{(3 - s)^2}$

No critical numbers

s-value	Endpoint $s = 0$	Endpoint $s = 2$
$h(s)$	$\frac{1}{3}$	1
Conclusion	Minimum	Maximum

26. $h(t) = \dfrac{t}{t - 2}, [3, 5]$

$h'(t) = \dfrac{(t - 2) - t}{(t - 2)^2} = -\dfrac{2}{(t - 2)^2}$

No critical numbers

t-value	Endpoint $t = 3$	Endpoint $t = 5$
$h(t)$	3	$\frac{5}{3}$
Conclusion	Maximum	Minimum

27. $g(t) = \dfrac{t^2}{t^2 + 3}, [-1, 1]$

$g'(t) = \dfrac{(t^2 + 3)(2t) - t^2(2t)}{(t^2 + 3)^2} = \dfrac{6t}{(t^2 + 3)^2}$

Critical number: $t = 0$

t-value	Endpoint $t = -1$	Critical $t = 0$	Endpoint $t = 1$
$g(t)$	$\frac{1}{4}$	0	$\frac{1}{4}$
Conclusion	Maximum	Minimum	Maximum

28. $g(x) = 4\left(1 + \dfrac{1}{x} + \dfrac{1}{x^2}\right) = 4(1 + x^{-1} + x^{-2}), [-4, 5]$

$g'(x) = 4(-x^{-2} - 2x^{-3}) = 4\left(-\dfrac{1}{x^2} - \dfrac{2}{x^3}\right) = -4\left(\dfrac{x + 2}{x^3}\right)$

Critical number: $x = -2$

Discontinuity: $x = 0$

x-value	Endpoint $x = -4$	Critical $x = -2$	Discontinuity $x = 0$	Endpoint $x = 5$
$g(x)$	3.25	3	Undefined	4.96
Conclusion		Minimum		

No maximum

29. $h(t) = (t - 1)^{2/3}, [-7, 2]$

$h'(t) = \dfrac{2}{3}(t - 1)^{-1/3} = \dfrac{2}{3(t - 1)^{1/3}}$

Critical number: $t = 1$

t-value	Endpoint $t = -7$	Critical $t = 1$	Endpoint $t = 2$
$h(t)$	4	0	1
Conclusion	Maximum	Minimum	

30. $g(x) = (x^2 - 4)^{2/3}, [-6, 3]$

$g'(x) = \dfrac{2}{3}(x^2 - 4)^{-1/3}(2x)$

$= \dfrac{4x}{3(x^2 - 4)^{1/3}}$

Critical numbers: $x = 0, \pm 2$

x-value	Endpoint $x = -6$	Critical $x = -2$	Critical $x = 0$	Critical $x = 2$	Endpoint $x = 3$
$f(x)$	$8\sqrt[3]{2} \approx 10.1$	0	$2\sqrt[3]{2} \approx 2.5$	0	$\sqrt[3]{25} \approx 2.9$
Conclusion	Maximum	Minimum		Minimum	

31. Critical number: $x = 2$

The function has an absolute maximum at the critical number.

32. Critical number: $x = 0$

The function has no maximum or minimum at the critical number.

33. Critical number: $x = 1$, absolute maximum and relative maximum

Critical number: $x = 2$, absolute minimum and relative minimum

Critical number: $x = 3$, absolute maximum and relative maximum

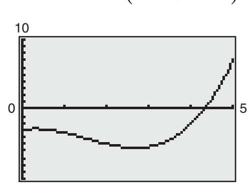
34. Critical number: $x = 2$, neither absolute nor relative extrema

Critical number: $x = 5$, absolute maximum and relative maximum

35. $f(x) = 0.4x^3 - 1.8x^2 + x - 3, [0, 5]$

Maximum: $(5, 7)$

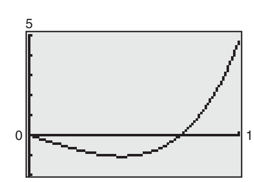
Minimum: $(2.69, 5.55)$



36. $f(x) = 3.2x^5 + 5x^3 - 3.5x, [0, 1]$

Maximum: $(1, 4.7)$

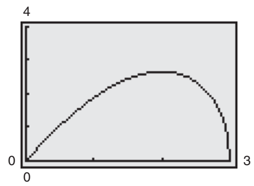
Minimum: $(0.44, -1.06)$



37. $f(x) = \frac{4}{3}x\sqrt{3 - x}, [0, 3]$

Maximum: $(2, 2.67)$

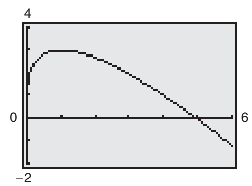
Minimum: $(0, 0), (3, 0)$



38. $f(x) = 4\sqrt{x} - 2x + 1, [0, 6]$

Maximum: $(1, 3)$

Minimum: $(6, -1.20)$



39. $f(x) = x^2 + 16x^{-1}, [0, \infty)$

$$f'(x) = 2x - 16x^{-2} = 2x - \frac{16}{x^2}$$

$$= \frac{2(x^3 - 8)}{x^2}$$

Critical number: $x = 2$

x-value	Critical $x = 2$
$f(x)$	12
Conclusion	Maximum

Absolute minimum: $(2, 12)$

40. $f(x) = \dfrac{8}{x + 1} = 8(x + 1)^{-1}, [0, \infty)$

$$f'(x) = -8(x + 1)^{-2} = -\frac{8}{(x + 1)^2}$$

No critical numbers

x-value	Endpoint $x = 0$
$f(x)$	8
Conclusion	Maximum

41. $f(x) = \dfrac{2x}{x^2 + 4}, [0, \infty)$

$$f'(x) = \frac{(x^2 + 4)(2) - 2x(2x)}{(x^2 + 4)^2}$$

$$= \frac{8 - 2x^2}{(x^2 + 4)^2}$$

$$= \frac{2(2 - x)(2 + x)}{(x^2 + 4)^2}$$

Critical number: $x = 2$

x-value	Endpoint $x = 0$	Critical $x = 2$
$f(x)$	0	$\frac{1}{2}$
Conclusion	Minimum	Maximum

42. $f(x) = 8 - \dfrac{4x}{x^2 + 1}, [0, \infty)$

$$f'(x) = -\frac{(x^2 + 1)(4) - 4x(2x)}{(x^2 + 1)^2}$$

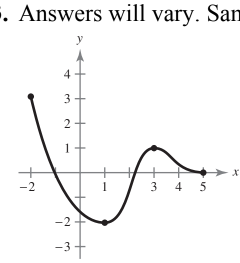
$$= \frac{4x^2 - 4}{(x^2 + 1)^2}$$

$$= \frac{4(x + 1)(x - 1)}{(x^2 + 1)^2}$$

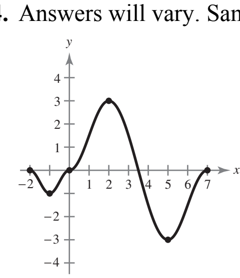
Critical number: $x = 1$

x-value	Endpoint $x = 0$	Critical $x = 1$
$f(x)$	8	6
Conclusion	Maximum	Minimum

43. Answers will vary. Sample answer:



44. Answers will vary. Sample answer:



45. (a) Population tends to increase each year, so the minimum population occurred in 1790 and the maximum population occurred in 2010.

(b) $P = 0.000006t^3 + 0.005t^2 + 0.14t + 4.6$

$\frac{dP}{dt} = P' = 0.000018t^2 + 0.01t + 0.14$

$P' = 0: 0.000018t^2 + 0.01t + 0.14 = 0$

$t \approx 14.4, \ -541.2$

No critical numbers in interval $[-10, 210]$.

t-value	Endpoint $t = -10$	Endpoint $t = 210$
$P(t)$	3.69 million	310.07 million
Conclusion	Minimum	Maximum

(c) The minimum population was about 3.69 million in 1790 and the maximum population was about 310.07 million in 2010.

46. (a) The fertility rate was the highest in 1970 $(t = 0)$ and was approximately 2475 births per 1000 women.

(b) During the time period from 1983 to 1991 $(13 \le t \le 21)$, the fertility rate was increasing most rapidly.

During the time period from 1997 to 2003 $(27 \le t \le 33)$, the fertility rate was increasing most slowly.

(c) During the time period from 1970 to 1975 $(0 \le t \le 5)$, the fertility rate was decreasing most rapidly.

During the time period from 1979 to 1982 $(9 \le t \le 12)$, the fertility rate was decreasing most slowly.

(d) Answers will vary.

47. $C = 3x + \frac{20{,}000}{x} = 3x + 20{,}000x^{-1}, \ 0 < x \le 200$

$C' = 3 - 20{,}000x^{-2} = \frac{3x^2 - 20{,}000}{x^2}$

Critical numbers: $x = \sqrt{\frac{20{,}000}{3}} \approx 82$ units

$C(82) \approx 489.90$, which is the minimum by the First-Derivative Test

48. $v = k(R - r)r^2, \ 0 \le r < R$

$= k(Rr^2 - r^3)$

$\frac{dv}{dr} = k(2Rr - 3r^2) = kr(2R - 3r)$

Critical numbers: $r = 0, \frac{2R}{3}$

The maximum air velocity occurs when $r = 2R/3$.

r-value	Endpoint $r = 0$	Critical $r = (2R)/3$
v	0	$(4kR^3)/27$
Conclusion		Maximum

49. Demand: (6000, 1.00), (5600, 1.20)

$m = \frac{1.20 - 1.00}{5600 - 6000} = \frac{0.2}{-400} = -0.0005$

$p - 1 = -0.0005(x - 6000)$

$p = -0.0005x + 4$

Cost $= C = 5000 + 0.50x$

Profit $= P = R - C$

$= xp - C$

$= x(-0.0005x + 4) - (5000 + 0.50x)$

$= -0.0005x^2 + 3.50x - 5000$

Selling the cans for \$2.25 will maximize the profit.

Section 3.3 Concavity and the Second-Derivative Test

Skills Warm Up

1. $f(x) = 4x^4 - 9x^3 + 5x - 1$

$f'(x) = 16x^3 - 27x^2 + 5$

$f''(x) = 48x^2 - 54x$

2. $g(s) = (s^2 - 1)(s^2 - 3s + 2)$

$= s^4 - 3s^3 + s^2 + 3s - 2$

$g'(s) = 4s^3 - 9s^2 + 2s + 3$

$g''(s) = 12s^2 - 18s + 2$

3. $g(x) = (x^2 + 1)^4$

$g'(x) = 4(x^2 + 1)^3(2x) = 8x(x^2 + 1)^3$

$g''(x) = 8x\left[3(x^2 + 1)^2(2x)\right] + (x^2 + 1)^3(8)$

$= 8(x^2 + 1)^2(7x^2 + 1)$

4. $f(x) = (x - 3)^{4/3}$

$f'(x) = \frac{4}{3}(x - 3)^{1/3}$

$f''(x) = \frac{4}{9}(x - 3)^{-2/3} = \frac{4}{9(x - 3)^{2/3}}$

5. $h(x) = \frac{4x + 3}{5x - 1}$

$h'(x) = \frac{(5x - 1)(4) - (4x + 3)(5)}{(5x - 1)^2}$

$= \frac{20x - 4 - 20x - 15}{(5x - 1)^2}$

$= -19(5x - 1)^{-2}$

$h''(x) = 18(5x - 1)^{-3}(5) = \frac{90}{(5x - 1)^3}$

6. $f(x) = \frac{2x - 1}{3x + 2}$

$f'(x) = \frac{(3x + 2)(2) - (2x - 1)(3)}{(3x + 2)^2}$

$= \frac{6x + 4 - 6x + 3}{(3x + 2)^2}$

$= 7(3x + 2)^{-2}$

$f''(x) = -14(3x + 2)^{-3}(3) = -\frac{42}{(3x + 2)^3}$

7. $f(x) = 5x^3 - 5x + 11$

$f'(x) = 15x^2 - 5 = 0$

$15x^2 = 5$

$x^2 = \frac{1}{3}$

$x = \pm\frac{1}{\sqrt{3}}$

Critical numbers: $x = \pm\frac{1}{\sqrt{3}}$

8. $f(x) = x^4 - 4x^3 - 10$

$f'(x) = 4x^3 - 12x^2 = 0$

$4x^2(x - 3) = 0$

$4x^2 = 0 \Rightarrow x = 0$

$x - 3 = 0 \Rightarrow x = 3$

Critical numbers: $x = 0,\ x = 3$

9. $g(t) = \frac{16 + t^2}{t} = \frac{16}{t} - t$

$g'(t) = -\frac{16}{t^2} - 1 = 0$

$t^2 = -16$

No critical numbers

10. $h(x) = \frac{x^4 - 50x^2}{8}$

$h'(x) = \frac{1}{2}x^3 - \frac{25}{2}x = 0$

$\frac{1}{2}x(x^2 - 25) = 0$

$\frac{1}{2}x(x + 5)(x - 5) = 0$

$\frac{1}{2}x = 0 \Rightarrow x = 0$

$x + 5 = 0 \Rightarrow x = -5$

$x - 5 = 0 \Rightarrow x = 5$

Critical numbers: $x = 0,\ x = \pm 5$

1. f is increasing so $f' > 0$.
f is concave upward so $f'' > 0$.

2. f is increasing so $f' > 0$.
f is concave downward so $f'' < 0$.

3. f is decreasing so $f' < 0$.
f is concave downward so $f'' < 0$.

4. f is decreasing so $f' < 0$.
f is concave upward so $f'' > 0$.

5. $f(x) = -3x^2$

$f'(x) = -6x$

$f''(x) = -6$

$f''(x) \neq 0$ for any value of x.

Interval	$-\infty < x < \infty$
Sign of f''	$f'' < 0$
Conclusion	Concave downward

6. $f(x) = -5x^{1/2}$, domain: $[0, \infty]$

$f'(x) = -\frac{5}{2}x$

$f''(x) = \frac{5}{4}x^{-3/2} = \frac{5}{4x^{3/2}}$

$f''(x) \neq 0$ for any value of x.

$f''(x)$ is undefined at $x = 0$.

Interval	$0 < x < \infty$
Sign of f''	$f'' > 0$
Conclusion	Concave upward

7. $f(x) = -x^3 + 3x^2 - 2$

$f'(x) = -3x^2 + 6x$

$f''(x) = -6x + 6$

$f''(x) = 0$ when $x = 1$.

Interval	$-\infty < x < 1$	$1 < x < \infty$
Sign of f''	$f'' > 0$	$f'' < 0$
Conclusion	Concave upward	Concave downward

8. $f(x) = -x^3 + 6x^2 - 9x - 1$

$f'(x) = -3x^2 + 12x - 9$

$f''(x) = -6x + 12$

$f''(x) = 0$ when $x = 2$.

Interval	$-\infty < x < 2$	$2 < x < \infty$
Sign of f''	$f'' > 0$	$f'' < 0$
Conclusion	Concave upward	Concave downward

9. $f(x) = \frac{x^2 - 1}{2x + 1}$

$$f'(x) = \frac{(2x + 1)(2x) - (x^2 - 1)(2)}{(2x + 1)^2} = \frac{2x^2 + 2x + 2}{(2x + 1)^2} = (2x^2 + 2x + 2)(2x + 1)^{-2}$$

$$\begin{aligned} f''(x) &= (2x^2 + 2x + 2)\left[-2(2x + 1)^{-3}(2)\right] + (2x + 1)^{-2}(4x + 2) \\ &= -8(x^2 + x + 1)(2x + 1)^{-3} + 2(2x + 1)^{-2}(2x + 1) \\ &= 2(2x + 1)^{-3}\left[-4(x^2 + x + 1) + (2x + 1)(2x + 1)\right] \\ &= 2(2x + 1)^{-3}\left[-4x^2 - 4x - 4 + 4x^2 + 4x + 1\right] \\ &= \frac{-6}{(2x + 1)^3} \end{aligned}$$

$f''(x) \neq 0$ for any value of x.

$x = -\frac{1}{2}$ is a discontinuity.

Interval	$-\infty < x < -\frac{1}{2}$	$-\frac{1}{2} < x < \infty$
Sign of f''	$f'' > 0$	$f'' < 0$
Conclusion	Concave upward	Concave downward

10. $f(x) = \frac{x^2 + 4}{4 - x^2}$

$f'(x) = \frac{(4 - x^2)(2x) - (x^2 + 4)(-2x)}{(4 - x^2)^2} = \frac{16x}{(4 - x^2)^2}$

$f''(x) = \frac{(4 - x^2)^2(16) - 16x\left[2(4 - x^2)(-2x)\right]}{(4 - x^2)^4}$

$= \frac{16(4 + 3x^2)}{(4 - x^2)^3}$

f'' is undefined at $x = \pm 2$.

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
Sign of f''	$f' < 0$	$f'' > 0$	$f'' < 0$
Conclusion	Concave downward	Concave upward	Concave downward

11. $f(x) = \frac{24}{x^2 + 12} = 24(x^2 + 12)^{-1}$

$f'(x) = -24(2x)(x^2 + 12)^{-2} = -48x(x^2 + 12)^{-2}$

$f''(x) = -48x\left[-2(x^2 + 12)^{-3}(2x)\right] + (x^2 + 12)^{-2}(-48) = \frac{-48(-4x^2 + x^2 + 12)}{(x^2 + 12)^3} = \frac{144(x^2 - 4)}{(x^2 + 12)^3}$

$f''(x) = 0$ when $x = \pm 2$.

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
Sign of f''	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion	Concave upward	Concave downward	Concave upward

12. $f(x) = \frac{x^2}{x^2 + 1}$

$f'(x) = \frac{(x^2 + 1)(2x) - x^2(2x)}{(x^2 + 1)^2} = \frac{2x}{(x^2 + 1)^2}$

$f''(x) = \frac{(x^2 + 1)^2(2) - 2x\left[2(x^2 + 1)(2x)\right]}{(x^2 + 1)^4} = \frac{-2(3x^2 - 1)}{(x^2 + 1)^3}$

$f''(x) = 0$ when $x = \pm\frac{1}{\sqrt{3}}$.

Interval	$-\infty < x < -\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}} < x < \infty$
Sign of f''	$f'' < 0$	$f'' > 0$	$f'' < 0$
Conclusion	Concave downward	Concave upward	Concave downward

13. $f(x) = x^3 - 9x^2 + 24x - 18$

$f'(x) = 3x^2 - 18x + 24$

$f''(x) = 6x - 18 = 6(x - 3)$

$f''(x) = 0$

$6(x - 3) = 0$

$x = 3$

Interval	$-\infty < x < 3$	$3 < x < \infty$
Sign of f''	$f'' < 0$	$f'' > 0$
Conclusion	Concave downward	Concave upward

Point of inflection: $(3, 0)$

14. $f(x) = -4x^3 - 8x^2 + 32$

$f'(x) = -12x^2 - 16x$

$f''(x) = -24x - 16 = -8(3x + 2)$

$f''(x) = 0$

$-8(3x + 2) = 0$

$x = -\frac{2}{3}$

Interval	$-\infty < x < -\frac{2}{3}$	$-\frac{2}{3} < x < \infty$
Sign of f''	$f'' > 0$	$f'' < 0$
Conclusion	Concave upward	Concave downward

Point of inflection: $\left(-\frac{2}{3}, \frac{800}{27}\right)$

15. $f(x) = 2x^3 - 3x^2 - 12x + 5$

$f'(x) = 6x^2 - 6x - 12$

$f''(x) = 12x - 6 = 6(2x - 1)$

$f''(x) = 0$

$6(2x - 1) = 0$

$x = \frac{1}{2}$

Interval	$-\infty < x < \frac{1}{2}$	$\frac{1}{2} < x < \infty$
Sign of f''	$f'' < 0$	$f'' > 0$
Conclusion	Concave downward	Concave upward

Point of inflection: $\left(\frac{1}{2}, -\frac{3}{2}\right)$

16. $f(x) = \frac{1}{2}x^4 + 2x^3$

$f'(x) = 2x^3 + 6x^2$

$f''(x) = 6x^2 + 12x = 6x(x + 2)$

$f''(x) = 0$

$6x(x + 2) = 0$

$x = 0$ or $x = -2$

Interval	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < \infty$
Sign of f''	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion	Concave upward	Concave downward	Concave upward

Points of inflection: $(-2, -8)$, $(0, 0)$

17. $g(x) = 2x^4 - 8x^3 + 12x^2 + 12x$

$g'(x) = 8x^3 - 24x^2 + 24x + 12$

$g''(x) = 24x^3 - 48x + 24 = 24(x^2 - 2x + 1) = 24(x - 1)^2$

$g''(x) = 0$

$24(x - 1)^2 = 0$

$x = 1$

Interval	$-\infty < x < -1$	$-1 < x < \infty$
Sign of $g''(x)$	$g'' > 0$	$g'' > 0$
Conclusion	Concave upward	Concave upward

No points of inflection

18. $g(x) = x^5 + 5x^4 - 40x^2$

$g'(x) = 5x^4 + 20x^3 - 80x$

$g''(x) = 20x^3 + 60x^2 - 80 = 20(x^3 + 3x^2 - 4)$

$$g''(x) = 0$$
$$20(x^3 + 3x^2 - 4) = 0$$
$$x = -2 \text{ or } x = 1$$

(**Note:** Use synthetic division to solve.)

Interval	$-\infty < x < -2$	$-2 < x < 1$	$1 < x < \infty$
Sign of $g''(x)$	$g'' < 0$	$g'' < 0$	$g'' > 0$
Conclusion	Concave downward	Concave downward	Concave upward

Point of inflection: $(1, -34)$

19. $f(x) = x(6 - x)^2 = x(36 - 12x + x^2) = 36x - 12x^2 + x^3$

$f'(x) = 3x^2 - 24x + 36$

$f''(x) = 6x - 24 = 6(x - 4)$

$$f''(x) = 0$$
$$6(x - 4) = 0$$
$$x = 4$$

Interval	$-\infty < x < 4$	$4 < x < \infty$
Sign of $f''(x)$	$f'' < 0$	$f'' > 0$
Conclusion	Concave downward	Concave upward

Point of inflection: $(4, 16)$

20. $f(x) = (x - 1)^3(x - 5)$

$f'(x) = 4(x - 1)^2(x - 4)$

$f''(x) = 12(x - 1)(x - 3)$

$$f''(x) = 0$$
$$12(x - 1)(x - 3) = 0$$
$$x = 1 \text{ or } x = 3$$

Interval	$-\infty < x < 1$	$1 < x < 3$	$3 < x < \infty$
Sign of $f''(x)$	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion	Concave upward	Concave downward	Concave upward

Points of inflection: $(1, 0)$, $(3, -16)$

21. $f(x) = 6x - x^2$

$f'(x) = 6 - 2x$

Critical number: $x = 3$

$f''(x) = -2$

$f''(3) = -2 < 0$

(3, 9) is a relative maximum.

22. $f(x) = 9x^2 - x^3$

$f'(x) = 18x - 3x^2 = -3x(x - 6)$

Critical numbers: $x = 0, x = 6$

$f''(x) = 18 - 6x$

$f''(0) = 18 > 0$

$f''(6) = -18 < 0$

So, $(0, 0)$ is a relative minimum and $(6, 108)$ is a relative maximum.

23. $f(x) = x^3 - 5x^2 + 7x$

$f'(x) = 3x^2 - 10x + 7 = (3x - 7)(x - 1)$

Critical numbers: $x = 1, x = \frac{7}{3}$

$f''(x) = 6x - 10$

$f''(1) = -4 < 0$

$f''\left(\frac{7}{3}\right) = 4 > 0$

So, (1, 3) is a relative maximum and $\left(\frac{7}{3}, \frac{49}{27}\right)$ is a relative minimum.

24. $f(x) = x^4 + 8x^3 - 6$

$f'(x) = 4x^3 + 24x^2 = 4x^2(x + 6)$

Critical numbers: $x = 0, x = -6$

$f''(x) = 12x^2 + 48x$

$f''(0) = 0 \Rightarrow$ Test fails.

Use the First-Derivative Test to conclude that there is no extrema at $x = 0$.

$f''(-6) = 144 > 0$

So, $(-6, -438)$ is a relative minimum.

25. $f(x) = x^{2/3} - 3$

$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$

Critical number: $x = 0$

The Second-Derivative Test does not apply, so use the First-Derivative Test to conclude that $(0, -3)$ is a relative minimum.

26. $f(x) = x + \frac{4}{x} = x + 4x^{-1}$

$f'(x) = 1 - 4x^{-2} = \frac{x^2 - 4}{x^2}$

Critical numbers: $x = \pm 2$

$f''(x) = \frac{8}{x^3}$

$f''(2) = 1 > 0$

$f''(-2) = -1 < 0$

So, (2, 4) is a relative minimum and $(-2, -4)$ is a relative maximum.

27. $f(x) = \sqrt{x^2 + 1} = (x + 1)^{1/2}$

$f'(x) = \frac{1}{2}(x^2 + 1)^{-1/2}(2x) = \frac{x}{(x^2 + 1)^{1/2}}$

Critical number: $x = 0$

$$f''(x) = \frac{(x^2 + 1)^{1/2} - x\left[\frac{1}{2}(x^2 + 1)^{-1/2}(2x)\right]}{x^2 + 1} = \frac{1}{(x^2 + 1)^{3/2}}$$

$f''(0) = 1 > 0$

So, (0, 1) is a relative minimum.

28. $f(x) = \sqrt{2x^2 + 6} = (2x^2 + 6)^{1/2}$

$$f'(x) = \frac{1}{2}(2x^2 + 6)^{-1/2}(4x) = \frac{2x}{(2x^2 + 6)^{1/2}}$$

Critical number: $x = 0$

$$f''(x) = \frac{(2x^2 + 6)^{1/2}(2) - 2x\left[\frac{1}{2}(2x^2 + 6)^{-1/2}(4x)\right]}{2x^2 + 6} = \frac{12}{(2x^2 + 6)^{3/2}}$$

$f''(0) = \frac{2}{\sqrt{6}} > 0$

So, $(0, \sqrt{6})$ is a relative minimum.

29. $f(x) = \sqrt{9 - x^2} = (9 - x^2)^{1/2}$

$f'(x) = \frac{1}{2}(9 - x^2)^{-1/2}(-2x) = -\frac{x}{(9 - x^2)^{1/2}}$

Critical number: $x = 0, x = \pm 3$

$$f''(x) = \frac{(9 - x^2)^{1/2}(-1) - (-x)\left[\frac{1}{2}(9 - x^2)^{-1/2}(-2x)\right]}{9 - x^2}$$

$$= -\frac{9}{(9 - x^2)^{3/2}}$$

$f''(0) = -\frac{1}{3} < 0$

So, (0, 3) is a relative maximum. There are absolute minima at $(\pm 3, 0)$.

30. $f(x) = \sqrt{4 - x^2} = (4 - x^2)^{1/2}$

$f'(x) = \frac{1}{2}(4 - x^2)^{-1/2}(-2x) = -\frac{x}{(4 - x^2)^{1/2}}$

Critical number: $x = 0, x = \pm 2$

$$f''(x) = \frac{(4 - x^2)^{1/2}(-1) - (-x)\left[\frac{1}{2}(4 - x^2)^{-1/2}(-2x)\right]}{4 - x^2}$$

$$= -\frac{4}{(4 - x^2)^{3/2}}$$

$f''(0) = -\frac{1}{2} < 0$

So, (0, 2) is a relative maximum.

There are absolute minima at $(\pm 2, 0)$.

31. $f(x) = \frac{8}{x^2 + 2} = 8(x^2 + 2)^{-1}$

$f'(x) = -8(x^2 + 2)^{-2}(2x)$

$= -\frac{16x}{(x^2 + 2)^2}$

Critical number: $x = 0$

$$f''(x) = \frac{(x^2 + 2)^2(-16) - (-16x)\left[(2)(x^2 + 2)(2x)\right]}{(x^2 + 2)^4}$$

$$= \frac{(x^2 + 2)\left[-16(x^2 + 2) + 64x^2\right]}{(x^2 + 2)^4}$$

$$= \frac{48x^2 - 32}{(x^2 + 2)^3}$$

$f''(0) = -4 < 0$

So, $(0, 4)$ is a relative maximum.

32. $f(x) = \frac{x}{x^2 + 16}$

$f'(x) = \frac{(x^2 + 16)(1) - x(2x)}{(x^2 + 16)^2} = \frac{16 - x^2}{(x^2 + 16)^2}$

Critical numbers: $x = \pm 4$

$$f''(x) = \frac{(x^2 + 16)^2(-2x) - (16 - x^2)(2)(x^2 + 16)(2x)}{(x^2 + 16)^4}$$

$$= \frac{2x(x^2 + 16)\left[-(x^2 + 16) - 2(16 - x^2)\right]}{(x^2 + 16)^4}$$

$$= \frac{2x(x^2 - 48)}{(x^2 + 16)^3}$$

$f''(4) = -\frac{1}{128} < 0$

$f''(-4) = -\frac{1}{128} > 0$

So, $\left(4, \frac{1}{8}\right)$ is a relative maximum and $\left(-4, -\frac{1}{8}\right)$ is a relative minimum.

33. $f(x) = \frac{x}{x - 1}$

$f'(x) = \frac{(x - 1) - (x)}{(x - 1)^2}$

$= -\frac{1}{(x - 1)^2}$

No critical numbers

No relative extrema

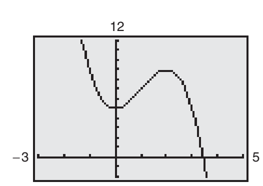
34. $f(x) = \frac{x}{x^2 - 1}$

$f'(x) = \frac{(x^2 - 1) - x(2x)}{(x^2 - 1)^2} = -\frac{x^2 + 1}{(x^2 - 1)^2}$

No critical numbers

No relative extrema

35. $f(x) = 5 + 3x^2 - x^3$



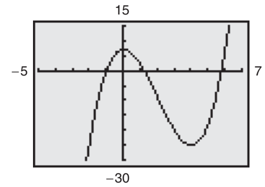
Relative minimum: (0, 5)

Relative maximum: (2, 9)

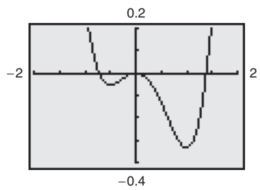
36. $f(x) = x^3 - 6x^2 + 7$

Relative maximum: $(0, 7)$

Relative minimum: $(4, -25)$



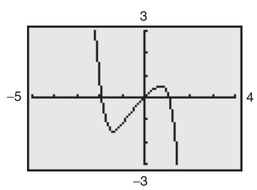
37. $f(x) = \frac{1}{2}x^4 - \frac{1}{3}x^3 - \frac{1}{2}x^2$



Relative maximum: (0, 0)

Relative minima: $(-0.5, -0.052)$, $(1, -0.333)$

38. $f(x) = -\frac{1}{3}x^5 - \frac{1}{2}x^4 + x$



Relative maximum: (0.683, 0.525)

Relative minimum: $(-1.413, -1.529)$

39. $f(x) = x^3 - 12x$

$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$

Critical numbers: $x = \pm 2$

$f''(x) = 6x$

$f''(2) = 12 > 0$

$f''(-2) = -12 < 0$

Relative maximum: $(-2, 16)$

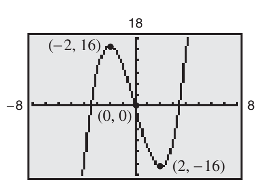
Relative minimum: $(2, -16)$

$f''(x) = 0$ when $x = 0$.

$f''(x) < 0$ on $(-\infty, 0)$.

$f''(x) > 0$ on $(0, \infty)$.

Point of inflection: (0, 0)



40. $f(x) = x^3 - 3x$

$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$

Critical numbers: $x = \pm 1$

$f''(x) = 6x$

$f''(-1) = -6 < 0$

$f''(1) = 6 > 0$

Relative maximum: $(-1, 2)$

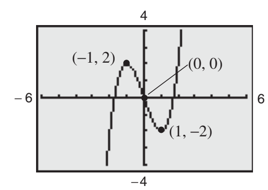
Relative minimum: $(1, -2)$

$f''(x) = 0$ when $x = 0$.

$f''(x) < 0$ on $(-\infty, 0)$.

$f''(x) > 0$ on $(0, \infty)$.

Point of inflection: (0, 0)



41. $g(x) = x^{1/2} + 4x^{-1/2}$

$g'(x) = \frac{1}{2}x^{-1/2} - 2x^{-3/2}$

$= \frac{1}{2x^{1/2}} - \frac{2}{x^{3/2}} = \frac{x - 4}{2x^{3/2}}$

Critical number: $x = 4$

$g''(x) = -\frac{1}{4}x^{-3/2} + 3x^{-5/2} = -\frac{1}{4x^{3/2}} + \frac{3}{x^{5/2}} = \frac{12 - x}{4x^{5/2}}$

$g''(4) = \frac{1}{16} > 0$

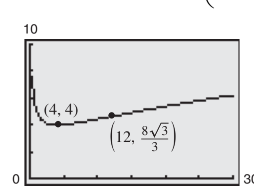
Relative minimum at $(4, 4)$

$g''(x) = 0$ when $x = 12$.

$g''(x) > 0$ on $(0, 12)$

$g''(x) < 0$ on $(12, \infty)$

Point of inflection: $\left(12, \frac{8\sqrt{3}}{3}\right)$



42. $f(x) = x^3 - \frac{3}{2}x^2 - 6x$

$f'(x) = 3x^2 - 3x - 6 = 3(x + 1)(x - 2)$

Critical numbers: $x = -1, x = 2$

$f''(x) = 6x - 3$

$f''(-1) = -9 < 0$

$f''(2) = 9 > 0$

Relative maximum: $\left(-1, \frac{7}{2}\right)$

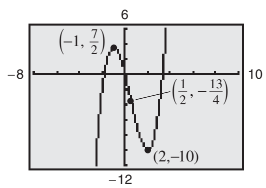
Relative minimum: $(2, -10)$

$f''(x) = 0$ when $x = \frac{1}{2}$.

$f''(x) < 0$ on $\left(-\infty, \frac{1}{2}\right)$.

$f''(x) > 0$ on $\left(\frac{1}{2}, \infty\right)$.

Point of inflection: $\left(\frac{1}{2}, -\frac{13}{4}\right)$



43. $f(x) = \frac{1}{4}x^4 - 2x^2$

$f'(x) = x^3 - 4x = x(x^2 - 4)$

Critical numbers: $x = \pm 2, x = 0$

$f''(x) = 3x^2 - 4$

$f''(-2) = 8 > 0$

$f''(0) = -4 < 0$

$f''(2) = 8 > 0$

Relative maximum: $(0, 0)$

Relative minima: $(\pm 2, -4)$

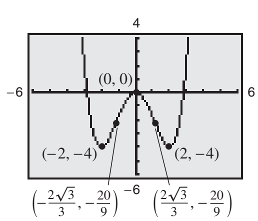
$f''(x) = 3x^2 - 4 = 0$ when $x = \pm\frac{2\sqrt{3}}{3}$.

