Solutions — Chapter 2

2.1.1. *Commutativity of Addition*:

$$(x + \mathrm{i}\,y) + (u + \mathrm{i}\,v) = (x + u) + \mathrm{i}\,(y + v) = (u + \mathrm{i}\,v) + (x + \mathrm{i}\,y).$$

Associativity of Addition:

$$\begin{aligned}(x + \mathrm{i}\,y) + \big[\,(u + \mathrm{i}\,v) + (p + \mathrm{i}\,q)\,\big] &= (x + \mathrm{i}\,y) + \big[\,(u + p) + \mathrm{i}\,(v + q)\,\big] \\ &= (x + u + p) + \mathrm{i}\,(y + v + q) \\ &= \big[\,(x + u) + \mathrm{i}\,(y + v)\,\big] + (p + \mathrm{i}\,q) = \big[\,(x + \mathrm{i}\,y) + (u + \mathrm{i}\,v)\,\big] + (p + \mathrm{i}\,q).\end{aligned}$$

Additive Identity: $\mathbf{0} = 0 = 0 + \mathrm{i}\,0$ and

$$(x + \mathrm{i}\,y) + 0 = x + \mathrm{i}\,y = 0 + (x + \mathrm{i}\,y).$$

Additive Inverse: $-(x + \mathrm{i}\,y) = (-x) + \mathrm{i}\,(-y)$ and

$$(x + \mathrm{i}\,y) + \big[\,(-x) + \mathrm{i}\,(-y)\,\big] = 0 = \big[\,(-x) + \mathrm{i}\,(-y)\,\big] + (x + \mathrm{i}\,y).$$

Distributivity:

$$(c + d)\,(x + \mathrm{i}\,y) = (c + d)\,x + \mathrm{i}\,(c + d)\,y = (c\,x + d\,x) + \mathrm{i}\,(c\,y + d\,y) = c\,(x + \mathrm{i}\,y) + d\,(x + \mathrm{i}\,y),$$

$$c[\,(x + \mathrm{i}\,y) + (u + \mathrm{i}\,v)\,] = c\,(x + u) + \,(y + v) = (c\,x + c\,u) + \mathrm{i}\,(c\,y + c\,v) = c\,(x + \mathrm{i}\,y) + c\,(u + \mathrm{i}\,v).$$

Associativity of Scalar Multiplication:

$$c\,[\,d\,(x + \mathrm{i}\,y)\,] = c\,[\,(d\,x) + \mathrm{i}\,(d\,y)\,] = (c\,d\,x) + \mathrm{i}\,(c\,d\,y) = (c\,d)\,(x + \mathrm{i}\,y).$$

Unit for Scalar Multiplication: $1\,(x + \mathrm{i}\,y) = (1\,x) + \mathrm{i}\,(1\,y) = x + \mathrm{i}\,y$.

Note: Identifying the complex number $x + \mathrm{i}\,y$ with the vector $(\,x, y\,)^T \in \mathbb{R}^2$ respects the operations of vector addition and scalar multiplication, and so we are in effect reproving that $\mathbb{R}^2$ is a vector space.

2.1.2. *Commutativity of Addition*:

$$(x_1, y_1) + (x_2, y_2) = (x_1\,x_2, y_1\,y_2) = (x_2, y_2) + (x_1, y_1).$$

Associativity of Addition:

$$(x_1, y_1) + \big[\,(x_2, y_2) + (x_3, y_3)\,\big] = (x_1\,x_2\,x_3, y_1\,y_2\,y_3) = \big[\,(x_1, y_1) + (x_2, y_2)\,\big] + (x_3, y_3).$$

Additive Identity: $\mathbf{0} = (1, 1)$, and

$$(x, y) + (1, 1) = (x, y) = (1, 1) + (x, y).$$

Additive Inverse:

$$-(x, y) = \left(\frac{1}{x}, \frac{1}{y}\right) \qquad \text{and} \qquad (x, y) + \big[\,-(x, y)\,\big] = (1, 1) = \big[\,-(x, y)\,\big] + (x, y).$$

Distributivity:

$$\begin{aligned}(c + d)\,(x, y) &= (x^{c+d}, y^{c+d}) = (x^c\,x^d, y^c\,y^d) = (x^c, y^c) + (x^d, y^d) = c\,(x, y) + d\,(x, y) \\ c\big[\,(x_1, y_1) + (x_2, y_2)\,\big] &= ((x_1\,x_2)^c, (y_1\,y_2)^c) = (x_1^c\,x_2^c, y_1^c\,y_2^c) \\ &= (x_1^c, y_1^c) + (x_2^c, y_2^c) = c\,(x_1, y_1) + c\,(x_2, y_2).\end{aligned}$$

Associativity of Scalar Multiplication:

$$c\,(d\,(x, y)) = c\,(x^d, y^d) = (x^{c\,d}, y^{c\,d}) = (c\,d)\,(x, y).$$

Unit for Scalar Multiplication: $1\,(x, y) = (x, y)$.

Note: We can uniquely identify a point $(x,y)\in Q$ with the vector $(\log x,\log y)^T\in\mathbb{R}^2$. Then the indicated operations agree with standard vector addition and scalar multiplication in $\mathbb{R}^2$, and so Q is just a disguised version of $\mathbb{R}^2$.

♢ 2.1.3. We denote a typical function in $\mathcal{F}(S)$ by $f(x)$ for $x\in S$.

Commutativity of Addition:

$$(f+g)(x)=f(x)+g(x)=(f+g)(x).$$

Associativity of Addition:

$$[f+(g+h)](x)=f(x)+(g+h)(x)=f(x)+g(x)+h(x)=(f+g)(x)+h(x)=[(f+g)+h](x).$$

Additive Identity: $0(x)=0$ for all x, and $(f+0)(x)=f(x)=(0+f)(x)$.

Additive Inverse: $(-f)(x)=-f(x)$ and

$$[f+(-f)](x)=f(x)+(-f)(x)=0=(-f)(x)+f(x)=[(-f)+f](x).$$

Distributivity:

$$[(c+d)f](x)=(c+d)f(x)=cf(x)+df(x)=(cf)(x)+(df)(x),$$
$$[c(f+g)](x)=cf(x)+cg(x)=(cf)(x)+(cg)(x).$$

Associativity of Scalar Multiplication:

$$[c(df)](x)=cdf(x)=[(cd)f](x).$$

Unit for Scalar Multiplication: $(1f)(x)=f(x)$.

2.1.4. (a) $(1,1,1,1)^T,\ (1,-1,1,-1)^T,\ (1,1,1,1)^T,\ (1,-1,1,-1)^T$. (b) Obviously not.

2.1.5. One example is $f(x)\equiv 0$ and $g(x)=x^3-x$.

2.1.6. (a) $f(x)=-4x+3$; (b) $f(x)=-2x^2-x+1$.

2.1.7.

(a) $\begin{pmatrix}x-y\\xy\end{pmatrix}$, $\begin{pmatrix}e^x\\\cos y\end{pmatrix}$, and $\begin{pmatrix}1\\3\end{pmatrix}$, which is a constant function.

(b) Their sum is $\begin{pmatrix}x-y+e^x+1\\xy+\cos y+3\end{pmatrix}$. Multiplied by -5 is $\begin{pmatrix}-5x+5y-5e^x-5\\-5xy-5\cos y-15\end{pmatrix}$.

(c) The zero element is the constant function $\mathbf{0}=\begin{pmatrix}0\\0\end{pmatrix}$.

♢ 2.1.8. This is the same as the space of functions $\mathcal{F}(\mathbb{R}^2,\mathbb{R}^2)$. Explicitly:

Commutativity of Addition:

$$\begin{pmatrix}v_1(x,y)\\v_2(x,y)\end{pmatrix}+\begin{pmatrix}w_1(x,y)\\w_2(x,y)\end{pmatrix}=\begin{pmatrix}v_1(x,y)+w_1(x,y)\\v_2(x,y)+w_2(x,y)\end{pmatrix}=\begin{pmatrix}w_1(x,y)\\w_2(x,y)\end{pmatrix}+\begin{pmatrix}v_1(x,y)\\v_2(x,y)\end{pmatrix}.$$

Associativity of Addition:

$$\begin{pmatrix}u_1(x,y)\\u_2(x,y)\end{pmatrix}+\left[\begin{pmatrix}v_1(x,y)\\v_2(x,y)\end{pmatrix}+\begin{pmatrix}w_1(x,y)\\w_2(x,y)\end{pmatrix}\right]=\begin{pmatrix}u_1(x,y)+v_1(x,y)+w_1(x,y)\\u_2(x,y)+v_2(x,y)+w_2(x,y)\end{pmatrix}$$
$$=\left[\begin{pmatrix}u_1(x,y)\\u_2(x,y)\end{pmatrix}+\begin{pmatrix}v_1(x,y)\\v_2(x,y)\end{pmatrix}\right]+\begin{pmatrix}w_1(x,y)\\w_2(x,y)\end{pmatrix}.$$

Additive Identity: $\mathbf{0}=(0,0)$ for all x,y, and

$$\begin{pmatrix}v_1(x,y)\\v_2(x,y)\end{pmatrix}+\mathbf{0}=\begin{pmatrix}v_1(x,y)\\v_2(x,y)\end{pmatrix}=\mathbf{0}+\begin{pmatrix}v_1(x,y)\\v_2(x,y)\end{pmatrix}.$$

Additive Inverse: $-\begin{pmatrix}v_1(x,y)\\v_2(x,y)\end{pmatrix}=\begin{pmatrix}-v_1(x,y)\\-v_2(x,y)\end{pmatrix}$, and

$$\begin{pmatrix}v_1(x,y)\\v_2(x,y)\end{pmatrix}+\begin{pmatrix}-v_1(x,y)\\-v_2(x,y)\end{pmatrix}=\mathbf{0}=\begin{pmatrix}-v_1(x,y)\\-v_2(x,y)\end{pmatrix}+\begin{pmatrix}v_1(x,y)\\v_2(x,y)\end{pmatrix}.$$

For the exclusive use of adopters of the book Applied Linear Algebra, by Peter J. Olver and Cheri Shakiban. ISBN 0-13-147382-4.

Distributivity:

$$(c+d)\begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix} = \begin{pmatrix} (c+d)\,v_1(x,y) \\ (c+d)\,v_2(x,y) \end{pmatrix} = c\begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix} + d\begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix},$$

$$c\left[\begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix} + \begin{pmatrix} w_1(x,y) \\ w_2(x,y) \end{pmatrix}\right] = \begin{pmatrix} c\,v_1(x,y) + c\,w_1(x,y) \\ c\,v_2(x,y) + c\,w_2(x,y) \end{pmatrix} = c\begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix} + c\begin{pmatrix} w_1(x,y) \\ w_2(x,y) \end{pmatrix}.$$

Associativity of Scalar Multiplication:

$$c\left[d\begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix}\right] = \begin{pmatrix} c\,d\,v_1(x,y) \\ c\,d\,v_2(x,y) \end{pmatrix} = (c\,d)\begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix}.$$

Unit for Scalar Multiplication:

$$1\begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix} = \begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix}.$$

♡ 2.1.9. We identify each sample value with the matrix entry $m_{ij} = f(i\,h, j\,k)$. In this way, every sampled function corresponds to a uniquely determined $m \times n$ matrix and conversely. Addition of sample functions, $(f+g)(i\,h, j\,k) = f(i\,h, j\,k) + g(i\,h, j\,k)$ corresponds to matrix addition, $m_{ij} + n_{ij}$, while scalar multiplication of sample functions, $c\,f(i\,h, j\,k)$, corresponds to scalar multiplication of matrices, $c\,m_{ij}$.

2.1.10. $\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots\,)$, $c\,\mathbf{a} = (c\,a_1, c\,a_2, c\,a_3, \ldots\,)$. Explicity verification of the vector space properties is straightforward. An alternative, smarter strategy is to identify $\mathbb{R}^\infty$ as the space of functions $f\colon \mathbb{N} \to \mathbb{R}$ where $\mathbb{N} = \{1, 2, 3, \ \ldots\ \}$ is the set of natural numbers and we identify the function f with its sample vector $\mathbf{f} = (f(1), f(2), \ldots\,)$.

2.1.11. (*i*) $\mathbf{v} + (-1)\mathbf{v} = 1\,\mathbf{v} + (-1)\mathbf{v} = \big(1 + (-1)\big)\mathbf{v} = 0\,\mathbf{v} = \mathbf{0}$.

(*j*) Let $\mathbf{z} = c\,\mathbf{0}$. Then $\mathbf{z} + \mathbf{z} = c\,(\mathbf{0} + \mathbf{0}) = c\,\mathbf{0} = \mathbf{z}$, and so, as in the proof of (*h*), $\mathbf{z} = \mathbf{0}$.

(*k*) Suppose $c \neq \mathbf{0}$. Then $\mathbf{v} = 1\,\mathbf{v} = \left(\dfrac{1}{c}\cdot c\right)\mathbf{v} = \dfrac{1}{c}\,(c\,\mathbf{v}) = \dfrac{1}{c}\,\mathbf{0} = \mathbf{0}$.

◇ 2.1.12. If $\mathbf{0}$ and $\widetilde{\mathbf{0}}$ both satisfy axiom (*c*), then $\mathbf{0} = \widetilde{\mathbf{0}} + \mathbf{0} = \mathbf{0} + \widetilde{\mathbf{0}} = \widetilde{\mathbf{0}}$.

