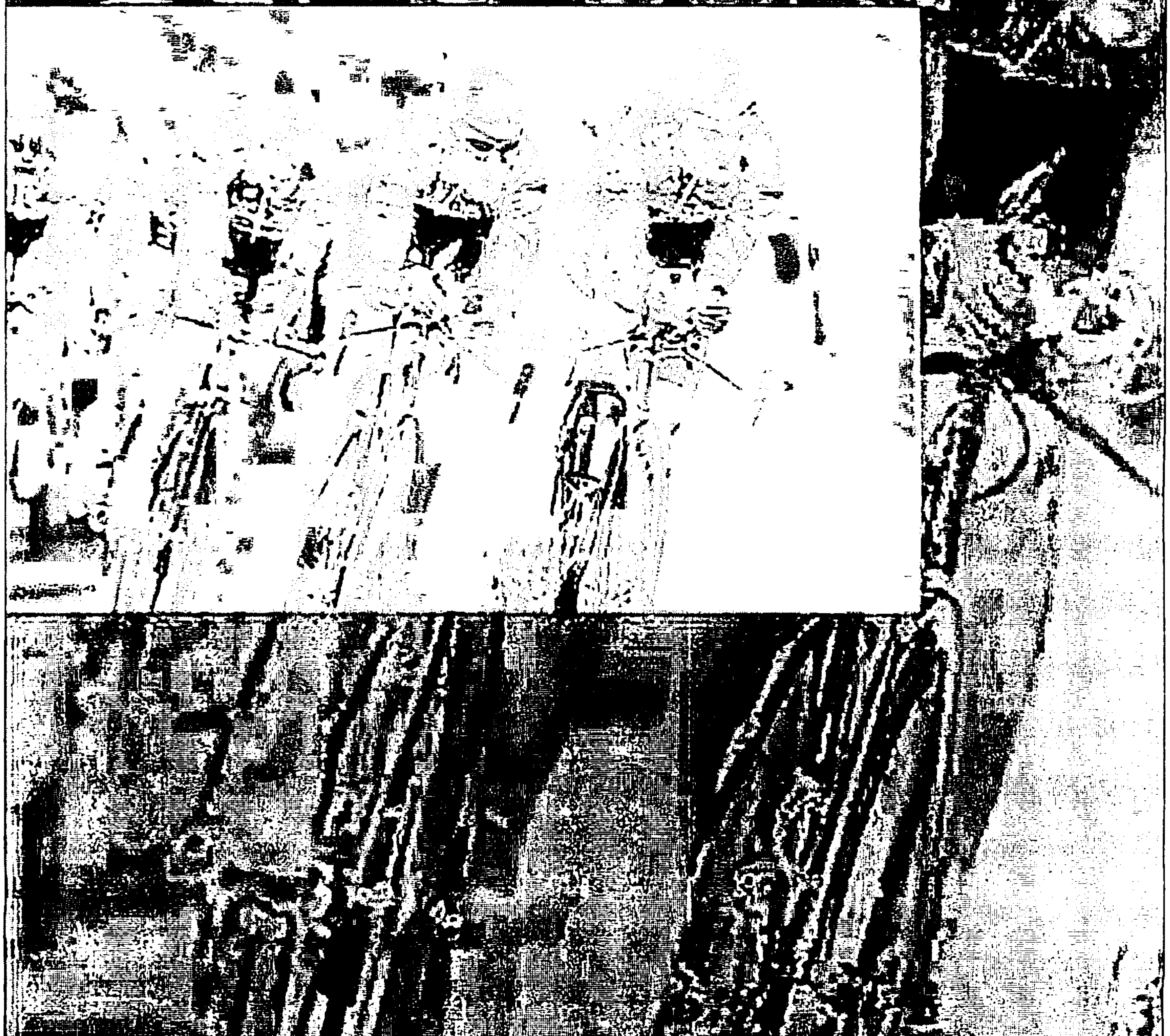
INSTRUCTOR'S SOLUTIONS MANUAL

SEVENTH EDITION

ANALYTICAL MECHANICS



FOWLES & CASSIDAY

CHAPTER 1
FUNDAMENTAL CONCEPTS: VECTORS

1.1 (a) $\vec{A}+\vec{B}=(\hat{i}+\hat{j})+(\hat{j}+\hat{k})=\hat{i}+2\hat{j}+\hat{k}$

$$\left|\vec{A}+\vec{B}\right|=(1+4+1)^{\frac{1}{2}}=\sqrt{6}$$

(b) $3\vec{A}-2\vec{B}=3(\hat{i}+\hat{j})-2(\hat{j}+\hat{k})=3\hat{i}+\hat{j}-2\hat{k}$