$f''(x) > 0$ on $\left(-\infty, -\frac{2\sqrt{3}}{3}\right)$.

$f''(x) < 0$ on $\left(-\frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right)$.

$f''(x) > 0$ on $\left(\frac{2\sqrt{3}}{3}, \infty\right)$.

Points of inflection: $\left(-\frac{2\sqrt{3}}{3}, -\frac{20}{9}\right), \left(\frac{2\sqrt{3}}{3}, -\frac{20}{9}\right)$



44. $f(x) = 2x^4 - 8x + 3$

$f'(x) = 8x^3 - 8 = 8(x^3 - 1)$

Critical number: $x = 1$

$f''(x) = 24x^2$

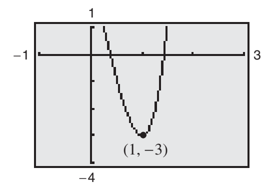
$f''(1) = 24 > 0$

Relative minimum: $(1, -3)$

$f''(x) = 24x^2 = 0$ when $x = 0$.

$f''(x) > 0$ on $(-\infty, 0)$ and on $(0, \infty)$

Because there are no inflection points.



45. $g(x) = (x - 2)(x + 1)^2 = x^3 - 3x - 2$

$g'(x) = (x - 2)[2(x + 1)] + (x + 1)^2 = 3(x^2 - 1)$

Critical numbers: $x = \pm 1$

$g''(x) = 6x$

$g''(-1) = -6 < 0$

$g''(1) = 6 > 0$

Relative maximum: $(-1, 0)$

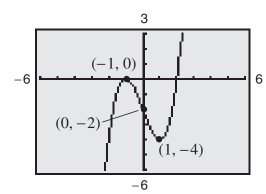
Relative minimum: $(1, -4)$

$g''(x) = 6x = 0$ when $x = 0$.

$g''(x) < 0$ on $(-\infty, 0)$.

$g''(x) > 0$ on $(0, \infty)$.

Point of inflection: $(0, -2)$



46. $g(x) = (x-6)(x+2)^3$

$g'(x) = (x-6)\left[3(x+2)^2\right] + (x+2)^3$

$= (x+2)^2\left[3(x-6)+(x+2)\right]$

$= (x+2)^2(4x-16)$

Critical numbers: $x = -2,\ x = 4$

$g''(x) = (x+2)^2(4) + (4x-16)\left[2(x+2)\right]$

$= 2(x+2)\left[2(x+2)+(4x-16)\right]$

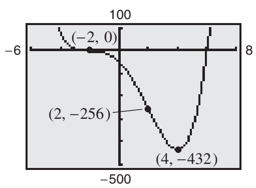
$= 2(x+2)(6x-12)$

$= 12(x+2)(x-2)$

$g''(-2) = 0$

$g''(4) = 144 > 0$

Relative minimum: $(4, -432)$



$g''(x) = 12(x+2)(x-2) = 0$ when $x = \pm 2$.

$g''(x) > 0$ on $(-\infty, -2)$.

$g''(x) < 0$ on $(-2, 2)$.

$g''(x) > 0$ on $(2, \infty)$.

So, $(-2, 0)$ and $(2, -256)$ are points of inflection.

47. $g(x) = x\sqrt{x+3} = x(x+3)^{1/2}$

The domain of g is $[-3, \infty)$.

$g'(x) = x\left[\frac{1}{2}(x+3)^{-1/2}\right] + (x+3)^{1/2} = \frac{3x+6}{2(x+3)^{1/2}}$

Critical numbers: $x = -3,\ x = -2$

$g''(x) = \frac{2(x+3)^{1/2}(3) - (3x+6)\left[2\left(\frac{1}{2}\right)(x+3)^{-1/2}\right]}{4(x+3)}$

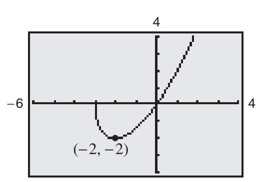
$= \frac{3(x+4)}{4(x+3)^{3/2}}$

$g''(-3)$ is undefined.

$g''(-2) = \frac{3}{2} > 0$

Relative minimum: $(-2, -2)$

$g''(x) > 0$ for all x in the domain, so there are no points of inflection.



48. $g(x) = x\sqrt{9-x} = x(9-x)^{1/2}$

The domain of g is $(-\infty, 9]$.

$g'(x) = x\left[\frac{1}{2}(9-x)^{-1/2}(-1)\right] + (9-x)^{1/2} = \frac{18-3x}{2(9-x)^{1/2}}$

Critical number: $x = 6$

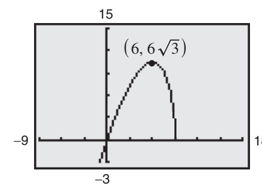
$g''(x) = \frac{2(9-x)^{1/2}(-3) - (18-3x)\left[2\left(\frac{1}{2}\right)(9-x)^{-1/2}(-1)\right]}{4(9-x)}$

$= \frac{3(x-12)}{4(9-x)^{3/2}}$

$g''(6) < 0$

Relative maximum: $\left(6, 6\sqrt{3}\right)$

$g''(x) < 0$ for all x in the domain, so there are no points of inflection.



49. $f(x) = \frac{4}{1+x^2} = 4(1+x^2)^{-1}$

$f'(x) = -4(1+x^2)^{-2}(2x) = -\frac{8x}{(1+x^2)^2}$

Critical number: $x = 0$

$f''(x) = \frac{(1+x^2)^2(-8) - (-8x)\left[2(1+x^2)(2x)\right]}{(1+x^2)^4}$

$= -\frac{8(1-3x^2)}{(1+x^2)^3}$

$f''(0) = -8 < 0$

Relative maximum: $(0, 4)$

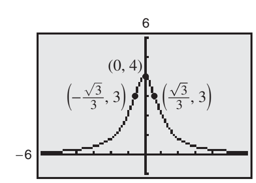
$f''(x) = 1 - 3x^2 = 0$ when $x = \pm\frac{\sqrt{3}}{3}$.

$f''(x) > 0$ on $\left(-\infty, -\frac{\sqrt{3}}{3}\right)$.

$f''(x) < 0$ on $\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$.

$f''(x) > 0$ on $\left(\frac{\sqrt{3}}{3}, \infty\right)$.

Points of inflection: $\left(\frac{\sqrt{3}}{3}, 3\right), \left(-\frac{\sqrt{3}}{3}, 3\right)$



50. $f(x) = \dfrac{2}{x^2 - 1} = 2(x^2 - 1)^{-1}$

$f'(x) = -2(x^2 - 1)^{-2}(2x) = -\dfrac{4x}{(x^2 - 1)^2}$

Critical number: $x = 0$

Discontinuities: $x = \pm 1$

$$f''(x) = \frac{(x^2 - 1)^2(-4) - (-4x)\left[2(x^2 - 1)(2x)\right]}{(x^2 - 1)^4} = \frac{4(3x^2 + 1)}{(x^2 - 1)^3}$$

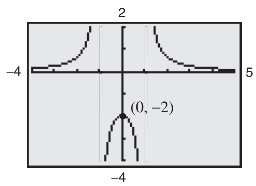
$f''(0) < 0$

$f''(1)$ is undefined.

$f''(-1)$ is undefined.

Relative maximum: $(0, -2)$

No points of inflection

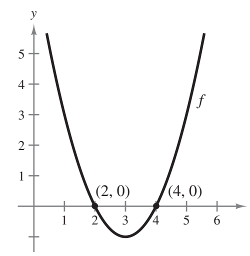


51. *Function* | *First Derivative* | *Second Derivative*

Function	*First Derivative*	*Second Derivative*
$f(2) = 0$	$f'(x) < 0, x < 3$	$f''(x) > 0$
$f(4) = 0$	$f'(3) = 0$	
	$f'(x) > 0, x > 3$	

The function has x-intercepts at (2, 0) and (4, 0). On $(-\infty, 3)$, f is decreasing, and on $(3, \infty)$, f is increasing. A relative minimum occurs when $x = 3$. The graph of f is concave upward.

Answers will vary. Sample answer:

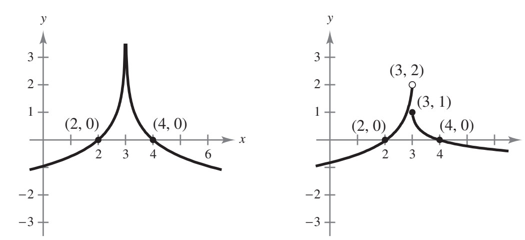


52.

Function	*First Derivative*	*Second Derivative*
$f(2) = 0$	$f'(x) > 0, x < 3$	$f''(x) > 0, x \neq 3$
$f(4) = 0$	$f'(3)$ is undefined.	
	$f'(x) < 0, x > 3$	

The function has x-intercepts at (2, 0) and (4, 0). On $(-\infty, 3)$, f is increasing and on $(3, \infty)$, f is decreasing. f has either a relative maximum at $x = 3$ or a discontinuity at $x = 3$. Also, f is concave upward on $(-\infty, 3)$ and $(3, \infty)$.

Answers will vary. Sample answer:

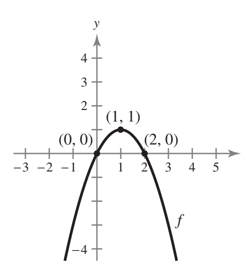


53.

Function	*First Derivative*	*Second Derivative*
$f(0) = 0$	$f'(x) > 0, x < 1$	$f''(x) < 0$
$f(2) = 0$	$f'(1) = 0$	
	$f'(x) < 0, x > 1$	

The function has x-intercepts at (0, 0) and (2, 0). On $(-\infty, 1)$, f is increasing and on $(1, \infty)$, f is decreasing. A relative maximum occurs when $x = 1$. The graph of f is concave downward.

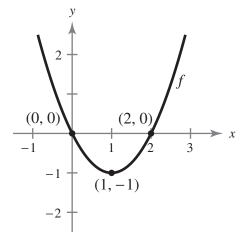
Answers will vary. Sample answer:



54.

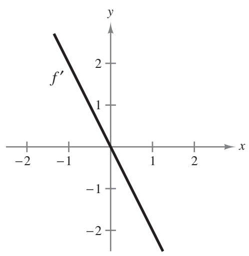
Function	*First Derivative*	*Second Derivative*
$f(0) = 0$	$f'(x) < 0,\ x < 1$	$f''(x) > 0$
$f(2) = 0$	$f'(1) = 0$	
	$f'(x) > 0,\ x > 1$	

Answers will vary. Sample answer:



The function has x-intercepts at (0, 0) and (2, 0). On $(-\infty, 1)$, f is decreasing and on $(1, \infty)$, f is increasing. A relative minimum occurs when $x = 1$. The graph of f is concave upward.

55.



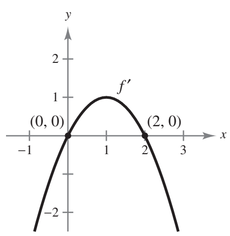
(a) $f'(x) > 0$ on $(-\infty, 0)$ where f is increasing.

(b) $f'(x) < 0$ on $(0, \infty)$ where f is decreasing.

(c) f' is not increasing. f is not concave upward.

(d) f' is decreasing on $(-\infty, \infty)$ where f is concave downward.

56.



(a) f' is positive on (0, 2) where f is increasing.

(b) f' is negative on $(-\infty, 0)$ and $(2, \infty)$ where f is decreasing.

(c) f' is increasing on $(-\infty, 1)$ where f is concave upward.

(d) f' is decreasing on $(1, \infty)$ where f is concave downward.

57. $f'(x) = 2x + 5$

$f''(x) = 2$

(a) Because the second derivative is positive for all x, f' is increasing on $(-\infty, \infty)$.

(b) Because the second derivative is positive for all x, f is concave upward.

(c) Critical number: $x = -\frac{5}{2}$

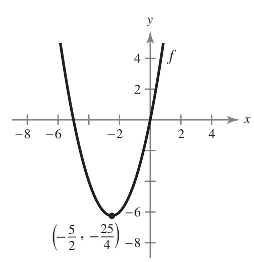
$$f''\left(-\frac{5}{2}\right) = 2 > 0$$

A relative minimum occurs when $x = -\frac{5}{2}$.

$f''(x) > 0$ on $(-\infty, \infty)$

No points of inflection

(d)



58. $f'(x) = 3x^2 - 2$; Critical numbers: $x = \pm\frac{\sqrt{6}}{3}$

$f''(x) = 6x$; $f''(x) = 0$ when $x = 0$

(a) The value of $f''(x)$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$. So, $f'(x)$ is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

(b) $f''(x) < 0$ on $(-\infty, 0)$.

$f''(x) > 0$ on $(0, \infty)$.

So, f is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$.

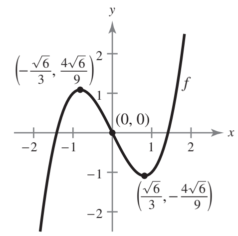
(c) $f''\left(-\frac{\sqrt{6}}{3}\right) = -2\sqrt{6} < 0$

$f''\left(\frac{\sqrt{6}}{3}\right) = 2\sqrt{6} > 0$

A relative maximum occurs when $x = -\frac{\sqrt{6}}{3}$ and a relative minimum occurs when $x = \frac{\sqrt{6}}{3}$.

Concavity changes at $x = 0$, so $(0, 0)$ is a point of inflection.

(d)



59. $f'(x) = -x^2 + 2x - 1$; Critical number: $x = 1$

$f''(x) = -2x + 2$; $f''(x) = 0$ when $x = 1$

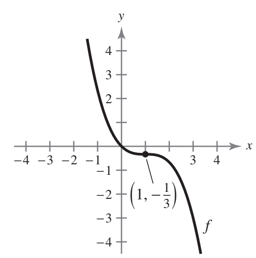
(a) The value of $f''(x)$ is positive on $(-\infty, 1)$ and negative on $(1, \infty)$. So, $f'(x)$ is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$.

(b) $f''(x) > 0$ on $(-\infty, 1)$.
$f''(x) < 0$ on $(1, \infty)$.
So, f is concave upward on $(-\infty, 1)$ and concave downward on $(1, \infty)$.

(c) $f''(1) = 0$

The Second-Derivative Test fails. Because $f'(x) \le 0$, f is never increasing and there are no relative extrema. Concavity changes at $x = 1$, so a point of inflection occurs when $x = 1$.

(d)



60. $f'(x) = x^2 + x - 6 = (x - 2)(x + 3)$; Critical numbers: $x = 2, x = -3$

$f''(x) = 2x + 1$; $f''(x) = 0$ when $x = -\frac{1}{2}$

(a) The value of $f''(x)$ is negative on $\left(-\infty, -\frac{1}{2}\right)$ and positive on $\left(-\frac{1}{2}, \infty\right)$. So, $f'(x)$ is increasing on $\left(-\frac{1}{2}, \infty\right)$ and decreasing on $\left(-\infty, -\frac{1}{2}\right)$.

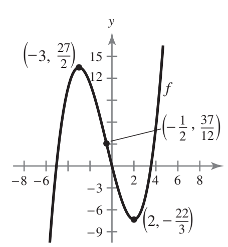
(b) $f''(x) < 0$ on $\left(-\infty, -\frac{1}{2}\right)$.
$f''(x) > 0$ on $\left(-\frac{1}{2}, \infty\right)$.
So, f is concave upward on $\left(-\frac{1}{2}, \infty\right)$ and concave downward on $\left(-\infty, -\frac{1}{2}\right)$.

(c) $f''(2) = 5 > 0$
$f''(-3) = -5 < 0$

A relative minimum occurs at $x = 2$, and a relative maximum occurs at $x = -3$.

Concavity changes at $x = -\frac{1}{2}$, so a point of inflection occurs when $x = -\frac{1}{2}$.

(d)



61. $R = \frac{1}{50{,}000}\left(600x^2 - x^3\right),\ 0 \le x \le 400$

$R' = \frac{1}{50{,}000}\left(1200x - 3x^2\right)$

$R'' = \frac{1}{50{,}000}(1200 - 6x) = 0$ when $x = 200$.

$R'' > 0$ on $(0, 200)$.

$R'' < 0$ on $(200, 400)$.

Point of diminishing returns: $(200, 320)$

62. $R = -\frac{4}{9}x^3 + 4x^2 + 12,\ 0 \le x \le 5$

$R' = -\frac{4}{3}x^2 + 8x$

$R'' = -\frac{8}{3}x + 8 = 0$ when $x = 3$.

$R'' > 0$ on $(0, 3)$.

$R'' < 0$ on $(3, 5)$.

Point of diminishing returns: $(3, 36)$

63. $N(t) = -0.12t^3 + 0.54t^2 + 8.22t,\ 0 \le t \le 4$

$N'(t) = -0.36t^2 + 1.08t + 8.22$

$N''(t) = -0.72t + 1.08 = 0$ when $t = 1.5$.

$N'''(t) = -0.72$

$N'''(1.5) = -0.72 < 0$

The student is assembling components at the greatest rate when $t = 1.5$, or 8:30 P.M.

64. $N(t) = \frac{20t^2}{4 + t^2},\ 0 \le t \le 4$

$$N'(t) = \frac{\left(4 + t^2\right)(40t) - \left(20t^2\right)(2t)}{\left(4 + t^2\right)^2} = 160t\left(4 + t^2\right)^{-2}$$

$$N''(t) = 160\left\{t\left[-2\left(4 + t^2\right)^{-3}(2t)\right] + \left(4 + t^2\right)^{-2}\right\} = 160\left(4 + t^2\right)^{-3}\left[-4t^2 + \left(4 + t^2\right)\right] = \frac{160\left(4 - 3t^2\right)}{\left(4 + t^2\right)^3} = 0$$

when $t = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.

$$N'''(t) = \frac{\left(4 + t^2\right)^3(-960t) - 160\left(4 - 3t^2\right)\left[3\left(4 + t^2\right)^2(2t)\right]}{\left(4 + t^2\right)^6} = \frac{\left(4 + t^2\right)^2\left(-3840t - 960t^3 - 3840t + 2880t^3\right)}{\left(4 + t^2\right)^6} = \frac{1920x\left(x^2 - 4\right)}{\left(4 + t^2\right)^4}$$

$$N'''\left(\frac{2\sqrt{3}}{3}\right) \approx -7.31 < 0$$

The student is assembling components at the greatest rate when $t = \left(2\sqrt{3}\right)/3$ or at approximately 8:09 P.M.

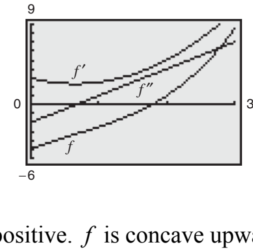
65. $f(x) = \frac{1}{2}x^3 - x^2 + 3x - 5,\ [0, 3]$

$f'(x) = \frac{3}{2}x^2 - 2x + 3$

$f''(x) = 3x - 2$

No relative extrema.

Point of inflection: $\left(\frac{2}{3}, -3.30\right)$



f is increasing when f' is positive. f is concave upward when f'' is positive and concave downward when f'' is negative.

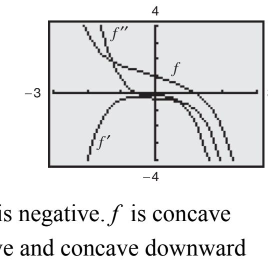
66. $f(x) = -\frac{1}{20}x^5 - \frac{1}{12}x^2 - \frac{1}{3}x + 1,\ [-2, 2]$

$f'(x) = -\frac{1}{4}x^4 - \frac{1}{6}x - \frac{1}{3}$

$f''(x) = -x^3 - \frac{1}{6}$

No relative extrema.

Point of inflection: $(-0.55, 1.16)$



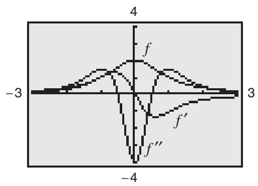
f is decreasing when f' is negative. f is concave upward when f'' is positive and concave downward when f'' is negative.

67. $f(x) = \dfrac{2}{x^2 + 1} = 2(x^2 + 1)^{-1}, [-3, 3]$

$f'(x) = -2(x^2 + 1)^{-2}(2x) = -\dfrac{4x}{(x^2 + 1)^2}$

$f''(x) = \dfrac{(x^2 + 1)^2(-4) - (-4x)[2(x^2 + 1)(2x)]}{(x^2 + 1)^4}$

$= \dfrac{12x^2 - 4}{(x^2 + 1)^3}$



Relative maximum: (0, 2)

Points of inflection: $(0.58, 1.5), (-0.58, 1.5)$

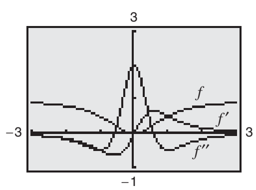
f is increasing when f' is positive and decreasing when f' is negative. f is concave upward when f'' is positive and concave downward when f'' is negative.

68. $f(x) = \dfrac{x^2}{x^2 + 1}, [-3, 3]$

$f'(x) = \dfrac{(x^2 + 1)(2x) - x^2(2x)}{(x^2 + 1)^2} = \dfrac{2x}{(x^2 + 1)^2}$

$f''(x) = \dfrac{(x^2 + 1)^2(2) - 2x[2(x^2 + 1)(2x)]}{(x^2 + 1)^4}$

$= \dfrac{2(1 - 3x^2)}{(x^2 + 1)^3}$



Relative minimum: $(0, 0)$

Points of inflection: $(0.58, 0.25), (-0.58, 0.25)$

f is increasing when f' is positive and decreasing when f' is negative. f is concave upward when f'' is positive and concave downward when f'' is negative.

69. $\overline{C} = 0.5x + 10 + \dfrac{7200}{x}$

$\overline{C}' = 0.5 - \dfrac{7200}{x^2}$

Critical number: $x = 120$

$\overline{C}'' = \dfrac{14{,}400}{x^3}$

$\overline{C}''(120) = \dfrac{1}{120} > 0$

So, producing 120 units minimizes the average cost per unit.

70. $C = 2x + \dfrac{300{,}000}{x}$

$C' = 2 - \dfrac{300{,}000}{x^2}$

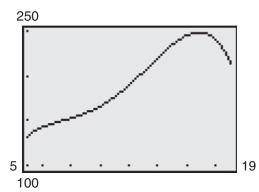
Critical number: $x \approx 387$

$C'' = \dfrac{600{,}000}{x^3}$

$C''(387) \approx 0.010 > 0$

So, ordering 387 units will produce a minimum cost.

71. (a)



(b) $t \approx 5$: 1995

(c) $t \approx 17$: 2007

(d) $t \approx 13$: 2003; the greatest rate of increase

$t \approx 19$: 2009; the least rate of increase

72. (a) Relative maximum: $(3, 2120)$

Relative minimum: $(1.5, 2050)$

Absolute maximum: $(0, 2250)$

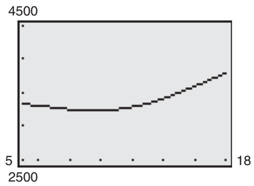
Absolute minimum: $(6.5, 1750)$

The market opened at the maximum for the day and closed at the minimum. At approximately 11:00 A.M. the market started to recover after falling, and at approximately 12:30 P.M. the market started to fall again.

(b) $(2, 2100)$ is the approximate point of inflection.

At approximately 11:30 A.M. the market began to increase at a greater rate.

73. (a)



(b) and (c) $v = -0.0687t^4 + 3.169t^3 - 45t^2 + 230.6t + 2950,\ 5 \le t \le 18$

$$v' = \frac{dv}{dt} = -0.2748t^3 + 9.507t^2 - 90t + 230.6$$

$$v'' = \frac{d^2v}{dt^2} = -0.8244t^2 + 19.014t - 90$$

$$v'' = 0$$

$$-8.244t^2 + 19.014t - 90 = 0$$

$$t \approx 16.4, 6.7$$

Interval	$5 < t < 6.7$	$6.7 < t < 16.4$	$16.4 < t < 18$
Sign of v''	$v'' < 0$	$v'' > 0$	$v'' < 0$
Conclusion	Concave downward	Concave upward	Concave downward

Points of inflection: $(6.7, 3289.6), (16.4, 3637.2)$

(d) The first inflection point is where the change in the number of veterans receiving benefits starts to increase after it has been decreasing. The second inflection point is where the change in the number of veterans receiving benefits starts to decrease again.

74. (a) S' is increasing, so $S'' > 0$.

(b) Both S and S' are increasing, so $S' > 0$ and $S'' > 0$.

(c) S' is not changing, so $S'' = 0$.

(d) S is not changing, so $S' = 0$.

(e) S is decreasing, but S' is increasing, so $S' < 0$ and $S'' > 0$.

(f) S is increasing, so $S' > 0$. There are no restrictions on S''.

75. Answers will vary.

Section 3.4 Optimization Problems

Skills Warm Up

1. Let x be the first number and y be the second number.

$x + \frac{1}{2}y = 12$

2. Let x be the first number and y be the second number.

$2xy = 24$

3. Let x be the length of the rectangle and y be the width of the rectangle.

$xy = 24$

4. Let (x_1, y_1) be the first point and (x_2, y_2) be the second point.

$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = 10$

5. $y = x^2 + 6x - 9$

$y' = 2x + 6 = 0$

Critical number: $x = -3$

6. $y = 2x^3 - x^2 - 4x$

$$y' = 6x^2 - 2x - 4 = 0$$

$$2(3x^2 - x - 2) = 0$$

$$2(3x + 2)(x - 1) = 0$$

Critical numbers $x = -\frac{2}{3}, x = 1$

Skills Warm Up —*continued*—

7. $y = 5x + \dfrac{125}{x}$

$y' = 5 - \dfrac{125}{x^2} = 0$

$\dfrac{5(x^2 - 25)}{x^2} = 0$

$5(x^2 - 25) = 0$

Critical numbers: $x = \pm 5$

8. $y = 3x + \dfrac{96}{x^2}$

$y' = 3 - \dfrac{192}{x^3} = 0$

$\dfrac{3(x^3 - 64)}{x^3} = 0$

$3(x^3 - 64) = 0$

Critical number: $x = 4$

9. $y = \dfrac{x^2 + 1}{x} = x + \dfrac{1}{x}$

$y' = 1 - \dfrac{1}{x^2} = 0$

$\dfrac{x^2 - 1}{x^2} = 0$

$x^2 - 1 = 0$

Critical numbers: $x = \pm 1$

10. $y = \dfrac{x}{x^2 + 9}$

$y' = \dfrac{x^2 + 9 - x(2x)}{(x^2 + 9)^2} = 0$

$\dfrac{9 - x^2}{(x^2 + 9)^2} = 0$

$9 - x^2 = 0$

Critical numbers: $x = \pm 3$

1. Let l be the length and w be the width of the rectangle. Then $2l + 2w = 100$ and $w = 50 - l$. The area is:

$A = lw = l(50 - l)$

$A' = 50 - 2l$

$A'' = -2$

$A' = 0$ when $x = 25$. Because $A''(25) = -2 < 0$, A is maximum when $l = 25$ meters and $w = 50 - 25 = 25$ meters.

2. Let l be the length and w be the width of the rectangle. Then $P = 2w + 2l$ and $y = \frac{1}{2}(P - 2l)$. The area is:

$A = lw = l\left(\dfrac{1}{2}\right)(P - 2l) = \dfrac{1}{2}Pl - l^2$

$A' = \dfrac{1}{2}P - 2l$

$A'' = -2$

$A' = 0$ when $l = P/4$. Because $A''(P/4) = -2 < 0$, A is a maximum when $l = P/4$ and $w = P/4$.

3. Let l and w be the length and width of the rectangle. Then the area is $lw = 64$ and $w = 64/l$. The perimeter is:

$P = 2l + 2w = 2l + 2\left(\dfrac{64}{l}\right)$

$P' = 2 - \dfrac{128}{l^2} = \dfrac{2(l^2 - 64)}{l^2}$

$P' = \dfrac{256}{l^3}$

$P' = 0$ when $l = 8$. Because $P''(8) = \frac{1}{2} > 0$, P is a minimum when $l = 8$ and $w = 64/8 = 8$ feet.

4. Let l and w be the length and width of the rectangle. The area is $A = lw$ and $w = A/l$. The perimeter is:

$P = 2l + 2w = 2l + 2\left(\dfrac{A}{l}\right)$

$P' = 2 - \dfrac{2A}{l^2} = \dfrac{2l^2 - 2A}{l^2}$

$P'' = \dfrac{4A}{l^3}$

$P' = 0$ when $l = \sqrt{A}$. Because $P''(\sqrt{A}) > 0$, P is minimum when $l = \sqrt{A}$ and $w = \sqrt{A}$.

5. The length of fencing is given by $4x + 3y = 200$ and $y = (200 - 4x)/3$. The area of the corrals is:

$$A = 2xy = 2x\left(\frac{200 - 4x}{3}\right) = \frac{8}{3}(50x - x^2)$$

$$A' = \frac{8}{3}(50 - 2x)$$

$$A'' = -\frac{16}{3}$$

$A' = 0$ when $x = 25$. Because $A''(25) = -\frac{16}{3} < 0$, A is maximum when $x = 25$ feet and $y = \frac{100}{3}$ feet.

6. Let x be the length of the fence parallel to the river and y be the length of fence perpendicular to the river.

Then, Area $= xy = 245{,}000 \Rightarrow y = \dfrac{245{,}000}{x}$.

Amount of fencing:

$$F = x + 2y = x + 2\left(\frac{245{,}000}{x}\right) = x + \frac{490{,}000}{x}$$

$$F' = 1 - \frac{490{,}000}{x^2}$$

$$F'' = \frac{980{,}000}{x^3}$$

$$F' = 0$$

$$1 - \frac{490{,}000}{x^2} = 0$$

$$x^2 = 490{,}000$$

$$x = 700$$

$F''(700) > 0 \Rightarrow F$ is minimum when $x = 700$ meters.

When $x = 700$, $y = \dfrac{245{,}000}{700} = 350$ meters.

7. (a) $9 + 9 + 4(3)(11) = 150 \text{ in.}^2$

$6(25) = 150 \text{ in.}^2$

$36 + 36 + 4(6)(3.25) = 150 \text{ in.}^2$

(b) $V = 3(3)(11) = 99 \text{ in.}^3$

$V = 5(5)(5) = 125 \text{ in.}^3$

$V = 6(6)(3.25) = 117 \text{ in.}^3$

(c) Let the base measure x by x, and the height measure y.

Then the surface area is $2x^2 + 4xy = 150$ and $y = \dfrac{1}{2x}(75 - x^2)$. The volume of the solid is:

$$V = x^2y = x^2\left[\frac{1}{2x}(75 - x^2)\right] = \frac{75}{2}x - \frac{x^3}{2}$$

$$V' = \frac{75}{2} - \frac{3x^2}{2}$$

$$V'' = -3x$$

$V' = 0$ when $x = 5$. Because $V''(5) = -15 < 0$, V is maximum when $x = 5$ inches and $y = 5$ inches.

8. (a) Let the base of the solid have dimensions x by x and let the height be y.

Then the surface area is given by

$$2x^2 + 4xy = 337.5 \Rightarrow y = \frac{1}{4x}(337.5 - 2x^2).$$

Volume of the solid: $V = x^2y$

$$V = x^2\left[\frac{1}{4x}(337.5 - 2x^2)\right]$$

$$= 84.375x - 0.5x^3$$

$$V' = 84.375 - 1.5x^2$$

$$V'' = -3x$$

$$V' = 0$$

$$84.375 - 1.5x^2 = 0$$

$$1.5x^2 = 84.375$$

$$x^2 = 56.25$$

$$x = 7.5 \text{ cm}$$

$V''(7.5) < 0 \Rightarrow V$ is a maximum when $x = 7.5$ cm.

When $x = 7.5$ cm, $y = \dfrac{1}{4(7.5)}\left(337.5 - 2(7.5)^2\right) = 7.5$ cm.

(b) $V = 84.375(7.5) - 0.5(7.5)^3 = 421.875 \text{ cm}^3$

9. (a) Let the base have dimensions x by x and the height have dimension y. Then the volume is given by

$$V = x^2y = 8000 \Rightarrow y = \frac{8000}{x^2}.$$

Surface area: $S = 2x^2 + 4xy$

$$= 2x^2 + 4x\left(\frac{8000}{x^2}\right)$$

$$= 2x^2 + \frac{32{,}000}{x}$$

$$S' = 4x - \frac{32{,}000}{x^2}$$

$$S'' = 4 + \frac{64{,}000}{x^3}$$

$$S' = 0$$

$$4x - \frac{32{,}000}{x^2} = 0$$

$$4x^3 = 32{,}000$$

$$x^3 = 8000$$

$$x = 20$$

$S''(20) > 0 \Rightarrow S$ is a minimum when $x = 20$.

When $x = 20$ in., $y = \dfrac{8000}{(20)^2} = 20$ in.

(b) $S = 2(20)^2 + \dfrac{32{,}000}{20} = 2400 \text{ in.}^2$

10. (a) Profit is increasing for advertising costs between \$0 and \$40 thousand, $0 \le x < 40$.

(b) Profit is decreasing for advertising costs between \$40 thousand and \$61 thousand, $40 < x \le 61$.

(c) To yield a maximum profit of approximately \$3.55 million $(P = 3550)$, the company should spend \$40 thousand in advertising costs.

(d) The point of diminishing returns is approximately $(20, 2000)$.

11. Let x and y be the length and width of the rectangle. The radius of the semicircle is $r = y/2$, and the perimeter is

$200 = 2x + 2\pi r = 2x + 2\pi\left(\dfrac{y}{2}\right) = 2x + \pi y$ and

$y = (200 - 2x)/\pi$. The area of the rectangle is:

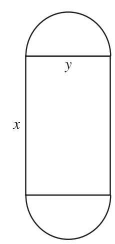
$$A = xy = x\left[\frac{200 - 2x}{\pi}\right] = \frac{2}{\pi}\left(100x - x^2\right)$$

$$A' = \frac{2}{\pi}(100 - 2x)$$

$$A'' = -\frac{4}{\pi}$$

$A' = 0$ when $x = 50$. Because $A'' < 0$, A is maximum when $x = 50$ meters and

$$y = \frac{200 - 2(50)}{\pi} = \frac{100}{\pi} \text{ meters.}$$



12. The perimeter of the window is $x + 2y + \dfrac{\pi}{2}x = 16$ and $y = 8 - \left(\dfrac{1}{2} + \dfrac{\pi}{4}\right)x$.