◇ 2.1.13. *Commutativity of Addition*:

$$(\mathbf{v}, \mathbf{w}) + (\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) = (\mathbf{v} + \widehat{\mathbf{v}}, \mathbf{w} + \widehat{\mathbf{w}}) = (\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) + (\mathbf{v}, \mathbf{w}).$$

Associativity of Addition:

$$(\mathbf{v}, \mathbf{w}) + \big[(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) + (\widetilde{\mathbf{v}}, \widetilde{\mathbf{w}})\big] = (\mathbf{v} + \widehat{\mathbf{v}} + \widetilde{\mathbf{v}}, \mathbf{w} + \widehat{\mathbf{w}} + \widetilde{\mathbf{w}}) = \big[(\mathbf{v}, \mathbf{w}) + (\widehat{\mathbf{v}}, \widehat{\mathbf{w}})\big] + (\widetilde{\mathbf{v}}, \widetilde{\mathbf{w}}).$$

Additive Identity: the zero element is $(\mathbf{0}, \mathbf{0})$, and

$$(\mathbf{v}, \mathbf{w}) + (\mathbf{0}, \mathbf{0}) = (\mathbf{v}, \mathbf{w}) = (\mathbf{0}, \mathbf{0}) + (\mathbf{v}, \mathbf{w}).$$

Additive Inverse: $-(\mathbf{v}, \mathbf{w}) = (-\mathbf{v}, -\mathbf{w})$ and

$$(\mathbf{v}, \mathbf{w}) + (-\mathbf{v}, -\mathbf{w}) = (\mathbf{0}, \mathbf{0}) = (-\mathbf{v}, -\mathbf{w}) + (\mathbf{v}, \mathbf{w}).$$

Distributivity:

$$(c+d)\,(\mathbf{v}, \mathbf{w}) = ((c+d)\,\mathbf{v}, (c+d)\,\mathbf{w}) = c\,(\mathbf{v}, \mathbf{w}) + d\,(\mathbf{v}, \mathbf{w}),$$

$$c\big[(\mathbf{v}, \mathbf{w}) + (\widehat{\mathbf{v}}, \widehat{\mathbf{w}})\big] = (c\,\mathbf{v} + c\,\widehat{\mathbf{v}}, c\,\mathbf{v} + c\,\widehat{\mathbf{w}}) = c\,(\mathbf{v}, \mathbf{w}) + c\,(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}).$$

Associativity of Scalar Multiplication:

$$c\,(d\,(\mathbf{v}, \mathbf{w})) = (c\,d\,\mathbf{v}, c\,d\,\mathbf{w}) = (c\,d)\,(\mathbf{v}, \mathbf{w}).$$

Unit for Scalar Multiplication: $1\,(\mathbf{v}, \mathbf{w}) = (1\,\mathbf{v}, 1\,\mathbf{w}) = (\mathbf{v}, \mathbf{w})$.

2.1.14. Here $V = \mathrm{C}^0$ while $W = \mathbb{R}$, and so the indicated pairs belong to the Cartesian product vector space $\mathrm{C}^0 \times \mathbb{R}$. The zero element is the pair $\mathbf{0} = (0, 0)$ where the first 0 denotes the identically zero function, while the second 0 denotes the real number zero. The laws of vector addition and scalar multiplication are

$$(f(x), a) + (g(x), b) = (f(x) + g(x), a + b), \qquad c\,(f(x), a) = (c\,f(x), c\,a).$$

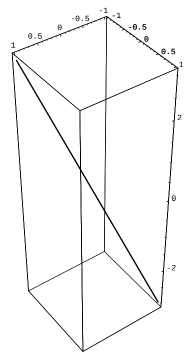
2.2.1.

(*a*) If $\mathbf{v} = (\,x, y, z\,)^T$ satisfies $x - y + 4z = 0$ and $\widetilde{\mathbf{v}} = (\,\widetilde{x}, \widetilde{y}, \widetilde{z}\,)^T$ also satisfies $\widetilde{x} - \widetilde{y} + 4\widetilde{z} = 0$, so does $\mathbf{v} + \widetilde{\mathbf{v}} = (\,x + \widetilde{x}, y + \widetilde{y}, z + \widetilde{z}\,)^T$ since $(x + \widetilde{x}) - (y + \widetilde{y}) + 4(z + \widetilde{z}) = (x - y + 4z) + (\widetilde{x} - \widetilde{y} + 4\widetilde{z}) = 0$, as does $c\mathbf{v} = (\,cx, cy, cz\,)^T$ since $(cx) - (cy) + 4(cz) = c(x - y + 4z) = 0$.

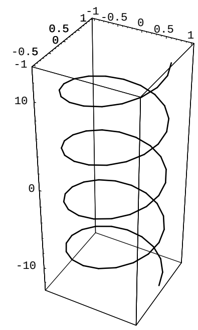
(*b*) For instance, the zero vector $\mathbf{0} = (\,0, 0, 0\,)^T$ does not satisfy the equation.

2.2.2. (*b*,*c*,*d*,*g*,*i*) are subspaces; the rest are not. Case (*j*) consists of the 3 coordinate axes and the line $x = y = z$.

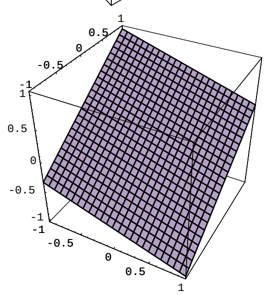
2.2.3. (*a*) Subspace:



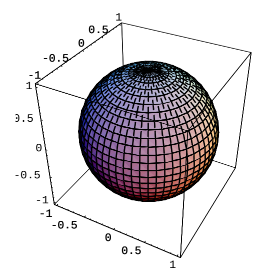
(*b*) Not a subspace:



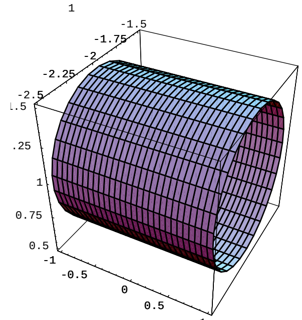
(*c*) Subspace:



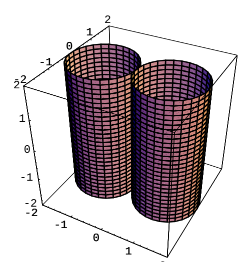
(*d*) Not a subspace:



(*e*) Not a subspace:



(*f*) Even though the cylinders are not subspaces, their intersection is the z axis, which is a subspace:



2.2.4. Any vector of the form $a\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + b\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + c\begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} a + 2b \\ 2a - c \\ -a + b + 3c \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ will belong to W. The coefficient matrix $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & -1 \\ -1 & 1 & 3 \end{pmatrix}$ is nonsingular, and so for any

$\mathbf{x} = (x, y, z)^T \in \mathbb{R}^3$ we can arrange suitable values of a, b, c by solving the linear system. Thus, every vector in $\mathbb{R}^3$ belongs to W and so $W = \mathbb{R}^3$.

2.2.5. False, with two exceptions: $[0,0] = \{0\}$ and $(-\infty, \infty) = \mathbb{R}$.

2.2.6.
(a) Yes. For instance, the set $S = \{(x,0\} \cup \{(0,y)\}$ consisting of the coordinate axes has the required property, but is not a subspace. More generally, any (finite) collection of 2 or more lines going through the origin satisfies the property, but is not a subspace.
(b) For example, $S = \{(x,y) \mid x, y \geq 0\}$ — the positive quadrant.

2.2.7. (*a*,*c*,*d*) are subspaces; (*b*,*e*) are not.

2.2.8. Since $\mathbf{x} = \mathbf{0}$ must belong to the subspace, this implies $\mathbf{b} = A\mathbf{0} = \mathbf{0}$. For a homogeneous system, if $\mathbf{x}, \mathbf{y}$ are solutions, so $A\mathbf{x} = \mathbf{0} = A\mathbf{y}$, so are $\mathbf{x}+\mathbf{y}$ since $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0}$, as is $c\mathbf{x}$ since $A(c\mathbf{x}) = cA\mathbf{x} = \mathbf{0}$.

2.2.9. L and M are strictly lower triangular if $l_{ij} = 0 = m_{ij}$ whenever $i \leq j$. Then $N = L + M$ is strictly lower triangular since $n_{ij} = l_{ij} + m_{ij} = 0$ whenever $i \leq j$, as is $K = cL$ since $k_{ij} = c\,l_{ij} = 0$ whenever $i \leq j$.

◇ 2.2.10. Note $\operatorname{tr}(A+B) = \sum_{i=1}^{n} (a_{ii} + b_{ii}) = \operatorname{tr} A + \operatorname{tr} B$ and $\operatorname{tr}(cA) = \sum_{i=1}^{n} c\,a_{ii} = c \sum_{i=1}^{n} a_{ii} = c \operatorname{tr} A$. Thus, if $\operatorname{tr} A = \operatorname{tr} B = 0$, then $\operatorname{tr}(A+B) = 0 = \operatorname{tr}(cA)$, proving closure.

2.2.11.
(a) No. The zero matrix is not an element.
(b) No if $n \geq 2$. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ satisfy $\det A = 0 = \det B$, but $\det(A+B) = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$, so $A + B$ does not belong to the set.

2.2.12. (*d*,*f*,*g*,*h*) are subspaces; the rest are not.

2.2.13. (*a*) Vector space; (*b*) not a vector space: $(0,0)$ does not belong; (*c*) vector space; (*d*) vector space; (*e*) not a vector space: If f is non-negative, then $-1\,f = -f$ is not (unless $f \equiv 0$); (*f*) vector space; (*g*) vector space; (*h*) vector space.

2.2.14. If $f(1) = 0 = g(1)$, then $(f+g)(1) = 0$ and $(cf)(1) = 0$, so both $f + g$ and cf belong to the subspace. The zero function does not satisfy $f0) = 1$. For a subspace, a can be anything, while $b = 0$.

2.2.15. All cases except (*e*,*g*) are subspaces. In (*g*), $|\,x\,|$ is not in C^1.

2.2.16. (*a*) Subspace; (*b*) subspace; (*c*) Not a subspace: the zero function does not satisfy the condition; (*d*) Not a subspace: if $f(0) = 0$, $f(1) = 1$, and $g(0) = 1$, $g(1) = 0$, then f and g are in the set, but $f + g$ is not; (*e*) subspace; (*f*) Not a subspace: the zero function does not satisfy the condition; (*g*) subspace; (*h*) subspace; (*i*) Not a subspace: the zero function does not satisfy the condition.

2.2.17. If $u'' = xu$, $v'' = xv$, are solutions, and c, d constants, then $(cu + dv)'' = cu'' + dv'' = cxu + dxv = x(cu + dv)$, and hence $cu + dv$ is also a solution.

2.2.18. For instance, the zero function $u(x) \equiv 0$ is not a solution.

2.2.19.
(a) It is a subspace of the space of all functions $\mathbf{f}\colon [a,b] \to \mathbb{R}^2$, which is a particular instance of Example 2.7. Note that $\mathbf{f}(t) = (f_1(t), f_2(t))^T$ is continuously differentiable if and

only if its component functions $f_1(t)$ and $f_2(t)$ are. Thus, if $\mathbf{f}(t) = (f_1(t), f_2(t))^T$ and $\mathbf{g}(t) = (g_1(t), g_2(t))^T$ are continuously differentiable, so are
$$(\mathbf{f}+\mathbf{g})(t) = (f_1(t)+g_1(t), f_2(t)+g_2(t))^T \text{ and } (c\mathbf{f})(t) = (c f_1(t), c f_2(t))^T.$$
(*b*) Yes: if $\mathbf{f}(0) = \mathbf{0} = \mathbf{g}(0)$, then $(c\mathbf{f} + d\mathbf{g})(0) = \mathbf{0}$ for any $c, d \in \mathbb{R}$.

2.2.20. $\nabla \cdot (c\mathbf{v} + d\mathbf{w}) = c\nabla\cdot\mathbf{v} + d\nabla\cdot\mathbf{w} = 0$ whenever $\nabla\cdot\mathbf{v} = \nabla\cdot\mathbf{w} = 0$ and $c, d, \in \mathbb{R}$.

2.2.21. Yes. The sum of two convergent sequences is convergent, as is any constant multiple of a convergent sequence.

2.2.22.
(*a*) If $\mathbf{v}, \mathbf{w} \in W \cap Z$, then $\mathbf{v}, \mathbf{w} \in W$, so $c\mathbf{v} + d\mathbf{w} \in W$ because W is a subspace, and $\mathbf{v}, \mathbf{w} \in Z$, so $c\mathbf{v} + d\mathbf{w} \in Z$ because Z is a subspace, hence $c\mathbf{v} + d\mathbf{w} \in W \cap Z$.
(*b*) If $\mathbf{w}+\mathbf{z}, \widetilde{\mathbf{w}}+\widetilde{\mathbf{z}} \in W + Z$ then $c(\mathbf{w}+\mathbf{z}) + d(\widetilde{\mathbf{w}}+\widetilde{\mathbf{z}}) = (c\mathbf{w} + d\widetilde{\mathbf{w}}) + (c\mathbf{z} + d\widetilde{\mathbf{z}}) \in W + Z$, since it is the sum of an element of W and an element of Z.
(*c*) Given any $\mathbf{w} \in W$ and $\mathbf{z} \in Z$, then $\mathbf{w}, \mathbf{z} \in W \cup Z$. Thus, if $W \cup Z$ is a subspace, the sum $\mathbf{w} + \mathbf{z} \in W \cup Z$. Thus, either $\mathbf{w} + \mathbf{z} = \widetilde{\mathbf{w}} \in W$ or $\mathbf{w} + \mathbf{z} = \widetilde{\mathbf{z}} \in Z$. In the first case $\mathbf{z} = \widetilde{\mathbf{w}} - \mathbf{w} \in W$, while in the second $\mathbf{w} = \widetilde{\mathbf{z}} - \mathbf{z} \in Z$. We conclude that for any $\mathbf{w} \in W$ and $\mathbf{z} \in Z$, either $\mathbf{w} \in Z$ or $\mathbf{z} \in W$. Suppose $W \not\subset Z$. Then we can find $\mathbf{w} \in W \setminus Z$, and so for any $\mathbf{z} \in Z$, we must have $\mathbf{z} \in W$, which proves $Z \subset W$.