(c) $\vec{A}\cdot\vec{B}=(1)(0)+(1)(1)+(0)(1)=1$

(d) $$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 0\\ 0 & 1 & 1\end{vmatrix}=\hat{i}(1-0)+\hat{j}(0-1)+\hat{k}(1-0)=\hat{i}-\hat{j}+\hat{k}$$

$$\left|\vec{A}\times\vec{B}\right|=(1+1+1)^{\frac{1}{2}}=\sqrt{3}$$

1.2 (a) $\vec{A}\cdot\left(\vec{B}+\vec{C}\right)=\left(2\hat{i}+\hat{j}\right)\cdot\left(\hat{i}+4\hat{j}+\hat{k}\right)=(2)(1)+(1)(4)+(0)(1)=6$

$$\left(\vec{A}+\vec{B}\right)\cdot\vec{C}=\left(3\hat{i}+\hat{j}+\hat{k}\right)\cdot4\hat{j}=(3)(0)+(1)(4)+(1)(0)=4$$

(b) $$\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=\begin{vmatrix}2 & 1 & 0\\ 1 & 0 & 1\\ 0 & 4 & 0\end{vmatrix}=-8$$

$$\left(\vec{A}\times\vec{B}\right)\cdot\vec{C}=\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=-8$$

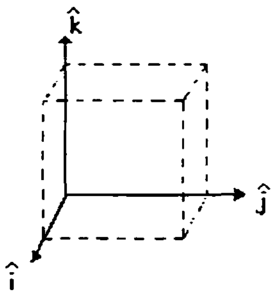
(c) $\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\left(\vec{A}\cdot\vec{C}\right)\vec{B}-\left(\vec{A}\cdot\vec{B}\right)\vec{C}=4\left(\hat{i}+\hat{k}\right)-2\left(4\hat{j}\right)=4\hat{i}-8\hat{j}+4\hat{k}$

$$\left(\vec{A}\times\vec{B}\right)\times\vec{C}=-\vec{C}\times\left(\vec{A}\times\vec{B}\right)=-\left[\left(\vec{C}\cdot\vec{B}\right)\vec{A}-\left(\vec{C}\cdot\vec{A}\right)\vec{B}\right]$$
$$=-\left[0\left(2\hat{i}+\hat{j}\right)-4\left(\hat{i}+\hat{k}\right)\right]=4\hat{i}+4\hat{k}$$

1.3 $\cos\theta = \frac{\vec{A}\cdot\vec{B}}{AB} = \frac{(a)(a)+(2a)(2a)+(0)(3a)}{\sqrt{5a^2}\sqrt{14a^2}} = \frac{5a^2}{a^2\sqrt{5}\sqrt{14}}$

$$\theta = \cos^{-1}\sqrt{\frac{5}{14}} \approx 53°$$

1.4



(a) $\vec{A} = \hat{i}+\hat{j}+\hat{k}$: *body diagonal*

$$A = \left|\vec{A}\cdot\vec{A}\right| = \sqrt{\hat{i}\cdot\hat{i}+\hat{j}\cdot\hat{j}+\hat{k}\cdot\hat{k}} = \sqrt{3}$$

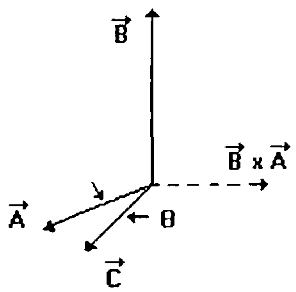
(b) $\vec{B} = \hat{i}+\hat{j}$: *face diagonal*

$$B = \left|\vec{B}\cdot\vec{B}\right| = \sqrt{2}$$

(c) $\vec{C} = \vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

(d) $\cos\theta = \frac{\vec{A}\cdot\vec{B}}{AB} = \frac{1-1}{\sqrt{3}\sqrt{2}} = 0 \quad \therefore \theta = 90°$

1.5



$B = \left|\vec{B}\right| = \left|\vec{A}\times\vec{C}\right| = AC\sin\theta \qquad \therefore C_y = C\sin\theta = \frac{B}{A}$

$\vec{A}\cdot\vec{C} = AC\cos\theta = u \qquad \therefore C_x = C\cos\theta = \frac{u}{A}$

$$\vec{C} = \frac{\vec{A}}{A}C_x + \frac{\vec{B}\times\vec{A}}{\left|\vec{B}\times\vec{A}\right|}C_y = \frac{u}{A^2}\vec{A} + \frac{\vec{B}\times\vec{A}}{AB}\left(\frac{B}{A}\right)$$

$$= \frac{u}{A^2}\vec{A} + \frac{1}{A^2}\vec{B}\times\vec{A}$$

1.6 $\frac{d\vec{A}}{dt} = \hat{i}\frac{d}{dt}(\alpha t) + \hat{j}\frac{d}{dt}\left(\beta t^2\right) + \hat{k}\frac{d}{dt}\left(\gamma t^3\right) = \hat{i}\alpha + \hat{j}2\beta t + \hat{k}3\gamma t^2$

$$\frac{d^2\vec{A}}{dt^2} = \hat{j}2\beta + \hat{k}6\gamma t$$

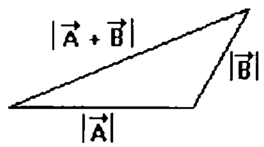
1.7 $0 = \vec{A}\cdot\vec{B} = (q)(q) + (3)(-q) + (1)(2) = q^2 - 3q + 2$