The area of the window is:

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2 = x\left[8 - \left(\frac{1}{2} + \frac{\pi}{4}\right)x\right] + \frac{\pi x^2}{8} = 8x - \left(\frac{1}{2} + \frac{\pi}{8}\right)x^2$$

$$A' = 8 - \left(1 + \frac{\pi}{4}\right)x$$

$$A'' = -\left(1 + \frac{\pi}{4}\right)$$

$A' = 0$ when $x = \dfrac{8}{1 + \pi/4}$. Because $A'' < 0$, A is maximum when $x = \dfrac{8}{1 + \pi/4} = \dfrac{32}{4 + \pi}$ feet and $y = \dfrac{16}{4 + \pi}$ feet.

13. Let the base measure x by x and the height measure y. Then the volume is $x^2y = 80$ and $y = \frac{80}{x^2}$. The cost of the box is:

$$C = (0.20)(2)x^2 + (0.10)(4)xy$$
$$= 0.40x^2 + 0.40x\left(\frac{80}{x^2}\right)$$
$$= 0.40x^2 + \frac{32}{x}$$
$$C' = 0.80x - \frac{32}{x^2}$$
$$= \frac{0.80x^3 - 32}{x^2}$$
$$C'' = 0.80 + \frac{64}{x^3}$$

$C' = 0$ when $x = \sqrt[3]{40} = 2\sqrt[3]{5}$. Because $C''\left(2\sqrt[3]{5}\right) > 0$, C is minimum when $x = 2\sqrt[3]{5}$ centimeters and $y = 4\sqrt[3]{5}$ centimeters.

14. The volume of the enclosure is $x^2y = 83\frac{1}{3} = \frac{250}{3}$ and $y = \frac{250}{3x^2}$.

The surface area of the enclosure is:

$$A = 3xy + x^2 = 3x\left(\frac{250}{3x^2}\right) + x^2 = \frac{250}{x} + x^2$$
$$A' = -\frac{250}{x^2} + 2x = \frac{2x^3 - 250}{x^2}$$
$$A'' = \frac{500}{x^3} + 2$$

$A' = 0$ when $x = 5$. Because $A''(5) > 0$, the surface area is minimum when $x = 5$ meters and

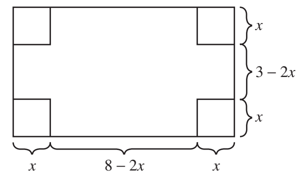
$$y = \frac{250}{3(5)^2} = \frac{10}{3} \text{ meters.}$$

15. The volume of the box is:

$$V = x(6 - 2x)^2,\ 0 < x < 3$$
$$V' = 12(x - 1)(x - 3)$$

$V' = 0$ when $x = 3$ and $x = 1$. Because $V = 0$ when $x = 3$ and $V = 16$ when $x = 1$, the volume is maximum when $x = 1$. The corresponding volume is $V = 16$ cubic inches.

16. Let x be the length of the cut from each corner of the 3-foot by 8-foot rectangle.



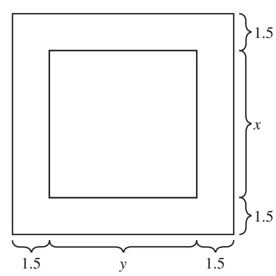
$$V = \text{Length} \cdot \text{Width} \cdot \text{Height}$$
$$= (3 - 2x)(8 - 2x)x$$
$$= 4x^3 - 22x^2 + 24x$$
$$V' = 12x^2 - 44x + 24$$
$$V'' = 24x - 44$$
$$V' = 0$$
$$12x^2 - 44x + 24 = 0$$
$$4(3x^2 - 11x + 6) = 0$$
$$4(3x - 2)(x - 3) = 0$$
$$x = \tfrac{2}{3} \text{ or } x = 3$$

(Note that $x = 3$ does not make sense in the context of the problem.)

$V''\left(\frac{2}{3}\right) < 0 \Rightarrow V$ is a maximum when $x = \frac{2}{3}$ ft.

When $x = \frac{2}{3}$ ft, $V = \frac{200}{27}\text{ ft}^3 \approx 7.41\text{ ft}^3$.

17. Let x and y be the lengths shown in the figure.



Then $xy = 36$ and $y = \frac{36}{x}$.

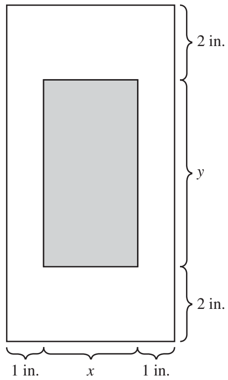
The area of the page is:

$$A = (x + 3)\left(\frac{36}{x} + 3\right) = 45 + \frac{108}{x} + 3x$$
$$A' = -\frac{108}{x^2} + 3 = \frac{3(x^2 - 36)}{x^2}$$
$$A'' = \frac{216}{x^3}$$

$A' = 0$ when $x = 6$. Because $A''(6) = 1 > 0$, A is minimum when $x = 6$ inches and $y = \frac{36}{6} = 6$ inches.

The dimensions of the page are 9 inches by 9 inches.

18. Let x and y be the dimensions of the printed area.



Printed area: $xy = 50 \Rightarrow y = \dfrac{50}{x}$

Area of the entire page:
$$A = (x + 2)(y + 4) = (x + 2)\left(\frac{50}{x} + 4\right) = 50 + 4x + \frac{100}{x} + 8$$

$$A' = 4 - \frac{100}{x^2}$$

$$A'' = \frac{200}{x^3}$$

$$A' = 0$$
$$4 - \frac{100}{x^2} = 0$$
$$4x^2 = 100$$
$$x^2 = 25$$
$$x = 5 \quad (x = -5 \text{ in not in the domain.})$$

$A''(5) > 0 \Rightarrow A$ is a minimum when $x = 5$ inches.

When $x = 5$ inches, $y = \dfrac{50}{5} = 10$ inches.

19. The area of the rectangle is

$$A = xy = x\left(\frac{6 - x}{2}\right) = \frac{1}{2}(6x - x^2)$$

$$A' = \frac{1}{2}(6 - 2x).$$

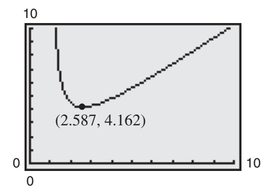
$$A'' = -1$$

$A' = 0$ when $x = 3$. Because $A'' < 0$, A is maximum when $x = 3$ units and $y = \dfrac{6 - 3}{2} = \dfrac{3}{2}$ units.

20. The triangles $(0, y)$, $(1, 2)$, $(0, 2)$ and $(1, 2)$, $(x, 0)$, $(1, 0)$ are similar.

(a)
$$\frac{y - 2}{0 - 1} = \frac{0 - 2}{x - 1}$$
$$y - 2 = \frac{2}{x - 1}$$
$$y = 2 + \frac{2}{x - 1}$$
$$L = \sqrt{x^2 + y^2} = \sqrt{x^2 + \left(2 + \frac{2}{x - 1}\right)^2} = \sqrt{x^2 + 4 + \frac{8}{x - 1} + \frac{4}{(x - 1)^2}}, \quad x > 1$$

(b)



L is a minimum when $x \approx 2.587$ units and $L \approx 4.162$ units.

(c) The area is:

$$A(x) = \frac{1}{2}xy = \frac{1}{2}x\left(2 + \frac{2}{x - 1}\right) = x + \frac{x}{x - 1}$$

$$A'(x) = 1 + \frac{(x - 1) - x}{(x - 1)^2} = \frac{(x - 1)^2 - 1}{(x - 1)^2} = \frac{x(x - 2)}{(x - 1)^2}$$

$$A''(x) = \frac{2}{(x - 1)^3}$$

$A'(x) = 0$ when $x = 2$. Because $A''(2) > 0$, $A(x)$ is minimum when $x = 2$ and $y = 4$.

Vertices: $(0, 0)$, $(2, 0)$, $(0, 4)$

21. The area is:

$$A = 2xy = 2x(25 - x^2)^{1/2}$$

$$A' = 2\left[x\left(\frac{1}{2}\right)(25 - x^2)^{-1/2}(-2x) + (25 - x^2)^{1/2}\right] = 2\left[\frac{25 - 2x^2}{(25 - x^2)^{1/2}}\right]$$

$$A'' = 2\left[\frac{(25 - x^2)^{1/2}(-4x) - (25 - 2x^2)\left(\frac{1}{2}\right)(25 - x^2)^{-1/2}(-2x)}{25 - x^2}\right] = 2x\left[\frac{2x^2 - 75}{(25 - x^2)^{3/2}}\right]$$

$A' = 0$ when $x = 5/\sqrt{2}$. Because $A''(5/\sqrt{2}) < 0$, A is maximum when the length is

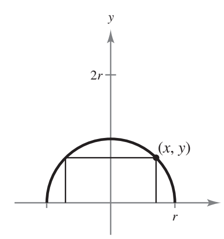
$2x = \dfrac{10}{\sqrt{2}} \approx 7.07$ units and the width is $y = \sqrt{25 - \left(\dfrac{5}{\sqrt{2}}\right)^2} = \dfrac{5}{\sqrt{2}} \approx 3.54$ units.

22. The area is:

$$A = 2xy = 2x(r^2 - x^2)^{1/2}$$

$$A' = 2\left[x\left(\frac{1}{2}\right)(r^2 - x^2)^{-1/2}(-2x) + (r^2 - x^2)^{1/2}\right] = 2\left[\frac{r^2 - 2x^2}{(r^2 - x^2)^{1/2}}\right]$$

$$A'' = 2\left[\frac{(r^2 - x^2)^{1/2}(-4x) - (r^2 - 2x^2)\left(\frac{1}{2}\right)(r^2 - x^2)^{-1/2}(-2x)}{r^2 - x^2}\right] = 2x\left[\frac{2x^2 - 3r^2}{(r^2 - x^2)^{3/2}}\right]$$



$A' = 0$ when $x = \dfrac{r}{\sqrt{2}} = \dfrac{\sqrt{2}r}{2}$. Because $A''\left(\dfrac{\sqrt{2}r}{2}\right) < 0$, A is maximum when the

length is $2x = \sqrt{2}r$ units and the width is $y = \left[r^2 - \left(\dfrac{r}{\sqrt{2}}\right)^2\right]^{1/2} = \dfrac{\sqrt{2}r}{2}$ units.

23. The volume of the cylinder is

$V = \pi r^2 h = (12 \text{ oz})(1.80469 \text{ in.}^3 \text{per oz}) \approx 21.66 \text{ in.}^3$

which implies that $h = 21.66/(\pi r^2)$. The surface area of the container is:

$$S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r\left(\frac{21.66}{\pi r^2}\right) = 2\left(\pi r^2 + \frac{21.66}{r}\right)$$

$$S' = 2\left(2\pi r - \frac{21.66}{r^2}\right) = 2\left(\frac{2\pi r^3 - 21.66}{r^2}\right)$$

$$S'' = 2\left(2\pi + \frac{43.32}{r^3}\right)$$

$S' = 0$ when $r = \sqrt[3]{\dfrac{21.66}{2\pi}} \approx 1.51$ inches. Because $S''(1.51) > 0$, S is minimum when $r \approx 1.51$ inches and

$h = \dfrac{21.66}{\pi(1.51)^2} \approx 3.02$ inches.

24. Let s represent the cost per square inch of the sides. The volume of the cylinder is

$V = \pi r^2 h = (16 \text{ oz})(1.80469 \text{ in.}^3 \text{ per oz}) \approx 28.88 \text{ in.}^3$

which implies that $h = 28.88/(\pi r^2)$. The cost of the container is given by the following.

$$C = 2s(2\pi r^2) + s(2\pi rh)$$
$$= 2s(2\pi r^2) + 2\pi rs\left(\frac{28.88}{\pi r^2}\right)$$
$$= 2s\left(2\pi r^2 + \frac{28.88}{r}\right)$$
$$C' = 2s\left(4\pi r - \frac{28.88}{r^2}\right) = 2s\left(\frac{4\pi r^3 - 28.88}{r^2}\right)$$
$$C'' = 2s\left(4\pi + \frac{57.76}{r^3}\right)$$

$C' = 0$ when $r = \sqrt[3]{\frac{7.22}{\pi}} \approx 1.32.$

Because $C''(1.32) > 0$, C is minimum when

$r \approx 1.32$ inches and $h = \dfrac{28.88}{\pi(1.32)^2} \approx 5.28$ inches.

25. The distance between a point (x, y) on the graph and the point $(2, 1/2)$ is

$$d = \sqrt{(x-2)^2 + \left(y - \frac{1}{2}\right)^2} = \sqrt{(x-2)^2 + \left(x^2 - \frac{1}{2}\right)^2}$$

and d can be minimized by minimizing its square $L = d^2$.

$$L = (x-2)^2 + \left(x^2 - \frac{1}{2}\right)^2 = x^4 - 4x + \frac{17}{4}$$
$$L' = 4x^3 - 4 = 4(x^3 - 1)$$
$$L'' = 12x^2$$

$L' = 0$ when $x = 1$. Because $L''(1) > 0$, L is minimum when $x = 1$ and $y = (1)^2 = 1$. The point nearest $(2, 1/2)$ is $(1, 1)$.

26. The distance between a point (x, y) on the graph and the point $(5, 3)$ is

$$d = \sqrt{(x-5)^2 + (y-3)^2}$$
$$= \sqrt{(x-5)^2 + \left[(x+1)^2 - 3\right]^2}$$

and d can be minimized by minimizing its square $L = d^2$.

$$L = (x-5)^2 + \left[(x+1)^2 - 3\right]^2$$
$$= x^4 + 4x^3 + x^2 - 18x + 29$$
$$L' = 4x^3 + 12x^2 + 2x - 18$$
$$L'' = 12x^2 + 24x + 2$$

$L' = 0$ when $x = 1$. Because $L''(1) > 0$, L is minimum when $x = 1$ and $y = (1+1)^2 = 4$. The point nearest $(5, 3)$ is $(1, 4)$.

27. The distance between a point (x, y) on the graph and the point $(4, 0)$ is

$$d = \sqrt{(x-4)^2 + y^2} = \sqrt{(x-4)^2 + x}$$

and d can be minimized by minimizing its square $L = d^2$.

$$L = (x-4)^2 + x = x^2 - 7x + 16$$
$$L' = 2x - 7$$
$$L'' = 2$$

$L' = 0$ when $x = \frac{7}{2}$. Because $L''\left(\frac{7}{2}\right) > 0$, L is minimum when $x = \frac{7}{2}$ and $y = \sqrt{\frac{7}{2}} = \frac{\sqrt{14}}{2}$. The point nearest $(4, 0)$ is $\left(\frac{7}{2}, \frac{\sqrt{14}}{2}\right)$.

28. The distance between a point (x, y) on the graph and the point $(12, 0)$ is

$d = \sqrt{(x-12)^2 + (y-0)^2} = \sqrt{(x-12)^2 + \left(\sqrt{x-8} - 0\right)^2}$ and d can be minimized by minimizing its square $L = d^2$.

$$L = (x-12)^2(x-8)$$
$$= x^2 - 24x + 144 + x - 8$$
$$= x^2 - 23x + 136$$
$$L' = 2x - 23$$
$$L'' = 2$$

$L' = 0$ when $x = \frac{23}{2}$. Because $L''\left(\frac{23}{2}\right) > 0$, L is a minimum when $x = \frac{23}{2}$ and $y = \sqrt{\frac{23}{2} - 8} = \sqrt{\frac{7}{2}}$.

The point nearest $(12, 0)$ is $\left(\frac{23}{2}, \sqrt{\frac{7}{2}}\right)$.

29. The length and girth is $4x + y = 108$ and $y = 108 - 4x$. The volume is:

$V = x^2 y = x^2(108 - 4x) = 108x^2 - 4x^3$

$V' = 216x - 12x^2 = 12x(18 - x)$.

$V'' = 216 - 24x$

$V' = 0$ when $x = 0$ and $x = 18$. Because $V''(18) < 0$, V is minimum when $x = 18$ inches and $y = 36$ inches. The dimensions are 18 inches by 18 inches by 36 inches.

30. The volume is $\frac{4}{3}\pi r^3 + \pi r^2 h = 12$ and $h = \frac{12}{\pi r^2} - \frac{4}{3}r$.

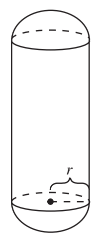
The surface area is:

$A = 4\pi r^2 + 2\pi rh$

$= 4\pi r^2 + 2\pi r\left(\frac{12}{\pi r^2} - \frac{4}{3}r\right)$

$= 4\pi r^2 + \frac{24}{r} - \frac{8}{3}\pi r^2$

$= \frac{4}{3}\pi r^2 + \frac{24}{r}$



$A' = \frac{8}{3}\pi r - \frac{24}{r^2} = \frac{8(\pi r^3 - 9)}{3r^2}$

$A'' = \frac{8}{3}\pi + \frac{48}{r^3}$

$A' = 0$ when $r = \sqrt[3]{9/\pi} \approx 1.42$ inches. Because $A''(1.42) > 0$, A is minimum when $r \approx 1.42$ inches.

31. Let p represent the cost per square foot of the material.

The volume is $\frac{4}{3}\pi r^3 + \pi r^2 h = 3000$ and

$h = \frac{3000}{\pi r^2} - \frac{4}{3}r$. The cost of the tank is:

$C = 2p(4\pi r^2) + p(2\pi rh)$

$= 2p(4\pi r^2) + 2\pi rp\left(\frac{3000}{\pi r^2} - \frac{4}{3}r\right)$

$= 2p\left(\frac{8}{3}\pi r^2 + \frac{3000}{r}\right)$

$C' = 2p\left(\frac{16}{3}\pi r - \frac{3000}{r^2}\right) = 2p\left(\frac{\frac{16}{3}\pi r^3 - 3000}{r^3}\right)$

$C'' = 2p\left(\frac{16}{3}\pi + \frac{6000}{r^3}\right)$

$C' = 0$ when $r = \sqrt[3]{\frac{562.5}{\pi}} \approx 5.64$ feet.

Because $C''(5.64) > 0$, C is minimum when $r \approx 5.64$ feet and

$h = \frac{3000}{\pi(5.64)^2} - \frac{4}{3}(5.64) \approx 22.50$ feet.

32. Using the formula Distance = (Rate)(Time), we have $T = D/R$.

$T = T_{\text{rowed}} + T_{\text{walked}}$

$= \frac{D_{\text{rowed}}}{R_{\text{rowed}}} + \frac{D_{\text{walked}}}{R_{\text{walked}}}$

$= \frac{\sqrt{x^2 + 4}}{2} + \frac{\sqrt{1 + (3 - x)^2}}{4}$

$T' = \frac{x}{2\sqrt{x^2 + 4}} - \frac{3 - x}{4\sqrt{1 + (3 - x)^2}}$

By setting $T' = 0$, we have:

$$\frac{x^2}{4(x^2 + 4)} = \frac{(3 - x)^2}{16\left[1 + (3 - x)^2\right]}$$

$$\frac{x^2}{x^2 + 4} = \frac{9 - 6x + x^2}{4(10 - 6x + x^2)}$$

$$4(x^4 - 6x^3 + 10x^2) = (x^2 + 4)(9 - 6x + x^2)$$

$x^4 - 6x^3 + 9x^2 + 8x - 12 = 0$

Possible rational roots: $\pm1, \pm2, \pm3, \pm4, \pm6, \pm12$

Using a graphing utility, the solution on $[0, 3]$ is $x = 1$ mile.

By testing, we find that $x = 1$ mile.

33. Let x be the length of a side of the square and r be the radius of the circle. Then the combined perimeter is $4x + 2\pi r = 16$ and

$x = \frac{16 - 2\pi r}{4} = 4 - \frac{\pi r}{2}$.

The total area is:

$A = x^2 + \pi r^2 = \left(4 - \frac{\pi r}{2}\right)^2 + \pi r^2$

$A' = 2\left(4 - \frac{\pi r}{2}\right)\left(-\frac{\pi}{2}\right) + 2\pi r = \frac{1}{2}(\pi^2 r + 4\pi r - 8\pi)$

$A'' = \frac{1}{2}(\pi^2 + 4\pi)$

$A' = 0$ when $r = \frac{8\pi}{\pi^2 + 4\pi} = \frac{8}{\pi + 4}$. Because $A'' > 0$, A is minimum when $r = \frac{8}{\pi + 4}$ units and

$x = 4 - \frac{\pi\left[8/(\pi + 4)\right]}{2} = \frac{16}{\pi + 4}$.

34. Let x be the length of a side of the triangle and y be the length of a side of the square.

The combined perimeter is $3x + 4y = 10$ and $y = \frac{1}{4}(10 - 3x)$. The total area is:

$$A = \frac{1}{2}bh + y^2 = \frac{1}{2}x\left(\frac{\sqrt{3}x}{2}\right) + \left[\frac{1}{4}(10 - 3x)\right]^2 = \frac{\sqrt{3}}{4}x^2 + \frac{1}{16}\left(100 - 60x + 9x^2\right)$$

$$A' = \frac{\sqrt{3}}{2}x + \frac{1}{16}(-60 + 18x) = x\left(\frac{\sqrt{3}}{2} + \frac{9}{8}\right) - \frac{15}{4}$$

$$A'' = \frac{\sqrt{3}}{2} + \frac{9}{8}$$

$$A' = 0 \text{ when } x = \frac{30}{4\sqrt{3} + 9}.$$

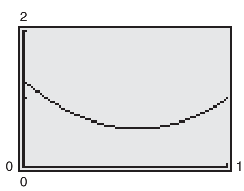
Because $A'' > 0$, A is minimum when $x = \dfrac{30}{4\sqrt{3} + 9}$ units and $y = \dfrac{1}{4}\left[10 - 3\left(\dfrac{30}{4\sqrt{3} + 9}\right)\right] = \dfrac{10\sqrt{3}}{4\sqrt{3} + 9}$.

35. (a) The perimeter is $4x + 2\pi r = 4$ and $r = \dfrac{2 - 2x}{\pi}$.

The area is: $A(x) = x^2 + \pi r^2 = x^2 + \pi\left(\dfrac{2 - 2x}{\pi}\right)^2 = x^2 + \dfrac{(2 - 2x)^2}{\pi} = \left(1 + \dfrac{4}{\pi}\right)x^2 - \dfrac{8}{\pi}x + \dfrac{4}{\pi}$

(b) Because $x \ge 0$ and $r = \dfrac{2 - 2x}{\pi} \ge 0$, the domain is $0 \le x \le 1$

(c)



(d) $A'(x) = \left(2 + \dfrac{8}{\pi}\right)x - \dfrac{8}{\pi}$

$A''(x) = 2 + \dfrac{8}{\pi}$

$A'(x) = 0$ when $x = \dfrac{8/\pi}{2 + 8/\pi} = \dfrac{4}{4 + \pi}$.

Because $A''(x) > 0$, $A(x)$ is minimum when $x = \dfrac{4}{4 + \pi} \approx 0.56$ feet and $r = \dfrac{2}{4 + \pi} \approx 0.28$ feet.

So the total area is minimum when 2.24 feet is used for the square and 1.76 feet is used for the circle.

From the graph, $A(x)$ is maximum when $x = 0$ feet and $r = \dfrac{2}{\pi} \approx 0.64$ feet. So the total area is maximum when all 4 feet is used for the circle.

36. Let x = each additional tree, then $x + 16$ is the total number of trees. Since 80 is the average yield of apples per tree, then $80 - 4x$ is the average yield if x trees are added.

The total yield, T, is the number of trees times the average yield per tree.

$$T = (x + 16)(80 - 4x) = -4x^2 + 16x + 1280$$

$$T' = -8x + 16$$

$$T'' = -8$$

$$T' = 0$$

$$-8x + 16 = 0$$

$$x = 2$$

$T''(2) < 0 \Rightarrow T$ is a maximum when $x = 2$.

The total number of trees is $16 + 2 = 18$ and the maximum yield is $T = 1296$ apples.

37. Let w be the number of weeks and p be the price per bushel. Use the points $(1, 30)$ and $(2, 29.20)$ to determine the linear equation relating the price per bushel to the number of weeks that pass.

$$p - 30 = \frac{29.20 - 30}{2 - 1}(w - 1)$$
$$p - 30 = -0.80w + 0.80$$
$$p = -0.80w + 30.80$$

Let b be the number of bushels in the field.

Use the points $(1, 120)$ and $(2, 124)$ to determine the linear equation relating the number of bushels in the field to the number of weeks that pass.

$$b - 120 = \frac{124 - 120}{2 - 1}(w - 1)$$
$$b - 120 = 4w - 4$$
$$b = 4w + 116$$

The total value of the crop is:

$$R = pb = (-0.80w + 30.80)(4w + 116) = -3.2w^2 + 30.4w + 3572.8$$
$$R' = -6.4w + 30.4$$
$$R'' = -6.4$$

$R' = 0$ when $w = 4.75$. Because $R'' < 0$, R is maximum when $w = 4.75$. So the farmer should harvest the strawberries after 4 weeks.

The total bushels harvested is $b(4.75) = 135$ bushels.

The maximum value of the strawberries is $R(4.75) = \$3645$.

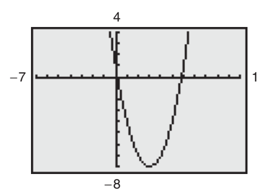
Chapter 3 Quiz Yourself

1. $f(x) = x^2 - 6x + 1$

$f'(x) = 2x - 6$

Critical number: $x = 3$

Interval	$-\infty < x < 3$	$3 < x < \infty$
Sign of f'	$f' < 0$	$f' > 0$
Conclusion	Decreasing	Increasing

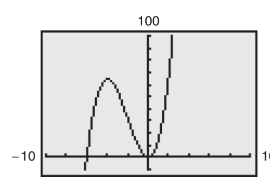


2. $f(x) = 2x^3 + 12x^2$

$f'(x) = 6x^2 + 24x = 6x(x + 4)$

Critical numbers: $x = 0$, $x = -4$

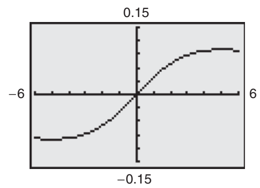
Interval	$-\infty < x < -4$	$-4 < x < 0$	$0 < x < \infty$
Sign of f'	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion	Increasing	Decreasing	Increasing



3. $f(x) = \dfrac{x}{x^2 + 25}$

$f'(x) = \dfrac{x^2 + 25(1) - x(2x)}{(x^2 + 25)^2}$

$f'(x) = \dfrac{25 - x^2}{(x^2 + 25)^2}$



Critical numbers: $x = \pm 5$

Interval	$-\infty < x < -5$	$-5 < x < 5$	$5 < x < \infty$
Sign of f'	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion	Decreasing	Increasing	Decreasing

4. $f(x) = x^3 + 3x^2 - 5$

$f'(x) = 3x^2 + 6x = 3x(x + 2)$

Critical numbers: $x = -2,\ x = 0$

Interval	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < \infty$
Sign of f'	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion	Increasing	Decreasing	Increasing

Relative maximum: $(-2, -1)$

Relative minimum: $(0, -5)$

5. $f(x) = x^4 - 8x^2 + 3$

$f'(x) = 4x^3 - 16x = 4x(x^2 - 4)$

Critical numbers: $x = \pm 2,\ x = 0$

Interval	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < 2$	$2 < x < \infty$
Sign of f'	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion	Decreasing	Increasing	Decreasing	Increasing

Relative maximum: $(0, 3)$

Relative minima: $(-2, -13)$, $(2, -13)$

6. $f(x) = 2x^{2/3}$

$f'(x) = \dfrac{4}{3}x^{-1/3} = \dfrac{4}{3x^{1/3}}$

Critical number: $x = 0$

Interval	$-\infty < x < 0$	$0 < x < \infty$
Sign of f'	$f' < 0$	$f' > 0$
Conclusion	Decreasing	Increasing

Relative minimum: $(0, 0)$

7. $f(x) = x^2 + 2x - 8, [-2, 1]$

$f'(x) = 2x + 2$

Critical number: $x = -1$

x-value	Endpoint $x = -2$	Critical $x = -1$	Endpoint $x = 1$
$f(x)$	-8	-9	-5
Conclusion		Minimum	Maximum

8. $f(x) = x^3 - 27x, [-4, 4]$

$f'(x) = 3x^2 - 27$

Critical numbers: $x = \pm 3$

x-value	Endpoint $x = -4$	Critical $x = -3$	Critical $x = 3$	Endpoint $x = 4$
$f(x)$	44	54	-54	44
Conclusion		Maximum	Minimum	

9. $f(x) = \frac{x}{x^2 + 1}, [0, 2]$

$$f'(x) = \frac{(x^2 - 1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

Critical number: $x = 1$

x-value	Endpoint $x = 0$	Critical $x = 1$	Endpoint $x = 2$
$f(x)$	0	$\frac{1}{2}$	$\frac{2}{5}$
Conclusion	Minimum	Maximum	

10. $f(x) = x^3 - 6x^2 + 7x$

$f'(x) = 3x^2 - 12x + 7$

$f''(x) = 6x - 12$

$f''(x) = 0$ when $x = 2$.

Point of inflection: $(2, -2)$

Interval	$-\infty < x < 2$	$2 < x < \infty$
Sign of $f''(x)$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave downward	Concave upward

11. $f(x) = x^4 - 24x^2$

$f'(x) = 4x^3 - 48x$

$f''(x) = 12x^2 - 48$

$f''(x) = 0$ when $x = \pm 2$.

Points of inflection: $(-2, -80), (2, -80)$

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
Sign of $f''(x)$	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

12. $f(x) = 2x^3 + 3x^2 - 12x + 16$

$f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1)$

Critical numbers: $x = -2, x = 1$

$f''(x) = 12x + 6$

$f''(-2) = -18 < 0$

$f''(1) = 18 > 0$

So, $(-2, 36)$ is a relative maximum and $(1, 9)$ is a relative minimum.

13. $f(x) = 2x + 18x^{-1}$

$$f'(x) = 2 - 18x^{-2} = 2 - \frac{18}{x^2} = \frac{2(x^2 - 9)}{x^2}$$

Critical numbers: $x = \pm 3$

$$f''(x) = 36x^{-3} = \frac{36}{x^3}$$

$f''(-3) < 0$ and $f''(3) > 0$

So, $(-3, -12)$ is a relative maximum and $(3, 12)$ is a relative minimum.

14. $S = \frac{1}{3600}(360x^2 - x^3),\ 0 \le x \le 240$

$S' = \frac{1}{3600}(720x - 3x^2)$

$S'' = \frac{1}{3600}(720 - 6x) = 0$ when $x = 120$.

$S'' > 0$ on $(0, 120)$.

$S'' < 0$ on $(120, 240)$.

Since $(120, 960)$ is a point of inflection, it is the point of diminishing returns.

15. The perimeter is $x + 2y = 200$ and

$y = 100 - \frac{1}{2}x$. The area is:

$A = xy = x\left(100 - \frac{1}{2}x\right) = 100x - \frac{1}{2}x^2$

$A' = 100 - x$

$A'' = -1$

$A' = 0$ when $x = 100$. Because $A'' = -1 < 0$, A is maximum when $x = 100$ feet and

$y = 100 - \frac{1}{2}(100) = 50$ feet.

16. $P = 0.001t^3 - 0.64t^2 + 10.3t + 1276,\ 0 \le t \le 9$

$P' = 0.003t^2 - 1.28t + 10.3$

$P' = 0$

$0.003t^2 - 1.28t + 10.3 = 0$

$t \approx 8.2$ or 418.5 $(t \approx 418.5$ is not in the domain.)

(a)

Interval	$0 < t < 8.2$	$8.2 < t < 9$
Sign of P'	$P' > 0$	$P' < 0$
Conclusion	Increasing	Decreasing

The population was increasing from 2000 to early 2008 $(0 < t < 8.2)$ and decreasing from early 2008 to 2009 $(8.2 < t < 9)$.

(b)

t-value	Endpoint $t = 0$	Critical number $t = 8.2$	Endpoint $t = 9$
$P(t)$	1276	1317.8	1317.4

The maximum population was 1317.8 thousand or 1,317,800 in early 2008 and the minimum population was 1276 thousand or 1,276,000 in 2000.

Section 3.5 Business and Economics Applications

Skills Warm Up

1. $\left|-\frac{300}{150} + 3\right| = |-2 + 3| = 1$

2. $\left|-\frac{600}{5(150)} + 2\right| = \left|-\frac{4}{5} + 2\right| = \frac{6}{5}$

3. $\left|\frac{20(150)^{-1/2}/150}{-10(150)^{-3/2}}\right| = \left|\frac{20(150)^{-3/2}}{-10(150)^{-3/2}}\right| = 2$

4. $\left|\frac{(4000/150^2)/150}{-8000(150)^{-3}}\right| = \left|\frac{4000(150)^{-3}}{-8000(150)^{-3}}\right| = -\frac{1}{2}$

5. $\frac{dC}{dx} = 1.2 + 0.006x$

6. $\frac{dP}{dx} = 0.02x + 11$

7. $\frac{dP}{dx} = -1.4x + 7$

8. $\frac{dC}{dx} = 4.2 + 0.003x^2$

9. $\frac{dR}{dx} = 14 - \frac{x}{1000}$

10. $\frac{dR}{dx} = 3.4 - \frac{x}{750}$

1. $R = 800x - 0.2x^2$

$R' = 800 - 0.4x$

$R'' = -0.4$

$R' = 0$ when $x = 2000$. Because $R'' < 0$, R is maximum when $x = 2000$ units.

2. $R = 48x^2 - 0.02x^3$

$R' = 96x - 0.06x^2 = x(96 - 0.06x)$

$R'' = 96 - 0.12x$

$R''(1600) < 0,$

$R' = 0$ when $x = 0$ or $x = 1600$. Because $R''(1600) < 0$, R is maximum when $x = 1600$ units.

3. $R = 400x - x^2$

$R' = 400 - 2x$

$R'' = -2$

$R' = 0$ when $x = 200$. Because $R'' < 0$, R is maximum when $x = 200$ units.

4. $R = 30x^{2/3} - 2x$

$R' = 20x^{-1/3} - 2 = \dfrac{20}{\sqrt[3]{x}} - 2$

$R'' = -\dfrac{20}{3}x^{-4/3} = -\dfrac{20}{3x^{4/3}}$

$R' = 0$ when $x = 1000$. Because $R''(1000) < 0$, R is maximum when $x = 1000$ units.

5. $\overline{C} = 0.125x + 20 + \dfrac{5000}{x}$

$\overline{C}' = 0.125 - \dfrac{5000}{x^2}$

$\overline{C}'' = \dfrac{10,000}{x^3}$

$\overline{C}' = 0$ when $x = 200$.

Because $\overline{C}''(200) > 0$, $\overline{C}$ is minimum when $x = 200$ units.