♢ 2.2.23. If $\mathbf{v}, \mathbf{w} \in \bigcap W_i$, then $\mathbf{v}, \mathbf{w} \in W_i$ for each i and so $c\mathbf{v} + d\mathbf{w} \in W_i$ for any $c, d \in \mathbb{R}$ because W_i is a subspace. Since this holds for all i, we conclude that $c\mathbf{v} + d\mathbf{w} \in \bigcap W_i$.

♡ 2.2.24.
(*a*) They clearly only intersect at the origin. Moreover, every $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix}$ can be written as a sum of vectors on the two axes.
(*b*) Since the only common solution to $x = y$ and $x = 3y$ is $x = y = 0$, the lines only intersect at the origin. Moreover, every $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix} + \begin{pmatrix} 3b \\ b \end{pmatrix}$, where $a = -\frac{1}{2}x + \frac{3}{2}y$, $b = \frac{1}{2}x - \frac{1}{2}y$, can be written as a sum of vectors on each line.
(*c*) A vector $\mathbf{v} = (a, 2a, 3a)^T$ in the line belongs to the plane if and only if $a + 2(2a) + 3(3a) = 14a = 0$, so $a = 0$ and the only common element is $\mathbf{v} = \mathbf{0}$. Moreover, every $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{14}\begin{pmatrix} x + 2y + 3z \\ 2(x + 2y + 3z) \\ 3(x + 2y + 3z) \end{pmatrix} + \frac{1}{14}\begin{pmatrix} 13x - 2y - 3z \\ -2x + 10y - 6z \\ -3x - 6y + 5z \end{pmatrix}$ can be written as a sum of a vector in the line and a vector in the plane.
(*d*) If $\mathbf{w} + \mathbf{z} = \widetilde{\mathbf{w}} + \widetilde{\mathbf{z}}$, then $\mathbf{w} - \widetilde{\mathbf{w}} = \widetilde{\mathbf{z}} - \mathbf{z}$. The left hand side belongs to W, while the right hand side belongs to Z, and so, by the first assumption, they must both be equal to $\mathbf{0}$. Therefore, $\mathbf{w} = \widetilde{\mathbf{w}}$, $\mathbf{z} = \widetilde{\mathbf{z}}$.

2.2.25.
(*a*) $(\mathbf{v}, \mathbf{w}) \in V_0 \cap W_0$ if and only if $(\mathbf{v}, \mathbf{w}) = (\mathbf{v}, \mathbf{0})$ and $(\mathbf{v}, \mathbf{w}) = (\mathbf{0}, \mathbf{w})$, which means $\mathbf{v} = \mathbf{0}, \mathbf{w} = \mathbf{0}$, and hence $(\mathbf{v}, \mathbf{w}) = (\mathbf{0}, \mathbf{0})$ is the only element of the intersection. Moreover, we can write any element $(\mathbf{v}, \mathbf{w}) = (\mathbf{v}, \mathbf{0}) + (\mathbf{0}, \mathbf{w})$.
(*b*) $(\mathbf{v}, \mathbf{w}) \in D \cap A$ if and only if $\mathbf{v} = \mathbf{w}$ and $\mathbf{v} = -\mathbf{w}$, hence $(\mathbf{v}, \mathbf{w}) = (\mathbf{0}, \mathbf{0})$. Moreover, we can write $(\mathbf{v}, \mathbf{w}) = (\frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w}, \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w}) + (\frac{1}{2}\mathbf{v} - \frac{1}{2}\mathbf{w}, -\frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w})$ as the sum of an element of D and an element of A.

2.2.26.
(*a*) If $f(-x) = f(x)$, $\widetilde{f}(-x) = \widetilde{f}(x)$, then $(cf + d\widetilde{f})(-x) = cf(-x) + d\widetilde{f}(-x) = cf(x) + d\widetilde{f}(x) = (cf + d\widetilde{f})(x)$ for any $c, d, \in \mathbb{R}$, and hence it is a subspace.
(*b*) If $g(-x) = -g(x)$, $\widetilde{g}(-x) = -\widetilde{g}(x)$, then $(cg + d\widetilde{g})(-x) = cg(-x) + d\widetilde{g}(-x) = -cg(x) - d\widetilde{g}(x) = -(cg + d\widetilde{g})(x)$, proving it is a subspace. If $f(x)$ is both even and

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odd, then $f(x) = f(-x) = -f(x)$ and so $f(x) \equiv 0$ for all x. Moreover, we can write any function $h(x) = f(x) + g(x)$ as a sum of an even function $f(x) = \frac{1}{2}\big[\, h(x) + h(-x) \,\big]$ and an odd function $g(x) = \frac{1}{2}\big[\, h(x) - h(-x) \,\big]$.

(c) This follows from part (b), and the uniqueness follows from Exercise 2.2.24(d).

2.2.27. If $A = A^T$ and $A = -A^T$ is both symmetric and skew-symmetric, then $A = \mathrm{O}$. Given any square matrix, write $A = S + J$ where $S = \frac{1}{2}\left(A + A^T \right)$ is symmetric and $J = \frac{1}{2}\left(A - A^T \right)$ is skew-symmetric. This verifies the two conditions for complementary subspaces. Uniqueness of the decomposition $A = S + J$ follows from Exercise 2.2.24(d).

♢ 2.2.28.

(a) By induction, we can show that

$$f^{(n)}(x) = P_n\left(\frac{1}{x}\right) e^{-1/x} = Q_n(x)\frac{e^{-1/x}}{x^n},$$

where $P_n(y)$ and $Q_n(x) = x^n P_n(1/x)$ are certain polynomials of degree n. Thus,

$$\lim_{x \to 0} f^{(n)}(x) = \lim_{x \to 0} Q_n(x)\frac{e^{-1/x}}{x^n} = Q_n(0) \lim_{y \to \infty} y^n\, e^{-y} = 0,$$

because the exponential e^{-y} goes to zero faster than any power of y goes to ∞.

(b) The Taylor series at $a = 0$ is $0 + 0x + 0x^2 + \cdots \equiv 0$, which converges to the zero function, not to $e^{-1/x}$.

2.2.29.

(a) The Taylor series is the geometric series $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$.

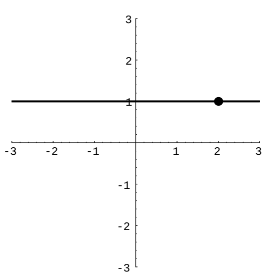
(b) The ratio test can be used to prove that the series converges precisely when $|\,x\,| < 1$.

(c) Convergence of the Taylor series to $f(x)$ for x near 0 suffices to prove analyticity of the function at $x = 0$.

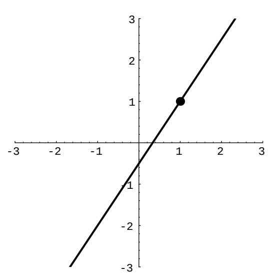
♡ 2.2.30.

(a) If $\mathbf{v}+\mathbf{a}, \mathbf{w}+\mathbf{a} \in A$, then $(\mathbf{v}+\mathbf{a})+(\mathbf{w}+\mathbf{a}) = (\mathbf{v}+\mathbf{w}+\mathbf{a})+\mathbf{a} \in A$ requires $\mathbf{v}+\mathbf{w}+\mathbf{a} = \mathbf{u} \in V$, and hence $\mathbf{a} = \mathbf{u} - \mathbf{v} - \mathbf{w} \in A$.

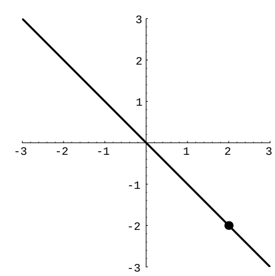
(b) (i)



(ii)



(iii)



(c) Every subspace $V \subset \mathbb{R}^2$ is either a point (the origin), or a line through the origin, or all of $\mathbb{R}^2$. Thus, the corresponding affine subspaces are the point $\{\,\mathbf{a}\,\}$; a line through $\mathbf{a}$, or all of $\mathbb{R}^2$ since in this case $\mathbf{a} \in V = \mathbb{R}^2$.

(d) Every vector in the plane can be written as $(\,x, y, z\,)^T = (\,\widetilde{x}, \widetilde{y}, \widetilde{z}\,)^T + (\,1, 0, 0\,)^T$ where $(\,\widetilde{x}, \widetilde{y}, \widetilde{z}\,)^T$ is an arbitrary vector in the subspace defined by $\widetilde{x} - 2\widetilde{y} + 3\widetilde{x} = 0$.

(e) Every such polynomial can be written as $p(x) = q(x) + 1$ where $q(x)$ is any element of the subspace of polynomials that satisfy $q(1) = 0$.

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2.3.1. $\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}$.

2.3.2. $\begin{pmatrix} -3 \\ 7 \\ 6 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ -3 \\ -2 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} -2 \\ 6 \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \\ 6 \\ -7 \end{pmatrix}$.

2.3.3.

(*a*) Yes, since $\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - 3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$;

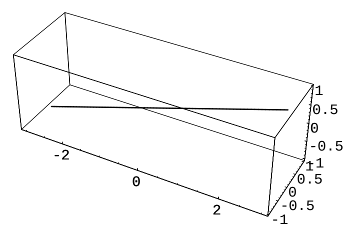
(*b*) Yes, since $\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \frac{3}{10} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \frac{7}{10} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} - \frac{4}{10} \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$;

(*c*) No, since the vector equation $\begin{pmatrix} 3 \\ 0 \\ -1 \\ -2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \\ 3 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ does not have a solution.

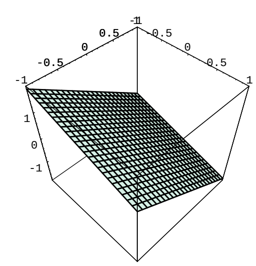
2.3.4. Cases (*b*), (*c*), (*e*) span $\mathbb{R}^2$.

2.3.5.

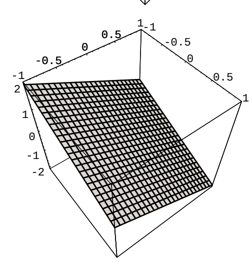
(*a*) The line $(3t, 0, t)^T$:



(*b*) The plane $z = -\frac{3}{5}x - \frac{6}{5}y$:



(*c*) The plane $z = -x - y$:



2.3.6. They are the same. Indeed, since $\mathbf{v}_1 = \mathbf{u}_1 + 2\mathbf{u}_2$, $\mathbf{v}_2 = \mathbf{u}_1 + \mathbf{u}_2$, every vector $\mathbf{v} \in V$ can be written as a linear combination $\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 = (c_1 + c_2)\mathbf{u}_1 + (2c_1 + c_2)\mathbf{u}_2$ and hence belongs to U. Conversely, since $\mathbf{u}_1 = -\mathbf{v}_1 + 2\mathbf{v}_2$, $\mathbf{u}_2 = \mathbf{v}_1 - \mathbf{v}_2$, every vector $\mathbf{u} \in U$ can be written as a linear combination $\mathbf{u} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 = (-c_1 + c_2)\mathbf{v}_1 + (2c_1 - c_2)\mathbf{v}_2$, and hence belongs to U.

2.3.7. (a) Every symmetric matrix has the form $\begin{pmatrix} a & b \\ b & c \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(*b*) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$

2.3.8.
(*a*) They span $\mathcal{P}^{(2)}$ since $ax^2+bx+c = \frac{1}{2}(a-2b+c)(x^2+1) + \frac{1}{2}(a-c)(x^2-1) + b(x^2+x+1)$.
(*b*) They span $\mathcal{P}^{(3)}$ since $ax^3+bx^2+cx+d = a(x^3-1)+b(x^2+1)+c(x-1)+(a-b+c+d)1$.
(*c*) They do not span $\mathcal{P}^{(3)}$ since $ax^3+bx^2+cx+d = c_1x^3+c_2(x^2+1)+c_3(x^2-x)+c_4(x+1)$ cannot be solved when $b+c-d \neq 0$.

2.3.9. (*a*) Yes. (*b*) No. (*c*) No. (*d*) Yes: $\cos^2 x = 1-\sin^2 x$. (*e*) No. (*f*) No.

2.3.10. (*a*) $\sin 3x = \cos\left(3x - \frac{1}{2}\pi\right)$; (*b*) $\cos x - \sin x = \sqrt{2}\,\cos\left(x + \frac{1}{4}\pi\right)$,
(*c*) $3\cos 2x + 4\sin 2x = 5\cos\left(2x - \tan^{-1}\frac{4}{3}\right)$, (*d*) $\cos x \sin x = \frac{1}{2}\sin 2x = \frac{1}{2}\cos\left(2x - \frac{1}{2}\pi\right)$.

2.3.11. (*a*) If u_1 and u_2 are solutions, so is $u = c_1u_1 + c_2u_2$ since $u'' - 4u' + 3u = c_1(u_1'' - 4u_1' + 3u_1) + c_2(u_2'' - 4u_2' + 3u_2) = 0$. (*b*) span $\{e^x, e^{3x}\}$; (*c*) 2.

2.3.12. Each is a solution, and the general solution $u(x) = c_1 + c_2\cos x + c_3 \sin x$ is a linear combination of the three independent solutions.

2.3.13. (*a*) e^{2x}; (*b*) $\cos 2x, \sin 2x$; (*c*) $e^{3x}, 1$; (*d*) e^{-x}, e^{-3x}; (*e*) $e^{-x/2}\cos\frac{\sqrt{3}}{2}x$, $e^{-x/2}\sin\frac{\sqrt{3}}{2}x$; (*f*) $e^{5x}, 1, x$; (*g*) $e^{x/\sqrt{2}}\cos\frac{x}{\sqrt{2}}, e^{x/\sqrt{2}}\sin\frac{x}{\sqrt{2}}, e^{-x/\sqrt{2}}\cos\frac{x}{\sqrt{2}}, e^{-x/\sqrt{2}}\sin\frac{x}{\sqrt{2}}$.