$(q-2)(q-1) = 0\,, \qquad q = 1 \text{ or } 2$

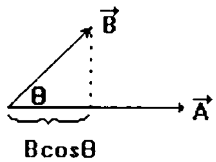
1.8 $\left|\vec{A}+\vec{B}\right|^2 = \left(\vec{A}+\vec{B}\right)\cdot\left(\vec{A}+\vec{B}\right) = A^2 + B^2 + 2\vec{A}\cdot\vec{B}$

$\left[\left|\vec{A}\right| + \left|\vec{B}\right|\right]^2 = A^2 + B^2 + 2AB$

Since $\vec{A}\cdot\vec{B} = AB\cos\theta \le AB\,, \quad \left|\vec{A}+\vec{B}\right| \le \left|\vec{A}\right| + \left|\vec{B}\right|$



$\left|\vec{A}\cdot\vec{B}\right| = \left|AB\cos\theta\right| = \left|\vec{A}\right|\left|\vec{B}\right|\left|\cos\theta\right| \le \left|\vec{A}\right|\left|\vec{B}\right|$



$B\cos\theta \le B$

1.9 Show $\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \left(\vec{A}\cdot\vec{C}\right)\vec{B} - \left(\vec{A}\cdot\vec{B}\right)\vec{C}$

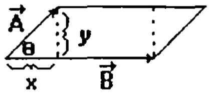
or $\vec{A}\times\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix} = \left(A_xC_x + A_yC_y + A_zC_z\right)\vec{B} - \left(A_xB_x + A_yB_y + A_zB_z\right)\vec{C}$

$$\begin{aligned}
&= \left(A_xB_xC_x + A_yB_xC_y + A_zB_xC_z - A_xB_xC_x - A_yB_yC_x - A_zB_zC_x\right)\hat{i}\\
&+ \left(A_xB_yC_x + A_yB_yC_y + A_zB_yC_z - A_xB_xC_y - A_yB_yC_y - A_zB_zC_y\right)\hat{j}\\
&+ \left(A_xB_zC_x + A_yB_zC_y + A_zB_zC_z - A_xB_xC_z - A_yB_yC_z - A_zB_zC_z\right)\hat{k}
\end{aligned}$$

$$\begin{aligned}
&\left(A_yB_xC_y + A_zB_xC_z - A_yB_yC_x - A_zB_zC_x\right)\hat{i}\\
&= +\left(A_xB_yC_x + A_zB_yC_z - A_xB_xC_y - A_zB_zC_y\right)\hat{j}\\
&\quad +\left(A_xB_zC_x + A_yB_zC_y - A_xB_xC_z - A_yB_yC_z\right)\hat{k}
\end{aligned}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_yC_z - B_zC_y & B_zC_x - B_xC_z & B_xC_y - B_yC_x \end{vmatrix} \begin{matrix} \hat{i}\left(A_yB_xC_y - A_yB_yC_x - A_zB_zC_x + A_zB_xC_z\right) \\ = +\hat{j}\left(A_zB_yC_z - A_zB_zC_y - A_xB_xC_y + A_xB_yC_x\right) \\ +\hat{k}\left(A_xB_zC_x - A_xB_xC_z - A_yB_yC_z + A_yB_zC_y\right) \end{matrix}$$

1.10



$$y = A\sin\theta$$

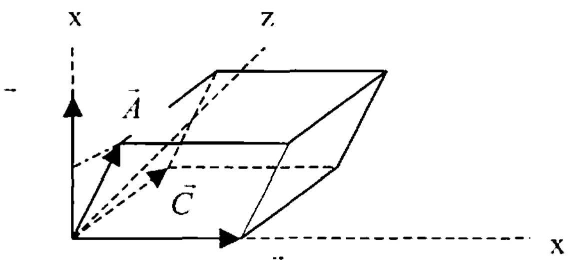
$$\text{A} = 2\left(\frac{1}{2}xy\right) + y\left(B - x\right) = xy + yB - xy = AB\sin\theta$$

$$\text{A} = \left|\vec{A} \times \vec{B}\right|$$

1.11

$$\vec{A}\cdot\left(\vec{B}\times\vec{C}\right) = \vec{A}\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = -\begin{vmatrix} B_x & B_y & B_z \\ A_x & A_y & A_z \\ C_x & C_y & C_z \end{vmatrix} = -\vec{B}\cdot\left(\vec{A}\times\vec{C}\right)$$