6. $\overline{C} = 0.001x^2 + 5 + \dfrac{250}{x}$

$\overline{C}' = 0.002x - \dfrac{250}{x^2} = \dfrac{0.002x^3 - 250}{x^2}$

$\overline{C}'' = 0.002 + \dfrac{500}{x^3}$

$\overline{C}' = 0$ when $x = 50$. Because $\overline{C}''(50) > 0$, $\overline{C}$ is minimum when $x = 50$ units.

7. $\overline{C} = 2x + 255 + \dfrac{5000}{x}$

$\overline{C}' = 2 - \dfrac{5000}{x^2}$

$\overline{C}'' = \dfrac{10,000}{x^3}$

$\overline{C}' = 0$ when $x = 50$. Because $\overline{C}''(50) > 0$, $\overline{C}$ is minimum when $x = 50$ units.

8. $\overline{C} = 0.02x^2 + 55x + \dfrac{1380}{x}$

$\overline{C}' = 0.04x + 55 - \dfrac{1380}{x^2}$

$\overline{C}'' = 0.04 + \dfrac{2760}{x^3}$

Using a graphing utility, $\overline{C}' = 0$ when $x = 5$. Because $C''(5) > 0$, $\overline{C}$ is minimum when $x = 5$ units.

9. $P = xp - C = x(90 - x) - (100 + 30x)$

$= -x^2 + 60x - 100$

$P' = -2x + 60$

$P'' = -2$

$P' = 0$ when $x = 30$. Because $P''(30) < 0$, P is maximum when $x = 30$ units and $p = 90 - 30 = \$60$ per unit.

10. $P = xp - C$

$= x(70 - 0.01x) - (8000 + 50x + 0.03x^2)$

$= -0.04x^2 + 20x - 8000$

$P' = -0.08x + 20$

$P'' = -0.08$

$P' = 0$ when $x = 250$. Because $P''(250) < 0$, P is maximum when $x = 250$ units and $p = 70 - 0.01(250) = \$67.50$ per unit.

11. $P = xp - C = x\left(50 - \dfrac{\sqrt{x}}{10}\right) - (35x + 500)$

$= 15x - \dfrac{1}{10}x^{3/2} - 500$

$P' = 15 - \dfrac{3}{20}x^{1/2}$

$P'' = -\dfrac{3}{40}x^{-1/2} = -\dfrac{3}{40x^{1/2}}$

$P' = 0$ when $x = 10,000$.

Because $P''(10,000) < 0$, P is maximum when $x = 10,000$ units and $p = 50 - \dfrac{\sqrt{10,000}}{10} = \$40$ per unit.

12. $P = xp - C$

$= x\left(\frac{24}{\sqrt{x}}\right) - (0.4x + 600)$

$= 24\sqrt{x} - 0.4x - 600$

$P' = 12x^{-1/2} - 0.4$

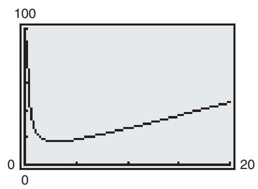
$P'' = -6x^{-3/2}$

$P' = 0$ when $x = 900$.

Because $P''(900) < 0$, P is a maximum when

$x = 900$ units and $p = \frac{24}{\sqrt{900}} = \$0.80$ per unit.

13. $\overline{C} = 2x + 5 + \frac{18}{x}$



$\overline{C}' = 2 - \frac{18}{x^2}$

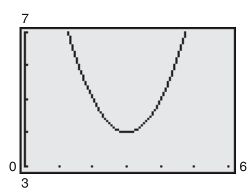
$\overline{C}'' = \frac{36}{x^3}$

$\overline{C}' = 0$ when $x = 3$. Because $\overline{C}''(3) > 0$, $\overline{C}$ is minimum when $x = 3$ units and $\overline{C}(3) = \$17$ per unit.

Marginal cost $= \frac{dx}{dc} = 4x + 5$

When $x = 3$, $\frac{dx}{dc} = \overline{C} = 17$.

14. $C = x^3 - 6x^2 + 13x$



$\overline{C} = x^2 - 6x + 13$

$\overline{C}' = 2x - 6$

$\overline{C}'' = 2$

$\overline{C}' = 0$ when $x = 3$.

Because $\overline{C}'' > 0$, $\overline{C}$ is minimum when $x = 3$ units and $\overline{C}(3) = \$4$ per unit.

Marginal cost $= \frac{dC}{dx} = 3x^2 - 12x + 13$

When $x = 3$, $\frac{dx}{dC} = \overline{C} = 4$.

15. (a) $P = xp - C$

$= x(80 - 0.2x) - (30x + 40)$

$= -0.2x^2 + 50x - 40$

$P' = -0.4x + 50$

$P'' = -0.4$

$P' = 0$

$-0.4x + 50 = 0$

$-0.4x = -50$

$x = 125$ units

$P''(125) < 0 \Rightarrow P$ is a maximum when

$x = 125$ units and

$p = 80 - 0.2(125) = \$55$ per unit.

(b) $\overline{C} = \frac{C}{x} = \frac{30x + 40}{x}$

$\overline{C} = (125) = \$30.32$ per unit

16. (a) $P = xp - C$

$= x(100 - 0.5x) - (50x + 37.5)$

$= -0.52x^2 + 50x - 37.5$

$P' = -x + 50$

$P'' = -1$

$P' = 0$

$-x + 50 = 0$

$x = 50$

$P''(50) < 0 \Rightarrow P$ is a maximum when

$x = 50$ units and

$p = 100 - 0.5(50) = \$75$ per unit.

(b) $\overline{C} = \frac{C}{x} = \frac{50x + 37.5}{x}$

$\overline{C}(50) = \$50.75$ per unit

17. $P = -2s^3 + 35s^2 - 100s + 200$

$P' = -6s^2 + 70s - 100$

$= -2(3s^2 - 35s + 50)$

$= -2(3s - 5)(s - 10)$

$P'' = -12x + 70$

$P' = 0$ when $s = \frac{5}{3}$ or $s = 10$.

Because $P''\left(\frac{5}{3}\right) = 50 > 0$, P is minimum when $s = \frac{5}{3}$.

Because $P''(10) = -50 < 0$, P is maximum when $s = 10$.

$P'' = -12s + 70 = 0$ when $s = \frac{35}{6}$. The point of diminishing returns occurs at $s = \frac{35}{6}$.

18. $P = -\frac{1}{10}s^3 + 6s^2 + 400$

$P' = -\frac{3}{10}s^2 + 12s = s\left(-\frac{3}{10}s + 12\right)$

$P'' = -\frac{3}{5}s + 12$

$P' = 0$ when $s = 0$ or $s = 40$.

Because $P''(0) = 12 > 0$, P is minimum when $s = 0$.

Because $P''(40) = -12 < 0$, P is maximum when $s = 40$.

$P'' = -\frac{3}{5}s + 12 = 0$ when $s = 20$.

The point of diminishing returns occurs at $s = 20$.

19. Let x = number of units purchased, p = price per unit, and P = profit.

$p = 150 - (0.10)(x - 100) = 160 - 0.10x, x \geq 100$

$$\begin{aligned} P &= xp - C \\ &= x(160 - 0.10x) - 90x \\ &= -0.10x^2 + 70x \end{aligned}$$

$P' = -0.2x + 70$

$P'' = -0.2$

$$\begin{aligned} P' &= 0 \\ -0.02x + 70 &= 0 \\ -0.2x &= -70 \\ x &= 350 \end{aligned}$$

$P''(350) < 0 \Rightarrow P$ is a maximum when $x = 350$ MP3 players.

20. Let x = the number of \$40 increases in rent and P = profit.

$$\begin{aligned} P &= (\text{Rent})(\text{Number of apartments}) - (\text{Cost})(\text{Number of apartments}) \\ &= (580 + 40x)(50 - x) - 45(50 - x) \\ &= -40x^2 + 1465x + 26{,}750 \end{aligned}$$

$P' = -80x + 1465$

$P'' = -8$

$P' = 0$ when $x = 18.3125$. Because $P'' < 0$, P is maximum when $x \approx 18$.

To maximize profit, the rent should be $P(18) = 580 + 40(18) = \$1300$.

21. Let x be the number of units sold per week, p be the price per unit, and R be the revenue. Use the points $(40, 300)$, and $(45, 275)$ to determine the linear equation relating units sold to price per unit.

$$\begin{aligned} x - 300 &= \frac{275 - 300}{45 - 40}(p - 40) \\ x &= -5p + 500 \end{aligned}$$

$$\begin{aligned} R = xp &= (-5p + 500)p \\ &= -5p^2 + 500p \end{aligned}$$

$R' = -10p + 500$

$R'' = -10$

$R' = 0$ when $p = 50$. Because $R'' < 0$, R is maximum when $p = \$50$.

22. $A = ki^2, k > 0$

$P = 0.12A - i(A) = 0.12ki^2 - ki^3$

$$\begin{aligned} P' = ki^2(-1) + (0.12 - i)(2ki) &= 0.24ki - 3ki^2 \\ &= ki(0.24 - 3i) \end{aligned}$$

$P' = 0$ when $i = 0$ and $i = 0.08$. The profit is maximum when $i = 8\%$.

$P' = 0.24ki - 3ki^2 = ki(0.24 - 3i)$

$P'' = 0.24k - 6ki$

$P' = 0$ when $i = 0$ and when $i = 0.08$. Because

$P''(0.08) = -0.24k < 0$

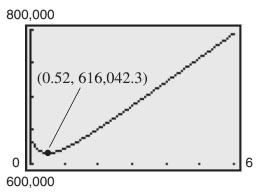
P is maximum when $i = 0.08$.

23. Let T be the total cost

r = cost under water + cost on land

$$T = 25(5280)\sqrt{x^2 + 0.25} + 18(5280)(6 - x)$$
$$= 132{,}000\sqrt{x^2 + 0.25} + 570{,}240 - 95{,}040x$$
$$x \approx 0.52$$

The line should run from the power station to a point across the river approximately 0.52 mile downstream.



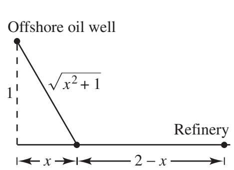
24. Let k = cost per mile to run the line overland. The total cost to buy the pipe is:

$$C = 2k\sqrt{x^2 + 1} + k(2 - x)$$
$$= k\left[2(x^2 + 1)^{1/2} + (2 - x)\right]$$
$$C' = k\left[2\left(\frac{1}{2}\right)(x^2 + 1)^{-1/2}(2x) - 1\right]$$
$$= k\left[\frac{2x}{(x^2 + 1)^{1/2}}\right]$$
$$C'' = \frac{(x^2 + 1)^{1/2}(2) - 2x\left[\frac{1}{2}(x^2 + 1)^{-1/2}(2x)\right]}{x^2 + 1}$$
$$= \frac{2k}{(x^2 + 1)^{3/2}}$$

$C' = 0$ when $\dfrac{2x}{\sqrt{x^2 + 1}} = 1$

$$2x = \sqrt{x^2 + 1}$$
$$4x^2 = x^2 + 1$$
$$3x^2 = 1$$
$$x = \sqrt{\frac{1}{3}} = \frac{\sqrt{3}}{3}$$

Because $C'' > 0$, C is minimum when $x = \dfrac{\sqrt{3}}{3}$ mile.



25. distance = (rate)(time)

$$110 = vt$$
$$\frac{110}{v} = t$$

Let T be the total cost.

$$T = \frac{v^2}{300}\left(\frac{110}{v}\right) + 12\left(\frac{110}{v}\right)$$
$$= \frac{11}{30}v + \frac{1320}{v}$$
$$T' = \frac{11}{30} - \frac{1320}{v^2}$$
$$T'' = \frac{2640}{v^3}$$

$T' = 0$ when $v = 60$. Since $T''(60) > 0$, T is minimum when $V = 60$ mi/h.

26. distance = (rate)(time)

$$110 = vt$$
$$\frac{110}{v} = t$$

Let T be the total cost.

$$T = \frac{V^2}{500}\left(\frac{110}{v}\right) + 9.5\left(\frac{110}{v}\right)$$
$$= \frac{11}{50}v + \frac{1045}{v}$$
$$T' = \frac{11}{50} - \frac{1045}{v^2}$$
$$T'' = \frac{2090}{v^3}$$

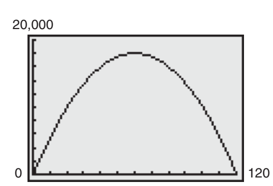
$T' = 0$ when $v = \sqrt{4750} \approx 68.9$. Because $T''\left(\sqrt{4750}\right) > 0$, T is minimum when $v \approx 68.9$ mi/h.

27. Because $dp/dx = -5$, the price elasticity of demand is

$$\eta = \frac{p/x}{dp/dx} = \frac{\dfrac{600 - 5x}{x}}{-5} = 1 - \frac{600}{5x}.$$

When $x = 60$, you have $\eta = 1 - \dfrac{600}{5(60)} = -1$.

Because $\left|\eta(60)\right| = 1$, the demand is unit elastic.



Elastic: $(0, 60)$

Inelastic: $(60, 120)$

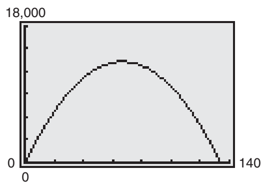
28. Because $dp/dx = -3$, the price elasticity of demand is

$$\eta = \frac{p/x}{dp/dx} = \frac{(400 - 3x)/x}{-3} = 1 - \frac{400}{3x}.$$

When $x = 20$, you have

$$\eta = 1 - \frac{400}{3(20)} = -\frac{17}{3}.$$

Because $|\eta(20)| = \frac{17}{3} > 1$, the demand is elastic.



Elastic: $\left(0, \frac{200}{3}\right)$

Inelastic: $\left(\frac{200}{3}, \frac{400}{3}\right)$

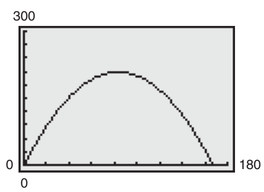
29. Because $dp/dx = -0.03$, the price elasticity of demand is

$$\eta = \frac{p/x}{dp/dx} = \frac{(5 - 0.03x)/x}{-0.03} = 1 - \frac{5}{0.03x}.$$

When $x = 100$, you have

$$\eta = 1 - \frac{5}{0.03(100)} = -\frac{2}{3}.$$

Because $|\eta(100)| = \frac{2}{3} < 1$, the demand is inelastic.



Elastic: $\left(0, \frac{250}{3}\right)$

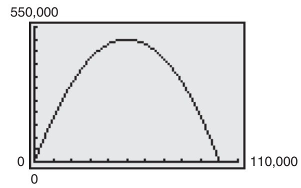
Inelastic: $\left(\frac{250}{3}, \frac{500}{3}\right)$

30. Because $dp/dx = -0.0002$, the price elasticity of demand is

$$\eta = \frac{p/x}{dp/dx} = \frac{(20 - 0.0002x)/x}{-0.0002} = \frac{-100{,}000}{x} + 1.$$

When $x = 30$, you have $\eta = \frac{-100{,}000}{30} + 1 = -\frac{9997}{3}$.

Because $|\eta(30)| > 1$, the demand is elastic.



Elastic: $(0, 50{,}000)$

Inelastic: $(50{,}000, 100{,}000)$

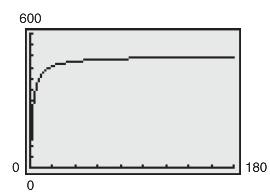
31. Because $\frac{dp}{dx} = -\frac{500}{x(x+2)^2}$, the price elasticity of demand is

$$\eta = \frac{p/x}{dp/dx} = \frac{500}{x(x+2)} \cdot \frac{(x+2)^2}{-500} = -\frac{x+2}{x}.$$

When $x = 23$, you have

$$\eta = -\frac{23 + 2}{23} = -\frac{25}{23}.$$

Because $|\eta(23)| = \frac{25}{23} > 1$, the demand is elastic.



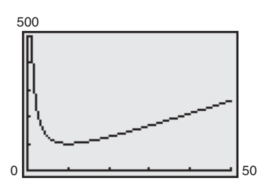
Elastic: $(0, \infty)$

32. Because $dp/dx = \frac{-1000}{x^3}$, the price elasticity of demand is

$$\eta = \frac{p/x}{dp/dx} = \frac{\dfrac{\dfrac{500}{x^2} + 5}{x}}{\dfrac{-1000}{x^3}} = -\frac{500 + 5x^2}{1000} = -\frac{100 + x^2}{200} = -\frac{1}{2} - \frac{x^2}{200}.$$

When $x = 5$, you have $\eta = -\frac{1}{2} - \frac{(5)^2}{200} = -\frac{5}{8}$.

Because $|\eta(5)| = \frac{5}{8} < 1$, the demand is inelastic.



Elastic: $(10, \infty)$

Inelastic: $(0, 10)$

33. (a) $p = 20 - 0.02x,\ 0 \le x \le 1000$

$$\frac{dp}{dx} = -0.02$$

$$\eta = \frac{p/x}{dp/dy} = \frac{\dfrac{20 - 0.02x}{x}}{-0.02} = -\frac{1000}{x} + 1.$$

In the interval $[0, 1000]$, the solution to $|\eta| = \left|-\frac{1000}{x} + 1\right| = 1$ is $x = 500$. So the demand is of unit elasticity when $x = 500$. For x-values in $[0, 500)$, $|\eta| > 1$, so demand is elastic. For x-values in $(500, 1000]$, $|\eta| < 1$, so demand is inelastic.

(b) The revenue function increases on the interval $[0, 500)$, then is flat at $x = 500$, and decreases on the interval $(500, 1000]$.

34. (a) $p = 800 - 4x,\ 0 \le x \le 200$

$$\frac{dp}{dx} = -4$$

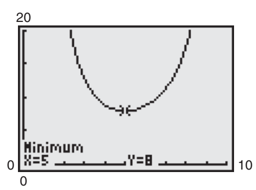
$$\eta = \frac{p/x}{dp/dx} = \frac{\dfrac{800 - 4x}{x}}{-4} = -\frac{200}{x} + 1.$$

In the interval $[0, 200]$, the solution to $\eta = \left|-\frac{200}{x} + 1\right| = 1$ is $x = 100$.

So the demand is of unit elasticity when $x = 100$. For x-values in $[0, 100)$, $|\eta| > 1$, so demand is elastic. For x-values in $(100, 200]$, $|\eta| < 1$, so demand is inelastic.

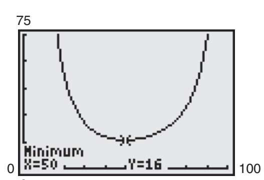
(b) The revenue function increases on the interval $[0, 100)$, then is flat at $x = 100$, and decreases on the interval $(100\ 200]$.

35. $C = 4\left(\dfrac{25}{x^2} - \dfrac{x}{x - 10}\right)$



C is minimum when $x = 5$, or 500 units shipped.

36. $C = 8\left(\dfrac{2500}{x^2} - \dfrac{x}{x - 100}\right)$



C is minimum when $x = 50$, or an order size of 5000.

37. $x = 900 - 45p \Rightarrow p = 20 - \dfrac{x}{45}$

$$\frac{dp}{dx} = -\frac{1}{45}$$

When $p = 8$, $x = 540$ and $\eta = \dfrac{p/x}{dp/dx} = \dfrac{\frac{8}{540}}{-\frac{1}{45}} = -\dfrac{2}{3}$.

Because $|\eta| = \dfrac{2}{3} < 1$, the demand is inelastic.

38. (a) Demand function; the number of units sold decreases as the price increases.

(b) Cost function; cost increases linearly with the number of units produced.

(c) Revenue function; a revenue function is greater than a profit function.

(d) Profit function; a profit function is less than a revenue function.

39. $S = -1.893t^3 + 41.03t^2 - 58.6t + 3972, 1 \le t \le 10$

$S' = -5.679t^2 + 82.06t - 58.6$

$S'' = -11.358t + 82.06$

(a)–(c) $S'' = 0$

$-11.358t + 82.06 = 0$

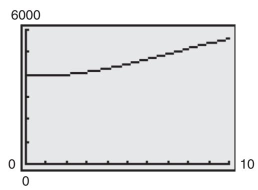
$t \approx 7.2 \Rightarrow t = 7$

t-value	Endpoint $t = 1$	Critical value of S'' $t = 7$	Endpoint $t = 10$
s-value	17.78	237.55	194.1
Conclusion	Minimum	Maximum	

Most rapidly: 2007; rate: \$237.55 million/yr

Slowest rate: 2001; rate: \$17.78 million/yr

(d)



40. $S = \dfrac{18.17 + 8.165t}{1 + 0.116t}, 1 \le t \le 10$

$$S' = \frac{(1 + 0.116t)(8.165) - (18.17 + 8.165t)(0.116)}{(1 + 0.116t)^2} = \frac{6.05728}{(1 + 0.116t)^2} \text{ or } 6.05728(1 + 0.116t)^{-2}$$

Note: $S' \ne 0$

$$S'' = (6.05728)(-2)(1 + 0.116t)^{-3}(0.116) = \frac{-1.40528896}{(1 + 0.116t)^3}$$

Note: $S'' \ne 0$

(a)

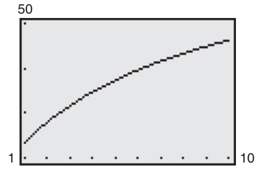
t-value	Endpoint $t = 1$	Endpoint $t = 10$
S-value	23.598	46.213
Conclusion	Minimum	Maximum

Sales were greatest in 2010 with \$46.213 billion and sales were the least in 2001 with \$23.598 billion.

(b)

t-value	Endpoint $t = 1$	Endpoint $t = 10$
S'-value	4.864	1.298
Conclusion	Maximum	Minimum

(c)



Sales were greatest in 2010 with \$46.213 billion and sales were the least in 2001 with \$23.598 billion.
Sales were increasing at the greatest rate in 2001 and sales were decreasing at the greatest rate in 2010.

41. $x = \frac{a}{p^m},\ m > 1$

$1 = -\frac{am}{p^{m+1}}\frac{dp}{dx}$

$\frac{dp}{dx} = -\frac{p^{m+1}}{am}$

$\eta = \frac{p/x}{dp/dx} = \frac{p}{x}\cdot\frac{-am}{p^{m+1}} = \frac{p}{a/p^m}\cdot\frac{-am}{p^{m+1}} = \frac{p^{m+1}}{a}\cdot\frac{-am}{p^{m+1}} = -m$

42. Answers will vary.

43. Answers will vary.

Section 3.6 Asymptotes

Skills Warm Up

1. $\lim_{x\to 2}(x + 1) = 2 + 1 = 3$

2. $\lim_{x\to -1}(3x + 4) = 3(-1) + 4 = 1$

3. $\lim_{x\to -3}\frac{2x^2 + x - 15}{x + 3} = \lim_{x\to -3}\frac{(2x - 5)(x + 3)}{x + 3}$
$= \lim_{x\to -3}(2x - 5) = 2(-3) - 5$
$= -11$

4. $\lim_{x\to 2}\frac{3x^2 - 8x + 4}{x - 2} = \lim_{x\to 2}\frac{(3x - 2)(x - 2)}{x - 2}$
$= \lim_{x\to 2}(3x - 2) = 3(2) - 2 = 4$

5. $\lim_{x\to 2^+}\frac{x^2 - 5x + 6}{x^2 - 4} = \lim_{x\to 2^+}\frac{(x - 3)(x - 2)}{(x + 2)(x - 2)}$
$= \lim_{x\to 2^+}\frac{x - 3}{x + 2} = \frac{2 - 3}{2 + 2} = -\frac{1}{4}$

6. $\lim_{x\to 1^-}\frac{x^2 - 6x + 5}{x^2 - 1} = \lim_{x\to 1^-}\frac{(x - 5)(x - 1)}{(x + 1)(x - 1)}$
$= \lim_{x\to 1^-}\frac{x - 5}{x + 1} = \frac{1 - 5}{1 + 1} = -2$

7. $\lim_{x\to 0^+}\sqrt{x} = \sqrt{0} = 0$

8. $\lim_{x\to 1^+}\left(x + \sqrt{x - 1}\right) = 1 + \sqrt{1 - 1} = 1$

9. $\overline{C} = \frac{C}{x} = \frac{150}{x} + 3$

$\frac{dC}{dx} = 3$

10. $\overline{C} = \frac{C}{x} = \frac{1900}{x} + 1.7 + 0.002x$

$\frac{dC}{dx} = 1.7 + 0.004x$

11. $\overline{C} = \frac{C}{x} = 0.005x + 0.5 + \frac{1375}{x}$

$\frac{dC}{dx} = 0.01x + 0.5$

12. $\overline{C} = \frac{C}{x} = \frac{760}{x} + 0.05$

$\frac{dC}{dx} = 0.05$

1. A horizontal asymptote occurs at $y = 1$ because

$\lim_{x\to\infty}\frac{x^2 + 1}{x^2} = 1,\ \lim_{x\to -\infty}\frac{x^2 + 1}{x^2} = 1.$

A vertical asymptote occurs at $x = 0$ because

$\lim_{x\to 0^-}\frac{x^2 + 1}{x^2} = \infty,\ \lim_{x\to 0^+}\frac{x^2 + 1}{x^2} = \infty.$

2. A horizontal asymptote occurs at $y = 0$ because

$\lim_{x\to\infty}\frac{4}{(x - 2)^3} = 0$ and $\lim_{x\to -\infty}\frac{4}{(x - 2)^3} = 0.$

A vertical asymptote occurs at $x = 2$ because

$\lim_{x\to 2^-}\frac{4}{(x - 2)^3} = -\infty$ and $\lim_{x\to 2^+}\frac{4}{(x - 2)^3} = \infty.$

3. A horizontal asymptote occurs at $y = 1$ because

$$\lim_{x\to\infty} \frac{x^2 - 2}{x^2 - x - 2} = 1, \lim_{x\to-\infty} \frac{x^2 - 2}{x^2 - x - 2} = 1.$$

Vertical asymptotes occur at $x = -1$ and $x = 2$ because

$$\lim_{x\to-1^-} \frac{x^2 - 2}{x^2 - x - 2} = -\infty, \lim_{x\to-1^+} \frac{x^2 - 2}{x^2 - x - 2} = \infty,$$

$$\lim_{x\to-2^-} \frac{x^2 - 2}{x^2 - x - 2} = -\infty, \lim_{x\to-2^+} \frac{x^2 - 2}{x^2 - x - 2} = \infty.$$

4. A horizontal asymptote occurs at $y = 1$ because

$$\lim_{x\to\infty} \frac{x + 1}{x + 2} = 1, \lim_{x\to-\infty} \frac{x + 1}{x + 2} = 1.$$

A vertical asymptote occurs at $x = -2$ because

$$\lim_{x\to-2^-} \frac{x + 1}{x + 2} = \infty, \lim_{x\to-2^+} \frac{x + 1}{x + 2} = -\infty.$$

5. A horizontal asymptote occurs at $y = \frac{3}{2}$ because

$$\lim_{x\to\infty} \frac{3x^2}{2(x^2 + 1)} = \frac{3}{2} \text{ and } \lim_{x\to-\infty} \frac{3x^2}{2(x^2 + 1)} = \frac{3}{2}.$$

The graph has no vertical asymptotes because the denominator is never zero.

6. A horizontal asymptote occurs at $y = 0$ because

$$\lim_{x\to\infty} \frac{-4x}{x^2 + 4} = 0 \text{ and } \lim_{x\to-\infty} \frac{-4x}{x^2 + 4} = 0.$$

The graph has no vertical asymptotes because the denominator is never zero.

7. A horizontal asymptote occurs at $y = \frac{1}{2}$ because

$$\lim_{x\to\infty} \frac{x^2 - 1}{2x^2 - 8} = \frac{1}{2} \text{ and } \lim_{x\to-\infty} \frac{x^2 - 1}{2x^2 - 8} = \frac{1}{2}.$$

Vertical asymptotes occur at $x = \pm 2$ because

$$\lim_{x\to2^-} \frac{x^2 - 1}{2x^2 - 8} = -\infty, \lim_{x\to2^+} \frac{x^2 - 1}{2x^2 - 8} = \infty,$$

$$\lim_{x\to-2^-} \frac{x^2 - 1}{2x^2 - 8} = \infty, \lim_{x\to-2^+} \frac{x^2 - 1}{2x^2 - 8} = -\infty.$$

8. A horizontal asymptote occurs at $y = 0$ because

$$\lim_{x\to\infty} \frac{x^2 + 1}{x^3 - 8} = 0 \text{ and } \lim_{x\to-\infty} \frac{x^2 + 1}{x^3 - 8} = 0.$$

A vertical asymptote occurs at $x = 2$ because

$$\lim_{x\to2^-} \frac{x^2 + 1}{x^3 - 8} = -\infty \text{ and } \lim_{x\to2^+} \frac{x^2 + 1}{x^3 - 8} = \infty.$$

9. $f(x) = \dfrac{x - 3}{x^2 + 3x} = \dfrac{x - 3}{x(x + 3)}$

$x(x + 3) = 0$

$x = 0$ or $x = -3$

Vertical asymptotes: $x = 0, x = -3$

10. $f(x) = \dfrac{x}{x^2 + 6x} = \dfrac{x}{x(x + 6)} = \dfrac{1}{x + 6}, x \neq 0$

$x + 6 = 0$

$x = -6$

Vertical asymptote: $x = -6$

11. $f(x) = \dfrac{x^2 - 8x + 15}{x^2 - 9} = \dfrac{(x - 3)(x - 5)}{(x - 3)(x + 3)} = \dfrac{x - 5}{x + 3}, x \neq 3$

$x + 3 = 0$

$x = -3$

Vertical asymptote: $x = -3$

12. $f(x) = \dfrac{x^2 + 2x - 35}{x^2 - 25} = \dfrac{(x + 7)(x - 5)}{(x + 5)(x - 5)} = \dfrac{x + 7}{x + 5}, x \neq 5$

$x + 5 = 0$

$x = -5$

Vertical asymptote: $x = -5$

13. $f(x) = \dfrac{2x^2 - x - 3}{2x^2 - 11x + 12} = \dfrac{(2x - 3)(x + 1)}{(2x - 3)(x - 4)} = \dfrac{x + 1}{x - 4}, x \neq \dfrac{3}{2}$

$x - 4 = 0$

$x = 4$

Vertical asymptote: $x = 4$

14. $f(x) = \dfrac{x^2 + x - 30}{4x^2 - 17x - 15} = \dfrac{(x + 6)(x - 5)}{(4x + 3)(x - 5)} = \dfrac{x + 6}{4x + 3}, x \neq 5$

$4x + 3 = 0$

$x = -\dfrac{3}{4}$

Vertical asymptote: $x = -\dfrac{3}{4}$

15. $\lim_{x \to 6^+} \dfrac{1}{(x - 6)^2} = \infty$

16. $\lim_{x \to -2^-} \dfrac{1}{x + 2} = -\infty$

17. $\lim_{x \to 3^+} \dfrac{x - 4}{x - 3} = -\infty$

18. $\lim_{x \to 1^+} \dfrac{2 + x}{1 - x} = -\infty$

19. $\lim_{x \to -1^-} \dfrac{x^2 + 1}{x^2 - 1} = \infty$

20. $\lim_{x \to 5^+} \dfrac{2x - 3}{x^2 - 25} = \infty$

21. $\lim_{x \to \infty} \left(1 + \dfrac{1}{x}\right) = 1 + 0 = 1$

22. $\lim_{x \to -\infty} \left(6 - \dfrac{3}{x}\right) = 6 + 0 = 6$

23. $\lim_{x \to -\infty} \left(7 + \dfrac{4}{x^2}\right) = 7 + 0 = 7$

24. $\lim_{x \to \infty} \left(10 - \dfrac{8}{x^2}\right) = 10 - 0 = 10$

25. $\lim_{x \to \pm\infty} \dfrac{4x - 3}{2x + 1} = \dfrac{4}{2} = 2$

Horizontal asymptote: $y = 2$

26. $\lim_{x \to \pm\infty} \dfrac{5x^2 + 1}{10x^3 - 3x^2 + 7} = 0$

Horizontal asymptote: $y = 0$ or x-axis

27. $\lim_{x \to \pm\infty} \dfrac{3x}{4x^2 - 1} = 0$

Horizontal asymptote: $y = 0$ or x-axis

28. $\lim_{x \to \pm\infty} \dfrac{2x^2 - 5x - 12}{1 - 6x - 8x^2} = -\dfrac{2}{8} = -\dfrac{1}{4}$

Horizontal asymptote: $y = -\dfrac{1}{4}$

29. $\lim_{x \to \pm\infty} \dfrac{5x^2}{x + 3} = \pm\infty$, does not exist.

No horizontal asymptote

30. $\lim_{x \to \pm\infty} \dfrac{x^3 - 2x^2 + 3x + 1}{x^2 - 3x + 2} = \pm\infty$, does not exist.

No horizontal asymptote

31. $\lim_{x \to \pm\infty} \left(\dfrac{2x}{x - 1} + \dfrac{3x}{x + 1}\right) = 2 + 3 = 5$

Horizontal asymptote: $y = 5$

32. $\lim_{x \to \pm\infty} \left(\dfrac{2x^2}{x - 1} + \dfrac{3x}{x + 1}\right) = \lim_{x \to \pm\infty} \left(\dfrac{2x^3 + 5x^2 - 3x}{x^2 - 1}\right) = \pm\infty$, does not exist.