2.3.14. (*a*) If u_1 and u_2 are solutions, so is $u = c_1u_1 + c_2u_2$ since $u'' + 4u = c_1(u_1'' + 4u_1) + c_2(u_2'' + 4u_2) = 0$, $u(0) = c_1u_1(0) + c_2u_2(0) = 0$, $u(\pi) = c_1u_1(\pi) + c_2u_2(\pi) = 0$.
(*b*) span $\{\sin 2x\}$

2.3.15. (*a*) $\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\,\mathbf{f}_1(x) + \mathbf{f}_2(x) - \mathbf{f}_3(x)$; (*b*) not in the span; (*c*) $\begin{pmatrix} 1-2x \\ -1-x \end{pmatrix} = \mathbf{f}_1(x) - \mathbf{f}_2(x) - \mathbf{f}_3(x)$; (*d*) not in the span; (*e*) $\begin{pmatrix} 2-x \\ 0 \end{pmatrix} = 2\,\mathbf{f}_1(x) - \mathbf{f}_3(x)$.

2.3.16. True, since $\mathbf{0} = 0\,\mathbf{v}_1 + \cdots + 0\,\mathbf{v}_n$.

2.3.17. False. For example, if $\mathbf{z} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, then $\mathbf{z} = \mathbf{u} + \mathbf{v}$, but the equation $\mathbf{w} = c_1\,\mathbf{u} + c_2\,\mathbf{v} + c_3\,\mathbf{z} = \begin{pmatrix} c_1 + c_3 \\ c_2 + c_3 \\ 0 \end{pmatrix}$ has no solution.

◇ 2.3.18. By the assumption, any $\mathbf{v} \in V$ can be written as a linear combination

$$\mathbf{v} = c_1\,\mathbf{v}_1 + \cdots + c_m\,\mathbf{v}_m = c_1\,\mathbf{v}_1 + \cdots + c_n\,\mathbf{v}_m + 0\,\mathbf{v}_{m+1} + \cdots + 0\,\mathbf{v}_n$$

of the combined collection.

◇ 2.3.19.
(*a*) If $\mathbf{v} = \sum_{j=1}^{m} c_j\,\mathbf{v}_j$ and $\mathbf{v}_j = \sum_{i=1}^{n} a_{ij}\,\mathbf{w}_i$, then $\mathbf{v} = \sum_{i=1}^{n} b_i\,\mathbf{v}_i$ where $b_i = \sum_{j=1}^{m} a_{ij}\,c_j$, or, in vector language, $\mathbf{b} = A\,\mathbf{c}$.
(*b*) Every $\mathbf{v} \in V$ can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$, and hence, by part (a), a linear combination of $\mathbf{w}_1, \ldots, \mathbf{w}_m$, which shows that $\mathbf{w}_1, \ldots, \mathbf{w}_m$ also span V.

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♢ 2.3.20.

(a) If $\mathbf{v} = \sum_{i=1}^{m} a_i \mathbf{v}_i$, $\mathbf{w} = \sum_{i=1}^{n} b_i \mathbf{v}_i$, are two finite linear combinations, so is

$$c\mathbf{v} + d\mathbf{w} = \sum_{i=1}^{\max\{m,n\}} (c a_i + d b_i)\mathbf{v}_i \quad \text{where we set } a_i = 0 \text{ if } i > m \text{ and } b_i = 0 \text{ if } i > n.$$

(b) The space $\mathcal{P}^{(\infty)}$ of all polynomials, since every polynomial is a finite linear combination of monomials and vice versa.

2.3.21. (a) Linearly independent; (b) linearly dependent; (c) linearly dependent; (d) linearly independent; (e) linearly dependent; (f) linearly dependent; (g) linearly dependent; (h) linearly independent; (i) linearly independent.

2.3.22. (a) The only solution to the homogeneous linear system

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 3 \\ -1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ -2 \\ 1 \\ -1 \end{pmatrix} = \mathbf{0} \qquad \text{is} \qquad c_1 = c_2 = c_3 = 0.$$

(b) All but the second lie in the span. (c) $a - c + d = 0$.

2.3.23.

(a) The only solution to the homogeneous linear system

$$A\mathbf{c} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix} = \mathbf{0}$$

with nonsingular coefficient matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$ is $\mathbf{c} = \mathbf{0}$.

(b) Since A is nonsingular, the inhomogeneous linear system

$$\mathbf{v} = A\mathbf{c} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}$$

has a solution $\mathbf{c} = A^{-1}\mathbf{v}$ for any $\mathbf{v} \in \mathbb{R}^4$.

(c)
$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \tfrac{3}{8} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \tfrac{1}{8} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} + \tfrac{3}{4} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} - \tfrac{1}{4} \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}$$

2.3.24. (a) Linearly dependent; (b) linearly dependent; (c) linearly independent; (d) linearly dependent; (e) linearly dependent; (f) linearly independent.

2.3.25. False:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \mathrm{O}.$$

2.3.26. False — the zero vector always belongs to the span.

2.3.27. Yes, when it is the zero vector.

2.3.28. Because $\mathbf{x}, \mathbf{y}$ are linearly independent, $\mathbf{0} = c_1 \mathbf{u} + c_2 \mathbf{v} = (a c_1 + c c_2)\mathbf{x} + (b c_1 + d c_2)\mathbf{y}$ if and only if $a c_1 + c c_2 = 0, b c_1 + d c_2 = 0$. The latter linear system has a nonzero solution $(c_1, c_2) \neq \mathbf{0}$, and so $\mathbf{u}, \mathbf{v}$ are linearly dependent, if and only if the determinant of the coefficient matrix is zero: $\det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = a d - b c = 0$, proving the result. The full collection $\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}$ is linearly dependent since, for example, $a\mathbf{x} + b\mathbf{y} - \mathbf{u} + 0\mathbf{v} = \mathbf{0}$ is a nontrivial linear combination.

2.3.29. The statement is false. For example, any set containing the zero element that does not span V is linearly dependent.

♢ 2.3.30. (*b*) If the only solution to $A\mathbf{c} = \mathbf{0}$ is the trivial one $\mathbf{c} = \mathbf{0}$, then the only linear combination which adds up to zero is the trivial one with $c_1 = \cdots = c_k = 0$, proving linear independence. (*c*) The vector $\mathbf{b}$ lies in the span if and only if $\mathbf{b} = c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k = A\mathbf{c}$ for some $\mathbf{c}$, which implies that the linear system $A\mathbf{c} = \mathbf{b}$ has a solution.

♢ 2.3.31.

(*a*) Since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent,
$$\mathbf{0} = c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k = c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k + 0\mathbf{v}_{k+1} + \cdots + 0\mathbf{v}_n$$
if and only if $c_1 = \cdots = c_k = 0$.

(*b*) This is false. For example, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, are linearly dependent, but the subset consisting of just $\mathbf{v}_1$ is linearly independent.

2.3.32.

(*a*) They are linearly dependent since $(x^2 - 3) + 2(2 - x) - (x - 1)^2 \equiv 0$.

(*b*) They do not span $\mathcal{P}^{(2)}$.

2.3.33. (*a*) Linearly dependent; (*b*) linearly independent; (*c*) linearly dependent; (*d*) linearly independent; (*e*) linearly dependent; (*f*) linearly dependent; (*g*) linearly independent; (*h*) linearly independent; (*i*) linearly independent.

2.3.34. When $x > 0$, we have $f(x) - g(x) \equiv 0$, proving linear dependence. On the other hand, if $c_1 f(x) + c_2 g(x) \equiv 0$ for all x, then at, say $x = 1$, we have $c_1 + c_2 = 0$ while at $x = -1$, we must have $-c_1 + c_2 = 0$, and so $c_1 = c_2 = 0$, proving linear independence.

♡ 2.3.35.

(*a*) $0 = \sum_{i=1}^{k} c_i p_i(x) = \sum_{j=0}^{n} \sum_{i=1}^{k} c_i a_{ij} x^j$ if and only if $\sum_{j=0}^{n} \sum_{i=1}^{k} c_i a_{ij} = 0$, $j = 0, \ldots, n$, or, in matrix notation, $A^T \mathbf{c} = \mathbf{0}$. Thus, the polynomials are linearly independent if and only if the linear system $A^T \mathbf{c} = \mathbf{0}$ has only the trivial solution $\mathbf{c} = \mathbf{0}$ if and only if its $(n+1) \times k$ coefficient matrix has rank $A^T = \operatorname{rank} A = k$.

(*b*) $q(x) = \sum_{j=0}^{n} b_j x^j = \sum_{i=1}^{k} c_i p_i(x)$ if and only if $A^T \mathbf{c} = \mathbf{b}$.

(*c*) $A = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 0 & 1 & 0 \\ 0 & -4 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 4 & -1 \end{pmatrix}$ has rank 4 and so they are linearly dependent.

(*d*) $q(x)$ is not in the span.

♢ 2.3.36. Suppose the linear combination $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n \equiv 0$ for all x. Thus, every real x is a root of $p(x)$, but the Fundamental Theorem of Algebra says this is only possible if $p(x)$ is the zero polynomial with coefficients $c_0 = c_1 = \cdots = c_n = 0$.

♡ 2.3.37.

(a) If $c_1 f_1(x) + \cdots + c_n f_n(x) \equiv 0$, then $c_1 f_1(x_i) + \cdots + c_n f_n(x_i) = 0$ at all sample points, and so $c_1 \mathbf{f}_1 + \cdots + c_n \mathbf{f}_n = \mathbf{0}$. Thus, linear dependence of the functions implies linear dependence of their sample vectors.

(b) Sampling $f_1(x) = 1$ and $f_2(x) = x^2$ at $-1, 1$ produces the linearly dependent sample vectors $\mathbf{f}_1 = \mathbf{f}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(c) Sampling at $0, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{3}{4}\pi, \pi$, leads to the linearly independent sample vectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ 1 \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

2.3.38.

(a) Suppose $c_1 \mathbf{f}_1(t) + \cdots + c_n \mathbf{f}_n(t) \equiv \mathbf{0}$ for all t. Then $c_1 \mathbf{f}_1(t_0) + \cdots + c_n \mathbf{f}_n(t_0) = \mathbf{0}$, and hence, by linear independence of the sample vectors, $c_1 = \cdots = c_n = 0$, which proves linear independence of the functions.

(b) $c_1 \mathbf{f}_1(t) + c_2 \mathbf{f}_1(t) = \begin{pmatrix} 2c_2 t + (c_1 - c_2) \\ 2c_2 t^2 + (c_1 - c_2)t \end{pmatrix} \equiv \mathbf{0}$ if and only if $c_2 = 0$, $c_1 - c_2 = 0$, and so $c_1 = c_2 = 0$, proving linear independence. However, at any t_0, the vectors $\mathbf{f}_2(t_0) = (2t_0 - 1)\mathbf{f}_1(t_0)$ are scalar multiples of each other, and hence linearly dependent.

♡ 2.3.39.

(a) Suppose $c_1 f(x) + c_2 g(x) \equiv 0$ for all x for some $\mathbf{c} = (c_1, c_2)^T \neq \mathbf{0}$. Differentiating, we find $c_1 f'(x) + c_2 g'(x) \equiv 0$ also, and hence $\begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbf{0}$ for all x. The homogeneous system has a nonzero solution if and only if the coefficient matrix is singular, which requires its determinant $W[f(x), g(x)] = 0$.

(b) This is the contrapositive of part (a), since if f, g were not linearly independent, then their Wronskian would vanish everywhere.

(c) Suppose $c_1 f(x) + c_2 g(x) = c_1 x^3 + c_2 |x|^3 \equiv 0$. then, at $x = 1$, $c_1 + c_2 = 0$, whereas at $x = -1$, $-c_1 + c_2 = 0$. Therefore, $c_1 = c_2 = 0$, proving linear independence. On the other hand, $W[x^3, |x|^3] = x^3(3x^2 \operatorname{sign} x) - (3x^2)|x|^3 \equiv 0$.

2.4.1. Only (a) and (c) are bases.

2.4.2. Only (b) is a basis.

2.4.3. (a) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$; (b) $\begin{pmatrix} \frac{3}{4} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{4} \\ 0 \\ 1 \end{pmatrix}$; (c) $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

2.4.4.

(a) They do not span $\mathbb{R}^3$ because the linear system $A\mathbf{c} = \mathbf{b}$ with coefficient matrix $A = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 0 & -1 & -1 & -1 \\ 2 & 1 & -1 & 3 \end{pmatrix}$ does not have a solution for all $\mathbf{b}$ since rank $A = 2$.

(b) 4 vectors in $\mathbb{R}^3$ are automatically linearly dependent.

(*c*) No, because if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ don't span $\mathbb{R}^3$, no subset of them will span it either.
(*d*) 2, because $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent and span the subspace, and hence form a basis.

2.4.5.
(a) They span $\mathbb{R}^3$ because the linear system $A\mathbf{c} = \mathbf{b}$ with coefficient matrix
$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & -2 & -2 & 3 \\ 2 & 5 & 1 & -1 \end{pmatrix}$ has a solution for all $\mathbf{b}$ since $\operatorname{rank} A = 3$.
(*b*) 4 vectors in $\mathbb{R}^3$ are automatically linearly dependent.
(*c*) Yes, because $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ also span $\mathbb{R}^3$ and so form a basis.
(*d*) 3 because they span all of $\mathbb{R}^3$.