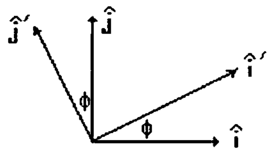
1.12



Let $\vec{A}$ = (Ax,Ay,Az), $\vec{B}$ = (0,By,0) and $\vec{C}$ = (0,Cy,Cz)
C_z is the perpendicular distance between the plane $\vec{A}$, $\vec{B}$ and its opposite. $\vec{u} = \vec{B}\,x\,\vec{C}$ is directed along the x-axis since the vectors $\vec{B}$, $\vec{C}$ are in the y,z plane. $u_x = \left|\vec{B}\,x\,\vec{C}\right|$ = B_yC_z is the area of the parallelogram formed by the vectors $\vec{B}$, $\vec{C}$. Multiply that area times the height of plane $\vec{A}$, $\vec{B}$ = A_x to get the volume of the parallopiped

$$V = A_x u_x = A_x\,B_y\,C_z = \vec{A}\bullet\left(\vec{B}\,x\,\vec{C}\right)$$

1.13 For rotation about the z axis:

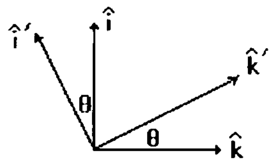


$\hat{i}\cdot\hat{i}' = \cos\phi = \hat{j}\cdot\hat{j}', \quad \hat{k}\cdot\hat{k}' = 1$

$\hat{i}\cdot\hat{j}' = -\sin\phi$

$\hat{j}\cdot\hat{i}' = \sin\phi$

For rotation about the y' axis:



$\hat{i}\cdot\hat{i}' = \cos\theta = \hat{k}\cdot\hat{k}', \quad \hat{j}\cdot\hat{j}' = 1$

$\hat{i}\cdot\hat{k}' = \sin\theta$

$\hat{k}\cdot\hat{i}' = -\sin\theta$

$$\vec{T} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix}$$

1.14

$\hat{i}\cdot\hat{i}' = \cos 30^\circ = \frac{\sqrt{3}}{2} \qquad \hat{j}\cdot\hat{i}' = \sin 30^\circ = \frac{1}{2} \qquad \hat{k}\cdot\hat{i}' = 0$

$\hat{i}\cdot\hat{j}' = -\sin 30^\circ = -\frac{1}{2} \qquad \hat{j}\cdot\hat{j}' = \cos 30^\circ = \frac{\sqrt{3}}{2} \qquad \hat{k}\cdot\hat{j}' = 0$

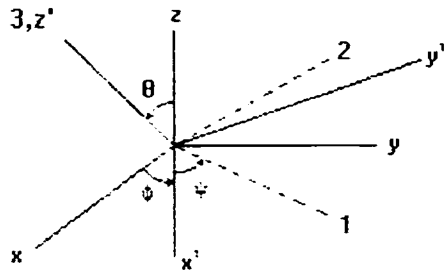
$\hat{i}\cdot\hat{k}' = 0 \qquad \hat{j}\cdot\hat{k}' = 0 \qquad \hat{k}\cdot\hat{k}' = 1$

$$\begin{bmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} \sqrt{3} + \frac{3}{2} \\ \frac{3}{2}\sqrt{3} - 1 \\ -1 \end{bmatrix}$$

$\vec{A} = 3.232\hat{i}' + 1.598\hat{j}' - \hat{k}'$

1.15

1. Rotate thru ϕ about z-axis $\qquad \phi = 45^\circ \qquad R_\phi$
2. Rotate thru θ about x'-axis $\qquad \theta = 45^\circ \qquad R_\theta$
3. Rotate thru ψ about z'-axis $\qquad \psi = 45^\circ \qquad R_\psi$



$$R_\phi = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad R_\psi = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R(\psi,\theta,\phi) = R_\psi R_\theta R_\phi = \begin{pmatrix} \frac{1}{2}-\frac{1}{2\sqrt{2}} & \frac{1}{2}+\frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2}-\frac{1}{2\sqrt{2}} & -\frac{1}{2}+\frac{1}{2\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = R(\psi,\theta,\phi) \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

Condition is: $\vec{x}' = R\vec{x}$ $\qquad$ where $\vec{x}' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$

Since $\vec{x}\cdot\vec{x} = 1$ $\qquad$ we have: $\psi^2 + \beta^2 + \alpha^2 = 1$

After a lot of algebra: $\alpha = \frac{1}{2} - \frac{\sqrt{2}}{4}, \ \beta = \frac{1}{2} + \frac{\sqrt{2}}{4}, \ \gamma = \frac{1}{2}$