No horizontal asymptote

33. The graph of f has a horizontal asymptote at $y = 3$. It matches graph (d).

34. The graph of f has a horizontal asymptote at $y = 0$. It matches graph (b).

35. The graph of f has a horizontal asymptote at $y = 2$. It matches graph (a).

36. The graph of f has a horizontal asymptote at $y = 5$. It matches graph (c).

37. (a) $h(x) = \dfrac{5x^3 - 3}{x^2}$

$\lim_{x \to \infty} h(x) = \infty$

(b) $h(x) = \dfrac{5x^3 - 3}{x^3}$

$\lim_{x \to \infty} h(x) = 5$

(c) $h(x) = \dfrac{5x^3 - 4}{x^4}$

$\lim_{x \to \infty} h(x) = 0$

38. (a) $h(x) = \dfrac{3x^2 + 7}{x}$

$\lim_{x\to\infty} h(x) = \infty$

(b) $h(x) = \dfrac{3x^2 + 7}{x^2}$

$\lim_{x\to\infty} h(x) = 3$

(c) $h(x) = \dfrac{3x^2 + 7}{x^3}$

$\lim_{x\to\infty} h(x) = 0$

39. (a) $\lim_{x\to\infty} \dfrac{x^2 + 2}{x^3 - 1} = 0$

(b) $\lim_{x\to\infty} \dfrac{x^2 + 2}{x^2 - 1} = 1$

(c) $\lim_{x\to\infty} \dfrac{x^2 + 2}{x - 1} = \infty$

40. (a) $\lim_{x\to\infty} \dfrac{4 - 5x}{2x^3 + 6} = 0$

(b) $\lim_{x\to\infty} \dfrac{4 - 5x}{2x + 6} = -\dfrac{5}{2}$

(c) $\lim_{x\to\infty} \dfrac{4 - 5x^2}{2x + 6} = -\infty$

41. $f(x) = \sqrt{x^3 + 6} - 2x$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	0.646	11.718	800.003	29,622.777	980,000	31,422,776.6	998,000,000

$\lim_{x\to\infty} \sqrt{x^3 + 6} - 2x = \infty$

42. $f(x) = x - \sqrt{x(x - 1)}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1	0.5132	0.5013	0.5001	0.5000	0.5000	0.5000

$\lim_{x\to\infty} x - \sqrt{x(x - 1)} = 0.5$

43. $f(x) = \dfrac{x + 1}{x\sqrt{x}}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	2.000	0.348	0.101	0.032	0.010	0.003	0.001

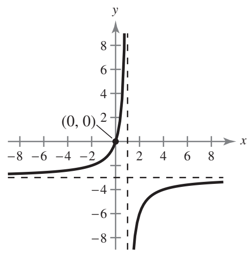
$\lim_{x\to\infty} f(x) = 0$

44. $f(x) = \dfrac{\sqrt{x}}{x^2 + 3}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	0.25	0.031	0.001	0.00003	0.000001	0.00000003	0.000000001

$\lim_{x\to\infty} \dfrac{\sqrt{x}}{x^2 + 3} = 0$

45. $y = \dfrac{3x}{1 - x}$



Intercept: $(0, 0)$

Horizontal asymptote: $y = -3$

Vertical asymptote: $x = 1$

$$y' = \frac{(1 - x)3 - 3x(-1)}{(1 - x)^2} = \frac{1}{(1 - x)^2}$$

$y' \neq 0$ so there are no relative extrema.

46. $y = \dfrac{x - 3}{x - 2}$

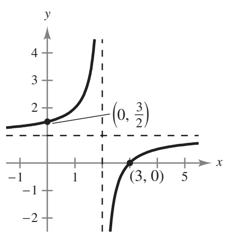
Intercepts: $(3, 0), \left(0, \frac{3}{2}\right)$

Horizontal asymptote: $y = 1$

Vertical asymptote: $x = 2$

$$y' = \frac{(x - 2) - (x - 3)}{(x - 2)^2} = \frac{1}{(x - 2)^2}$$

$y' \neq 0$ so there are no relative extrema.



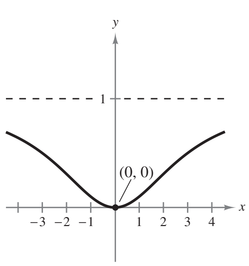
47. $f(x) = \dfrac{x^2}{x^2 + 9}$

Intercept: $(0, 0)$

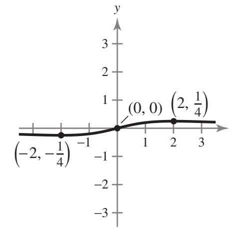
Horizontal asymptote: $y = 1$

$$f'(x) = \frac{(x^2 + 9)(2x) - x^2(2x)}{(x^2 + 9)^2} = \frac{18x}{(x^2 + 9)^2}$$

The critical number is $x = 0$ and by the First-Derivative Test $(0, 0)$ is a relative minimum.



48. $f(x) = \dfrac{x}{x^2 + 4}$



Intercept: $(0, 0)$

Horizontal asymptote: $y = 0$

$$f'(x) = \frac{(x^2 + 4) - x(2x)}{(x^2 + 4)^2} = \frac{4 - x^2}{(x^2 + 4)^2}$$

The critical numbers are $x = \pm 2$ and by the First-Derivative Test $\left(2, \frac{1}{4}\right)$ is a relative maximum and $\left(-2, -\frac{1}{4}\right)$ is a relative minimum.

49. $g(x) = \dfrac{x^2}{x^2 - 16}$

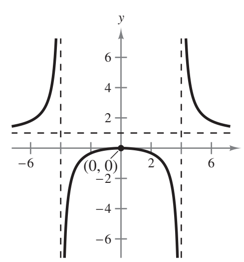
Intercept: $(0, 0)$

Horizontal asymptote: $y = 1$

Vertical asymptotes: $x = \pm 4$

$$g'(x) = \frac{(x^2 - 16)(2x) - x^2(2x)}{(x^2 - 16)^2} = \frac{-32x}{(x^2 - 16)^2}$$

The critical number is $x = 0$ and by the First-Derivative Test $(0, 0)$ is a relative maximum.



50. $g(x) = \dfrac{x}{x^2 - 36}$

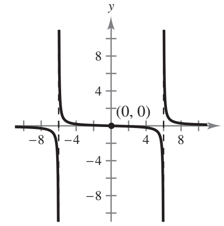
Intercept: $(0, 0)$

Horizontal asymptote: $y = 0$

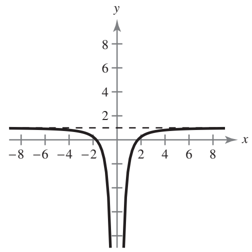
Vertical asymptotes: $x = \pm 6$

$$g'(x) = \frac{(x^2 - 36)(1) - x(2x)}{(x^2 - 36)^2} = \frac{-(x^2 + 36)}{(x^2 - 36)^2}$$

No critical numbers because $g'(x) \neq 0$, so there are no relative extrema.



51. $y = 1 - \dfrac{3}{x^2}$



Intercepts: $\left(\pm\sqrt{3}, 0\right)$

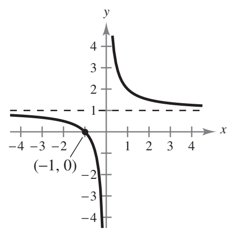
Horizontal asymptote: $y = 1$

Vertical asymptote: $x = 0$

$y' = \dfrac{6}{x^3}$

No critical numbers because $y' \neq 0$, so there are no relative extrema.

52. $y = 1 + \dfrac{1}{x}$



Intercept: $(-1, 0)$

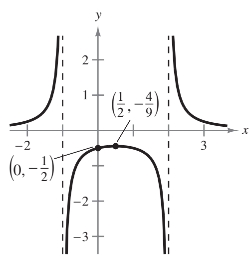
Horizontal asymptote: $y = 1$

Vertical asymptote: $x = 0$

$y' = -\dfrac{1}{x^2}$

No critical numbers because $y' \neq 0$, so there are no relative extrema.

53. $f(x) = \dfrac{1}{x^2 - x - 2} = \dfrac{1}{(x+1)(x-2)}$



Intercept: $\left(0, -\dfrac{1}{2}\right)$

Horizontal asymptote: $y = 0$

Vertical asymptotes: $x = -1$, $x = 2$

$$f'(x) = -\left(x^2 - x - 2\right)^{-2}(2x - 1)$$
$$= -\frac{2x - 1}{\left(x^2 - x - 2\right)^2}$$

The critical number is $x = \dfrac{1}{2}$ and by the First-Derivative Test $\left(\dfrac{1}{2}, -\dfrac{4}{9}\right)$ is a relative maximum.

54. $f(x) = \dfrac{x - 2}{x^2 - 4x + 3} = \dfrac{x - 2}{(x - 1)(x - 3)}$

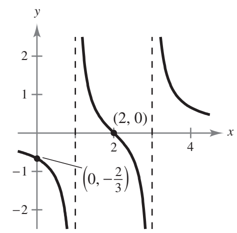
Intercepts: $(2, 0)$, $(0, -2/3)$

Horizontal asymptote: $y = 0$

Vertical asymptotes: $x = 1$ and $x = 3$

$$f'(x) = \frac{\left(x^2 - 4x + 3\right) - (x - 2)(2x - 4)}{\left(x^2 - 4x + 3\right)^2}$$
$$= -\frac{\left(x^2 - 4x + 5\right)}{\left(x^2 - 4x + 3\right)^2}$$

$f'(x) \neq 0$ so there are no relative extrema.



55. $g(x) = \dfrac{x^2 - x - 2}{x - 2} = \dfrac{(x - 2)(x + 1)}{x - 2}$

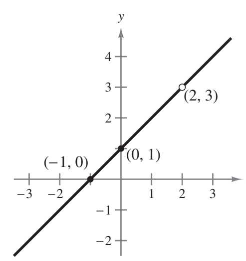
$= x + 1$ for $x \neq 2$

Intercepts: $(-1, 0)$, $(0, 1)$

No asymptotes

$g'(x) = 1$ for $x \neq 2$

$g'(x) \neq 0$ so there are no relative extrema.



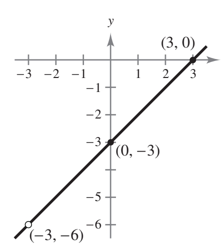
56. $g(x) = \dfrac{x^2 - 9}{x + 3} = \dfrac{(x + 3)(x - 3)}{x + 3} = x - 3$ for $x \neq 3$

Intercepts: $(3, 0)$, $(0, -3)$

No asymptotes

$g'(x) = 1$ for $x \neq -3$

$g'(x) \neq 0$ so there are no relative extrema.



57. $y = \dfrac{2x^2 - 6}{x^2 - 2x + 1} = \dfrac{2(x^2 - 3)}{(x - 1)^2}$

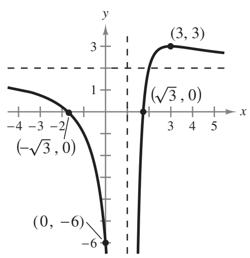
Intercepts: $(\pm\sqrt{3}, 0), (0, -6)$

Vertical asymptote: $x = 1$

Horizontal asymptote: $y = 2$

$$y' = \frac{(x^2 - 2x + 1)(4x) - (2x^2 - 6)(2x - 2)}{(x - 1)^4}$$

$$= \frac{4(3 - x)}{(x - 1)^3}$$



The critical number is $x = 3$ and by the First-Derivative Test $(3, 3)$ is a relative maximum.

58. $y = \dfrac{x}{(x + 1)^2}$

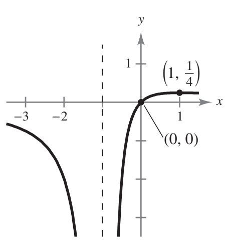
Intercept: $(0, 0)$

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = -1$

$$y' = \frac{(x + 1)^2 - x[2(x + 1)]}{(x + 1)^4} = \frac{1 - x}{(x + 1)^3}$$

The critical number is $x = 1$ and by the First-Derivative Test $\left(1, \frac{1}{4}\right)$ is a relative maximum.



59. $y = \dfrac{x}{\sqrt{x^2 + 1}}$

Intercept: $(0, 0)$

Horizontal asymptotes: $y = -1$ as $x \to -\infty$

$y = 1$ as $x \to \infty$

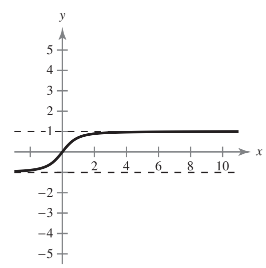
Vertical asymptote: None

$$y' = \frac{(x^2 + 1)^{1/2}(1) - x\left(\frac{1}{2}\right)(x^2 + 1)^{-1/2}(2x)}{\left(\sqrt{x^2 + 1}\right)^2}$$

$$= \frac{(x^2 + 1)^{-1/2}\left[(x^2 + 1) - x^2\right]}{(x^2 + 1)}$$

$$= \frac{1}{(x^2 + 1)^{3/2}}$$

No critical numbers because $y' \neq 0$, so there are no relative extrema.



60. $y = \dfrac{2x}{\sqrt{x^2 + 4}}$

Intercept: $(0, 0)$

Horizontal asymptotes: $y = -2$ as $x \to -\infty$

$y = 2$ as $x \to \infty$

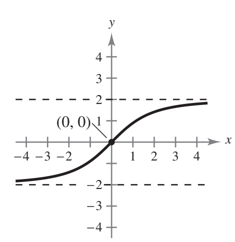
Vertical asymptote: None

$$y' = \frac{(x^2 + 4)^{1/2}(2) - 2x\left(\frac{1}{2}\right)(x^2 + 4)^{-1/2}(2x)}{\left(\sqrt{x^2 + 4}\right)^2}$$

$$= \frac{2(x^2 + 4)^{-1/2}\left[(x^2 + 4) - x^2\right]}{(x^2 + 4)}$$

$$= \frac{8}{(x^2 + 4)^{3/2}}$$

No critical numbers because $y' \neq 0$, so there are no relative extrema.



61. $C = 1.15x + 6000$

(a) $\overline{C} = \frac{C}{x} = \frac{1.15x + 6000}{x}$

(b) $\overline{C}(600) = \$11.15/\text{unit}$

$\overline{C}(6000) = \$2.15/\text{unit}$

(c) $\lim_{x\to\infty} \overline{C} = \lim_{x\to\infty} \frac{1.15x + 6000}{x} = \$1.15/\text{unit}$

The cost approaches \$1.15 as the number of units produced increases.

62. $C = 1.25x + 10{,}500$

(a) $\overline{C} = 1.25 + \frac{10{,}500}{x}$

(b) $\overline{C}(100) = \$106.25/\text{ton}$

$\overline{C}(1000) = \$11.75/\text{ton}$

(c) $\lim_{x\to\infty}\left(1.25 + \frac{10{,}500}{x}\right) = \$1.25$

As the amount of material recycled increases, the average cost of recycling one ton of material approaches \$1.25.

63. $C = 34.5x + 15{,}000$, $R = 69.9x$

(a) $\overline{P} = \frac{R - C}{x} = \frac{69.9x - (34.5x + 15{,}000)}{x}$

$= 35.4 - \frac{15{,}000}{x}$

(b) $\overline{P}(1000) = \$20.40/\text{unit}$

$\overline{P}(10{,}000) = \$33.90/\text{unit}$

$\overline{P}(100{,}000) = \$35.25/\text{unit}$

(c) $\lim_{x\to\infty}\left(35.4 - \frac{15{,}000}{x}\right) = \$35.40$

As the number of products produced increases, the average profit approaches \$35.40.

64. (a) $\lim_{t\to 0^+} T = 425°\text{F}$. When the apple pie is removed from the oven, it has a temperature of 425°F.

(b) $\lim_{t\to\infty} T = 72°\text{F}$. As time passes after the apple pie is removed from the oven, its temperature cools toward 72°F (possibly room temperature).

65. $C = \frac{528p}{(100 - p)}$, $0 \le p < 100$

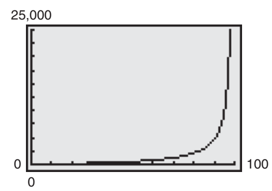
(a) $C(25) = \frac{528(25)}{100 - 25} = \$176 \text{ million}$

$C(50) = \frac{528(50)}{100 - 50} = \$528 \text{ million}$

$C(75) = \frac{528(75)}{100 - 75} = \$1584 \text{ million}$

(b) $\lim_{p\to 100^-} \frac{528p}{100 - p} = \infty$

As the percent of illegal drugs seized approaches 100%, the cost increases without bound.



66. $C = \frac{85{,}000p}{100 - p}$, $0 \le p \le 100$

(a) $C(15) = \$15{,}000$

$C(50) = \$85{,}000$

$C(95) = \$1{,}615{,}000$

(b) $\lim_{p\to 100^-} C = \lim_{p\to 100^-} \frac{85{,}000p}{100 - p} = \infty.$

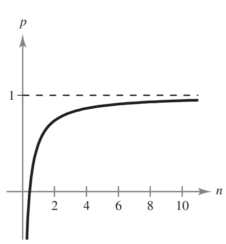
As the percent of air pollutants removed approaches 100%, the cost increases without bound.

67. (a)

n	1	2	3	4	5	6	7	8	9	10
P	0.50	0.74	0.82	0.86	0.89	0.91	0.92	0.93	0.94	0.95

(b) $\lim_{n\to\infty} \frac{0.5 + 0.9(n - 1)}{1 + 0.9(n - 1)} = 1$

(c)



As the number of times the task is performed increases, the percent of correct responses approaches 100%.

68. Answers will vary.

Section 3.7 Curve Sketching: A Summary

Skills Warm Up

1. A vertical asymptote occurs at $x = 0$ because

$$\lim_{x \to 0^-} \frac{1}{x^2} = \infty \text{ and } \lim_{x \to 0^+} \frac{1}{x^2} = \infty.$$

No horizontal asymptotes.

2. A vertical asymptote occurs at $x = 2$ because

$$\lim_{x \to 2^-} \frac{8}{(x-2)^2} = \infty \text{ and } \lim_{x \to 2^+} \frac{8}{(x-2)^2} = \infty.$$

No horizontal asymptotes.

3. A vertical asymptote occurs at $x = -3$ because

$$\lim_{x \to -3^-} \frac{40x}{x+3} = \infty \text{ and } \lim_{x \to -3^+} \frac{40x}{x+3} = -\infty.$$

A horizontal asymptote occurs at $y = 40$ because

$$\lim_{x \to \infty} \frac{40x}{x+3} = 40 \text{ and } \lim_{x \to -\infty} \frac{40x}{x+3} = 40.$$

4. Vertical asymptotes occur at $x = 1$ and $x = 3$ because

$$\lim_{x \to 1^-} \frac{x^2 - 3}{x^2 - 4x + 3} = -\infty, \ \lim_{x \to 1^+} \frac{x^2 - 3}{x^2 - 4x + 3} = \infty,$$

$$\lim_{x \to 3^-} \frac{x^2 - 3}{x^2 - 4x + 3} = -\infty, \text{ and}$$

$$\lim_{x \to 3^+} \frac{x^2 - 3}{x^2 - 4x + 3} = \infty.$$ A horizontal asymptote

occurs at $y = 1$ because $\lim_{x \to \infty} \frac{x^2 - 3}{x^2 - 4x + 3} = 1$ and

$$\lim_{x \to -\infty} \frac{x^2 - 3}{x^2 - 4x + 3} = 1.$$

5. $f(x) = x^2 + 4x + 2$

$f'(x) = 2x + 4$

Critical number: $x = -2$

Interval	$-\infty < x < -2$	$-2 < x < \infty$
Sign of f'	$f' < 0$	$f' > 0$
Conclusion	Decreasing	Increasing

6. $f(x) = -x^2 - 8x + 1$

$f'(x) = -2x - 8$

Critical number: $x = -4$

Interval	$-\infty < x < -4$	$-4 < x < \infty$
Sign of f'	$f' > 0$	$f' < 0$
Conclusion	Increasing	Decreasing

7. $f(x) = x^3 - 3x + 1$

$f'(x) = 3x^2 - 3$

Critical numbers: $x = \pm 1$

Interval	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of f'	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion	Increasing	Decreasing	Increasing

8. $f(x) = \dfrac{-x^3 + x^2 - 1}{x^2} = -x + 1 - \dfrac{1}{x^2}$

$f'(x) = -1 + \dfrac{2}{x^3} = \dfrac{-x^3 + 2}{x^3}$

Critical number: $x = \sqrt[3]{2}$

Discontinuity: $x = 0$

Interval	$-\infty < x < 0$	$0 < x < \sqrt[3]{2}$	$\sqrt[3]{2} < x < \infty$
Sign of f'	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion	Decreasing	Increasing	Decreasing

9. $f(x) = \dfrac{x - 2}{x - 1}$

$f'(x) = \dfrac{(x-1) - (x-2)}{(x-1)^2} = \dfrac{1}{(x-1)^2}$

No critical numbers

Discontinuity: $x = 1$

Interval	$-\infty < x < 1$	$1 < x < \infty$
Sign of f'	$f' > 0$	$f' > 0$
Conclusion	Increasing	Increasing

Skills Warm Up —*continued*—

10. $f(x) = -x^3 - 4x^2 + 3x + 2$

$f'(x) = -3x^2 - 8x + 3$

Critical numbers: $x = -3,\ x = \dfrac{1}{3}$

Interval	$-\infty < x < -3$	$-3 < x < \frac{1}{3}$	$\frac{1}{3} < x < \infty$
Sign of f'	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion	Decreasing	Increasing	Decreasing

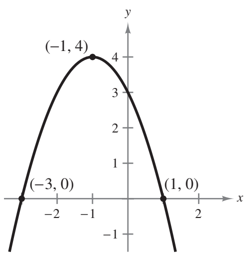
1. $y = -x^2 - 2x + 3 = -(x + 3)(x - 1)$

$y' = -2x - 2 = -2(x + 1)$

$y'' = -2$

Intercepts: $(0, 3), (1, 0), (-3, 0)$

Relative maximum: $(-1, 4)$



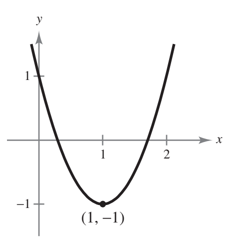
2. $y = 2x^2 - 4x + 1$

$y' = 4x - 4 = 4(x - 1)$

$y'' = 4$

Intercepts: $(0, 1), \left(1 \pm \sqrt{2}/2, 0\right)$

Relative minimum: $(1, -1)$



3. $y = x^3 - 4x^2 + 6$

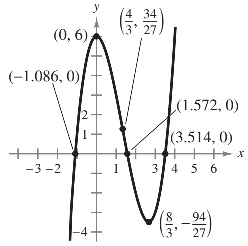
$y' = 3x^2 - 8x = x(3x - 8)$

$y'' = 6x - 8 = 2(3x - 4)$

Relative maximum: $(0, 6)$

Relative minimum: $\left(\dfrac{8}{3}, -\dfrac{94}{27}\right)$

Point of inflection: $\left(\dfrac{4}{3}, \dfrac{34}{27}\right)$



4. $y = -x^3 + x - 2$

$y' = -3x^2 + 1$

$y'' = -6x$

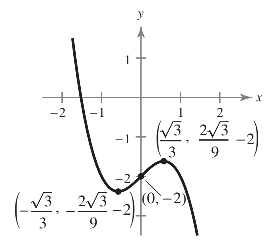
Relative minimum:

$\left(-\dfrac{\sqrt{3}}{3}, -\dfrac{2\sqrt{3}}{9} - 2\right)$

Relative maximum:

$\left(\dfrac{\sqrt{3}}{3}, \dfrac{2\sqrt{3}}{9} - 2\right)$

Point of inflection: $(0, -2)$



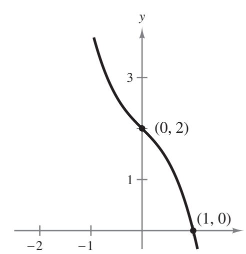
5. $y = 2 - x - x^3$

$y' = -1 - 3x^2$

$y'' = -6x$

No relative extrema

Point of inflection: $(0, 2)$



6. $y = x^3 + 3x^2 + 3x + 2$

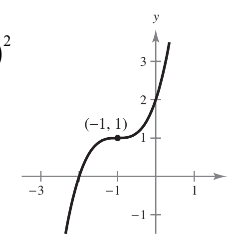
$y' = 3x^2 + 6x + 3 = 3(x + 1)^2$

$y'' = 6x + 6 = 6(x + 1)$

Intercepts: $(0, 2), (-2, 0)$

No relative extrema

Point of inflection: $(-1, 1)$



7. $y = 3x^4 + 4x^3 = x^3(3x + 4)$

$y' = 12x^3 + 12x^2 = 12x^2(x + 1)$

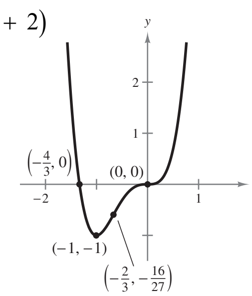
$y'' = 36x^2 + 24x = 12x(3x + 2)$

Intercepts: $(0, 0), \left(-\dfrac{4}{3}, 0\right)$

Relative minimum: $(-1, -1)$

Points of inflection:

$(0, 0), \left(-\dfrac{2}{3}, -\dfrac{16}{27}\right)$



8. $y = x^4 - 2x^2 = x^2(x^2 - 2)$

$y' = 4x^3 - 4x = 4x(x^2 - 1)$

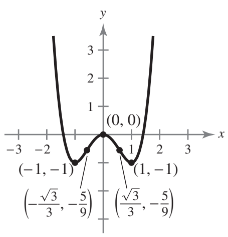
$y'' = 12x^2 - 4 = 4(3x^2 - 1)$

Intercepts: $(0, 0), (\pm\sqrt{2}, 0)$

Relative minima: $(-1, -1), (1, -1)$

Relative maximum: $(0, 0)$

Points of inflection: $\left(-\frac{\sqrt{3}}{3}, -\frac{5}{9}\right), \left(\frac{\sqrt{3}}{3}, -\frac{5}{9}\right)$



9. $y = x^4 - 8x^3 + 18x^2 - 16x + 5 = (x - 5)(x - 1)^3$

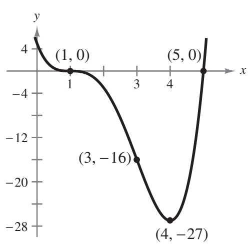
$y' = 4x^3 - 24x^2 + 36x - 16 = 4(x - 4)(x - 1)^2$

$y'' = 12x^2 - 48x + 36 = 12(x - 1)(x - 3)$

Intercepts: $(0, 5), (1, 0), (5, 0)$

Relative minimum: $(4, -27)$

Points of inflection: $(1, 0), (3, -16)$



10. $f(x) = x^4 - 4x^3 + 16x - 16 = (x + 2)(x - 2)^3$

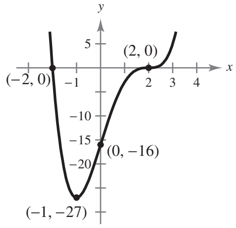
$f'(x) = 4x^3 - 12x^2 + 16 = 4(x + 1)(x - 2)^2$

$f''(x) = 12x^2 - 24x = 12x(x - 2)$

Intercepts: $(0, -16), (-2, 0), (2, 0)$

Relative minimum: $(-1, -27)$

Points of inflection: $(0, -16), (2, 0)$



11. $y = \frac{x^2 + 1}{x} = x + \frac{1}{x}$

$y' = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$

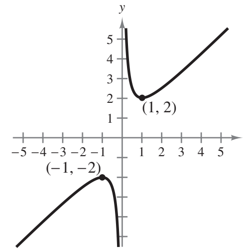
$y'' = \frac{2}{x^3}$

No intercepts

Relative maximum: $(-1, -2)$

Relative minimum: $(1, 2)$

No points of inflection



12. $y = \frac{x + 2}{x} = 1 + \frac{2}{x}$

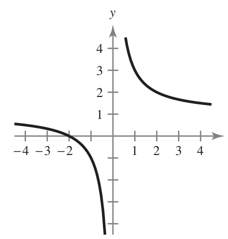
$y' = -\frac{2}{x^2}$

$y'' = \frac{4}{x^3}$

Intercept: $(-2, 0)$

No relative extrema

No points of inflection



13. $y = \frac{x^2 - 6x + 12}{x - 4}$

$y' = \frac{(x - 4)(2x - 6) - (x^2 - 6x + 12)}{(x - 4)^2}$

$= \frac{x^2 - 8x + 12}{(x - 4)^2}$

$= \frac{(x - 2)(x - 6)}{(x - 4)^2}$

$y'' = \frac{(x - 4)^2(2x - 8) - (x^2 - 8x + 12)[2(x - 4)]}{(x - 4)^4}$

$= \frac{8}{(x - 4)^3}$

Intercept: $(0, -3)$

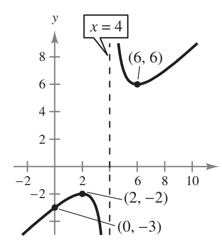
Relative maximum: $(6, 6)$

Relative minimum: $(2, -2)$

No points of inflection

Vertical asymptote: $x = 4$

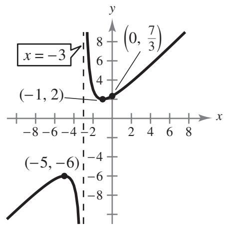
Domain: $(-\infty, 4) \cup (4, \infty)$



14. $y = \dfrac{x^2 + 4x + 7}{x + 3}$

$y' = \dfrac{x^2 + 6x + 5}{(x + 3)^2}$

$y'' = \dfrac{8}{(x + 3)^3}$



No x-intercepts; y-intercept: $\left(0, \frac{7}{3}\right)$

Vertical asymptote: $x = -3$; No horizontal asymptote

Relative minimum: $(-1, 2)$

Relative maximum: $(-5, -6)$

No points of inflection

15. $y = \dfrac{x^2 + 1}{x^2 - 9}$

$y' = \dfrac{(x^2 - 9)(2x) - (x^2 + 1)(2x)}{(x^2 - 9)^2} = -\dfrac{20x}{(x^2 - 9)^2}$

$y'' = \dfrac{(x^2 - 9)^2(-20) - (-20x)\left[2(x^2 - 9)(2x)\right]}{(x^2 - 9)^4}$

$= \dfrac{60(x^2 + 3)}{(x^2 - 9)^3}$

No intercepts

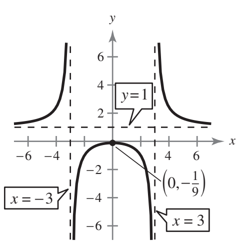
Relative maximum: $\left(0, -\dfrac{1}{9}\right)$

No points of inflection

Horizontal asymptote: $y = 1$

Vertical asymptote: $x = \pm 3$

Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$



16. $y = \dfrac{2x}{x^2 - 1}$

$y' = \dfrac{(x^2 - 1)(2) - 2x(2x)}{(x^2 - 1)^2} = -\dfrac{2(x^2 + 1)}{(x^2 - 1)^2}$

$y'' = \dfrac{(x^2 - 1)^2(-4x) - (-2x^2 - 2)\left[2(x^2 - 1)(2x)\right]}{(x^2 - 1)^4}$

$= \dfrac{4x(x^2 + 3)}{(x^2 - 1)^3}$

Intercept: $(0, 0)$

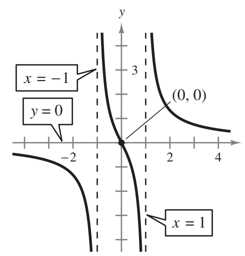
No relative extrema

Point of inflection: $(0, 0)$

Horizontal asymptote: $y = 0$

Vertical asymptotes: $x = \pm 1$

Domain: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$



17. $y = 3x^{2/3} - x^2$

$y' = \dfrac{2}{x^{1/3}} - 2x$

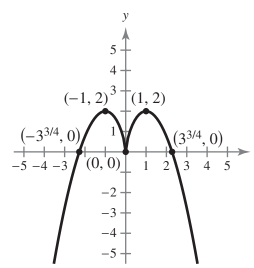
$y'' = -2\left(\dfrac{1}{3x^{4/3}} + 1\right)$

Intercepts: $\left(\pm\sqrt[4]{27}, 0\right), (0, 0)$

Relative maxima: $(\pm 1, 2)$

Relative minimum: $(0, 0)$

No points of inflection



18. $y = x^{5/3} - 5x^{2/3}$

$y' = \dfrac{5(x - 2)}{3x^{1/3}}$

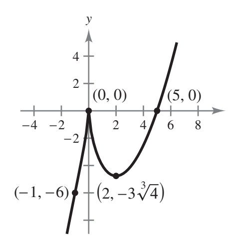
$y'' = \dfrac{10(x + 1)}{9x^{4/3}}$

Intercepts: $(5, 0), (0, 0)$

Relative maximum: $(0, 0)$

Relative minimum: $\left(2, -3\sqrt[3]{4}\right)$

Point of inflection: $(-1, -6)$



19. $y = x\sqrt{9 - x}$

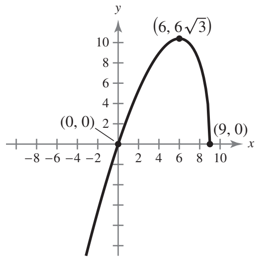
$y' = \dfrac{3(6 - x)}{2\sqrt{9 - x}}$

$y'' = 3\dfrac{(x - 12)}{4(9 - x)^{3/2}}$

Intercepts: $(0, 0), (9, 0)$

Relative maximum: $\left(6, 6\sqrt{3}\right)$

No points of inflection



20. $y = x\sqrt{4 - x^2}$

$y' = \dfrac{2(2 - x^2)}{\sqrt{4 - x^2}}$

$y'' = \dfrac{2x(x^2 - 6)}{(4 - x^2)^{3/2}}$

Intercepts: $(0, 0)$, $(2, 0), (-2, 0)$

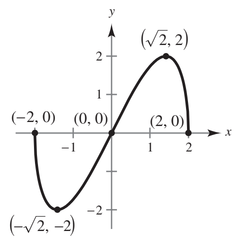
Relative maximum: $\left(\sqrt{2}, 2\right)$

Relative minimum: $\left(-\sqrt{2}, -2\right)$

Point of inflection: $(0, 0)$

No asymptotes

Domain: $[-2, 2]$



21. $y = \begin{cases} x^2 + 1, & x \le 0 \\ 1 - 2x, & x > 0 \end{cases}$

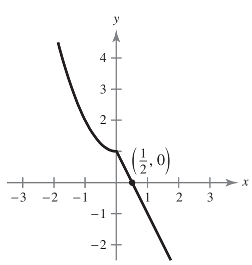
$y' = \begin{cases} 2x, & x < 0 \\ -2 & x > 0 \end{cases}$

$y'' = \begin{cases} 2, & x < 0 \\ 0 & x > 0 \end{cases}$

Intercept: $\left(\frac{1}{2}, 0\right)$

No relative extrema

No points of inflection



22. $y = \begin{cases} x^2 + 4, & x < 0 \\ 4 - x, & x \ge 0 \end{cases}$

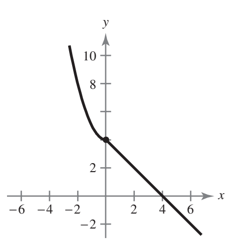
$y' = \begin{cases} 2x, & x < 0 \\ -1, & x > 0 \end{cases}$