2.4.6.
(*a*) Solving the defining equation, the general vector in the plane is $\mathbf{x} = \begin{pmatrix} 2y+4z \\ y \\ z \end{pmatrix}$ where y, z are arbitrary. We can write $\mathbf{x} = y\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} = (y+2z)\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + (y+z)\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ and hence both pairs of vectors span the plane. Both pairs are linearly independent since they are not parallel, and hence both form a basis.
(*b*) $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = (-1)\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix};$
(*c*) Any two linearly independent solutions, e.g., $\begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 10 \\ 1 \\ 2 \end{pmatrix}$, will form a basis.

♡ 2.4.7. (*a*) (*i*) Left handed basis; (*ii*) right handed basis; (*iii*) not a basis; (*iv*) right handed basis. (*b*) Switching two columns or multiplying a column by -1 changes the sign of the determinant. (*c*) If $\det A = 0$, its columns are linearly dependent and hence can't form a basis.

2.4.8.
(a) $\left(-\frac{2}{3}, \frac{5}{6}, 1, 0\right)^T, \left(\frac{1}{3}, -\frac{2}{3}, 0, 1\right)^T$; $\dim = 2$.
(*b*) The condition $p(1) = 0$ says $a + b + c = 0$, so $p(x) = (-b-c)x^2 + bx + c = b(-x^2+x) + c(-x^2+1)$. Therefore $-x^2+x, \ -x^2+1$ is a basis, and so $\dim = 2$.
(*c*) $e^x, \ \cos 2x, \ \sin 2x$, is a basis, so $\dim = 3$.

2.4.9. (a) $\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$, $\dim = 1$; (*b*) $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$, $\dim = 2$; (*c*) $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}$, $\dim = 3$.

2.4.10. (a) We have $a + bt + ct^2 = c_1(1+t^2) + c_2(t+t^2) + c_3(1+2t+t^2)$ provided $a = c_1 + c_3$, $b = c_2 + 2c_3$, $c = c_1 + c_2 + c_3$. The coefficient matrix of this linear system, $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$, is nonsingular, and hence there is a solution for any a, b, c, proving that they span the space of quadratic polynomials. Also, they are linearly independent since the linear combination is zero if and only if c_1, c_2, c_3 satisfy the corresponding homogeneous linear system $c_1 + c_3 = 0$, $c_2 + 2c_3 = 0$, $c_1 + c_2 + c_3 = 0$, and hence $c_1 = c_2 = c_3 = 0$. (Or, you can use the fact that $\dim \mathcal{P}^{(2)} = 3$ and the spanning property to conclude that they form a basis.)

(*b*) $1+4t+7t^2 = 2(1+t^2)+6(t+t^2)-(1+2t+t^2)$

2.4.11. (*a*) $a+bt+ct^2+dt^3 = c_1+c_2(1-t)+c_3(1-t)^2+c_4(1-t)^3$ provided $a = c_1+c_2+c_3+c_4$, $b = -c_2-2c_3-3c_4$, $c = c_3+3c_4$, $d = -c_4$. The coefficient matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ is nonsingular, and hence they span $\mathcal{P}^{(3)}$. Also, they are linearly independent since the linear combination is zero if and only if $c_1 = c_2 = c_3 = c_4 = 0$ satisfy the corresponding homogeneous linear system. (Or, you can use the fact that $\dim \mathcal{P}^{(3)} = 4$ and the spanning property to conclude that they form a basis.) (*b*) $1+t^3 = 2-3(1-t)+3(1-t)^2-(1-t)^3$.

2.4.12. (*a*) They are linearly dependent because $2p_1 - p_2 + p_3 \equiv 0$. (*b*) The dimension is 2, since p_1, p_2 are linearly independent and span the subspace, and hence form a basis.

2.4.13.

(*a*) The sample vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$ are linearly independent and hence form a basis for $\mathbb{R}^4$ — the space of sample functions.

(*b*) Sampling x produces $\begin{pmatrix} 0 \\ \frac14 \\ \frac12 \\ \frac34 \end{pmatrix} = \frac12 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2+\sqrt{2}}{8} \begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{pmatrix} - \frac{2-\sqrt{2}}{8} \begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$.

2.4.14.

(*a*) $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is a basis since we can uniquely write any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a E_{11} + b E_{12} + c E_{21} + d E_{22}$.

(*b*) Similarly, the matrices E_{ij} with a 1 in position (i,j) and all other entries 0, for $i = 1, \ldots, m,\ j = 1, \ldots, n$, form a basis for $\mathcal{M}_{m \times n}$, which therefore has dimension mn.

2.4.15. $k \neq -1, 2$.

2.4.16. A basis is given by the matrices E_{ii}, $i = 1, \ldots, n$ which have a 1 in the i^{th} diagonal position and all other entries 0.

2.4.17.

(*a*) $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$; dimension $= 3$.

(*b*) A basis is given by the matrices E_{ij} with a 1 in position (i,j) and all other entries 0 for $1 \le i \le j \le n$, so the dimension is $\frac12 n(n+1)$.

2.4.18. (*a*) Symmetric: $\dim = 3$; skew-symmetric: $\dim = 1$; (*b*) symmetric: $\dim = 6$; skew-symmetric: $\dim = 3$; (*c*) symmetric: $\dim = \frac12 n(n+1)$; skew-symmetric: $\dim = \frac12 n(n-1)$.

♡ 2.4.19.

(*a*) If a row (column) of A adds up to a and the corresponding row (column) of B adds up to b, then the corresponding row (column) of $C = A + B$ adds up to $c = a + b$. Thus, if all row and column sums of A and B are the same, the same is true for C. Similarly, the row (column) sums of $c\,A$ are c times the row (column) sums of A, and hence all the same if A is a semi-magic square.

(*b*) A matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$ is a semi-magic square if and only if

$$a+b+c = d+e+f = g+h+j = a+d+e = b+e+h = c+f+j.$$

The general solution to this system is

$$A = e\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + f\begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + g\begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + h\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + j\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= (e-g)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (g+j-e)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + g\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} +$$

$$+ f\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + (h-f)\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

which is a linear combination of permutation matrices.

(*c*) The dimension is 5, with any 5 of the 6 permutation matrices forming a basis.

(*d*) Yes, by the same reasoning as in part (*a*). Its dimension is 3, with basis

$$\begin{pmatrix} 2 & 2 & -1 \\ -2 & 1 & 4 \\ 3 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & 2 \\ 1 & 1 & 1 \\ 0 & 3 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 2 & 2 \\ 4 & 1 & -2 \\ 0 & 0 & 3 \end{pmatrix}.$$

(*e*) $A = c_1\begin{pmatrix} 2 & 2 & -1 \\ -2 & 1 & 4 \\ 3 & 0 & 0 \end{pmatrix} + c_2\begin{pmatrix} 2 & -1 & 2 \\ 1 & 1 & 1 \\ 0 & 3 & 0 \end{pmatrix} + c_3\begin{pmatrix} -1 & 2 & 2 \\ 4 & 1 & -2 \\ 0 & 0 & 3 \end{pmatrix}$ for any c_1, c_2, c_3.

♢ 2.4.20. For instance, take $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then $\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_1 + \mathbf{v}_3$. In fact, there are infinitely many different ways of writing this vector as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

♢ 2.4.21.

(a) By Theorem 2.31, we only need prove linear independence. If $\mathbf{0} = c_1 A\mathbf{v}_1 + \cdots + c_n A\mathbf{v}_n = A(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n)$, then, since A is nonsingular, $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$, and hence $c_1 = \cdots = c_n = 0$.

(*b*) $A\mathbf{e}_i$ is the i^{th} column of A, and so a basis consists of the column vectors of the matrix.

♢ 2.4.22. Since $V \neq \{\mathbf{0}\}$, at least one $\mathbf{v}_i \neq \mathbf{0}$. Let $\mathbf{v}_{i_1} \neq \mathbf{0}$ be the first nonzero vector in the list $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Then, for each $k = i_1 + 1, \ldots, n-1$, suppose we have selected linearly independent vectors $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_j}$ from among $\mathbf{v}_1, \ldots, \mathbf{v}_k$. If $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_j}, \mathbf{v}_{k+1}$ form a linearly independent set, we set $\mathbf{v}_{i_{j+1}} = \mathbf{v}_{k+1}$; otherwise, $\mathbf{v}_{k+1}$ is a linear combination of $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_j}$, and is not needed in the basis. The resulting collection $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_m}$ forms a basis for V since they are linearly independent by design, and span V since each $\mathbf{v}_i$ either appears in the basis, or is a linear combination of the basis elements that were selected before it. We have $\dim V = n$ if and only if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent and so form a basis for V.

♢ 2.4.23. This is a special case of Exercise 2.3.31(*a*).

♢ 2.4.24.

(*a*) $m \leq n$ as otherwise $\mathbf{v}_1, \ldots, \mathbf{v}_m$ would be linearly dependent. If $m = n$ then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent and hence, by Theorem 2.31 span all of $\mathbb{R}^n$. Since every vector in their span also belongs to V, we must have $V = \mathbb{R}^n$.

(*b*) Starting with the basis $\mathbf{v}_1, \ldots, \mathbf{v}_m$ of V with $m < n$, we choose any $\mathbf{v}_{m+1} \in \mathbb{R}^n \setminus V$. Since $\mathbf{v}_{m+1}$ does not lie in the span of $\mathbf{v}_1, \ldots, \mathbf{v}_m$, the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{m+1}$ are linearly independent and span an $m+1$ dimensional subspace of $\mathbb{R}^n$. Unless $m+1 = n$ we can then choose another vector $\mathbf{v}_{m+2}$ not in the span of $\mathbf{v}_1, \ldots, \mathbf{v}_{m+1}$, and so $\mathbf{v}_1, \ldots, \mathbf{v}_{m+2}$ are also linearly independent. We continue on in this fashion until we arrive at n linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ which necessarily form a basis of $\mathbb{R}^n$.

(c) (i) $\left(1, 1, \frac{1}{2}\right)^T, (1, 0, 0)^T, (0, 1, 0)^T$; (ii) $(1, 0, -1)^T, (0, 1, -2)^T, (1, 0, 0)^T$.

♢ 2.4.25.

(a) If $\dim V = \infty$, then the inequality is trivial. Also, if $\dim W = \infty$, then one can find infinitely many linearly independent elements in W, but these are also linearly independent as elements of V and so $\dim V = \infty$ also. Otherwise, let $\mathbf{w}_1, \ldots, \mathbf{w}_n$ form a basis for W. Since they are linearly independent, Theorem 2.31 implies $n \le \dim V$.

(b) Since $\mathbf{w}_1, \ldots, \mathbf{w}_n$ are linearly independent, if $n = \dim V$, then by Theorem 2.31, they form a basis for V. Thus every $\mathbf{v} \in V$ can be written as a linear combination of $\mathbf{w}_1, \ldots, \mathbf{w}_n$, and hence, since W is a subspace, $\mathbf{v} \in W$ too. Therefore, $W = V$.

(c) Example: $V = \mathrm{C}^0[a, b]$ and $W = \mathcal{P}^{(\infty)}$.

♢ 2.4.26. (a) Every $\mathbf{v} \in V$ can be uniquely decomposed as $\mathbf{v} = \mathbf{w} + \mathbf{z}$ where $\mathbf{w} \in W, \mathbf{z} \in Z$. Write $\mathbf{w} = c_1 \mathbf{w}_1 + \ldots + c_j \mathbf{w}_j$ and $\mathbf{z} = d_1 \mathbf{z}_1 + \cdots + d_k \mathbf{z}_k$. Then $\mathbf{v} = c_1 \mathbf{w}_1 + \ldots + c_j \mathbf{w}_j + d_1 \mathbf{z}_1 + \cdots + d_k \mathbf{z}_k$, proving that $\mathbf{w}_1, \ldots, \mathbf{w}_j, \mathbf{z}_1, \ldots, \mathbf{z}_k$ span V. Moreover, by uniqueness, $\mathbf{v} = \mathbf{0}$ if and only if $\mathbf{w} = \mathbf{0}$ and $\mathbf{z} = \mathbf{0}$, and so the only linear combination that sums up to $\mathbf{0} \in V$ is the trivial one $c_1 = \cdots = c_j = d_1 = \cdots = d_k = 0$, which proves linear independence of the full collection. (b) This follows immediately from part (a): $\dim V = j + k = \dim W + \dim Z$.

♢ 2.4.27. Suppose the functions are linearly independent. This means that for every $\mathbf{0} \ne \mathbf{c} = (c_1, c_2, \ldots, c_n)^T \in \mathbb{R}^n$, there is a point $x_{\mathbf{c}} \in \mathbb{R}$ such that $\displaystyle\sum_{i=1}^{n} c_i f_i(x_{\mathbf{c}}) \ne 0$. The assumption says that $\{\mathbf{0}\} \ne V_{x_1, \ldots, x_m}$ for all choices of sample points. Recursively define the following sample points. Choose x_1 so that $f_1(x_1) \ne 0$. (This is possible since if $f_1(x) \equiv 0$, then the functions are linearly dependent.) Thus $V_{x_1} \subsetneq \mathbb{R}^m$ since $\mathbf{e}_1 \notin V_{x_1}$. Then, for each $m = 1, 2, \ldots$, given $x_1, \ldots, x_m$, choose $\mathbf{0} \ne \mathbf{c}_0 \in V_{x_1, \ldots, x_m}$, and set $x_{m+1} = x_{\mathbf{c}_0}$. Then $\mathbf{c}_0 \notin V_{x_1, \ldots, x_{m+1}} \subsetneq V_{x_1, \ldots, x_m}$ and hence, by induction, $\dim V_m \le n - m$. In particular, $\dim V_{x_1, \ldots, x_n} = 0$, so $V_{x_1, \ldots, x_n} = \{\mathbf{0}\}$, which contradicts our assumption and proves the result. Note that the proof implies we only need check linear dependence at all possible collections of n sample points to conclude that the functions are linearly dependent.