1.16 $\vec{v} = v\hat{\tau} = ct\hat{\tau}$

$$\vec{a} = \dot{v}\hat{\tau} + \frac{v^2}{\rho}\hat{n} = c\hat{\tau} + \frac{c^2t^2}{b}\hat{n}$$

at $t = \sqrt{\frac{b}{c}}$, $\vec{v} = \hat{\tau}\sqrt{bc}$ and $\vec{a} = c\hat{\tau} + c\hat{n}$

$$\cos\theta = \frac{\vec{v}\cdot\vec{a}}{va} = \frac{c\sqrt{bc}}{\sqrt{bc}\sqrt{2c^2}} = \frac{1}{\sqrt{2}}$$

$$\theta = 45^\circ$$

1.17 $\vec{v}(t) = -\hat{i}b\omega\sin(\omega t) + \hat{j}2b\omega\cos(\omega t)$

$$|\vec{v}| = \left(b^2\omega^2\sin^2\omega t + 4b^2\omega^2\cos^2\omega t\right)^{\frac{1}{2}} = b\omega\left(1 + 3\cos^2\omega t\right)^{\frac{1}{2}}$$

$$\vec{a}(t) = -\hat{i}b\omega^2\cos\omega t - \hat{j}2b\omega^2\sin\omega t$$

$$|\vec{a}| = b\omega^2\left(1 + 3\sin^2\omega t\right)^{\frac{1}{2}}$$

at $t = 0$, $|\vec{v}| = 2b\omega$; at $t = \frac{\pi}{2\omega}$, $|\vec{v}| = b\omega$

1.18 $\vec{v}(t) = \hat{i}b\omega\cos\omega t - \hat{j}b\omega\sin\omega t + \hat{k}2ct$

$$\vec{a}(t) = -\hat{i}b\omega^2\sin\omega t - \hat{j}b\omega^2\cos\omega t + \hat{k}2c$$

$$|\vec{a}| = \left(b^2\omega^4\sin^2\omega t + b^2\omega^4\cos^2\omega t + 4c^2\right)^{\frac{1}{2}} = \left(b^2\omega^4 + 4c^2\right)^{\frac{1}{2}}$$

1.19 $\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta = bke^{kt}\hat{e}_r + bce^{kt}\hat{e}_\theta$

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta = b\left(k^2 - c^2\right)e^{kt}\hat{e}_r + 2bcke^{kt}\hat{e}_\theta$$

$$\cos\phi = \frac{\vec{v}\cdot\vec{a}}{va} = \frac{b^2k\left(k^2 - c^2\right)e^{2kt} + 2b^2c^2ke^{2kt}}{be^{kt}\left(k^2 + c^2\right)^{\frac{1}{2}}be^{kt}\left[\left(k^2 - c^2\right)^2 + 4c^2k^2\right]^{\frac{1}{2}}}$$

$$\cos\phi = \frac{k\left(k^2 + c^2\right)}{\left(k^2 + c^2\right)^{\frac{1}{2}}\left(k^2 + c^2\right)} = \frac{k}{\left(k^2 + c^2\right)^{\frac{1}{2}}}, \text{ a constant}$$

1.20 $\vec{a} = \left(\ddot{R} - R\dot{\phi}\right)\hat{e}_R + \left(2\dot{R}\dot{\phi} + R\ddot{\phi}\right)\hat{e}_\phi + \ddot{z}\hat{e}_z$

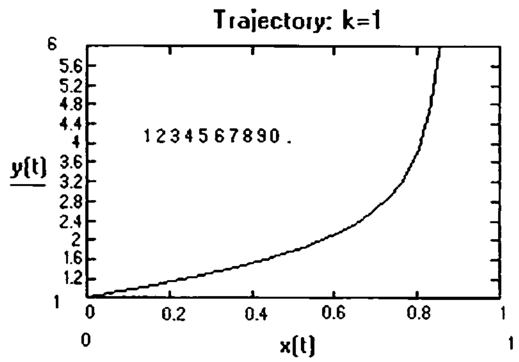
$$\vec{a} = -b\omega^2\hat{e}_R + 2c\hat{e}_z$$

$$|\vec{a}| = \left(b^2\omega^4 + 4c^2\right)^{\frac{1}{2}}$$

1.21 $\vec{r}(t) = \hat{i}\left(1 - e^{-kt}\right) + \hat{j}e^{kt}$

$\vec{r}(t) = \hat{i}ke^{-kt} + \hat{j}ke^{kt}$

$\vec{r}(t) = -\hat{i}k^2e^{-kt} + \hat{j}k^2e^{kt}$



1.22 $\vec{v} = \hat{e}_r\dot{r} + \hat{e}_\phi r\dot{\phi}\sin\theta + \hat{e}_\theta r\dot{\theta}$

$$\vec{v} = \hat{e}_\phi b\omega\sin\left\{\frac{\pi}{2}\left[1 + \frac{1}{4}\cos(4\omega t)\right]\right\} - \hat{e}_\theta b\frac{\pi}{2}\omega\sin(4\omega t)$$

$$\vec{v} = \hat{e}_\phi b\omega\cos\left[\frac{\pi}{8}\cos(4\omega t)\right] - \hat{e}_\theta b\omega\frac{\pi}{2}\sin(4\omega t)$$

$$|\vec{v}| = b\omega\left[\cos^2\left(\frac{\pi}{8}\cos 4\omega t\right) + \frac{\pi^2}{4}\sin^2 4\omega t\right]^{\frac{1}{2}}$$

Path is sinusoidal oscillation about the equator.