$y'' = \begin{cases} 2, & x < 0 \\ 0, & x > 0 \end{cases}$

Intercept: $(4, 0)$

No relative extrema

No points of inflection



23. $y = 3x^3 - 9x + 1$

$y' = 9x^2 - 9 = 9(x^2 - 1)$

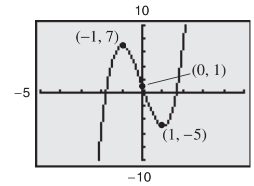
$y'' = 18x$

Intercept: $(0, 1)$

Relative maximum: $(-1, 7)$

Relative minimum: $(1, -5)$

Point of inflection: $(0, 1)$



24. $y = -4x^3 + 6x^2$

$y' = -12x^2 + 12x = -12x(x - 1)$

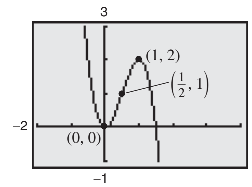
$y'' = -24x + 12 = -12(2x - 1)$

Intercept: $(0, 0)$

Relative minimum: $(0, 0)$

Relative maximum: $(1, 2)$

Point of inflection: $\left(\frac{1}{2}, 1\right)$



25. $y = x^5 - 5x$

$y' = 5x^4 - 5 = 5(x^4 - 1) = 5(x^2 - 1)(x^2 + 1)$

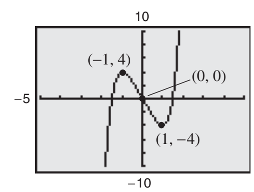
$y'' = 20x^3$

Intercept: $(0, 0)$

Relative maximum: $(-1, 4)$

Relative minimum: $(1, -4)$

Point of inflection: $(0, 0)$



26. $y = (x - 1)^5$

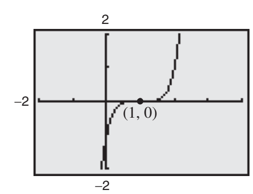
$y' = 5(x - 1)^4$

$y'' = 20(x - 1)^3$

Intercepts: $(0, -1), (1, 0)$

No relative extrema

Point of inflection: $(1, 0)$



27. $y = \dfrac{5 - 3x}{x - 2}$

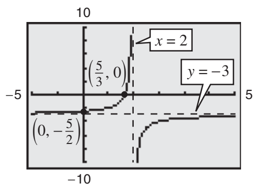
$y' = \dfrac{(x - 2)(-3) - (5 - 3x)(1)}{(x - 2)^2} = \dfrac{1}{(x - 2)^2}$

$y'' = -2(x - 2)^{-3}(1) = \dfrac{-2}{(x - 2)^3}$

Intercepts: $\left(0, -\dfrac{5}{2}\right), \left(\dfrac{5}{3}, 0\right)$

No relative extrema

No points of inflection



28. $y = \dfrac{x}{x^2 + 1}$

$y' = \dfrac{(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \dfrac{1 - x^2}{(x^2 + 1)^2}$

$y'' = \dfrac{(x^2 + 1)^2(-2x) - (1 - x^2)\left[2(x^2 + 1)(2x)\right]}{(x^2 + 1)^4}$

$= \dfrac{2x(x^2 - 3)}{(x^2 + 1)^3}$

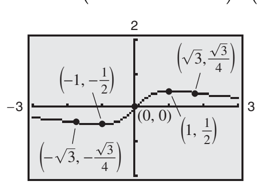
Intercept: $(0, 0)$

Relative maximum: $(1, 1/2)$

Relative minimum: $(-1, -1/2)$

Points of inflection:

$(0, 0), \left(-\sqrt{3}, -\sqrt{3}/4\right), \left(\sqrt{3}, \sqrt{3}/4\right)$



29. $y = 1 - x^{2/3}$

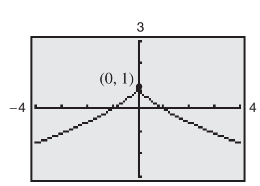
$y' = -\dfrac{2}{3}x^{-1/3} = -\dfrac{2}{3x^{1/3}}$

$y'' = \dfrac{2}{9}x^{-4/3} = \dfrac{2}{9x^{4/3}}$

Intercepts: $(0, 1), (\pm 1, 0)$

Relative maximum: $(0, 1)$

No points of inflection



30. $y = (1 - x)^{2/3}$

$y' = -\dfrac{2}{3}(1 - x)^{-1/3}$

$= -\dfrac{2}{3(1 - x)^{1/3}}$

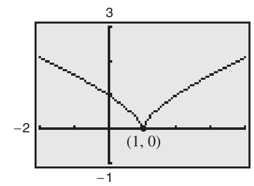
$y'' = -\dfrac{2}{9}(1 - x)^{-4/3}$

$= -\dfrac{2}{9(1 - x)^{4/3}}$

Intercepts: $(0, 1), (1, 0)$

Relative minimum: $(1, 0)$

No points of inflection



31. $y = x^{4/3}$

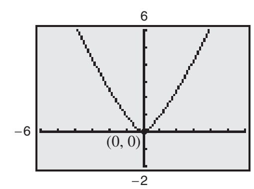
$y' = \dfrac{4}{3}x^{1/3}$

$y'' = \dfrac{4}{9}x^{-2/3} = \dfrac{4}{9x^{2/3}}$

Intercept: $(0, 0)$

Relative minimum: $(0, 0)$

No points of inflection



32. $y = x^{-1/3} = \dfrac{1}{x^{1/3}}$

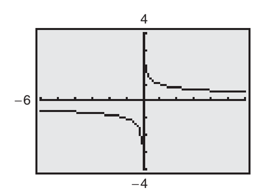
$y' = -\dfrac{1}{3}x^{-4/3} = -\dfrac{1}{3x^{4/3}}$

$y'' = \dfrac{4}{9}x^{-7/3} = \dfrac{4}{9x^{7/3}}$

No intercepts

No relative extrema

No points of inflection



33. $y = \dfrac{x}{\sqrt{x^2 - 4}}, \ |x| > 2$

$y' = \dfrac{(x^2 - 4)^{1/2} - x\left[\frac{1}{2}(x^2 - 4)^{-1/2}(2x)\right]}{x^2 - 4}$

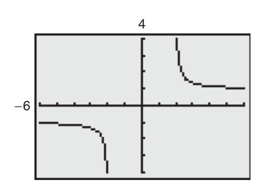
$= -\dfrac{4}{(x^2 - 4)^{3/2}}$

$y'' = 6(x^2 - 4)^{-5/2}(2x) = \dfrac{12x}{(x^2 - 4)^{5/2}}$

No intercepts

No relative extrema

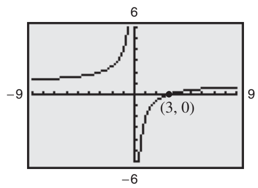
No points of inflection



34. $y = \dfrac{x-3}{x}$

$y' = \dfrac{3}{x^2}$

$y'' = -\dfrac{6}{x^3}$



Intercept: $(3, 0)$

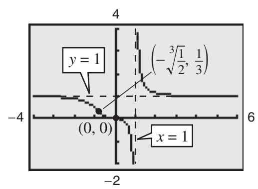
No relative extrema

No points of inflection

35. $y = \dfrac{x^3}{x^3 - 1}$

$y' = \dfrac{-3x^2}{\left(x^3 - 1\right)^2}$

$y'' = \dfrac{6x\left(2x^3 + 1\right)}{\left(x^3 - 1\right)^3}$



Intercept: $(0, 0)$

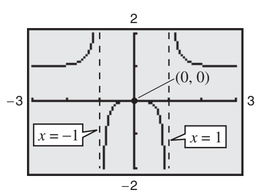
No relative extrema

Points of inflection: $\left(-\sqrt[3]{\dfrac{1}{2}}, \dfrac{1}{3}\right), (0, 0)$

36. $y = \dfrac{x^4}{x^4 - 1}$

$y' = \dfrac{-4x^3}{\left(x^4 - 1\right)^2}$

$y'' = \dfrac{4x^2\left(5x^4 + 3\right)}{\left(x^4 - 1\right)^3}$



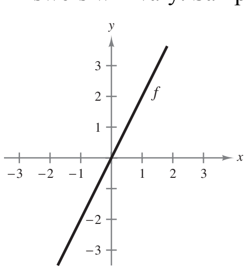
Intercept: $(0, 0)$

Relative maximum: $(0, 0)$

No points of inflection

37. Because $f'(x) = 2$, the graph of f is a line with a slope of 2.

Answers will vary. Sample answer:

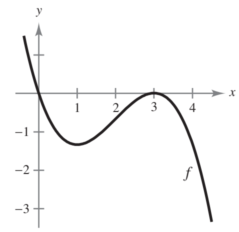


38. $f'(x)$ is negative and $f(x)$ is decreasing on $(-\infty, 1)$ and $(3, \infty)$.

$f'(x)$ is positive and $f(x)$ is increasing on $(1, 3)$.

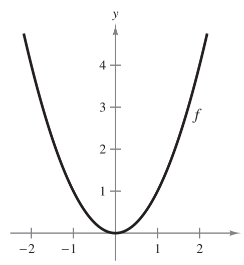
$f'(x) = 0$ when $x = 1$ and $x = 3$, so $f(x)$ has a relative minimum when $x = 1$ and $f(x)$ has a relative maximum when $x = 3$.

Answers will vary. Sample answer:



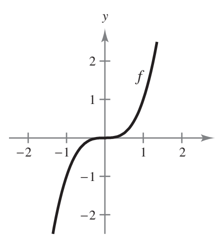
39. $f''(x) = 2 > 0$ so $f(x)$ is a concave upward parabola.

Answers will vary. Sample answer:



40. When $x < 0$, $f''(x) < 0$ and $f(x)$ is concave downward. When $x > 0$, $f''(x) > 0$ and $f(x)$ is concave upward.

Answers will vary. Sample answer:



41.

Interval	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < \infty$
Sign of f'	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion	Increasing	Decreasing	Increasing

Intercepts: $(-2, 0), (0, 0)$

Relative maximum: $(-1, f(-1))$

Relative minimum: $(0, f(0)) = (0, 0)$

42.

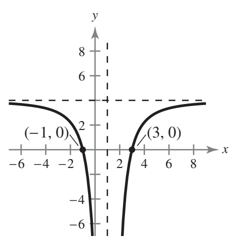
Interval	$-\infty < x < 1$	$1 < x < \infty$
Sign of f'	$f' < 0$	$f' > 0$
Conclusion	Decreasing	Increasing

Intercepts: $(-1, 0), (3, 0)$

Horizontal asymptote: $y = 4$

Vertical asymptote: $x = 1$

Answers will vary. Sample answer:



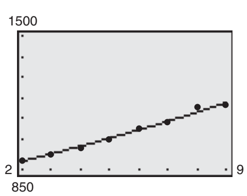
43. Answers will vary. Sample answer:

$$f(x) = \frac{1}{x - 5}$$

44. Answers will vary. Sample answer:

$$f(x) = \frac{x^2}{x + 3}$$

45. (a)

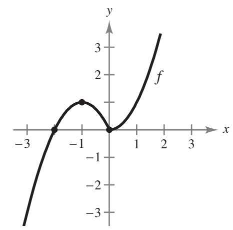


The model fits the data well.

(b) 2014: $B(14) \approx \$1468.54$

(c) No, because the benefits increase without bound as time approaches the year 2035 $(x = 35)$, and the benefits are negative for the years past 2035.

Answers will vary. Sample answer:

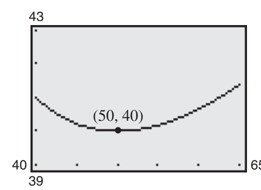


46. (a) distance = (rate)(time) so the total time traveled is $t = \frac{100}{s}$. The number of gallons of gasoline used is $\frac{100}{700/s} = \frac{s}{7}$.

Let C be the total cost.

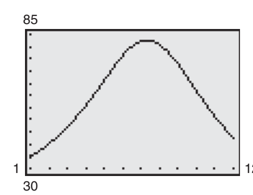
$$C = 10\left(\frac{100}{s}\right) + 2.80\left(\frac{s}{7}\right)$$
$$= \frac{1000}{s} + \frac{2}{5}s$$

(b)



When $s = 50$, cost is minimum. So, the most economical speed is 50 miles per hour.

47.



Absolute maximum: $(7, 82.28)$

Absolute minimum: $(1, 34.84)$

The maximum temperature of 82.28°F occurs in July.

The minimum temperature of 34.84°F occurs in January.

48. (a) $P' = 0$ at $t = -4$ and $t = 4$

$P' > 0$ for $-10 < t < -4$ and $4 < t < 10$

$P' < 0$ for $-4 < t < 4$

$P' > 0$ for $-10 < t < -4$ means that the profit was increasing from 1990 to 1994. Then in 1994 $(t = -4)$, when $P' = 0$, profit reached a (relative) maximum. $P' < 0$ for $-4 < t < 4$ means that profit was decreasing from 1994 to 2004. Then in 2004 $(t = 4)$, when $P' = 0$, profit dropped to a (relative) minimum.

$P' > 0$ for $4 < t < 10$ means that profit was increasing from 2004 to 2010.

(b) $P'' = 0$ at $t = 0$

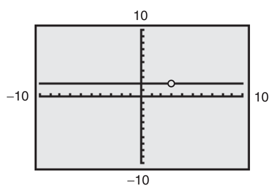
$P'' > 0$ for $0 < t < 10$

$P'' < 0$ for $-10 < t < 0$

$P'' < 0$ for $-10 < t < 0$ means that the rate at which profit was changing was decreasing from 1990 to 2000. Then in 2000 $(t = 0)$ when $P'' = 0$, the rate at which profit was changing reached a (relative) minimum.

$P'' > 0$ for $0 < t < 10$, means that the rate at which profit was changing was increasing from 2000 to 2010.

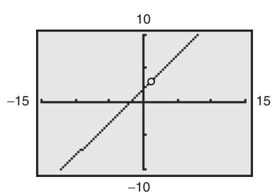
49.



$$h(x) = \frac{6 - 2x}{3 - x} = \frac{2(3 - x)}{3 - x} = 2, \ x \neq 3$$

The rational function simplifies to a constant function that is undefined at $x = 3$.

50.

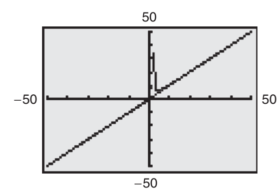


$$g(x) = \frac{x^2 + x - 2}{x - 1} = \frac{(x - 1)(x + 2)}{x - 1} = x + 2, \ x \neq 1$$

The rational function simplifies to a linear function that is undefined at $x = 1$.

51. (a) $$f(x) = \frac{x^2 - 2x + 4}{x - 2} = \frac{x^2 - 2x}{x - 2} + \frac{4}{x - 2} = \frac{x(x - 2)}{x - 2} + \frac{4}{x - 2} = x + \frac{4}{x - 2}$$

(b)



The graphs become almost identical as you zoom out.

(c) A slant asymptote is neither horizontal nor vertical. It is diagonal, following $y = x$.

Section 3.8 Differentials and Marginal Analysis

Skills Warm Up

1. $C = 44 + 0.09x^2$

$$\frac{dC}{dx} = 0.18x$$

2. $C = 250 + 0.15x$

$$\frac{dC}{dx} = 0.15$$

3. $R = x(1.25 + 0.02\sqrt{x}) = 1.25x + 0.02x^{3/2}$

$$\frac{dR}{dx} = 1.25 + 0.03\sqrt{x}$$

4. $R = x(15.5 - 1.55x) = 15.5x - 1.55x^2$

$$\frac{dR}{dx} = 15.5 - 3.1x$$

Skills Warm Up —*continued*—

5. $P = -0.03x^{1/3} + 1.4x - 2250$

$\frac{dP}{dx} = \frac{-0.01}{x^{2/3}} + 1.4$

6. $P = -0.02x^2 + 25x - 1000$

$\frac{dP}{dx} = -0.04x + 25$

7. $A = \frac{1}{4}\sqrt{3}x^2$

$\frac{dA}{dx} = \frac{1}{2}\sqrt{3}x$

8. $A = 6x^2$

$\frac{dA}{dx} = 12x$

9. $C = 2\pi r$

$\frac{dC}{dr} = 2\pi$

10. $P = 4w$

$\frac{dP}{dw} = 4$

11. $S = 4\pi r^2$

$\frac{dS}{dr} = 8\pi r$

12. $P = 2x + \sqrt{2}x$

$\frac{dP}{dx} = 2 + \sqrt{2}$

13. $A = \pi r^2$

14. $A = x^2$

15. $V = x^3$

16. $V = \frac{4}{3}\pi r^3$

1. $y = 0.5x^3,\ x = 2,\ \Delta x = dx = 0.1$

$$\begin{aligned} dy &= 1.5x^2\ dx \\ &= 1.5(2)^2(0.1) \\ &= 0.6 \\ \Delta y &= 0.5(2 + 0.1)^3 - 0.5(2)^3 \\ &= 0.6305 \\ dy &\approx \Delta y \end{aligned}$$

2. $y = 1 - 2x^2,\ x = 0,\ \Delta x = dx = -0.1$

$$\begin{aligned} dy &= -4x\ dx \\ &= -4(0)(-0.1) \\ &= 0 \\ \Delta y &= 1 - 2[0 + (-0.1)]^2 - [1 - 2(0)^2] \\ &= -0.02 \\ dy &\approx \Delta y \end{aligned}$$

3. $y = x^4 + 1,\ x = -1,\ \Delta x = dx = 0.01$

$$\begin{aligned} dy &= 4x^3\ dx \\ &= 4(-1)^3(0.01) \\ &= -0.04 \\ \Delta y &= (-1 + 0.01)^4 + 1 - [(-1)^4 + 1] \\ &\approx -0.0394 \\ dy &\approx \Delta y \end{aligned}$$

4. $y = 2x + 1, x = 1, \Delta x = dx = 0.01$

$$\begin{aligned} dy &= 2\ dx \\ &= 2(0.01) \\ &= 0.02 \\ \Delta y &= [2(1 + 0.01) + 1] - [2(1) + 1] \\ &= 0.02 \\ dy &= \Delta y \end{aligned}$$

5. $y = 3x^{1/2}, x = 4, \Delta x = dx = 0.1$

$$\begin{aligned} dy &= \frac{3}{2x^{1/2}}\ dx \\ &= \frac{3}{2(4)^{1/2}}(0.1) \\ &= 0.075 \\ \Delta y &= 3(4 + 0.1)^{1/2} - 3(4)^{1/2} \\ &\approx 0.0745 \\ dy &\approx \Delta y \end{aligned}$$

6. $y = 6x^{4/3},\ x = -1,\ \Delta x = dx = 0.01$

$$\begin{aligned} dy &= 8x^{1/3}\,dx \\ &= 8(-1)^{1/3}(0.01) \\ &= -0.08 \\ \Delta y &= 6(-1 + 0.01)^{4/3} - 6(-1)^{4/3} \\ &\approx -0.0799 \\ dy &\approx \Delta y \end{aligned}$$

7. $dy = 2x\,dx,\ x = 2,\ \Delta y = (x + \Delta x)^2 - x^2$

$dx = \Delta x$	dy	Δy	$\Delta y - dy$	$dy/\Delta y$
1.000	4	5	1	0.8
0.500	2	2.25	0.25	0.889
0.100	0.4	0.41	0.010	0.976
0.010	0.04	0.040	0.000	0.998
0.001	0.004	0.004	0.000	1.000

8. $dy = 5x^4\,dx,\ x = 2,\ \Delta y = (x + \Delta x)^5 - x^5$

$dx = \Delta x$	dy	Δy	$\Delta y - dy$	$dy/\Delta y$
1.000	80	211	131	0.379
0.500	40	65.656	25.656	0.609
0.100	8	8.841	0.841	0.905
0.010	0.8	0.808	0.008	0.990
0.001	0.08	0.080	0.000	0.999

9. $dy = -\dfrac{2}{x^3}\,dx,\ x = 2,\ \Delta y = \dfrac{1}{(x + \Delta x)^2} - \dfrac{1}{x^2}$

$dx = \Delta x$	dy	Δy	$\Delta y - dy$	$dy/\Delta y$
1.000	–0.25	–0.139	0.111	1.8
0.500	–0.125	–0.09	0.035	1.389
0.100	–0.025	–0.023	0.002	1.076
0.010	–0.003	–0.002	0.000	1.008
0.001	0.000	0.000	0.000	1.001

10. $dy = -\dfrac{1}{x^2}\,dx,\ x = 2,\ \Delta y = \dfrac{1}{x + \Delta x} - \dfrac{1}{x}$

$dx = \Delta x$	dy	Δy	$\Delta y - dy$	$dy/\Delta y$
1.000	–0.25	–0.167	0.083	1.5
0.500	–0.125	–0.1	0.025	1.25
0.100	–0.025	–0.024	0.001	1.05
0.010	–0.003	–0.002	0.000	1.005
0.001	0.000	0.000	0.000	1.001

11. $dy = \dfrac{1}{4}x^{-3/4}\,dx = \dfrac{1}{4x^{3/4}}\,dx,\ x = 2,$

$\Delta y = (x + \Delta x)^{1/4} - x^{1/4}$

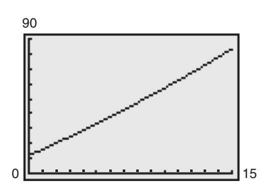
$dx = \Delta x$	dy	Δy	$\Delta y - dy$	$dy/\Delta y$
1.000	0.149	0.127	–0.022	1.172
0.500	0.074	0.068	–0.006	1.089
0.100	0.015	0.015	0.000	1.019
0.010	0.001	0.001	0.000	1.002
0.001	0.000	0.000	0.000	1.000

12. $dy = \dfrac{1}{2\sqrt{x}}\,dx,\ x = 2,\ \Delta y = \sqrt{x + \Delta x} - \sqrt{x}$

$dx = \Delta x$	dy	Δy	$\Delta y - dy$	$dy/\Delta y$
1.000	0.354	0.318	–0.036	1.112
0.500	0.177	0.167	–0.010	1.059
0.100	0.035	0.035	0.000	1.012
0.010	0.004	0.004	0.000	1.001
0.001	0.000	0.000	0.000	1.000

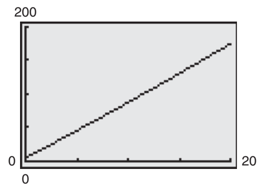
13. $x = 12,\ dx = \Delta x = 1$

$$\begin{aligned} \Delta C \approx dC &= (0.10x + 4)\,dx \\ &= [0.10(12) + 4](1) \\ &= \$5.20 \end{aligned}$$



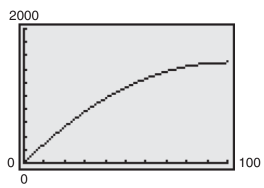
14. $x = 10,\ dx = \Delta x = 1$

$$\Delta C \approx dC = (0.05x + 8)\ dx$$
$$= [0.05(10) + 8](1)$$
$$= 8.5 = \$8.50$$



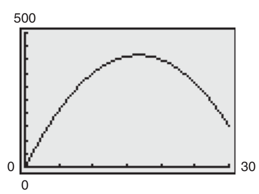
15. $x = 75,\ dx = \Delta x = 1$

$$\Delta R \approx dR = (30 - 0.30x)\ dx$$
$$= [30 - 0.30(75)](1) = \$7.50$$



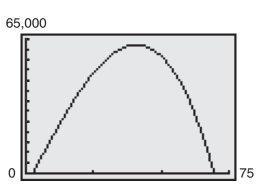
16. $x = 15,\ dx = \Delta x = 1$

$$\Delta R \approx dR = (50 - 3x)\ dx$$
$$= [50 - 3(15)](1)$$
$$= \$5.00$$



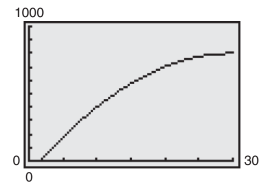
17. $x = 50,\ dx = \Delta x = 1$

$$\Delta P \approx dP = (-1.5x^2 + 2500)\ dx$$
$$= [-1.5(50)^2 + 2500](1)$$
$$= -\$1250$$



18. $x = 25,\ dx = \Delta x = 1$

$$\Delta P \approx dP = (-2x + 60)\ dx$$
$$= [-2(25) + 60](1)$$
$$= \$10$$



19. $y = 6x^4$

$$\frac{dy}{dx} = 24x^3$$
$$dy = 24x^3\ dx$$

20. $y = \dfrac{8 - 4x}{3} = \dfrac{8}{3} - \dfrac{4}{3}x$

$$\frac{dy}{dx} = -\frac{4}{3}$$
$$dy = \left(-\frac{4}{3}\right) dx$$

21. $y = 3x^2 - 4$

$$\frac{dy}{dx} = 6x$$
$$dy = 6x\ dx$$

22. $y = 3x^{2/3}$

$$\frac{dy}{dx} = 2x^{-1/3}$$
$$dy = \frac{2}{x^{1/3}}\ dx$$

23. $y = (4x - 1)^3$

$$\frac{dy}{dx} = 3(4x - 1)^2(4)$$
$$dy = 12(4x - 1)^2\ dx$$

24. $y = (x^2 + 3)(2x + 4)^2$

$$\frac{dy}{dx} = \left[(x^2 + 3)(2)(2x + 4)(2) + (2x + 4)^2(2x)\right]$$
$$= 2(2x + 4)\left[2(x^2 + 3) + x(2x + 4)\right]$$
$$dy = 8(x + 2)(2x^2 + 2x + 3)\ dx$$

25. $y = \dfrac{x + 1}{2x - 1}$

$$\frac{dy}{dx} = \frac{(2x - 1) - (x + 1)(2)}{(2x - 1)^2}$$

$$dy = -\frac{3}{(2x - 1)^2}\,dx$$

26. $y = \dfrac{x}{x^2 + 1}$

$$\frac{dy}{dx} = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2}$$

$$dy = -\frac{(x^2 - 1)}{(x^2 + 1)^2}\,dx$$

27. $y = \sqrt{9 - x^2}$

$$\frac{dy}{dx} = \frac{1}{2}(9 - x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{9 - x^2}}$$

$$dy = -\frac{x}{\sqrt{9 - x^2}}\,dx$$

28. $y = \sqrt[3]{6x^2}$

$$\frac{dy}{dx} = \frac{1}{3}(6x^2)^{-2/3}(12x)$$

$$dy = \frac{1}{3}(6x^2)^{-2/3}(12x)\,dx = \frac{4x}{\sqrt[3]{36x^4}}\,dx = \frac{4}{\sqrt[3]{36x}}\,dx$$

29. $f(x) = 2x^3 - x^2 + 1,\ (-2, -19)$

$f'(x) = 6x^2 - 2x$

$f'(-2) = 24 + 4 = 28$

$y + 19 = 28(x + 2)$

$y = 28x + 37$ Tangent line

$f(-2 + 0.01) \approx -18.72$

$y(-2 + 0.01) = -18.72$

$f(-2 - 0.01) \approx -19.28$

$y(-2 - 0.01) = -19.28$

30. $f(x) = 3x^2 - 1,\ (2, 11)$

$f'(x) = 6x,\ f'(2) = 12$

$y - 11 = 12(x - 2)$

$y = 12x - 13$ Tangent line

$f(2 + 0.01) \approx 11.1203$

$y(2 + 0.01) = 11.12$

$f(2 - 0.01) \approx 10.8803$

$y(2 - 0.01) = 10.88$

31. $f(x) = \dfrac{x}{x^2 + 1},\ (0, 0)$

$$f'(x) = \frac{(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

$f'(0) = 1$

$y - 0 = 1(x - 0)$

$y = x$ Tangent line

$f(0 + 0.01) \approx 0.009999$

$y(0 + 0.01) = 0.01$

$f(0 - 0.01) \approx -0.009999$

$y(0 - 0.01) = -0.01$

32. $f(x) = \sqrt{25 - x^2},\ (3, 4)$

$$f'(x) = \frac{1}{2}(25 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{25 - x^2}}$$

$$f'(3) = \frac{-3}{\sqrt{25 - 9}} = -\frac{3}{4}$$

$$y - 4 = -\frac{3}{4}(x - 3)$$

$$y = -\frac{3}{4}x + \frac{25}{4} \quad \text{Tangent line}$$

$f(3 + 0.01) \approx 3.99248$

$y(3 + 0.01) = 3.9925$

$f(3 - 0.01) \approx 4.00748$

$y(3 - 0.01) = 4.0075$

33. $P = (500x - x^2) - \left(\tfrac{1}{2}x^2 - 77x + 3000\right)$

$= -\tfrac{3}{2}x^2 + 577x - 3000$

(a) $dP = (-3x + 577)\,dx$

Find dP for $x = 115$ and $dx = 5$.

$dP = [-3(115) + 577](5) = \$1160$

(b) Find ΔP for $x = 115$ and $\Delta x = 5$.

$\Delta P = P(x + \Delta x) - P(x)$

$= P(115 + 5) - P(115)$

$= \$1122.50$

34. $R = 900x - 0.1x^2$

(a) $dR = (900 - 0.2x)\,dx$

Find dR for $x = 3000$ and $dx = 100$.

$$dR = [900 - 0.2(3000)](100) = \$30,000$$

(b) Find ΔR for $x = 3000$ and $\Delta x = 100$.

$$\begin{aligned}\Delta R &= R(x + \Delta x) - R(x)\\ &= R(3000 + 100) - R(3000)\\ &= \$29,000\end{aligned}$$

35.
$$\begin{aligned}R &= xp\\ &= x(75 - 0.25x)\\ &= 75x - 0.25x^2\end{aligned}$$

(a) $dR = (75 - 0.5x)\,dx$

Find dR for $x = 7$ and $dx = 1$.

$$dR = [75 - 0.5(7)](1) = \$71.50$$

(b) Find dR for $x = 70$ and $dx = 1$.

$$dR = [75 - 0.5(70)](1) = \$40$$

36. The change in profit is greater when the production level changes from 900 to 901 units. The slope of the tangent line (differential) is greater at 900 units.

37. $N = \dfrac{10(5 + 3t)}{1 + 0.04t}$

$$\begin{aligned}dN &= \frac{(1 + 0.04t)(30) - 10(5 + 3t)(0.04)}{(1 + 0.04t)^2}\,dt\\ &= \frac{28}{(1 + 0.04t)^2}\,dt\end{aligned}$$

When $t = 5$ and $dt = 6 - 5 = 1$, you have the following.

$$dN = \frac{28}{[1 + 0.04(5)]^2}(1) = \frac{28}{1.44} \approx 19.44$$

The change in herd size will be approximately 19 deer.

38. $C = \dfrac{3t}{27 + t^3}$

$$dC = \frac{(27 + t^3)(3) - (3t)(3t^2)}{(27 + t^3)^2} = \frac{3(27 - 2t^3)}{(27 + t^3)^2}\,dt$$

When $t = 1$ and $dt = \dfrac{1}{2}$, you have

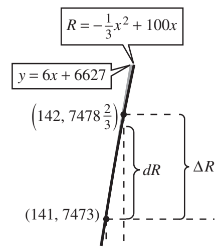
$$dC = \frac{3(25)}{(28)^2}\left(\frac{1}{2}\right) \approx 0.0478.$$

39. $(150, 50)$, $(120, 60)$

$$\begin{aligned}m &= \frac{60 - 50}{120 - 150} = -\frac{1}{3}\\ p - 50 &= -\frac{1}{3}(x - 150)\\ p &= -\frac{1}{3}x + 100\\ R &= xp = -\frac{1}{3}x^2 + 100x\end{aligned}$$

When $x = 141$ and $dx = \Delta x = 1$,

$$\begin{aligned}\Delta R \approx dR &= \left(-\frac{2}{3}x + 100\right)dx\\ &= \left[-\frac{2}{3}(141) + 100\right](1) = \$6.00.\end{aligned}$$

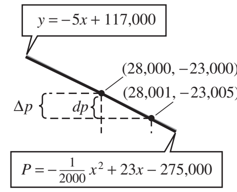


40. $(30,000, 25)$, $(40,000, 20)$

$$\begin{aligned}m &= \frac{20 - 25}{40,000 - 30,000} = \frac{-5}{10,000} = -\frac{1}{2000}\\ p - 25 &= -\frac{1}{2000}(x - 30,000)\\ p &= -\frac{1}{2000}x + 40\\ C &= 275,000 + 17x\\ P &= R - C = xp - C\\ &= \left(-\frac{1}{2000}x^2 + 40x\right) - (275,000 + 17x)\\ &= -\frac{1}{2000}x^2 + 23x - 275,000\end{aligned}$$

When $x = 28,000$ and $dx = \Delta x = 1$,

$$\begin{aligned}\Delta P \approx dP &= -\frac{1}{1000}x + 23\\ &= -\frac{1}{1000}(28,000) + 23\\ &= -\$5.00.\end{aligned}$$



41. $A = x^2$

$dA = 2x\,dx$

When $x = 6$ in. and $dx = \pm\frac{1}{16}$ in.,

$dA = 2(6)\left(\pm\frac{1}{16}\right) = \pm\frac{3}{4}$ in.^{2}.

When $x = 6$ in. and $A = 36$ in.^{2}, the relative error is

$\frac{dA}{A} = \frac{\pm\frac{3}{4}}{36} \approx 0.0208 \Rightarrow 2.08\%$.

42. $V = \frac{4}{3}\pi r^3$

$dV = 4\pi r^2\,dr$

When $r = 6$ in. and $dr = \pm 0.02$ in.,

$dV = 4\pi(6)^2(\pm 0.02) = \pm 2.88\pi$ in.^{3}

$\approx \pm 9.05$ in.^{3}

When $r = 6$ in. and $V = 288\pi$ in.^{3}, the relative error is

$\frac{dV}{V} = \frac{2.88\pi}{288\pi} = 0.01 \Rightarrow 1.0\%$.

43. True; $\frac{dy}{dx} = 1$ and $dy = dx$.

44. True; $\Delta y = y(x + \Delta x) - y(x)$

$= a(x + \Delta x) + b - (ax + b)$

$= a\Delta x$ and $\frac{\Delta y}{\Delta x} = \frac{a\Delta x}{\Delta x} = a = \frac{dy}{dx}$.

Review Exercises for Chapter 3

1. $f(x) = -x^2 + 2x + 4$

$f'(x) = -2x + 2$

Critical number: $x = 1$

2. $y = 3x^2 + 18x$

$y' = 6x + 18 = 6(x + 3)$

Critical number: $x = -3$

3. $y = 4x^3 - 108x$

$y' = 12x^2 - 108 = 12(x^2 - 9)$

Critical numbers: $x = \pm 3$

4. $f(x) = x^4 - 8x^2 + 13$

$f'(x) = 4x^3 - 16x = 4x(x^2 - 4)$

Critical numbers: $x = 0, \pm 2$

5. $g(x) = (x - 1)^2(x - 3)$

$g'(x) = (x - 3)(2)(x - 1) + (x - 1)^2$

$= (x - 1)(3x - 7)$

Critical numbers: $x = 1$, $x = \frac{7}{3}$

6. $f(x) = x^{3/2} - 3x^{1/2}$

$f'(x) = \frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2} = \frac{3}{2}x^{-1/2}(x - 1) = \frac{3(x - 1)}{2\sqrt{x}}$

Critical numbers: $x = 0$, $x = 1$

7. $f(x) = x^2 + x - 2$

$f'(x) = 2x + 1$

Set $f'(x) = 0$.