2.5.1.

(a) Range: all $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ such that $\frac{3}{4} b_1 + b_2 = 0$; kernel spanned by $\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$.

(b) Range: all $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ such that $2 b_1 + b_2 = 0$; kernel spanned by $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$.

(c) Range: all $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ such that $-2 b_1 + b_2 + b_3 = 0$; kernel spanned by $\begin{pmatrix} -\frac{5}{4} \\ -\frac{7}{8} \\ 1 \end{pmatrix}$.

(d) Range: all $\mathbf{b} = (b_1, b_2, b_3, b_4)^T$ such that $-2 b_1 - b_2 + b_3 = 2 b_1 + 3 b_2 + b_4 = 0$; kernel spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

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2.5.2. (a) $\begin{pmatrix} -\frac{5}{2} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$: plane; (b) $\begin{pmatrix} \frac{1}{4} \\ \frac{3}{8} \\ 1 \end{pmatrix}$: line; (c) $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$: plane;
(d) $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$: line; (e) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$: point; (f) $\begin{pmatrix} \frac{1}{3} \\ \frac{5}{3} \\ 1 \end{pmatrix}$: line.

2.5.3.
(a) Kernel spanned by $\begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$; range spanned by $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$;
(b) compatibility: $-\frac{1}{2}a + \frac{1}{4}b + c = 0$.

2.5.4. (a) $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$; (b) $\mathbf{x} = \begin{pmatrix} 1+t \\ 2+t \\ 3+t \end{pmatrix}$ where t is arbitrary.

2.5.5. In each case, the solution is $\mathbf{x} = \mathbf{x}^\star + \mathbf{z}$, where $\mathbf{x}^\star$ is the particular solution and $\mathbf{z}$ belongs to the kernel:
(a) $\mathbf{x}^\star = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{z} = y\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$; (b) $\mathbf{x}^\star = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{z} = z\begin{pmatrix} -\frac{2}{7} \\ \frac{1}{7} \\ 1 \end{pmatrix}$;
(c) $\mathbf{x}^\star = \begin{pmatrix} -\frac{7}{9} \\ \frac{2}{9} \\ \frac{10}{9} \end{pmatrix}$, $\mathbf{z} = z\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$; (d) $\mathbf{x}^\star = \begin{pmatrix} \frac{5}{6} \\ 1 \\ -\frac{2}{3} \end{pmatrix}$, $\mathbf{z} = \mathbf{0}$; (e) $\mathbf{x}^\star = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $\mathbf{z} = v\begin{pmatrix} 2 \\ 1 \end{pmatrix}$;
(f) $\mathbf{x}^\star = \begin{pmatrix} \frac{11}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{z} = r\begin{pmatrix} -\frac{13}{2} \\ -\frac{3}{2} \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$; (g) $\mathbf{x}^\star = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{z} = z\begin{pmatrix} 6 \\ 2 \\ 1 \\ 0 \end{pmatrix} + w\begin{pmatrix} -4 \\ -1 \\ 0 \\ 1 \end{pmatrix}$.

2.5.6. The i^{th} entry of $A\,(1,1,\ldots,1)^T$ is $a_{i1} + \ldots + a_{in}$ which is n times the average of the entries in the i^{th} row. Thus, $A\,(1,1,\ldots,1)^T = \mathbf{0}$ if and only if each row of A has average 0.

2.5.7. The kernel has dimension $n-1$, with basis $-r^{k-1}\mathbf{e}_1 + \mathbf{e}_k = \left(-r^{k-1}, 0, \ldots, 0, 1, 0, \ldots, 0\right)^T$ for $k = 2, \ldots n$. The range has dimension 1, with basis $(1, r^n, r^{2n} \ldots, r^{(n-1)n})^T$.

◇ 2.5.8. (a) If $\mathbf{w} = P\mathbf{w}$, then $\mathbf{w} \in \operatorname{rng} P$. On the other hand, if $\mathbf{w} \in \operatorname{rng} P$, then $\mathbf{w} = P\mathbf{v}$ for some $\mathbf{v}$. But then $P\mathbf{w} = P^2\mathbf{v} = P\mathbf{v} = \mathbf{w}$. (b) Given $\mathbf{v}$, set $\mathbf{w} = P\mathbf{v}$. Then $\mathbf{v} = \mathbf{w} + \mathbf{z}$ where $\mathbf{z} = \mathbf{v} - \mathbf{w} \in \ker P$ since $P\mathbf{z} = P\mathbf{v} - P\mathbf{w} = P\mathbf{v} - P^2\mathbf{v} = P\mathbf{v} - P\mathbf{v} = \mathbf{0}$. Moreover, if $\mathbf{w} \in \ker P \cap \operatorname{rng} P$, then $\mathbf{0} = P\mathbf{w} = \mathbf{w}$, and so $\ker P \cap \operatorname{rng} P = \{\mathbf{0}\}$, proving complementarity.

2.5.9. False. For example, if $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ then $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is in both $\ker A$ and $\operatorname{rng} A$.

◇ 2.5.10. Let $\mathbf{r}_1, \ldots, \mathbf{r}_{m+k}$ be the rows of C, so $\mathbf{r}_1, \ldots, \mathbf{r}_m$ are the rows of A. For $\mathbf{v} \in \ker C$, the i^{th} entry of $C\mathbf{v} = \mathbf{0}$ is $\mathbf{r}_i\,\mathbf{v} = 0$, but then this implies $A\mathbf{v} = \mathbf{0}$ and so $\mathbf{v} \in \ker A$. As an example, $A = (\,1 \quad 0\,)$ has kernel spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, while $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has $\ker C = \{\mathbf{0}\}$.

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♢ 2.5.11. If $\mathbf{b} = A\mathbf{x} \in \operatorname{rng} A$, then $\mathbf{b} = C\mathbf{z}$ where $\mathbf{z} = \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix}$, and so $\mathbf{b} \in \operatorname{rng} C$. As an example, $A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ has $\operatorname{rng} A = \{\mathbf{0}\}$, while the range of $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is the x axis.

2.5.12. $\mathbf{x}_1^\star = \begin{pmatrix} -2 \\ \frac{3}{2} \end{pmatrix}$, $\mathbf{x}_2^\star = \begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix}$; $\mathbf{x} = \mathbf{x}_1^\star + 4\mathbf{x}_2^\star = \begin{pmatrix} -6 \\ \frac{7}{2} \end{pmatrix}$.

2.5.13. $\mathbf{x}^\star = 2\mathbf{x}_1^\star + \mathbf{x}_2^\star = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$.

2.5.14.
(a) By direct matrix multiplication: $A\mathbf{x}_1^\star = A\mathbf{x}_2^\star = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$.
(b) The general solution is $\mathbf{x} = \mathbf{x}_1^\star + t(\mathbf{x}_2^\star - \mathbf{x}_1^\star) = (1-t)\mathbf{x}_1^\star + t\mathbf{x}_2^\star = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix}$.

2.5.15. 5 meters.

2.5.16. The mass will move 6 units in the horizontal direction and -6 units in the vertical direction.

2.5.17. $\mathbf{x} = c_1\mathbf{x}_1^\star + c_2\mathbf{x}_2^\star$ where $c_1 = 1 - c_2$.

2.5.18. False: in general, $(A+B)\mathbf{x}^\star = (A+B)\mathbf{x}_1^\star + (A+B)\mathbf{x}_2^\star = \mathbf{c} + \mathbf{d} + B\mathbf{x}_1^\star + A\mathbf{x}_2^\star$, and the third and fourth terms don't necessarily add up to $\mathbf{0}$.

♢ 2.5.19. $\operatorname{rng} A = \mathbb{R}^n$, and so A must be a nonsingular matrix.

♢ 2.5.20.
(a) If $A\mathbf{x}_i = \mathbf{e}_i$, then $\mathbf{x}_i = A^{-1}\mathbf{e}_i$ which, by (2.13), is the i^{th} column of the matrix A^{-1}.
(b) The solutions to $A\mathbf{x}_i = \mathbf{e}_i$ in this case are $\mathbf{x}_1 = \begin{pmatrix} \frac{1}{2} \\ 2 \\ -\frac{1}{2} \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ -1 \end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{pmatrix}$,
which are the columns of $A^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 2 & -1 & -1 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

2.5.21.
(a) range: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$; corange: $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$; kernel: $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$; cokernel: $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.
(b) range: $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -8 \\ -1 \\ 6 \end{pmatrix}$; corange: $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -8 \end{pmatrix}$; kernel: $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$; cokernel: $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.
(c) range: $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$; corange: $\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -3 \\ 2 \end{pmatrix}$; kernel: $\begin{pmatrix} 1 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$; cokernel: $\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$.

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(*d*) range: $\begin{pmatrix} 1\\0\\2\\3\\1 \end{pmatrix}, \begin{pmatrix} -3\\3\\-3\\-3\\0 \end{pmatrix}, \begin{pmatrix} 1\\-2\\0\\3\\3 \end{pmatrix}$; corange: $\begin{pmatrix} 1\\-3\\2\\2\\1 \end{pmatrix}, \begin{pmatrix} 0\\3\\-6\\0\\-2 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\0\\4 \end{pmatrix}$;

kernel: $\begin{pmatrix} 4\\2\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} -2\\0\\0\\1\\0 \end{pmatrix}$; cokernel: $\begin{pmatrix} -2\\-1\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 2\\1\\0\\-1\\1 \end{pmatrix}$.

2.5.22. $\begin{pmatrix} -1\\2\\-3 \end{pmatrix}, \begin{pmatrix} 0\\1\\2 \end{pmatrix}, \begin{pmatrix} -3\\1\\0 \end{pmatrix}$, which are its first, third and fourth columns;

Second column: $\begin{pmatrix} 2\\-4\\6 \end{pmatrix} = 2\begin{pmatrix} -1\\2\\-3 \end{pmatrix}$; fifth column: $\begin{pmatrix} 5\\-4\\8 \end{pmatrix} = -2\begin{pmatrix} -1\\2\\-3 \end{pmatrix} + \begin{pmatrix} 0\\1\\2 \end{pmatrix} - \begin{pmatrix} -3\\1\\0 \end{pmatrix}$.

2.5.23. range: $\begin{pmatrix} 1\\2\\-3 \end{pmatrix}, \begin{pmatrix} 0\\4\\1 \end{pmatrix}$; corange: $\begin{pmatrix} 1\\-3\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\4 \end{pmatrix}$; second column: $\begin{pmatrix} -3\\-6\\9 \end{pmatrix} = -3\begin{pmatrix} 1\\2\\-3 \end{pmatrix}$;

second and third rows: $\begin{pmatrix} 2\\-6\\4 \end{pmatrix} = 2\begin{pmatrix} 1\\-3\\0 \end{pmatrix} + \begin{pmatrix} 0\\0\\4 \end{pmatrix}$, $\begin{pmatrix} -3\\9\\1 \end{pmatrix} = -3\begin{pmatrix} 1\\-3\\0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0\\0\\4 \end{pmatrix}$.

2.5.24.

(*i*) rank $= 1$; $\dim \operatorname{rng} A = \dim \operatorname{corng} A = 1$, $\dim \ker A = \dim \operatorname{coker} A = 1$;
kernel basis: $\begin{pmatrix} -2\\1 \end{pmatrix}$; cokernel basis: $\begin{pmatrix} 2\\1 \end{pmatrix}$; compatibility conditions: $2b_1 + b_2 = 0$;
example: $\mathbf{b} = \begin{pmatrix} 1\\-2 \end{pmatrix}$, with solution $\mathbf{x} = \begin{pmatrix} 1\\0 \end{pmatrix} + z\begin{pmatrix} -2\\1 \end{pmatrix}$.

(*ii*) rank $= 1$; $\dim \operatorname{rng} A = \dim \operatorname{corng} A = 1$, $\dim \ker A = 2$, $\dim \operatorname{coker} A = 1$; kernel basis: $\begin{pmatrix} \frac{1}{3}\\1\\0 \end{pmatrix}, \begin{pmatrix} \frac{2}{3}\\0\\1 \end{pmatrix}$; cokernel basis: $\begin{pmatrix} 2\\1 \end{pmatrix}$; compatibility conditions: $2b_1 + b_2 = 0$;
example: $\mathbf{b} = \begin{pmatrix} 3\\-6 \end{pmatrix}$, with solution $\mathbf{x} = \begin{pmatrix} 1\\0\\0 \end{pmatrix} + y\begin{pmatrix} \frac{1}{3}\\1\\0 \end{pmatrix} + z\begin{pmatrix} \frac{2}{3}\\0\\1 \end{pmatrix}$.

(*iii*) rank $= 2$; $\dim \operatorname{rng} A = \dim \operatorname{corng} A = 2$, $\dim \ker A = 0$, $\dim \operatorname{coker} A = 1$;
kernel: $\{\mathbf{0}\}$; cokernel basis: $\begin{pmatrix} -\frac{20}{13}\\ \frac{3}{13}\\1 \end{pmatrix}$; compatibility conditions: $-\frac{20}{13}b_1 + \frac{3}{13}b_2 + b_3 = 0$;
example: $\mathbf{b} = \begin{pmatrix} 1\\-2\\2 \end{pmatrix}$, with solution $\mathbf{x} = \begin{pmatrix} 1\\0\\0 \end{pmatrix}$.

(*iv*) rank $= 2$; $\dim \operatorname{rng} A = \dim \operatorname{corng} A = 2$, $\dim \ker A = \dim \operatorname{coker} A = 1$;
kernel basis: $\begin{pmatrix} -2\\-1\\1 \end{pmatrix}$; cokernel basis: $\begin{pmatrix} -2\\1\\1 \end{pmatrix}$; compatibility conditions:
$-2b_1 + b_2 + b_3 = 0$; example: $\mathbf{b} = \begin{pmatrix} 2\\1\\3 \end{pmatrix}$, with solution $\mathbf{x} = \begin{pmatrix} 1\\0\\0 \end{pmatrix} + z\begin{pmatrix} -2\\-1\\1 \end{pmatrix}$.