1.23 $\vec{v}\cdot\vec{v} = v^2$

$$\frac{d\vec{v}}{dt}\cdot\vec{v} + \vec{v}\cdot\frac{d\vec{v}}{dt} = 2v\dot{v}$$

$2\vec{v}\cdot\vec{a} = 2v\dot{v}$

$\vec{v}\cdot\vec{a} = v\dot{v}$

1.24 $$\frac{d}{dt}\left[\vec{r}\cdot(\vec{v}\times\vec{a})\right]=\frac{d\vec{r}}{dt}\cdot(\vec{v}\times\vec{a})+\vec{r}\cdot\frac{d}{dt}(\vec{v}\times\vec{a})$$

$$=\vec{v}\cdot(\vec{v}\times\vec{a})+\vec{r}\cdot\left[\left(\frac{d\vec{v}}{dt}\times\vec{a}\right)+\left(\vec{v}\times\frac{d\vec{a}}{dt}\right)\right]$$

$$=0+\vec{r}\cdot\left[0+\left(\vec{v}\times\dot{\vec{a}}\right)\right]$$

$$\frac{d}{dt}\left[\vec{r}\cdot(\vec{v}\times\vec{a})\right]=\vec{r}\cdot\left(\vec{v}\times\dot{\vec{a}}\right)$$

1.25 $\vec{v}=v\hat{\tau}$ and $\vec{a}=a_\tau\hat{\tau}+a_n\hat{n}$

$\vec{v}\cdot\vec{a}=va_\tau$, so $a_\tau=\dfrac{\vec{v}\cdot\vec{a}}{v}$

$a^2=a_\tau^2+a_n^2$, so $a_n=\left(a^2-a_\tau^2\right)^{\frac{1}{2}}$

1.26 For 1.14, $$a_\tau=\frac{-b^2\omega^3\cos\omega t\cdot\sin\omega t+b^2\omega^3\sin\omega t\cdot\cos\omega t+4c^2t}{\left(b^2\omega^2\cos^2\omega t+b^2\omega^2\sin^2\omega t+4c^2t^2\right)^{\frac{1}{2}}}$$

$$a_\tau=\frac{4c^2t}{\left(b^2\omega^2+4c^2t^2\right)^{\frac{1}{2}}}$$

$$a_n=\left(b^2\omega^2+4c^2-\frac{16c^4t^2}{b^2\omega^2+4c^2t^2}\right)^{\frac{1}{2}}$$

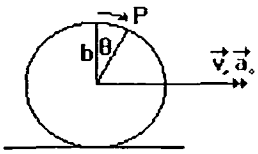
For 1.15, $$a_\tau=\frac{b^2k\left(k^2-c^2\right)e^{2kt}+2b^2c^2ke^{2kt}}{be^{kt}\left(k^2+c^2\right)^{\frac{1}{2}}}=bke^{kt}\left(k^2+c^2\right)^{\frac{1}{2}}$$

$$a_n=\left[b^2e^{2kt}\left(k^2+c^2\right)^2-b^2k^2e^{2kt}\left(k^2+c^2\right)\right]^{\frac{1}{2}}=bce^{kt}\left(k^2+c^2\right)^{\frac{1}{2}}$$

1.27 $\vec{v}=v\hat{\tau}$, $\vec{a}=\dot{v}\hat{\tau}+\dfrac{v^2}{\rho}\hat{n}$

$$\left|\vec{v}\times\vec{a}\right|=v\cdot a_n=v\frac{v^2}{\rho}=\frac{v^3}{\rho}$$

1.28



$$\bar{r}_{oP} = \hat{i} b \sin\theta + \hat{j} b \cos\theta$$

$$\bar{v}_{rel} = \hat{i} b\dot{\theta}\cos\theta - \hat{j} b\dot{\theta}\sin\theta$$

$$\bar{a}_{rel} = \hat{i} b\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) - \hat{j} b\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)$$

at the point $\theta = \frac{\pi}{2}$, $\bar{v}_{rel} = -\bar{v}$

So, $|\bar{v}_{rel}| = b\dot{\theta} = v$

$$\dot{\theta} = \frac{v}{b} \qquad \ddot{\theta} = \frac{\dot{v}}{b} = \frac{a_o}{b}$$