$2x + 1 = 0$

$x = -\frac{1}{2} \Rightarrow$ Critical number

Interval	$-\infty < x < -\frac{1}{2}$	$-\frac{1}{2} < x < \infty$
Sign of f'	$f' < 0$	$f' > 0$
Conclusion	Decreasing	Increasing

8. $g(x) = (x + 2)^3$

$g'(x) = 3(x + 2)^2(1) = 3(x + 2)^2$

Set $g'(x) = 0$.

$3(x + 2)^2 = 0$

$x = -2 \Rightarrow$ Critical number

Interval	$-\infty < x < -2$	$-2 < x < \infty$
Sign of g'	$g' > 0$	$g' > 0$
Conclusion	Increasing	Increasing

9. $f(x) = x^3 + 6x^2 - 2$

$f'(x) = -3x^2 + 12x = -3x(x - 4)$

Set $f'(x) = 0$.

$-3x(x - 4) = 0$

$x = 0, 4 \Rightarrow$ Critical numbers

Interval	$-\infty < x < 0$	$0 < x < 4$	$4 < x < \infty$
Sign of f'	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion	Decreasing	Increasing	Decreasing

10. $y = x^3 - 12x^2$

$y' = 3x^2 - 24x = 3x(x - 8)$

Set $y' = 0$.

$3x(x - 8) = 0$

$x = 0, 8 \Rightarrow$ Critical numbers

Interval	$-\infty < x < 0$	$0 < x < 8$	$8 < x < \infty$
Sign of y'	$y' > 0$	$y' < 0$	$y' > 0$
Conclusion	Increasing	Decreasing	Increasing

11. $y = (x - 1)^{2/3}$

$y' = \frac{2}{3}(x - 1)^{-1/3}(1) = \frac{2}{3(x - 1)^{1/3}}$

y' is undefined at $x = 1$.

Interval	$-\infty < x < 1$	$1 < x < \infty$
Sign of y'	$y' < 0$	$y' > 0$
Conclusion	Decreasing	Increasing

12. $y = 2x^{1/3} - 3$

$y' = \frac{2}{3}x^{-2/3} = \frac{2}{3x^{2/3}}$

y' is undefined at $x = 0$.

Interval	$-\infty < x < 0$	$0 < x < \infty$
Sign of y'	$y' > 0$	$y' > 0$
Conclusion	Increasing	Increasing

13. $R = 6.268t^2 + 136.07t - 191.3, 4 \le t \le 9$

$R' = 12.536t + 136.07 = \frac{dR}{dt}$

Set $R' = 0$.

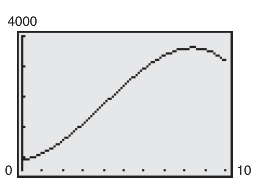
$12.536t + 136.07 = 0$

$t \approx -10.85$

The only critical number is $t \approx -10.85$. Any $t > -10.85$ produces a positive dR/dt, so the sales were increasing from 2004 to 2009.

14. $R = -5.5778t^3 + 67.524t^2 + 45.22t + 1969.2, 0 \le t \le 10$

(a)



The revenue was increasing from 2000 to about 2008.
The revenue was decreasing from about 2008 to 2010.

(b) $R' = \frac{dR}{dt} = -16.7334t^2 + 135.048t + 44.22$

Setting $R' = 0$ produces a solution of $t \approx 8.4$.

Interval	$0 < t < 8.4$	$8.4 < t < 10$
Sign of R'	$R' > 0$	$R' < 0$
Conclusion	Increasing	Decreasing

Revenue were increasing from 2000 to mid-2008, and decreasing from mid-2008 to 2010.

15. $f(x) = 4x^3 - 6x^2 - 2$

$f'(x) = 12x^2 - 12x = 12x(x - 1)$

Critical numbers: $x = 0, x = 1$

Relative maximum: $(0, -2)$

Relative minimum: $(1, -4)$

Interval	$-\infty < x < 0$	$0 < x < 1$	$1 < x < \infty$
Sign of f'	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion	Increasing	Decreasing	Increasing

16. $f(x) = \frac{1}{4}x^4 - 8x$

$f'(x) = x^3 - 8$

Critical number: $x = 2$

Interval	$-\infty < x < 2$	$2 < x < \infty$
Sign of f'	$f' < 0$	$f' > 0$
Conclusion	Decreasing	Increasing

Relative minimum: $(2, -12)$

17. $g(x) = x^2 - 16x + 12$

$g'(x) = 2x - 16 = 2(x - 8)$

Critical number: $x = 8$

Relative minimum: $(8, -52)$

Interval	$-\infty < x < 8$	$8 < x < \infty$
Sign of g'	$g' < 0$	$g' > 0$
Conclusion	Decreasing	Increasing

18. $h(x) = 4 + 10x - x^2$

$h'(x) = 10 - 2x = 2(5 - x)$

Critical number: $x = 5$

Relative maximum: $(5, 29)$

Interval	$-\infty < x < 5$	$5 < x < \infty$
Sign of h'	$h' > 0$	$h' < 0$
Conclusion	Increasing	Decreasing

19. $h(x) = 2x^2 - x^4$

$h'(x) = 4x - 4x^3$

$= 4x(1 - x)(1 + x)$

Critical numbers: $x = 0, x = \pm 1$

Relative maxima: $(-1, 1), (1, 1)$

Relative minimum: $(0, 0)$

Interval	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
Sign of h'	$h' > 0$	$h' < 0$	$h' > 0$	$h' < 0$
Conclusion	Increasing	Decreasing	Increasing	Decreasing

20. $s(x) = x^4 - 8x^2 + 3$

$s'(x) = 4x^3 - 16x = 4x(x - 2)(x + 2)$

Critical numbers: $x = 0, x = \pm 2$

Relative minima: $(-2, -13), (2, -13)$

Relative maximum: $(0, 3)$

Interval	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < 2$	$2 < x < \infty$
Sign of s'	$s' < 0$	$s' > 0$	$s' < 0$	$s' > 0$
Conclusion	Decreasing	Increasing	Decreasing	Increasing

21. $f(x) = \frac{6}{x^2 + 1}$

$f'(x) = -6(x^2 + 1)^{-2}(2x) = -\frac{12x}{(x^2 + 1)^2}$

Critical number: $x = 0$

Relative maximum: $(0, 6)$

Interval	$-\infty < x < 0$	$0 < x < \infty$
Sign of f'	$f' > 0$	$f' < 0$
Conclusion	Increasing	Decreasing

22. $f(x) = \dfrac{2}{x^2 - 1}$

$f'(x) = -2(x^2 - 1)^{-2}(2x) = -\dfrac{4x}{(x^2 - 1)^2}$

Critical number: $x = 0$

Discontinuities: $x = \pm 1$

Relative maximum: $(0, -2)$

Interval	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
Sign of f'	$f' > 0$	$f' > 0$	$f' < 0$	$f' < 0$
Conclusion	Increasing	Increasing	Decreasing	Decreasing

23. $h(x) = \dfrac{x^2}{x - 2}$

$h'(x) = \dfrac{(x - 2)(2x) - x^2}{(x - 2)^2} = \dfrac{x(x - 4)}{(x - 2)^2}$

Critical numbers: $x = 0, x = 4$

Discontinuity: $x = 2$

Relative maximum: $(0, 0)$

Relative minimum: $(4, 8)$

Interval	$-\infty < x < 0$	$0 < x < 2$	$2 < x < 4$	$4 < x < \infty$
Sign of h'	$h' > 0$	$h' < 0$	$h' < 0$	$h' > 0$
Conclusion	Increasing	Decreasing	Decreasing	Increasing

24. $g(x) = x - 6\sqrt{x}, x > 0$

$g'(x) = 1 - 3x^{-1/2} = \dfrac{\sqrt{x} - 3}{\sqrt{x}}$

Critical number: $x = 9$

Relative minimum: $(9, -9)$

Interval	$0 < x < 9$	$9 < x < \infty$
Sign of g'	$g' < 0$	$g' > 0$
Conclusion	Decreasing	Increasing

25. $f(x) = x^2 + 5x + 6, [-3, 0]$

$f'(x) = 2x + 5$

Critical number: $x = -\dfrac{5}{2}$

x-value	Endpoint $x = -3$	Critical $x = -\frac{5}{2}$	Endpoint $x = 0$
$f(x)$	0	$-\frac{1}{4}$	6
Conclusion		Minimum	Maximum

26. $f(x) = x^4 - 2x^3, [0, 2]$

$f'(x) = 4x^3 - 6x^2 = 2x^2(2x - 3)$

Critical numbers: $x = 0$ (endpoint), $x = \dfrac{3}{2}$

x-value	Endpoint $x = 0$	Critical $x = \frac{3}{2}$	Endpoint $x = 2$
$f(x)$	0	$-\frac{27}{16}$	0
Conclusion	Maximum	Minimum	Maximum

27. $f(x) = x^3 - 12x + 1, [-4, 4]$

$f'(x) = 3x^2 - 12 = 3(x - 2)(x + 2)$

Critical numbers: $x = \pm 2$

x-value	Endpoint $x = -4$	Critical $x = -2$	Critical $x = 2$	Endpoint $x = 4$
$f(x)$	-15	17	-15	17
Conclusion	Minimum	Maximum	Minimum	Maximum

28. $f(x) = x^3 + 2x^2 - 3x + 4, [-3, 2]$

$f'(x) = 3x^2 + 4x - 3$

Critical numbers: $x = \frac{-4 \pm \sqrt{52}}{6} = \frac{-2 \pm \sqrt{13}}{3}$

x-value	Endpoint $x = -3$	Critical $x = \frac{-2 - \sqrt{13}}{2}$	Critical $x = \frac{-2 + \sqrt{13}}{2}$	Endpoint $x = 2$
$f(x)$	4	≈ 10.06	≈ 3.12	14
Conclusion			Minimum	Maximum

29. $f(x) = 2\sqrt{x} - x, [0, 9]$

$f'(x) = x^{-1/2} - 1 = \frac{1 - \sqrt{x}}{\sqrt{x}}$

Critical numbers: $x = 0$ (endpoint), $x = 1$

x-value	Endpoint $x = 0$	Critical $x = 1$	Endpoint $x = 9$
$f(x)$	0	1	-3
Conclusion		Maximum	Minimum

30. $f(x) = \frac{x}{\sqrt{x^2 + 1}}, [0, 2]$

$$f'(x) = \frac{(x^2 + 1)^{1/2} - x\left[\frac{1}{2}(x^2 + 1)^{-1/2}(2x)\right]}{x^2 + 1}$$

$$= \frac{1}{(x^2 + 1)^{3/2}}$$

No critical numbers

x-value	Endpoint $x = 0$	Endpoint $x = 2$
$f(x)$	0	$\frac{2\sqrt{5}}{5}$
Conclusion	Minimum	Maximum

31. $f(x) = \frac{2x}{x^2 + 1}, [-1, 2]$

$$f'(x) = \frac{(x^2 + 1)(2) - 2x(2x)}{(x^2 + 1)^2} = -\frac{2(x^2 - 1)}{(x^2 + 1)^2}$$

Critical numbers: $x = 1$, $x = -1$ (endpoint)

x-value	Endpoint $x = -1$	Critical $x = 1$	Endpoint $x = 2$
$f(x)$	-1	1	$\frac{4}{5}$
Conclusion	Minimum	Maximum	

32. $f(x) = \frac{8}{x} + x, [1, 4]$

$f'(x) = -\frac{8}{x^2} + 1 = \frac{x^2 - 8}{x^2}$

Critical number: $x = 2\sqrt{2}$

x-value	Endpoint $x = 1$	Critical $x = 2\sqrt{2}$	Endpoint $x = 4$
$f(x)$	9	$4\sqrt{2}$	6
Conclusion	Maximum	Minimum	

33. $S = 2\pi r^2 + 50r^{-1}$

$S' = 4\pi r - 50r^{-2} = 4\pi r - \frac{50}{r^2}$

Set $S' = 0$.

$$4\pi r - \frac{50}{r^2} = 0$$

$$4\pi r = \frac{50}{r^2}$$

$$4\pi r^3 = 50$$

$$r^3 = \frac{25}{2\pi}$$

$$r = \sqrt[3]{\frac{25}{2\pi}} \approx 1.58 \text{ in.}$$

Interval	$0 < r < 1.58$	$1.58 < r < \infty$
Sign of S'	$S' < 0$	$S' > 0$
Conclusion	Decreasing	Increasing

S is a minimum when $r = 1.58$ inches.

34. $P = 1.64x - \dfrac{x^2}{15{,}000} - 2500$

$P' = 1.64 - \dfrac{1}{7500}x$

Set $P' = 0$.

$1.64 - \dfrac{1}{7500}x = 0$

$-\dfrac{1}{7500}x = -1.64$

$x = 12{,}300$

$P'' = -\dfrac{1}{7500}$

$P''(12{,}300) < 0 \Rightarrow P$ is a maximum when $x = 12{,}300$ units.

The maximum profit is $P(12{,}300) = \$7586$.

35. $f(x) = (x-2)^3$

$f'(x) = 3(x-2)^2$

$f''(x) = 6(x-2)$

$f''(x) = 0$ when $x = 2$.

Interval	$-\infty < x < 2$	$2 < x < \infty$
Sign of f''	$f'' < 0$	$f'' > 0$
Conclusion	Concave downward	Concave upward

36. $h(x) = x^5 - 10x^2$

$h'(x) = 5x^4 - 20x$

$h''(x) = 20x^3 - 20 = 20(x^3 - 1)$

$h''(x) = 0$ when $x = 1$.

Interval	$-\infty < x < 1$	$1 < x < \infty$
Sign of h''	$h'' < 0$	$h'' > 0$
Conclusion	Concave downward	Concave upward

37. $g(x) = \dfrac{1}{4}(-x^4 + 8x^2 - 12)$

$g'(x) = -x^3 + 4x$

$g''(x) = -3x^2 + 4$

$g''(x) = 0$ when $x = \pm\dfrac{2}{\sqrt{3}}$.

Interval	$-\infty < x < -\dfrac{2}{\sqrt{3}}$	$-\dfrac{2}{\sqrt{3}} < x < \dfrac{2}{\sqrt{3}}$	$\dfrac{2}{\sqrt{3}} < x < \infty$
Sign of g''	$g'' < 0$	$g'' > 0$	$g'' < 0$
Conclusion	Concave downward	Concave upward	Concave downward

38. $h(x) = x^3 - 6x$

$h'(x) = 3x^2 - 6$

$h''(x) = 6x$

$h''(x) = 0$ when $x = 0$.

Interval	$-\infty < x < 0$	$0 < x < \infty$
Sign of h''	$h'' < 0$	$h'' > 0$
Conclusion	Concave downward	Concave upward

39. $f(x) = \frac{1}{2}x^4 - 4x^3$

$f'(x) = 2x^3 - 12x^2$

$f''(x) = 6x^2 - 24x = 6x(x-4)$

$f''(x) = 0$

$6x(x-4) = 0$

$x = 0$ or $x = 4$

Interval	$-\infty < x < 0$	$0 < x < 4$	$4 < x < \infty$
Sign of f''	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion	Concave upward	Concave downward	Concave upward

Points of inflection: $(0, 0)$, $(4, -128)$

40. $f(x) = \frac{1}{4}x^4 - 2x^2 - x$

$f'(x) = x^3 - 4x - 1$

$f''(x) = 3x^2 - 4$

$f''(x) = 0$

$3x^2 - 4 = 0$

$x^2 = \frac{4}{3}$

$x = \frac{\pm 2}{\sqrt{3}} = \frac{\pm 2\sqrt{3}}{3}$

Interval	$-\infty < x < -\frac{2\sqrt{3}}{3}$	$-\frac{2\sqrt{3}}{3} < x < \frac{2\sqrt{3}}{3}$	$\frac{2\sqrt{3}}{3} < x < \infty$
Sign of f''	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion	Concave upward	Concave downward	Concave upward

Points of inflection: $\left(-\frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3} - \frac{20}{9}\right), \left(\frac{2\sqrt{3}}{3}, -\frac{2\sqrt{3}}{3} - \frac{20}{9}\right)$

41. $f(x) = x^3(x - 3)^2$

$f'(x) = x^3[2(x - 3)(1)] + (x - 3)^2(3x^2) = 5x^4 - 24x^3 + 27x^2$

$f''(x) = 20x^3 - 72x^2 + 54x = 2x(10x^2 - 36x + 27)$

$f''(x) = 0$

$2x(10x^2 - 36x + 27) = 0$

$x = 0$

$x = \frac{36 \pm \sqrt{36^2 - 4(10)(27)}}{2(10)}$

$\approx 1.0652, 2.5348$

Interval	$-\infty < x < 0$	$0 < x < 1.0652$	$1.0652 < x < 2.5348$	$2.5348 < x < \infty$
Sign of f''	$f'' < 0$	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion	Concave downward	Concave upward	Concave downward	Concave upward

Points of inflection: $(0, 0), (1.6052, 4.5244), (2.5348, 3.5246)$

42. $f(x) = (x - 1)^2(x - 3)$

$f'(x) = (x - 1)^2(1) + (x - 3)[(2)(x - 1)(1)]$

$= 3x^2 - 10x + 7$

$f''(x) = 6x - 10 = 2(3x - 5)$

$f''(x) = 0$

$2(3x - 5) = 0$

$x = \frac{5}{3}$

Interval	$-\infty < x < \frac{5}{3}$	$\frac{5}{3} < x < \infty$
Sign of f''	$f'' < 0$	$f'' > 0$
Conclusion	Concave downward	Concave upward

Point of inflection: $\left(\frac{5}{3}, -\frac{16}{27}\right)$

43. $f(x) = x^3 - 6x^2 + 12x$

$f'(x) = 3x^2 - 12x + 12$

$f''(x) = 6x - 12$

Critical number: $x = 2$

$f''(2) = 0$

By the First-Derivative Test, $(2, 8)$ is not a relative extremum.

44. $f(x) = x^4 - 32x^2 + 12$

$f'(x) = 4x^3 - 64x$

$f''(x) = 12x^2 - 64$

Critical numbers: $x = 0, x = \pm 4$

$f''(0) = -64 < 0$

Relative maximum: $(0, 12)$

$f''(-4) = 128 > 0$

Relative minimum: $(-4, -244)$

$f''(4) = 128 > 0$

Relative minimum: $(4, -244)$

45. $f(x) = x^5 - 5x^3$

$f'(x) = 5x^4 - 15x^2 = 5x^2(x^2 - 3)$

$f''(x) = 20x^3 - 30x$

Critical numbers: $x = 0,\ x = \pm\sqrt{3}$

$f''(\sqrt{3}) = 30\sqrt{3} > 0$

Relative minimum: $(\sqrt{3}, -6\sqrt{3})$

$f''(-\sqrt{3}) = -30\sqrt{3} < 0$

Relative maximum: $(-\sqrt{3}, 6\sqrt{3})$

$f''(0) = 0$

By the First-Derivative Test, $(0, 0)$ is not a relative extremum.

46. $f(x) = x(x^2 - 3x - 9) = x^3 - 3x^2 - 9x$

$$f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x - 3)(x + 1)$$

$f''(x) = 6x - 6$

Critical numbers: $x = 3,\ x = -1$

$f''(3) = 12 > 0$

Relative minimum: $(3, -27)$

$f''(-1) = -12 < 0$

Relative maximum: $(-1, 5)$

47. $f(x) = 2x^2(1 - x^2)$

$$f'(x) = (1 - x^2)(4x) + 2x^2(-2x) = 4x - 8x^3 = 4x(1 - 2x^2)$$

$f''(x) = 4 - 24x^2$

Critical numbers: $x = 0,\ x = \pm\dfrac{1}{\sqrt{2}}$

$f''(0) = 4 > 0$

Relative minimum: $(0, 0)$

$f''\left(-\dfrac{1}{\sqrt{2}}\right) = -8 < 0$

$f''\left(\dfrac{1}{\sqrt{2}}\right) = -8 < 0$

Relative maxima: $\left(\pm\dfrac{1}{\sqrt{2}}, \dfrac{1}{2}\right)$

48. $f(x) = x - 4\sqrt{x + 1}$

$$f'(x) = 1 - 4\left(\frac{1}{2}\right)(x + 1)^{-1/2} = 1 - \frac{2}{(x + 1)^{1/2}}$$

$$f''(x) = 2\left(\frac{1}{2}\right)(x + 1)^{-3/2} = \frac{1}{(x + 1)^{3/2}}$$

Critical number: $x = 3$

$f''(3) = \dfrac{1}{8} > 0$

Relative minimum: $(3, -5)$

49. $R = \frac{1}{1500}\left(150x^2 - x^3\right),\ 0 \le x \le 100$

$$R' = \frac{1}{1500}\left(300x - 3x^2\right)$$

$$R'' = \frac{1}{1500}(300 - 6x)$$

$$R'' = 0$$

$$300 - 6x = 0$$

$$x = 50$$

Interval	$0 < x < 50$	$50 < x < 100$
Sign of R''	$R'' > 0$	$R'' < 0$
Conclusion	Concave upward	Concave downward

Point of diminishing returns: $\left(50, \frac{500}{3}\right)$

50. $R = -\frac{2}{3}\left(x^3 - 12x^2 - 6\right),\ 0 \le x \le 8$

$$R' = -\frac{2}{3}\left(3x^2 - 24x\right)$$

$$R'' = -\frac{2}{3}(6x - 24)$$

$$R'' = 0$$

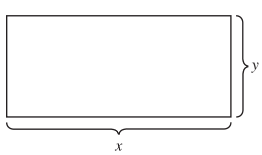
$$6x - 24 = 0$$

$$x = 4$$

Interval	$0 < x < 4$	$4 < x < 8$
Sign of R''	$R'' > 0$	$R'' < 0$
Conclusion	Concave upward	Concave downward

Point of diminishing returns: $\left(4, \frac{268}{3}\right)$

51.



Let x be the length and y be the width of the rectangle. Then $A = xy = 225$, and $P = 2x + 2y$.

$$P = 2x + 2\left(\frac{225}{x}\right) = 2x + \frac{550}{x} = 2x + 550x^{-1}$$

$$P' = 2 - 550x^{-2} = \frac{2 - 550}{x^2}$$

$$P'' = 1010x^{-3} = \frac{1010}{x^3}$$

$$P' = 0$$

$$\frac{2 - 550}{x^2} = 0$$

$$2x^2 = 550$$

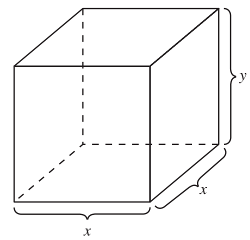
$$x^2 = 225$$

$$x = \pm 15$$

$P''(15) > 0$, so P is a minimum when $x = 15$ meters,

and $y = \frac{225}{15} = 15$ meters.

52. (a)



Let x be the dimensions of each side of the square base and y be the height.

Then $2x^2 + 4xy = 432$, and $V = x^2y$.

Solving for y in the surface area equation yields

$$y = \frac{432 - 2x^2}{4x} = \frac{108}{x} - \frac{1}{2}x.$$

Then $V = x^2\left(\frac{108}{x} - \frac{1}{2}x\right) = 108x - \frac{1}{2}x^3.$

$$V' = 108 - \frac{3}{2}x^2$$

$$V'' = -3x$$

$$V' = 0$$

$$108 - \frac{3}{2}x^2 = 0$$

$$\frac{3}{2}x^2 = 108$$

$$x^2 = 72$$

$$x = 6\sqrt{2} \approx 8.49$$

$V''\left(6\sqrt{2}\right) < 0$, so V is a maximum

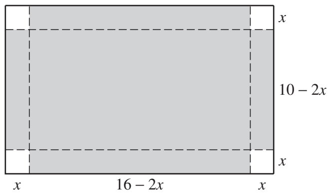
when $x = 6\sqrt{2}$ centimeters, and

$y = \frac{108}{6\sqrt{2}} - \frac{1}{2}\left(6\sqrt{2}\right) = 6\sqrt{2}$ centimeters.

(b) $V = \left(6\sqrt{2}\right)^2\left(6\sqrt{2}\right)$

$= 432\sqrt{2}\ \text{cm}^3$

$= \$610.94\ \text{cm}^3$

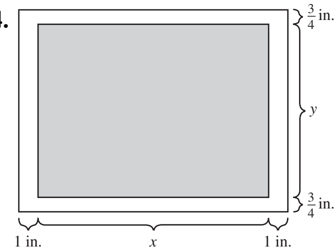
53. 

$$V = (16 - 2x)(10 - 2x)x = 4x^3 - 52x^2 + 160x$$
$$V' = 12x^2 - 104x + 160$$
$$V'' = 24x - 104$$
$$V' = 0$$
$$12x^2 - 104x + 160 = 0$$
$$4(3x^2 - 26x + 40) = 0$$
$$4(3x - 20)(x - 2) = 0$$
$$x = \tfrac{20}{3} \text{ or } x = 2$$

(**Note:** $x = \frac{20}{3}$ is not in the domain of V.)

$V''(2) < 0$, so V is a maximum when $x = 2$ in.

$V(2) = 144$ in.^{3}

54. 

Printed region: $108 = xy \rightarrow y = \dfrac{108}{x}$

Entire page: $A = (x + 2)(y + 1.5)$

$$A = (x + 2)\left(\frac{108}{x} + 1.5\right) = 108 + \frac{216}{x} + 1.5x + 3$$
$$A' = -216x^{-2} + 1.5 = \frac{-216}{x^2} + 1.5$$
$$A'' = \frac{432}{x^3}$$
$$A' = 0$$
$$\frac{-216}{x^2} + 1.5 = 0$$
$$1.5 = \frac{216}{x^2}$$
$$1.5x^2 = 216$$
$$x^2 = 144$$
$$x = \pm 12$$

(**Note:** $x = -12$ is not in the domain of A.)

$A''(12) > 0$, so A is a minimum when $x = 12$ in., and

$y = \dfrac{108}{12} = 9$ in.

The entire page is $12 + 2$ or 14 inches wide by $9 + 1.5 = 10\frac{1}{2}$ inches tall.

55. $R = 450x - 0.25x^2$

$$R' = \frac{dR}{dx} = 450 - 0.5x$$
$$R'' = -0.5$$
$$R' = 0$$
$$450 - 0.5x = 0$$
$$-0.5x = -450$$
$$x = 900$$

$R''(900) < 0$, so R is a maximum when $x = 900$, and $R(900) = \$202{,}500$.

56. $R = 36x^2 - 0.05x^3$

$R' = \frac{dR}{dx} = 72x - 0.15x^2 = x(72 - 0.15x)$

$R'' = 72 - 0.3x$

$R' = 0$

$x(72 - 0.15x) = 0$

$x = 0$ or $72 - 0.15x = 0$

$-0.15x = -72$

$x = 480$

(**Note:** $x = 0$ is not in the domain of R.)

$R''(480) < 0$, so R is a maximum when $x = 480$, and $R(480) = \$2{,}764{,}800$.

57. $\overline{C} = \frac{C}{x} = \frac{0.2x^2 + 10x + 4500}{x} = 0.2x + 10 + 4500x^{-1}$

$\overline{C}' = 0.2 - 4500x^{-2} = 0.2 - \frac{4500}{x^2}$

$\overline{C}'' = \frac{9000}{x^3}$

$\overline{C}' = 0$

$0.2 - \frac{4500}{x^2} = 0$

$0.2x^2 = 4500$

$x^2 = 22{,}500$

$x = \pm 150$

(**Note:** $x = -150$ is not in the domain of $\overline{C}$.)

$\overline{C}''(150) > 0$, so $\overline{C}$ is a minimum when $x = 150$ units, and $\overline{C}(150) = \$70/\text{unit}$.

58. $\overline{C} = \frac{C}{x} = \frac{0.03x^3 + 30x + 3840}{x}$

$= 0.03x^2 + 30 + 3840x^{-1}$

$\overline{C}' = 0.06x - 3840x^{-2} = 0.06x - \frac{3840}{x^2}$

$\overline{C}'' = 0.06 + \frac{7680}{x^3}$

$\overline{C}' = 0$

$0.06x - \frac{3840}{x^2} = 0$

$0.06x^3 = 3840$

$x^3 = 64{,}000$

$x = 40$

$\overline{C}''(40) > 0$, so $\overline{C}$ is a minimum when $x = 40$ units, and $\overline{C}(40) = \$174/\text{unit}$.

59. $P = R - C$ and $R = xp$

$R = x(36 - 4x) = 36x - 4x^2$

$P = (36x - 4x^2) - (2x^2 + 6)$

$P = -6x^2 + 36x - 6$

$\frac{dP}{dx} = P' = -12x + 36$

$P'' = -12$

$P' = 0$

$-12x + 36 = 0$

$x = 3$

$P''(3) < 0$, so P is a maximum when $x = 3$ units.

(a) When $x = 3$, $p = 36 - 4(3) = \$24/\text{unit}$.

(b) $\overline{C} = \frac{C}{x} = \frac{2x^2 + 6}{x} = 2x + \frac{6}{x}$

When $x = 3$, $\overline{C} = 2(3) + \frac{6}{3} = \$8/\text{unit}$.

60. $P = -4s^3 + 72s^2 - 240s + 500$

$\frac{dP}{ds} = P' = -12s^2 + 144s - 240$

$P'' = -24s + 144$

$P' = 0$

$-12s^2 + 144s - 240 = 0$

$-12(s^2 - 12s + 20) = 0$

$(s - 10)(s - 2) = 0$

$s = 10$ or $s = 2$

$P''(10) < 0$, so P is a maximum when $s = \$10$ thousand.

$P'' = 0$

$-24s + 144 = 0$

$s = 6$

Interval	$0 < s < 6$	$6 < s < \infty$
Sign of P''	$P'' > 0$	$P'' < 0$
Conclusion	Concave upward	Concave downward

Point of diminishing returns: $(6, 788)$

61. (a) $60 - 0.04x,\ 0 \le x \le 1500$

$\frac{dp}{dx} = -0.04$

$\eta = \frac{p/x}{dp/dx} = \frac{\frac{60 - 0.04x}{x}}{-0.04} = \frac{x - 1500}{x}$

$|\eta| = 1 = \left|\frac{x - 1500}{x}\right| \Rightarrow x = |x - 1500| \Rightarrow x = 750$

For $0 < x < 750$, $|\eta| > 1$ and demand is elastic.

For $750 < x < 1500$, $|\eta| < 1$ and demand is inelastic.

For $x = 750$, demand is of unit elasticity.

(b) $R = xp = x(60 - 0.04x) = 60x - 0.04x^2$

$R' = \frac{dR}{dx} = 60 - 0.08x$

$R' = 0$

$60 - 0.08x = 0$

$x = 750$

Interval	$0 < x < 750$	$750 < x < 1500$
Sign of R'	$R' > 0$	$R' < 0$
Conclusion	Increasing	Decreasing

From 0 to 750 units, revenue is increasing.

From 750 to 1500 units, revenue does not increase.

62. (a) $P = 960 - x,\ 0 \le x \le 960$

$\frac{dp}{dx} = -1$

$\eta = \frac{p/x}{dp/dx} = \frac{\frac{960 - x}{x}}{-1} = \frac{x - 960}{x}$

$|\eta| = 1 = \left|\frac{x - 960}{x}\right| \Rightarrow x = |x - 960| \Rightarrow x = 480$

For $0 < x < 480$, $|\eta| > 1$ and demand is elastic.

For $480 < x < 960$, $|\eta| < 1$ and demand is inelastic.

For $x = 480$, demand is of unit elasticity.

(b) $R = xp = x(960 - x) = 960x - x^2$

$R' = \frac{dR}{dx} = 960 - 2x$

$R' = 0$

$960 - 2x = 0$

$x = 480$

Interval	$0 < x < 480$	$480 < x < 960$
Sign of R'	$R' > 0$	$R' < 0$
Conclusion	Increasing	Decreasing

From 0 to 480 units, revenue is increasing.

From 480 to 960 units, revenue is decreasing.

63. $f(x) = \frac{x + 4}{x^2 + 7x} = \frac{x + 4}{x(x + 7)}$

Set denominator equal to 0.

$x(x + 7) = 0$

$x = 0$ or $x + 7 = 0$

$x = -7$

Vertical asymptotes: $x = 0, x = -7$

64. $f(x) = \frac{x - 1}{x^2 - 4} = \frac{x - 1}{(x + 2)(x - 2)}$

Set denominator equal to 0.

$(x + 2)(x - 2) = 0$

$x + 2 = 0$ or $x - 2 = 0$

$x = -2, x = 2$

Vertical asymptotes: $x = -2, x = 2$

65. $f(x) = \frac{x^2 - 16}{2x^2 + 9x + 4} = \frac{(x + 4)(x - 4)}{(2x + 1)(x + 4)} = \frac{x - 4}{2x + 1},\ x \ne -4$

Set denominator equal to 0.

$2x + 1 = 0$

$x = -\frac{1}{2}$

Vertical asymptote: $x = -\frac{1}{2}$

66. $f(x) = \dfrac{x^2 + 6x + 9}{x^2 - 5x - 24} = \dfrac{(x+3)^2}{(x-8)(x+3)} = \dfrac{x+3}{x-8}, x \neq -3$

Set denominator equal to 0.

$x - 8 = 0$

$x = 8$

Vertical asymptote: $x = 8$

67. $\lim_{x\to 0^+}\left(x - \dfrac{1}{x^3}\right) = 0 - \infty = -\infty$

68. $\lim_{x\to 0^-}\left(3 + \dfrac{1}{x}\right) = 3 - \infty = -\infty$

69. $\lim_{x\to -1^+}\dfrac{x^2 - 2x + 1}{x + 1} = \infty$

70. $\lim_{x\to 3^-}\dfrac{3x^2 + 1}{x^2 - 9} = \lim_{x\to 3^-}\dfrac{3x^2 + 1}{(x+3)(x-3)} = -\infty$

71. $\lim_{x\to\infty}\dfrac{2x^2}{3x^2 + 5} = \dfrac{2}{3}$

$y = \dfrac{2}{3}$ is a horizontal asymptote.

72. $\lim_{x\to\infty}\dfrac{2x^2 - 2x + 3}{x + 1} = \infty$

There is no horizontal asymptote.

73. $\lim_{x\to\infty}\dfrac{3x}{x^2 + 1} = 0$

$y = 0$ is a horizontal asymptote.

74. $\lim_{x\to\infty}\dfrac{x}{x - 2} + \dfrac{2x}{x + 2} = 1 + 2 = 3$

$y = 3$ is a horizontal asymptote.