(*v*) rank $= 2$; $\dim \operatorname{rng} A = \dim \operatorname{corng} A = 2$, $\dim \ker A = 1$, $\dim \operatorname{coker} A = 2$; kernel basis: $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$; cokernel basis: $\begin{pmatrix} -\frac{9}{4} \\ \frac{1}{4} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 0 \\ 1 \end{pmatrix}$; compatibility: $-\frac{9}{4}\, b_1 + \frac{1}{4}\, b_2 + b_3 = 0$, $\frac{1}{4}\, b_1 - \frac{1}{4}\, b_2 + b_4 = 0$; example: $\mathbf{b} = \begin{pmatrix} 2 \\ 6 \\ 3 \\ 1 \end{pmatrix}$, with solution $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$.

(*vi*) rank $= 3$; $\dim \operatorname{rng} A = \dim \operatorname{corng} A = 3$, $\dim \ker A = \dim \operatorname{coker} A = 1$; kernel basis: $\begin{pmatrix} \frac{13}{4} \\ \frac{13}{8} \\ -\frac{7}{2} \\ 1 \end{pmatrix}$; cokernel basis: $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$; compatibility conditions: $-b_1 - b_2 + b_3 + b_4 = 0$; example: $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 3 \end{pmatrix}$, with solution $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + w \begin{pmatrix} \frac{13}{4} \\ \frac{13}{8} \\ -\frac{7}{2} \\ 1 \end{pmatrix}$.

(*vii*) rank $= 4$; $\dim \operatorname{rng} A = \dim \operatorname{corng} A = 4$, $\dim \ker A = 1$, $\dim \operatorname{coker} A = 0$; kernel basis: $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$; cokernel is $\{\mathbf{0}\}$; no conditions;
example: $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ -3 \end{pmatrix}$, with $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

2.5.25. (*a*) dim $= 2$; basis: $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$; (*b*) dim $= 1$; basis: $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$;

(*c*) dim $= 3$; basis: $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}$; (*d*) dim $= 3$; basis: $\begin{pmatrix} 1 \\ 0 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -8 \\ 7 \end{pmatrix}$;

(*e*) dim $= 3$; basis: $\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \\ 1 \end{pmatrix}$.

2.5.26. It's the span of $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ -1 \\ -1 \end{pmatrix}$; the dimension is 3.

2.5.27. (*a*) $\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$; (*b*) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$; (*c*) $\begin{pmatrix} -1 \\ 3 \\ 0 \\ 1 \end{pmatrix}$.

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2.5.28. First method: $\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -4 \\ 5 \end{pmatrix}$; second method: $\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -8 \\ 3 \end{pmatrix}$. The first vectors are the same, while $\begin{pmatrix} 2 \\ 3 \\ -4 \\ 5 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \\ -8 \\ 3 \end{pmatrix}$; $\begin{pmatrix} 0 \\ 3 \\ -8 \\ 3 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ -4 \\ 5 \end{pmatrix}$.

2.5.29. Both sets are linearly independent and hence span a three-dimensional subspace of $\mathbb{R}^4$. Moreover, $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_3, \mathbf{w}_2 = \mathbf{v}_1 + \mathbf{v}_2 + 2\mathbf{v}_3, \mathbf{w}_3 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$ all lie in the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and hence, by Theorem 2.31(d) also form a basis for the subspace.

2.5.30.
(a) If $A = A^T$, then $\ker A = \{ A\mathbf{x} = \mathbf{0} \} = \{ A^T\mathbf{x} = \mathbf{0} \} = \operatorname{coker} A$, and $\operatorname{rng} A = \{ A\mathbf{x} \} = \{ A^T\mathbf{x} \} = \operatorname{corng} A$.
(b) $\ker A = \operatorname{coker} A$ has basis $(2, -1, 1)^T$; $\operatorname{rng} A = \operatorname{corng} A$ has basis $(1, 2, 0)^T, (2, 6, 2)^T$.
(c) No. For instance, if A is any nonsingular matrix, then $\ker A = \operatorname{coker} A = \{\mathbf{0}\}$ and $\operatorname{rng} A = \operatorname{corng} A = \mathbb{R}^3$.

2.5.31.
(a) Yes. This is our method of constructing the basis for the range, and the proof is outlined in the text.
(b) No. For example, if $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, then $U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and the first three rows of U form a basis for the three-dimensional $\operatorname{corng} U = \operatorname{corng} A$. but the first three rows of A only span a two-dimensional subspace.
(c) Yes, since $\ker U = \ker A$.
(d) No, since $\operatorname{coker} U \neq \operatorname{coker} A$ in general. For the example in part (b), $\operatorname{coker} A$ has basis $(-1, 1, 0, 0)^T$ while $\operatorname{coker} A$ has basis $(0, 0, 0, 1)^T$.

2.5.32. (a) Example: $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. ($b$) No, since then the first r rows of U are linear combinations of the first r rows of A. Hence these rows span $\operatorname{corng} A$, which, by Theorem 2.31c, implies that they form a basis for the corange.

2.5.33. Examples: any symmetric matrix; any permutation matrix since the row echelon form is the identity. Yet another example is the complex matrix $\begin{pmatrix} 0 & 0 & 1 \\ 1 & \mathrm{i} & \mathrm{i} \\ 0 & \mathrm{i} & \mathrm{i} \end{pmatrix}$.

♢ 2.5.34. The rows $\mathbf{r}_1, \ldots, \mathbf{r}_m$ of A span the corange. Reordering the rows — in particular interchanging two — will not change the span. Also, multiplying any of the rows by nonzero scalars, $\widetilde{\mathbf{r}}_i = a_i\,\mathbf{r}_i$, for $a_i \neq 0$, will also span the same space, since

$$\mathbf{v} = \sum_{i=1}^{n} c_i\,\mathbf{r}_i = \sum_{i=1}^{n} \frac{c_i}{a_i}\,\widetilde{\mathbf{r}}_i.$$

2.5.35. We know $\operatorname{rng} A \subset \mathbb{R}^m$ is a subspace of dimension $r = \operatorname{rank} A$. In particular, $\operatorname{rng} A = \mathbb{R}^m$ if and only if it has dimension $m = \operatorname{rank} A$.

2.5.36. This is false. If $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ then $\operatorname{rng} A$ is spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ whereas the range of its row echelon form $U = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

♢ 2.5.37.
(a) Method 1: choose the nonzero rows in the row echelon form of A. Method 2: choose the columns of A^T that correspond to pivot columns of its row echelon form.
(b) Method 1: $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}$. Method 2: $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ -7 \\ -7 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$. Not the same.

♢ 2.5.38. If $\mathbf{v} \in \ker A$ then $A\mathbf{v} = \mathbf{0}$ and so $BA\mathbf{v} = B\mathbf{0} = \mathbf{0}$, so $\mathbf{v} \in \ker(BA)$. The first statement follows from setting $B = A$.

♢ 2.5.39. If $\mathbf{v} \in \operatorname{rng} AB$ then $\mathbf{v} = AB\mathbf{x}$ for some vector $\mathbf{x}$. But then $\mathbf{v} = A\mathbf{y}$ where $\mathbf{y} = B\mathbf{x}$, and so $\mathbf{v} \in \operatorname{rng} A$. The first statement follows from setting $B = A$.

2.5.40. First note that BA and AC also have size $m \times n$. To show $\operatorname{rank} A = \operatorname{rank} BA$, we prove that $\ker A = \ker BA$, and so $\operatorname{rank} A = n - \dim \ker A = n - \dim \ker BA = \operatorname{rank} BA$. Indeed, if $\mathbf{v} \in \ker A$, then $A\mathbf{v} = \mathbf{0}$ and hence $BA\mathbf{v} = \mathbf{0}$ so $\mathbf{v} \in \ker BA$. Conversely, if $\mathbf{v} \in \ker BA$ then $BA\mathbf{v} = \mathbf{0}$. Since B is nonsingular, this implies $A\mathbf{v} = \mathbf{0}$ and hence $\mathbf{v} \in \ker A$, proving the first result. To show $\operatorname{rank} A = \operatorname{rank} AC$, we prove that $\operatorname{rng} A = \operatorname{rng} AC$, and so $\operatorname{rank} A = \dim \operatorname{rng} A = \dim \operatorname{rng} AC = \operatorname{rank} AC$. Indeed, if $\mathbf{b} \in \operatorname{rng} AC$, then $\mathbf{b} = AC\mathbf{x}$ for some $\mathbf{x}$ and so $\mathbf{b} = A\mathbf{y}$ where $\mathbf{y} = C\mathbf{x}$, and so $\mathbf{b} \in \operatorname{rng} A$. Conversely, if $\mathbf{b} \in \operatorname{rng} A$ then $\mathbf{b} = A\mathbf{y}$ for some $\mathbf{y}$ and so $\mathbf{b} = AC\mathbf{x}$ where $\mathbf{x} = C^{-1}\mathbf{y}$, so $\mathbf{b} \in \operatorname{rng} AC$, proving the second result. The final equality is a consequence of the first two: $\operatorname{rank} A = \operatorname{rank} BA = \operatorname{rank}(BA)C$.

♢ 2.5.41. (a) Since they are spanned by the columns, the range of $(A \ \ B)$ contains the range of A. But since A is nonsingular, $\operatorname{rng} A = \mathbb{R}^n$, and so $\operatorname{rng}(A \ \ B) = \mathbb{R}^n$ also, which proves $\operatorname{rank}(A \ B) = n$. (b) Same argument, using the fact that the corange is spanned by the rows.

2.5.42. True if the matrices have the same size, but false in general.

♢ 2.5.43. Since we know $\dim \operatorname{rng} A = r$, it suffices to prove that $\mathbf{w}_1, \ldots, \mathbf{w}_r$ are linearly independent. Given

$$\mathbf{0} = c_1 \mathbf{w}_1 + \cdots + c_r \mathbf{w}_r = c_1 A\mathbf{v}_1 + \cdots + c_r A\mathbf{v}_r = A(c_1 \mathbf{v}_1 + \cdots + c_r \mathbf{v}_r),$$

we deduce that $c_1 \mathbf{v}_1 + \cdots + c_r \mathbf{v}_r \in \ker A$, and hence can be written as a linear combination of the kernel basis vectors:

$$c_1 \mathbf{v}_1 + \cdots + c_r \mathbf{v}_r = c_{r+1} \mathbf{v}_{r+1} + \cdots + c_n \mathbf{v}_n.$$

But $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, and so $c_1 = \cdots = c_r = c_{r+1} = \cdots = c_n = 0$, which proves linear independence of $\mathbf{w}_1, \ldots, \mathbf{w}_r$.

♢ 2.5.44.
(a) Since they have the same kernel, their ranks are the same. Choose a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $\mathbb{R}^n$ such that $\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n$ form a basis for $\ker A = \ker B$. Then $\mathbf{w}_1 = A\mathbf{v}_1, \ldots, \mathbf{w}_r = A\mathbf{v}_r$ form a basis for $\operatorname{rng} A$, while $\mathbf{y}_1 = B\mathbf{v}_1, \ldots, \mathbf{y}_r = B\mathbf{v}_r$ form a basis for $\operatorname{rng} B$. Let M be any nonsingular $m \times m$ matrix such that $M\mathbf{w}_j = \mathbf{y}_j$, $j = 1, \ldots, r$, which exists since both sets of vectors are linearly independent. We claim $MA = B$. Indeed, $MA\mathbf{v}_j = B\mathbf{v}_j$, $j = 1, \ldots, r$, by design, while $MA\mathbf{v}_j = \mathbf{0} = B\mathbf{v}_j$, $j = r+1, \ldots, n$, since these vectors lie in the kernel. Thus, the matrices agree on a basis of $\mathbb{R}^n$ which is enough to conclude that $MA = B$.
(b) If the systems have the same solutions $\mathbf{x}^\star + \mathbf{z}$ where $\mathbf{z} \in \ker A = \ker B$, then $B\mathbf{x} = MA\mathbf{x} = M\mathbf{b} = \mathbf{c}$. Since M can be written as a product of elementary matrices, we conclude that one can get from the augmented matrix $\left(A \mid \mathbf{b} \right)$ to the augmented matrix

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$\left(B \mid \mathbf{c} \right)$ by applying the elementary row operations that make up M.

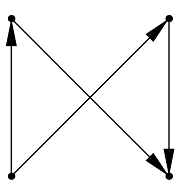
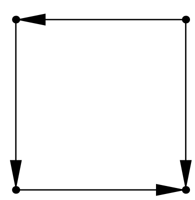
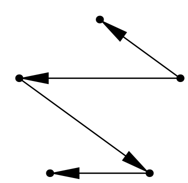
♢ 2.5.45. (*a*) First, $W \subset \operatorname{rng} A$ since every $\mathbf{w} \in W$ can be written as $\mathbf{w} = A\mathbf{v}$ for some $\mathbf{v} \in V \subset \mathbb{R}^n$, and so $\mathbf{w} \in \operatorname{rng} A$. Second, if $\mathbf{w}_1 = A\mathbf{v}_1$ and $\mathbf{w}_2 = A\mathbf{v}_2$ are elements of W, then so is $c\mathbf{w}_1 + d\mathbf{w}_2 = A(c\mathbf{v}_1 + d\mathbf{v}_2)$ for any scalars c, d because $c\mathbf{v}_1 + d\mathbf{v}_2 \in V$, proving that W is a subspace. (*b*) First, using Exercise 2.4.25, $\dim W \le r = \dim \operatorname{rng} A$ since it is a subspace of the range. Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_k$ form a basis for V, so $\dim V = k$. Let $\mathbf{w} = A\mathbf{v} \in W$. We can write $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$, and so, by linearity, $\mathbf{w} = c_1 A\mathbf{v}_1 + \cdots + c_k A\mathbf{v}_k$. Therefore, the k vectors $\mathbf{w}_1 = A\mathbf{v}_1, \ldots, \mathbf{w}_k = A\mathbf{v}_k$ span W, and therefore, by Proposition 2.33, $\dim W \le k$.