Now, $\vec{a}_{rel} = \dot{v}_{rel}\hat{\tau} + \frac{v_{rel}^2}{\rho}\hat{n} = a_o\hat{\tau} + \frac{v^2}{b}\hat{n}$

$$|\vec{a}_{rel}| = \left(a_o^2 + \frac{v^4}{b^2}\right)^{\frac{1}{2}}$$

$\bar{v}_P = \bar{v} + \bar{v}_{rel}$ and $\bar{a}_P = \bar{a}_o + \bar{a}_{rel}$

$$\bar{a}_P = \hat{i}\left[a_o + b\left(\frac{a_o}{b}\cos\theta - \frac{v^2}{b^2}\sin\theta\right)\right] - \hat{j} b\left(\frac{a_o}{b}\sin\theta + \frac{v^2}{b^2}\cos\theta\right)$$

$$|\bar{a}_P| = a_o\left(2 + 2\cos\theta + \frac{v^4}{a_o^2 b^2} - \frac{2v^2}{a_o b}\sin\theta\right)^{\frac{1}{2}}$$

$\bar{a}_P$ is a maximum at $\theta = 0$, i.e., at the top of the wheel.

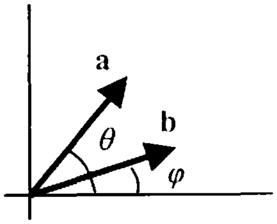
$$-2\sin\theta - \frac{2v^2}{a_o b}\cos\theta = 0$$

$$\theta = \tan^{-1}\left(-\frac{v^2}{a_o b}\right)$$

1.29 $\tilde{R}R = \begin{pmatrix} x & -x & 0 \\ x & x & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x & x & 0 \\ -x & x & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2x^2 & 0 & 0 \\ 0 & 2x^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Therefore, $x = \frac{1}{\sqrt{2}}$

The transformation represents a rotation of 45° about the z-axis (see Example 1.8.2)

1.30



(a) $a = \hat{i}\cos\theta + \hat{j}\sin\theta$

$b = \hat{i}\cos\varphi + \hat{j}\sin\varphi$

$a \cdot b = \cos(\theta-\varphi) = \left(\hat{i}\cos\theta + \hat{j}\sin\theta\right)\cdot\left(\hat{i}\cos\varphi + \hat{j}\sin\varphi\right)$

$\cos(\theta-\varphi) = \cos\theta\cos\varphi + \sin\theta\sin\varphi$

(b) $b \times a = \left|\hat{k}\right|\sin(\theta-\varphi) = \left|\left(\hat{i}\cos\theta + \hat{j}\sin\theta\right)\times\left(\hat{i}\cos\varphi + \hat{j}\sin\varphi\right)\right|$

$\sin(\theta-\varphi) = \sin\theta\cos\varphi - \cos\theta\sin\varphi$

CHAPTER 2
NEWTONIAN MECHANICS: RECTILINEAR MOTION OF A PARTICLE

2.1 (a) $\ddot{x} = \frac{1}{m}(F_o + ct)$

$$\dot{x} = \int_0^t \frac{1}{m}(F_o + ct)\,dt = \frac{F_o}{m}t + \frac{c}{2m}t^2$$

$$x = \int_0^t \left(\frac{F_o}{m}t + \frac{c}{2m}t^2\right)dt = \frac{F_o}{m}t^2 + \frac{c}{6m}t^3$$

(b) $\ddot{x} = \frac{F_o}{m}\sin ct$

$$\dot{x} = \int_0^t \frac{F_o}{m}\sin ct\,dt = -\frac{F_o}{cm}\cos ct\Big|_0^t = \frac{F_o}{cm}(1 - \cos ct)$$

$$x = \int_0^t \frac{F_o}{cm}(1 - \cos ct)\,dt = \frac{F_o}{cm}\left(1 - \frac{1}{c}\sin ct\right)$$

(c) $\ddot{x} = \frac{F_o}{m}e^{ct}$

$$\dot{x} = \frac{F_o}{cm}e^{ct}\Big|_0^t = \frac{F_o}{cm}\left(e^{ct} - 1\right)$$

$$x = \frac{F_o}{cm}\left(\frac{1}{c}e^{ct} - \frac{1}{c} - t\right) = \frac{F_o}{c^2 m}\left(e^{ct} - 1 - ct\right)$$

2.2 (a) $\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx}\cdot\frac{dx}{dt} = \dot{x}\frac{d\dot{x}}{dx}$

$$\dot{x}\frac{d\dot{x}}{dx} = \frac{1}{m}(F_o + cx)$$

$$\dot{x}d\dot{x} = \frac{1}{m}(F_o + cx)\,dx$$

$$\frac{1}{2}\dot{x}^2 = \frac{1}{m}\left(F_o x + \frac{cx^2}{2}\right)$$

$$\dot{x} = \left[\frac{x}{m}(2F_o + cx)\right]^{\frac{1}{2}}$$

(b) $\ddot{x} = \dot{x}\frac{d\dot{x}}{dx} = \frac{1}{m}F_o e^{-cx}$

$$\dot{x}d\dot{x} = \frac{1}{m}F_{\circ}e^{-cx}dx$$

$$\frac{1}{2}\dot{x}^2 = -\frac{F_{\circ}}{cm}\left(e^{-cx} - 1\right) = \frac{F_{\circ}}{cm}\left(1 - e^{-cx}\right)$$

$$\dot{x} = \left[\frac{2F_{\circ}}{cm}\left(1 - e^{-cx}\right)\right]^{\frac{1}{2}}$$