75. $C = 0.75x + 4000$

(a) $\overline{C} = \dfrac{C}{x} = \dfrac{0.75x + 4000}{x} = 0.75 + \dfrac{4000}{x}$

(b) $\overline{C}(100) = \$40.75/\text{unit}.$

$\overline{C}(1000) = \$4.75/\text{unit}.$

(c) $\lim_{x\to\infty}\overline{C} = \lim_{x\to\infty} 0.75 + \dfrac{4000}{x} = \$0.75/\text{unit}$

The limit is 0.75. As more and more units are produced, the average cost per unit will approach \$0.75.

76. $C = 1.50x + 8000$

(a) $\overline{C} = \dfrac{C}{x} = \dfrac{1.50x + 8000}{x} = 1.50 + \dfrac{8000}{x}$

(b) $\overline{C}(1000) = \$9.50/\text{unit}.$

$\overline{C}(10{,}000) = \$2.30/\text{unit}.$

(c) $\lim_{x\to\infty}\overline{C} = \lim_{x\to\infty} 1.50 + \dfrac{8000}{x} = \$1.50/\text{unit}$

The limit is 1.50. As more and more units are produced, the average cost per unit will approach \$1.50.

77. $C = \dfrac{250p}{100 - p}, 0 \le p < 100$

(a) $C(20) = \$62.5$ million

$C(50) = \$250$ million

$C(90) = \$2250$ million

(b) $\lim_{p\to 100^-} C = \lim_{p\to 100^-}\dfrac{250p}{100 - p} = \infty$

The limit is ∞, meaning that as the percent approaches 100%, the cost increases without bound.

78. $C = \dfrac{160{,}000p}{100 - p}, 0 \le p < 100$

(a) $C(25) \approx \$53{,}333.33$

$C(50) = \$160{,}000$

$C(75) = \$480{,}000$

(b) $\lim_{p\to 100^-} C = \lim_{p\to 100^-}\dfrac{160{,}000p}{100 - p} = \infty$

The limit is ∞, meaning that as the percent approaches 100%, the cost increases without bound.

79. $f(x) = 4x - x^2$

$f'(x) = 4 - 2x$

$f''(x) = -2$

x-intercepts: $f(x) = 0$

$4x - x^2 = 0$

$x(4 - x) = 0$

$x = 0 \text{ or } x = 4$

y-intercept: $f(0) = 0$

$y = 0$

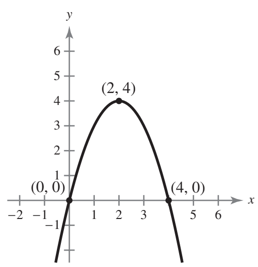
Critical number: $f'(x) = 0$

$4 - 2x = 0$

$x = 2$

$f''(2) < 0 \Rightarrow$ relative maximum at $(2, 4)$

$f'' \neq 0 \Rightarrow$ no points of inflection



80. $f(x) = 4x^3 - x^4$

$f'(x) = 12x^2 - 4x^3$

$f''(x) = 24x - 12x^2$

x-intercept: $f(x) = 0$

$4x^3 - x^4 = 0$

$x^3(4 - x) = 0$

$x = 0 \text{ or } x = 4$

y-intercept: $f(0) = 0$

$y = 0$

Critical numbers: $f'(x) = 0$

$12x^2 - 4x^3 = 0$

$4x^2(3 - x) = 0$

$x = 0 \text{ or } x = 3$

$f''(3) < 0 \Rightarrow$ relative maximum at $(3, 27)$

Use the First-Derivative Test: no extrema at $x = 0$

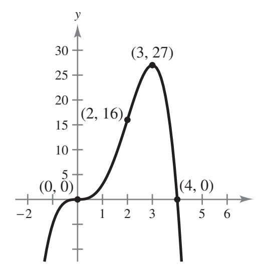
$f'' = 0$

$24x - 12x^2 = 0$

$12x(2 - x) = 0$

$x = 0 \text{ or } x = 2$

Points of inflection:
$(0, 0), (2, 16)$



81. $f(x) = x^3 - 6x^2 + 3x + 10$

$f'(x) = 3x^2 - 12x + 3$

$f''(x) = 6x - 12$

x-intercepts: $f(x) = 0$

$x^3 - 6x^2 + 3x + 10 = 0$

$(x - 5)(x - 2)(x + 1) = 0$

$x = 5 \text{ or } x = 2 \text{ or } x = -1$

y-intercept: $f(0) = 10$

$y = 10$

Critical numbers: $f'(x) = 0$

$3x^2 - 12x + 3 = 0$

$3(x^2 - 4x + 1) = 0$

$x = 2 \pm \sqrt{3}$

$f''(2 + \sqrt{3}) > 0 \Rightarrow$ relative minimum at $(2 + \sqrt{3}, -10.39)$

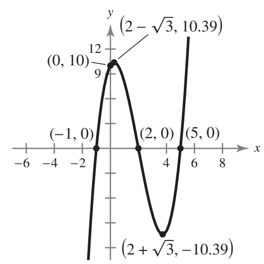
$f''(2 - \sqrt{3}) < 0 \Rightarrow$ relative maximum at $(2 - \sqrt{3}, 10.39)$

$f''(x) = 0$

$6x - 12 = 0$

$x = 2$

Point of inflection: $(2, 0)$



82. $f(x) = -x^3 + 3x^2 + 9x - 2$

$f'(x) = -3x^2 + 6x + 9$

$f''(x) = -6x + 6$

x-intercepts: $f(x) = 0$

$$-x^3 + 3x^2 + 9x - 2 = 0$$

$$x = -2 \text{ or } x = \frac{5 \pm \sqrt{21}}{2}$$

y-intercept: $f(0) = -2$

$$y = -2$$

Critical numbers: $f'(x) = 0$

$$-3x^2 + 6x + 9 = 0$$

$$-3(x^2 - 2x - 3) = 0$$

$$x = 3 \text{ or } x = -1$$

$f''(-1) > 0 \Rightarrow$ relative minimum at $(-1, -7)$

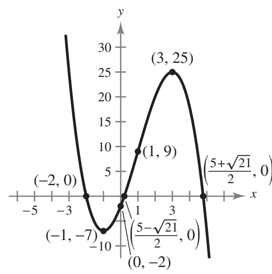
$f''(3) < 0 \Rightarrow$ relative maximum at $(3, 25)$

$$f''(x) = 0$$

$$-6x + 6 = 0$$

$$x = 1$$

Point of inflection: $(1, 9)$



83. $f(x) = x^4 - 4x^3 + 16x - 16$

$f'(x) = 4x^3 - 12x^2 + 16$

$f''(x) = 12x^2 - 24x$

x-intercepts: $f(x) = 0$

$$x^4 - 4x^3 + 16x - 16 = 0$$

$$x = 2 \text{ or } x = -2$$

y-intercept: $f(0) = -16$

$$y = -16$$

Critical numbers: $f'(x) = 0$

$$4x^3 - 12x^2 + 16 = 0$$

$$x = 2 \text{ or } x = -1$$

$f''(-1) > 0 \Rightarrow$ relative minimum at $(-1, -27)$

$f''(2) < 0 \Rightarrow$ Use First-Derivative Test $\Rightarrow$ no extrema

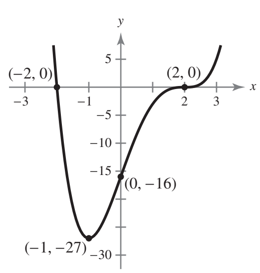
$$f''(x) = 0$$

$$12x^2 - 24x = 0$$

$$12x(x - 2) = 0$$

$$x = 0 \text{ or } x = 2$$

Points of inflection:
$(0, -16), (2, 0)$



84. $f(x) = x^5 + 1$

$f'(x) = 5x^4$

$f''(x) = 20x^3$

x-intercept: $f(x) = 0$

$$x^5 + 1 = 0$$

$$x = -1$$

y-intercept: $f(0) = 1$

$$y = 1$$

Critical number: $f'(x) = 0$

$$5x^4 = 0$$

$$x = 0$$

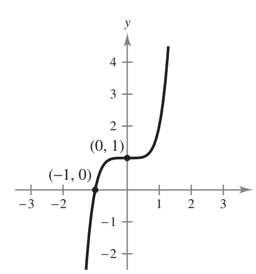
$f''(0) = 0 \Rightarrow$ Use First-Derivative Test $\Rightarrow$ no extrema

$$f''(x) = 0$$

$$20x^3 = 0$$

$$x = 0$$

Point of inflection: $(0, 1)$



85. $f(x) = x\sqrt{16 - x^2}, \quad -4 \le x \le 4$

$$f'(x) = \frac{16 - 2x^2}{\sqrt{16 - x^2}} = \frac{-2(x^2 - 8)}{\sqrt{16 - x^2}}$$

$$f''(x) = \frac{2x^3 - 48x}{(16 - x^2)^{3/2}} = \frac{2x(x^2 - 24)}{(16 - x^2)^{3/2}}$$

x-intercepts: $f(x) = 0$

$$x\sqrt{16 - x^2} = 0$$

$$x = 0 \text{ or } x = \pm 4$$

y-intercept: $f(0) = 0$

$$y = 0$$

Critical numbers: $f'(x) = 0$

$$\frac{-2(x^2 - 8)}{\sqrt{16 - x^2}} = 0$$

$$x = \pm 2\sqrt{2}$$

$f''(-2\sqrt{2}) > 0 \Rightarrow$ relative minimum at $(-2\sqrt{2}, -8)$

$f''(2\sqrt{2}) < 0 \Rightarrow$ relative maximum at $(2\sqrt{2}, 8)$

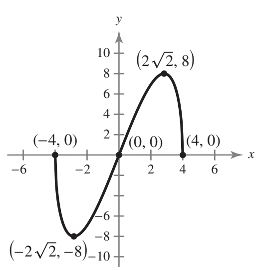
$$f''(x) = 0$$

$$\frac{2x(x^2 - 24)}{(16 - x^2)^{3/2}} = 0$$

$$x = 0 \text{ or } x = \pm 2\sqrt{6}$$

(**Note:** $x = \pm 2\sqrt{6}$ not in domain.)

Point of inflection: $(0, 0)$



86. $f(x) = x^2\sqrt{9 - x^2}, \quad -3 \le x \le 3$

$$f'(x) = \frac{18x - 3x^3}{\sqrt{9 - x^2}} = \frac{-3x(x^2 - 6)}{\sqrt{9 - x^2}}$$

$$f''(x) = \frac{6x^4 - 81x^2 + 162}{(9 - x^2)^{3/2}} = \frac{-3(2x^4 - 27x^2 + 54)}{(9 - x^2)^{3/2}}$$

x-intercepts: $f(x) = 0$

$$x\sqrt{9 - x^2} = 0$$

$$x = 0 \text{ or } x = \pm 3$$

y-intercept: $f(0) = 0$

$$y = 0$$

Critical numbers: $f'(x) = 0$

$$\frac{-3x(x^2 - 6)}{\sqrt{9 - x^2}} = 0$$

$$x = 0 \text{ or } \pm\sqrt{6}$$

$f''(0) > 0 \Rightarrow$ relative minimum at $(0, 0)$

$f''(\sqrt{6}) < 0 \Rightarrow$ relative maximum at $(\sqrt{6}, 6\sqrt{3})$

$f''(-\sqrt{6}) < 0 \Rightarrow$ relative maximum at $(-\sqrt{6}, 6\sqrt{3})$

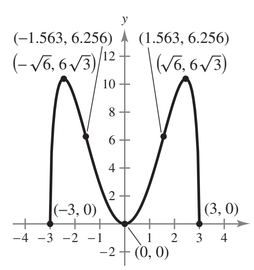
$$f''(x) = 0$$

$$\frac{-3(2x^4 - 27x^2 + 54)}{(9 - x^2)^{3/2}} = 0$$

$$x \approx \pm 1.563 \text{ or } x \approx \pm 3.325$$

(**Note:** $x = \pm 3.325$ are not in the domain.)

Points of inflection: $(\pm 1.563, 6.256)$



87. $f(x) = \dfrac{x+1}{x-1}$

$f'(x) = -\dfrac{2}{(x-1)^2}$

$f''(x) = \dfrac{4}{(x-1)^3}$

x-intercept: $f(x) = 0$ $\quad$ y-intercept: $f(0) = -1$

$\dfrac{x+1}{x-1} = 0 \qquad y = -1$

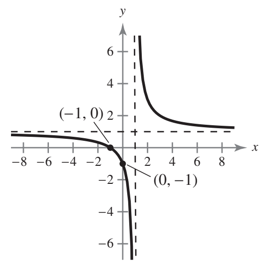
$x = -1$

Vertical asymptote: $x = 1$

Horizontal asymptote: $y = 1$

Critical numbers: $f'(x) \neq 0 \Rightarrow$ no extrema

$f''(x) \neq 0 \Rightarrow$ no points of inflection



88. $f(x) = \dfrac{x-1}{3x^2+1}$

$f'(x) = \dfrac{-(3x^2 - 6x - 1)}{(3x^2+1)^2}$

$f''(x) = \dfrac{6(3x^3 - 9x^2 - 3x + 1)}{(3x^2+1)^3}$

x-intercept: $f(x) = 0$ $\quad$ y-intercept: $f(0) = -1$

$\dfrac{x-1}{3x^2+1} = 0 \qquad y = -1$

$x = 1$

Horizontal asymptote: $y = 0$

$f'(x) = 0$

$\dfrac{-(3x^2 - 6x - 1)}{(3x^2+1)^2} = 0$

$-3x^2 + 6x + 1 = 0$

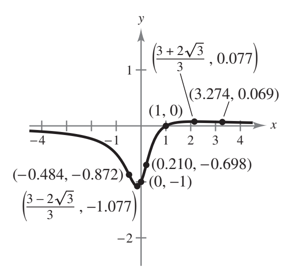
Critical numbers: $x = \dfrac{3 \pm 2\sqrt{3}}{3}$

$f''\left(\dfrac{3 + 2\sqrt{3}}{3}\right) < 0 \Rightarrow$ relative maximum at $\left(\dfrac{3 + 2\sqrt{3}}{3}, 0.077\right)$

$f''\left(\dfrac{3 - 2\sqrt{3}}{3}\right) > 0 \Rightarrow$ relative minimum at $\left(\dfrac{3 - 2\sqrt{3}}{3}, -1.077\right)$

$f''(x) = 0$: $x \approx -0.484, 0.210, 3.274$

Points of inflection: $(-0.484, -0.872)$, $(0.210, -0.698)$, $(3.274, 0.069)$



89. $f(x) = 3x^{2/3} - 2x$

$f'(x) = 2x^{-1/3} - 2$

$f''(x) = -\frac{2}{3}x^{-4/3}$

x-intercepts: $f(x) = 0$

$3x^{2/3} - 2x = 0$

$x(3x^{-1/3} - 2) = 0$

$x = 0$ or $x = \frac{27}{8}$

y-intercept: $f(0) = 0$

$y = 0$

Critical number: $f'(x) = 0$

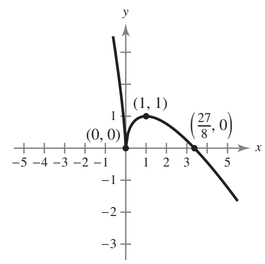
$2x^{-1/3} - 2 = 0$

$x = 1$

$f''(1) < 0 \Rightarrow$ relative maximum at $(1, 1)$

$f''(x) = 0 \Rightarrow$ Use First-Derivative Test $\Rightarrow$ relative minimum at $(0, 0)$

No points of inflection



90. $f(x) = x^{4/5}$

$f'(x) = \frac{4}{5}x^{-1/5}$

$f''(x) = -\frac{4}{25}x^{-6/5}$

x-intercept: $f(x) = 0$

$x = 0$

y-intercept: $f(0) = 0$

$y = 0$

Critical number: $f'(x) = 0$

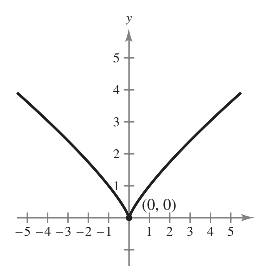
$\frac{4}{5}x^{-1/5} = 0$

f' is underfined at $x = 0$.

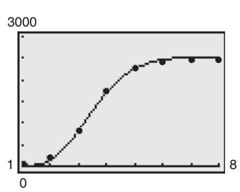
Interval	$-\infty < x < 0$	$0 < x < \infty$
Sign of f'	$f' < 0$	$f' > 0$
Conclusion	Decreasing	Increasing

Relative minimum at $(0, 0)$

$f''(x) \neq 0 \Rightarrow$ no points of inflection



91. (a)



The model fits the data well.

(b) $N(10) \approx 2434$ bacteria

(c) Answers will vary. Sample answer: The limit of N as t approaches infinity is $\frac{13{,}250}{7} \approx 1893$ bacteria; however, many different factors can have many different effects on the culture thus affecting the number of bacteria.

92. $T = \dfrac{31.6 - 1.822t + 0.0984t^2}{1 - 0.194t + 0.0131t^2}, 1 \le t \le 12$

$$T' = \frac{(1 - 0.194t + 0.0131t^2)(-1.822 + 0.1968t) - (31.6 - 1.822t + 0.0984t^2)(-0.194 + 0.0262t)}{(1 - 0.194t + 0.0131t)^2}$$

$$= \frac{0.0047786t^2 - 0.63112t + 4.3084}{(1 - 0.194t + 0.0131t^2)^2}$$

$T' = 0$: $t \approx 7.2, 124.9$ (**Note:** $t = 124.9$ is not in the domain.)

t-value	Endpoint $t = 1$	Critical number $t = 7.2$	Endpoint $t = 12$
T-value	36.5	83.5	42.8
Conclusion	Minimum	Maximum	

The minimum average temperature is 36.5°F in January and the maximum average temperature is 83.5°F in July.

93. $f(x) = 2x^2, x = 2, \Delta x = dx = 0.01$

$\Delta y = f(x + \Delta x) - f(x) = 2(2 + 0.01)^2 - 2(2)^2 = 0.0802$

$dy = f'(x)\,dx$

$dy = 4x\,dx$

$dy = 4(2)(0.01) = 0.08$

$dy \approx \Delta y$

94. $f(x) = x^4 + 3, x = 1, \Delta x = dx = 0.1$

$\Delta y = f(x + \Delta y) - f(x) = \left[(1 + 0.1)^4 + 3\right] - \left[(1)^4 + 3\right] = 0.4641$

$dy = f'(x)\,dx$

$dy = 4x^3\,dx$

$dy = 4(1)^3(0.1) = 0.4$

$dy \approx \Delta y$

95. $f(x) = 6x - x^3, x = 3, \Delta x = dx = 0.1$

$\Delta y = f(x + \Delta x) - f(x) = \left[6(3 + 0.1) - (3 + 0.1)^3\right] - \left[6(3) - (3)^3\right] = -2.191$

$dy = f'(x)\,dx$

$dy = (6 - 3x^2)\,dx$

$= (6 - 3(3)^2)(0.1) = -2.1$

$dy \approx \Delta y$

96. $f(x) = 5x^{3/2}, x = 9, \Delta x = dx = 0.01$

$\Delta y = f(x + \Delta x) - f(x) = 5(9 + 0.01)^{3/2} - 5(9)^{3/2} \approx 0.22506249$

$dy = f'(x)\,dx$

$dy = \frac{15}{2}x^{1/2}\,dx = \frac{15}{2}(9)^{1/2}(0.01) = 0.225$

$dy \approx \Delta y$

97. $C = 40x^2 + 1225, x = 10, \Delta x = dx = 1$

$dC = 80x\,dx$

$dC = 80(10)(1) = \$800$

98. $C = 1.5\sqrt[3]{x} + 500, x = 125, \Delta x = dx = 1$

$dC = 0.5x^{-2/3}\,dx = 0.5(125)^{-2/3}(1) = \$0.02$

99. $R = 6{,}25x - 0.4x^{3/2}, x = 225, \Delta x = dx = 1$

$dR = \left(6.25 + 0.6x^{1/2}\right)dx$

$= \left[6.25 + 0.6(225)^{1/2}\right](1) = \$15.25$

100. $R = 80x - 0.35x^2, x = 80, \Delta x = dx = 1$

$dR = (80 - 0.7x)\,dx = [80 - 0.7(80)](1) = \$24$

101. $P = 0.003x^2 + 0.019x - 1200, x = 750, \Delta x = dx = 1$

$dP = (0.006x + 0.019)\,dx$

$= [0.006(750) + 0.019](1) = 4.519 \approx \$4.52$

102. $P = -0.2x^3 + 3000x - 7500, x = 50, \Delta x = dx = 1$

$dP = (-0.6x + 3000)\,dx$

$= [-0.6(50) + 3000](1) = \$2970$

103. $y = 0.5x^2$

$dy = 1.5x^2\,dx$

104. $y = 7x^4 + 2x^2$

$dy = \left(28x^3 + 4x\right)dx$

105. $y = \left(3x^2 - 2\right)^3$

$\frac{dy}{dx} = 3\left(3x^2 - 2\right)^2(6x)$

$dy = 18x\left(3x^2 - 2\right)^2\,dx$

106. $y = \sqrt{36 - x^2}$

$\frac{dy}{dx} = \frac{1}{2}\left(36 - x^2\right)^{-1/2}(-2x)$

$dy = -\frac{x}{\sqrt{36 - x^2}}\,dx$

107. $y = \frac{2 - x}{x + 5}$

$\frac{dy}{dx} = \frac{(x + 5)(-1) - (2 - x)}{(x + 5)^2}$

$dy = -\frac{7}{(x + 5)^2}\,dx$

108. $y = \frac{3x^2}{x - 4}$

$\frac{dy}{dx} = \frac{(x - 4)(6x) - 3x^2(1)}{(x - 4)^2}$

$dy = \frac{3x^2 - 24x}{(x - 4)^2}\,dx$

109. $P = -0.8x^2 + 324x - 2000, x = 100, \Delta x = dx = 1$

(a) $dP = (-1.6x + 324)\,dx$

$dP = [-1.6(100) + 324](1) = \$164$

(b) $\Delta P = P(x + \Delta x) = P(x)$

$= \left[-0.8(100 + 1)^2 + 324(100 + 1) - 2000\right] - \left[-0.8(100)^2 + 324(100) - 2000\right]$

$= \$163.20$

This is $0.80 less than the answer in part (a).

110. $R = xp, p = 108 - 0.2x, x = 20, \Delta x = dx = 1$

$R = x(108 - 0.2x) = -0.2x^2 + 108x$

(a) $dR = (-0.4x + 108)\,dx$

$dR = [-0.4(20) + 108](1) = \$100$

(b) $x = 40, \Delta x = dx = 1$

$dR = (-0.4x + 108)\,dx$

$dR = [-0.4(40) + 108](1) = \$92$

111. $B = 0.1\sqrt{5w} = 0.1\sqrt{5}w^{1/2}, w = 90, \Delta w = dw = 5$

$dB = \frac{0.1\sqrt{5}}{2}w^{-1/2}\,dw = \frac{0.05\sqrt{5}}{\sqrt{w}}\,dw$

$dB = \frac{0.05\sqrt{5}}{\sqrt{90}}(5) \approx 0.059\text{ m}^2$

Chapter 3 Test Yourself

1. $f(x) = 3x^2 - 4$

$f'(x) = 6x$

Critical number: $x = 0$

Interval	$-\infty < x < 0$	$0 < x < \infty$
Sign of f'	$f' < 0$	$f' > 0$
Conclusion	Decreasing	Increasing

2. $f(x) = x^3 - 12x$

$f'(x) = 3x^2 - 12 = 3(x - 2)(x + 2)$

Critical numbers: $x = \pm 2$

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
Sign of f'	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion	Increasing	Decreasing	Increasing

3. $f(x) = (x - 5)^4$

$f'(x) = 4(x - 5)^3$

Critical number: $x = 5$

Interval	$-\infty < x < 5$	$5 < x < \infty$
Sign of f'	$f' < 0$	$f' > 0$
Conclusion	Decreasing	Increasing

4. $f(x) = \frac{1}{3}x^3 - 9x + 4$

$f'(x) = x^2 - 9 = (x + 3)(x - 3)$

Critical numbers: $x = \pm 3$

Relative minimum: $(3, -14)$

Relative maximum: $(-3, 22)$

Interval	$-\infty < x < -3$	$-3 < x < 3$	$3 < x < \infty$
Sign of f'	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion	Increasing	Decreasing	Increasing

5. $f(x) = 2x^4 - 4x^2 - 5$

$f'(x) = 8x^3 - 8x = 8x(x + 1)(x - 1)$

Critical numbers: $x = 0$, $x = \pm 1$

Relative minima: $(-1, -7)$, $(1, -7)$

Relative maximum: $(0, -5)$

Interval	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
Sign of f'	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion	Decreasing	Increasing	Decreasing	Increasing

6. $f(x) = \dfrac{5}{x^2 + 2}$

$f'(x) = -5(x^2 + 2)^{-2}(2x) = -\dfrac{10x}{(x^2 + 2)^2}$

Critical number: $x = 0$

Relative maximum: $\left(0, \frac{5}{2}\right)$

Interval	$-\infty < x < 0$	$0 < x < \infty$
Sign of f'	$f' > 0$	$f' < 0$
Conclusion	Increasing	Decreasing

7. $f(x) = x^2 + 6x + 8$, $[-4, 0]$

$f'(x) = 2x + 6$

Critical number: $x = -3$

x-value	Endpoint $x = -4$	Critical $x = -3$	Endpoint $x = 0$
$f(x)$	0	-1	8
Conclusion		Minimum	Maximum

8. $f(x) = 12\sqrt{x} - 4x, \ [0, 5]$

$f'(x) = 12\left(\frac{1}{2}x^{-1/2}\right) - 4 = \frac{6}{x^{1/2}} - 4$

Critical number: $x = \frac{9}{4}$

x-value	Endpoint $x = 0$	Critical $x = \frac{9}{4}$	Endpoint $x = 5$
$f(x)$	0	9	$12\sqrt{5} - 20$
Conclusion	Minimum	Maximum	

9. $f(x) = \frac{6}{x} + \frac{x}{2}, \ [1, 6]$

$f'(x) = -\frac{6}{x^2} + \frac{1}{2}$

Critical number: $x = 2\sqrt{3}$

x-value	Endpoint $x = 1$	Critical $x = 2\sqrt{3}$	Endpoint $x = 6$
$f(x)$	$\frac{13}{2}$	$2\sqrt{3}$	4
Conclusion	Maximum	Minimum	

10. $f(x) = x^5 - 80x^2$

$f'(x) = 5x^4 - 160x$

$f''(x) = 20x^3 - 160 = 20(x^3 - 8)$

$f''(x) = 0$ when $x = 2$

Interval	$-\infty < x < 2$	$2 < x < \infty$
Sign of $f''(x)$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave downward	Concave upward

11. $f(x) = \frac{20}{3x^2 + 8}$

$f'(x) = -20(3x^2 + 8)^{-2}(6x) = -\frac{120x}{(3x^2 + 8)^2}$

$$f''(x) = \frac{(3x^2 + 8)^2(-120) - (-120x)\left[2(3x^2 + 8)(6x)\right]}{(3x^2 + 8)^4} = \frac{120(9x^2 - 8)}{(3x^2 + 8)^3}$$

$f''(x) = 0$ when $x = \pm\frac{2\sqrt{2}}{3}$

Interval	$-\infty < x < -\frac{2\sqrt{2}}{3}$	$-\frac{2\sqrt{2}}{3} < x < \frac{2\sqrt{2}}{3}$	$\frac{2\sqrt{2}}{3} < x < \infty$
Sign of $f''(x)$	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

12. $f(x) = x^4 + 6$

$f'(x) = 4x^3$

$f''(x) = 12x^2 = 0$ when $x = 0$.

$f''(x) > 0$ on $(-\infty, 0)$.

$f''(x) > 0$ on $(0, \infty)$.

There are no points of inflection.

13. $f(x) = x^4 - 54x^2 + 230$

$f'(x) = 4x^3 - 108x$

$f''(x) = 12x^2 - 108 = 12(x^2 - 9)$

$f''(x) = 0$ when $x = \pm 3$

$f''(x) > 0$ on $(-\infty, -3) \Rightarrow$ concave upward

$f''(x) < 0$ on $(-3, 3) \Rightarrow$ concave downward

$f''(x) > 0$ on $(3, \infty) \Rightarrow$ concave upward

Points of inflection: $(\pm 3, -175)$

14. $f(x) = x^3 - 6x^2 - 36x + 50$

$$f'(x) = 3x^2 - 12x - 36$$
$$= 3(x^2 - 4x - 12)$$
$$= 3(x - 6)(x + 2)$$

Critical numbers: $x = -2, x = 6$

$$f''(x) = 6x - 12$$
$$f''(-2) < 0$$
$$f''(6) > 0$$

Relative maximum: $(-2, 90)$

Relative minimum: $(6, -166)$

15. $f(x) = \frac{3}{5}x^5 - 9x^3$

$$f'(x) = 3x^4 - 27x^2 = 3x^2(x + 3)(x - 3)$$

Critical numbers: $x = 0, x = \pm 3$

$$f''(x) = 12x^3 - 54x$$
$$f''(0) = 0$$
$$f''(-3) = -162 < 0$$
$$f''(3) = 162 > 0$$

Relative maximum: $\left(-3, \frac{486}{5}\right)$

Relative minimum: $\left(3, -\frac{486}{5}\right)$

By the First-Derivative Test, $(0, 0)$ is not a relative extremum.

16. A vertical asymptote occurs at $x = 5$ because $\lim_{x \to 5^-} \frac{3x + 2}{x - 5} = -\infty$ and $\lim_{x \to 5^+} \frac{3x + 2}{x - 5} = \infty$.

A horizontal asymptote occurs at $y = 3$ because $\lim_{x \to \infty} \frac{3x + 2}{x - 5} = 3$ and $\lim_{x \to -\infty} \frac{3x + 2}{x - 5} = 3$.

17. There are no vertical asymptotes because the denominator is never zero. A horizontal asymptote occurs at $y = 2$ because $\lim_{x \to \infty} \frac{2x^2}{x^2 + 3} = 2$ and $\lim_{x \to -\infty} \frac{2x^2}{x^2 + 3} = 2$.

18. A vertical asymptote occurs at $x = 1$ because $\lim_{x \to 1^-} \frac{2x^2 - 5}{x - 1} = \infty$ and $\lim_{x \to 1^+} \frac{2x^2 - 5}{x - 1} = -\infty$.

There are no horizontal asymptotes because $\lim_{x \to \infty} \frac{2x^2 - 5}{x - 1} = \infty$ and $\lim_{x \to -\infty} \frac{2x^2 - 5}{x - 1} = -\infty$.

19. $y = -x^3 + 3x^2 + 9x - 2$

$$y' = -3x^2 + 6x + 9$$
$$y'' = -6x + 6$$

x-intercepts: $-x^3 + 3x^2 + 9x - 2 = 0$

$$x = -2, \frac{5 \pm \sqrt{21}}{2}$$

y-intercept: $f(0) = -2$

$$y = -2$$

Critical numbers: $y' = 0$

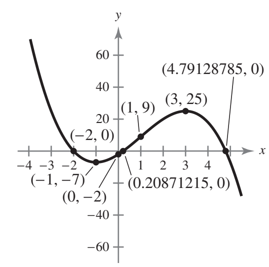
$$-3x^2 + 6x + 9 = 0$$
$$x = -1, 3$$

$y''(-1) > 0 \Rightarrow$ relative minimum at $(-1, -7)$

$y''(3) < 0 \Rightarrow$ relative maximum at $(3, 25)$

$$y'' = 0$$
$$-6x + 6 = 0$$
$$x = 1$$

Point of inflection: $(1, 9)$



20. $y = x^5 - 5x$

$$y' = 5x^4 - 5$$
$$y'' = 20x^3$$

x-intercepts: $x^5 - 5x = 0$

$$x(x^4 - 5) = 0$$
$$x = 0, \pm\sqrt[4]{5}$$

y-intercept: $f(0) = 0$

$$y = 0$$

Critical numbers: $y' = 0$

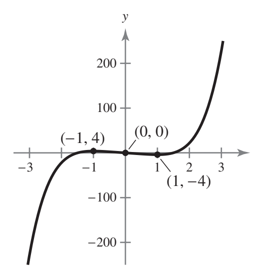
$$5x^4 - 5 = 0$$
$$5(x^4 - 1) = 0$$
$$x = \pm 1$$

$y''(-1) < 0 \Rightarrow$ relative maximum at $(-1, 4)$

$y''(1) > 0 \Rightarrow$ relative minimum at $(1, -4)$

$$y'' = 0$$
$$20x^3 = 0$$
$$x = 0$$

Point of inflection: $(0, 0)$



21. $y = \dfrac{x}{x^2 - 4}$

$y' = -\dfrac{(x^2 + 4)}{(x^2 - 4)^2}$

$y'' = \dfrac{2x(x^2 + 12)}{(x^2 - 4)^3}$

x-intercepts: $\dfrac{x}{x^2 - 4} = 0$

$x = 0$

y-intercept: $f(0) = 0$

$y = 0$

Vertical asymptotes: $x^2 - 4 = 0$

$x = \pm 2$

Horizontal asymptote: $y = 0$

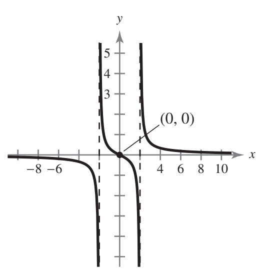
No critical numbers: $y' \neq 0$

$y'' = 0$

$\dfrac{2x(x^2 + 12)}{(x^2 - 4)^3} = 0$

$x = 0$

Point of inflection: $(0, 0)$



22. $y = 5x^2 - 3$

$\dfrac{dy}{dx} = 10x$

$dy = 10x\,dx$

23. $y = \dfrac{1 - x}{x + 3}$

$\dfrac{dy}{dx} = \dfrac{(x + 3)(-1) - (1 - x)}{(x + 3)^2} = -\dfrac{4}{(x + 3)^2}$

$dy = -\dfrac{4}{(x + 3)^2}\,dx$

24. $y = (x + 4)^3$

$\dfrac{dy}{dx} = 3(x + 4)^2$

$dy = 3(x + 4)^2\,dx$

25. $p = 280 - 0.4x,\ 0 \le x \le 700$

$\dfrac{dp}{dx} = -0.4$

$\eta = \dfrac{p/x}{dp/dx} = \dfrac{\dfrac{280 - 0.4x}{x}}{0.4} = \dfrac{-700}{x} + 1 = \dfrac{x - 700}{x}$

$|\eta| = 1 = \left|\dfrac{x - 700}{x}\right| \Rightarrow x = |x - 700| \Rightarrow x = 350$

For $0 \le x < 350$, $|\eta| > 1$ and demand is elastic.

For $350 < x \le 700$, $|\eta| < 1$ and demand is inelastic.

For $x = 350$, $|\eta| = 1$ and demand is of unit elasticity.