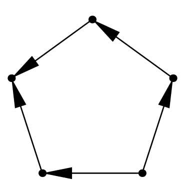
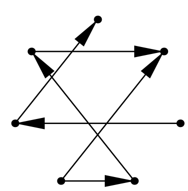
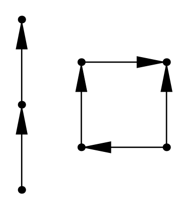
♢ 2.5.46.

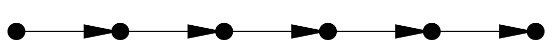
(*a*) To have a left inverse requires an $n \times m$ matrix B such that $BA = \mathrm{I}$. Suppose $\dim \operatorname{rng} A = \operatorname{rank} A < n$. Then, according to Exercise 2.5.45, the subspace $W = \{ B\mathbf{v} \mid \mathbf{v} \in \operatorname{rng} A \}$ has $\dim W \le \dim \operatorname{rng} A < n$. On the other hand, $\mathbf{w} \in W$ if and only if $\mathbf{w} = B\mathbf{v}$ where $\mathbf{v} \in \operatorname{rng} A$, and so $\mathbf{v} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$. But then $\mathbf{w} = B\mathbf{v} = BA\mathbf{x} = \mathbf{x}$, and therefore $W = \mathbb{R}^n$ since every vector $\mathbf{x} \in \mathbb{R}^n$ lies in it; thus, $\dim W = n$, contradicting the preceding result. We conclude that having a left inverse implies $\operatorname{rank} A = n$. (The rank can't be larger than n.)

(*b*) To have a right inverse requires an $m \times n$ matrix C such that $AC = \mathrm{I}$. Suppose $\dim \operatorname{rng} A = \operatorname{rank} A < m$ and hence $\operatorname{rng} A \subsetneq \mathbb{R}^m$. Choose $\mathbf{y} \in \mathbb{R}^m \setminus \operatorname{rng} A$. Then $\mathbf{y} = AC\mathbf{y} = A\mathbf{x}$, where $\mathbf{x} = C\mathbf{y}$. Therefore, $\mathbf{y} \in \operatorname{rng} A$, which is a contradiction. We conclude that having a right inverse implies $\operatorname{rank} A = m$.

(*c*) By parts (a–*b*), having both inverses requires $m = \operatorname{rank} A = n$ and A must be square and nonsingular.

2.6.1. (*a*)  (*b*)  (*c*) 

(*d*)  (*e*)  or, equivalently, 

2.6.2. (*a*) 

(*b*) $(1,1,1,1,1,1,1)^T$ is a basis for the kernel. The cokernel is trivial, containing only the zero vector, and so has no basis. (*c*) Zero.

2.6.3. (a) $\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$; (b) $\begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$; (c) $\begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$;

(d) $\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$; (e) $\begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}$;

(f) $\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$.

2.6.4. (a) 1 circuit: $\begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}$; (b) 2 circuits: $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$; (c) 2 circuits: $\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$;

(d) 3 circuits: $\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$; (e) 2 circuits: $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$;

(f) 3 circuits: $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$.

♡ 2.6.5. (a) $\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$; (b) rank $= 3$; (c) $\dim \operatorname{rng} A = \dim \operatorname{corng} A = 3$,

$\dim \ker A = 1$, $\dim \operatorname{coker} A = 2$; (d) kernel: $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$; cokernel: $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$;

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(e) $b_1 - b_2 + b_4 = 0,\ \ b_1 - b_3 + b_5 = 0$; (f) example: $\mathbf{b} = \begin{pmatrix} 1\\1\\1\\0\\0 \end{pmatrix};\quad \mathbf{x} = \begin{pmatrix} 1+t\\t\\t\\t \end{pmatrix}.$

♢ 2.6.6.
(a)

$$\begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0\\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{pmatrix}$$

Cokernel basis: $\mathbf{v}_1 = \begin{pmatrix} -1\\1\\0\\-1\\0\\1\\0\\0\\0\\0\\0\\0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1\\0\\1\\0\\-1\\0\\0\\1\\0\\0\\0\\0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0\\-1\\1\\0\\0\\0\\-1\\0\\1\\0\\0\\0 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 0\\0\\0\\-1\\1\\0\\0\\0\\0\\-1\\1\\0 \end{pmatrix}, \mathbf{v}_5 = \begin{pmatrix} 0\\0\\0\\0\\0\\-1\\1\\0\\0\\-1\\0\\1 \end{pmatrix}.$

These vectors represent the circuits around 5 of the cube's faces.

(b) Examples: $\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\-1\\1\\0\\-1\\1 \end{pmatrix} = \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 - \mathbf{v}_4 + \mathbf{v}_5, \quad \begin{pmatrix} 0\\1\\-1\\-1\\1\\1\\0\\-1\\0\\0\\0\\0 \end{pmatrix} = \mathbf{v}_1 - \mathbf{v}_2, \quad \begin{pmatrix} 0\\-1\\1\\1\\-1\\0\\-1\\0\\1\\1\\-1\\0 \end{pmatrix} = \mathbf{v}_3 - \mathbf{v}_4.$

♡ 2.6.7.

(a) Tetrahedron: $\begin{pmatrix}
1 & -1 & 0 & 0\\
1 & 0 & -1 & 0\\
1 & 0 & 0 & -1\\
0 & 1 & -1 & 0\\
0 & 1 & 0 & -1\\
0 & 0 & 1 & -1
\end{pmatrix}$

number of circuits $= \dim \operatorname{coker} A = 3$, number of faces $= 4$;

(*b*) Octahedron:

$$\begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & -1
\end{pmatrix}$$

number of circuits $= \dim \operatorname{coker} A = 7$, number of faces $= 8$.

(*c*) Dodecahedron:

$$\begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1
\end{pmatrix}$$

number of circuits $= \dim \operatorname{coker} A = 11$, number of faces $= 12$.

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(*d*) Icosahedron:

$$\begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{pmatrix}$$

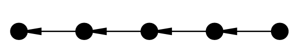
number of circuits $= \dim \operatorname{coker} A = 19$, number of faces $= 20$.

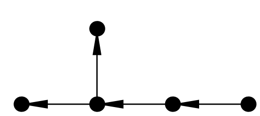
♡ 2.6.8.

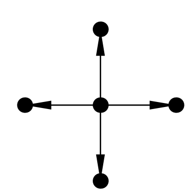
(a) (*i*) $\begin{pmatrix} -1 & 1 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & 1 & 0 & -1 \end{pmatrix}$, (*ii*) $\begin{pmatrix} -1 & 1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0\\ 0 & 0 & -1 & 1 & 0\\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}$,

(*iii*) $\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0\\ 0 & 1 & -1 & 0 & 0 & 0\\ 0 & 0 & 1 & -1 & 0 & 0\\ 0 & 1 & 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$, (*iv*) $\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0 & 0\\ 0 & 0 & -1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & -1 & 0\\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$.

(*b*)





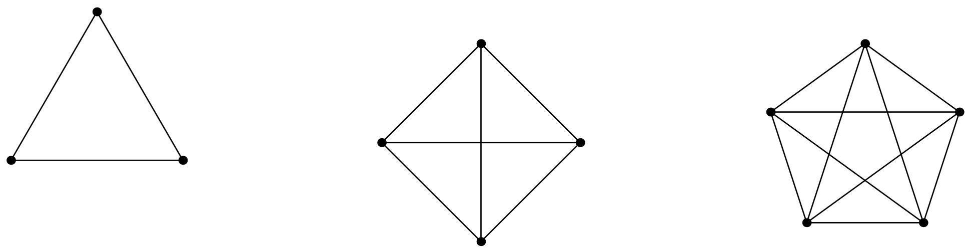


$\begin{pmatrix} -1 & 1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0\\ 0 & 0 & -1 & 1 & 0\\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0\\ 0 & 0 & -1 & 1 & 0\\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & 1 & 0 & 0 & 0\\ 0 & 1 & -1 & 0 & 0\\ 0 & 1 & 0 & -1 & 0\\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}$.

(*c*) Let m denote the number of edges. Since the graph is connected, its incidence matrix A has rank $n-1$. There are no circuits if and only if $\operatorname{coker} A = \{0\}$, which implies $0 = \dim \operatorname{coker} A = m - (n-1)$, and so $m = n - 1$.

♡ 2.6.9.

(*a*)

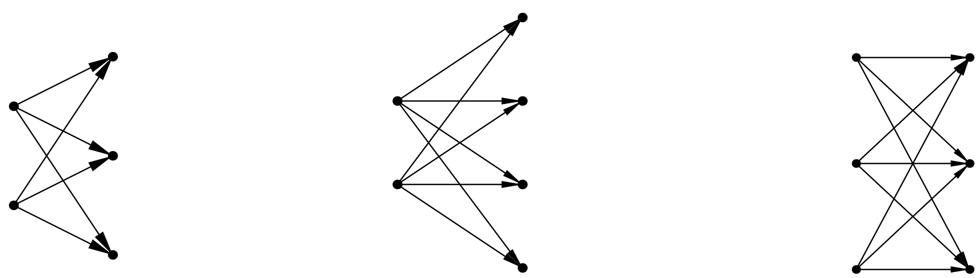


(*b*) $$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

(*c*) $\frac{1}{2} n(n-1)$; (*d*) $\frac{1}{2}(n-1)(n-2)$.

♡ 2.6.10.

(*a*)



(*b*) $$\begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}.$$

(*c*) $m\,n$; (*d*) $(m-1)(n-1)$.

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♡ 2.6.11.

(a) $A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$

(b) The vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ form a basis for $\ker A$.

(c) The entries of each $\mathbf{v}_i$ are indexed by the vertices. Thus the nonzero entries in $\mathbf{v}_1$ correspond to the vertices 1,2,4 in the first connected component, $\mathbf{v}_2$ to the vertices 3,6 in the second connected component, and $\mathbf{v}_3$ to the vertices 5,7,8 in the third connected component.

(d) Let A have k connected components. A basis for $\ker A$ consists of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ where $\mathbf{v}_i$ has entries equal to 1 if the vertex lies in the i^{th} connected component of the graph and 0 if it doesn't. To prove this, suppose $A\mathbf{v} = \mathbf{0}$. If edge $\#\ell$ connects vertex a to vertex b, then the ℓ^{th} component of the linear system is $v_a - v_b = 0$. Thus, $v_a = v_b$ whenever the vertices are connected by an edge. If two vertices are in the same connected component, then they can be connected by a path, and the values $v_a = v_b = \cdots$ at each vertex on the path must be equal. Thus, the values of v_a on all vertices in the connected component are equal, and hence $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$ can be written as a linear combination of the basis vectors, with c_i being the common value of the entries v_a corresponding to vertices in the i^{th} connected component. Thus, $\mathbf{v}_1, \ldots, \mathbf{v}_k$ span the kernel. Moreover, since the coefficients c_i coincide with certain entries v_a of $\mathbf{v}$, the only linear combination giving the zero vector is when all c_i are zero, proving their linear independence.

◇ 2.6.12. If the incidence matrix has rank r, then # circuits

$$= \dim \operatorname{coker} A = n - r = \dim \ker A \geq 1,$$

since $\ker A$ always contains the vector $(1, 1, \ldots, 1)^T$.

2.6.13. Changing the direction of an edge is the same as multiplying the corresponding row of the incidence matrix by -1. The dimension of the cokernel, being the number of independent circuits, does not change. Each entry of a cokernel vector that corresponds to an edge that has been reversed is multiplied by -1. This can be realized by left multiplying the incidence matrix by a diagonal matrix whose diagonal entries are -1 is the corresponding edge has been reversed, and $+1$ if it is unchanged.

♡ 2.6.14.

(a) Note that P permutes the rows of A, and corresponds to a relabeling of the vertices of the digraph, while Q permutes its columns, and so corresponds to a relabeling of the edges.

(b) (i),(ii),(v) represent equivalent digraphs; none of the others are equivalent.

(c) $\mathbf{v} = (v_1, \ldots, v_m) \in \operatorname{coker} A$ if and only if $\widehat{\mathbf{v}} = P\mathbf{v} = (\, v_{\pi(1)} \;\cdots\; v_{\pi(m)} \,) \in \operatorname{coker} B$. Indeed, $\widehat{\mathbf{v}}^T B = (P\mathbf{v})^T PAQ = \mathbf{v}^T AQ = \mathbf{0}$ since, according to Exercise 1.6.14, $P^T = P^{-1}$ is the inverse of the permutation matrix P.

2.6.15. False. For example, any two inequivalent trees, cf. Exercise 2.6.8, with the same number of nodes have incidence matrices of the same size, with trivial cokernels: $\operatorname{coker} A = \operatorname{coker} B = \{\mathbf{0}\}$. As another example, the incidence matrices

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

both have cokernel basis $(1,1,1,0,0)^T$, but do not represent equivalent digraphs.

2.6.16.

(a) If the first k vertices belong to one component and the last $n-k$ to the other, then there is no edge between the two sets of vertices and so the entries $a_{ij} = 0$ whenever $i = 1,\ldots,k$, $j = k+1,\ldots,n$, or when $i = k+1,\ldots,n$, $j = 1,\ldots,k$, which proves that A has the indicated block form.

(b) The graph consists of two disconnected triangles. If we use $1,2,3$ to label the vertices in one triangle and $4,5,6$ for those in the second, the resulting incidence matrix has the indicated block form $\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$, with each block a 3×3 submatrix.