(c) $\ddot{x} = \dot{x}\dfrac{d\dot{x}}{dx} = \dfrac{1}{m}\left(F_{\circ}\cos cx\right)$

$$\dot{x}d\dot{x} = \frac{F_{\circ}}{m}\cos cx\, dx$$

$$\frac{1}{2}\dot{x}^2 = \frac{F_{\circ}}{cm}\sin cx$$

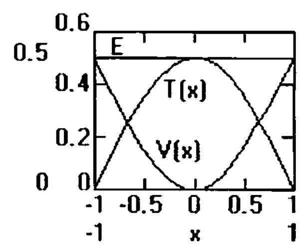
$$\dot{x} = \left(\frac{2F_{\circ}}{cm}\sin cx\right)^{\frac{1}{2}}$$

2.3 (a) $V(x) = -\int_{x_0}^{x}\left(F_{\circ} + cx\right)dx = -F_{\circ}x - \dfrac{cx^2}{2} + C$

(b) $V(x) = -\int_{x_0}^{x} F_{\circ}e^{-cx}dx = \dfrac{F_{\circ}}{c}e^{-cx} + C$

(c) $V(x) = -\int_{x_0}^{x} F_{\circ}\cos cx\, dx = -\dfrac{F_{\circ}}{c}\sin cx + C$

2.4 (a) $F(x) = -\dfrac{dV(x)}{dx} = -kx$



$$V(x) = \int_0^x kx\, dx = \frac{1}{2}kx^2$$

(b) $T_{\circ} = T(x) + V(x)$

$$T(x) = T_{\circ} - V(x) = \frac{1}{2}k\left(A - x^2\right)$$

(c) $E = T_{\circ} = \dfrac{1}{2}kA^2$

(d) *turning points* @ $T(x_1) \rightarrow 0 \quad \therefore x_1 = \pm A$

2.5 (a) $F(x) - kx + \dfrac{kx^3}{A^2}$ so $V(x) = \int_0^x\left(kx - \dfrac{kx^3}{A^2}\right)dx = \dfrac{1}{2}kx^2 - \dfrac{1}{4}\dfrac{kx^4}{A^2}$

(b) $T(x) = T_{\circ} - V(x) = T_{\circ} - \dfrac{1}{2}kx^2 + \dfrac{1}{4}\dfrac{kx^4}{A^2}$

(c) $E = T_{\circ}$

(d) $V(x)$ has maximum at $|F(x_m)| \to 0$

$$kx_m - \frac{kx_m^{\ 3}}{A^2} = 0 \qquad x_m = \pm A$$

$$V(x_m) = \frac{1}{2}kA^2 - \frac{1}{4}\frac{kA^4}{A^2} = \frac{1}{4}kA^2$$

If $E < V(x_m)$ turning points exist.

Turning points @ $T(x_1) \to 0$ let $u = x_1^{\ 2}$

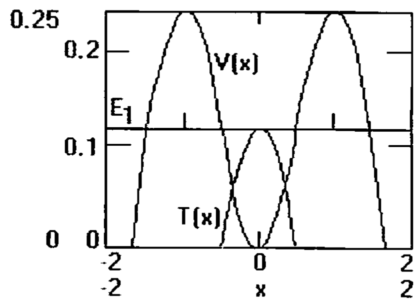
$$E - \frac{1}{2}ku + \frac{1}{4}\frac{ku^2}{A^2} = 0$$

solving for u, we obtain

$$u = A^2\left[1 \pm \left(1 - \frac{4E}{kA^2}\right)^{\frac{1}{2}}\right]$$

or

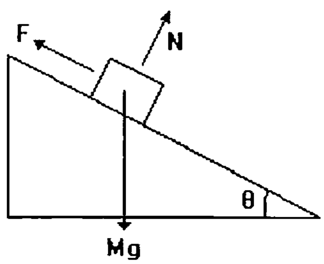
$$x_1 = \pm A\left[1 - \sqrt{\left(1 - \frac{4E}{kA^2}\right)}\right]^{\frac{1}{2}}$$



2.6 $\dot{x} = v(x) = \frac{\alpha}{x} \qquad \ddot{x} = -\frac{\alpha}{x^2}\dot{x} = -\frac{\alpha^2}{x^3}$

$$F(x) = m\ddot{x} = -\frac{m\alpha^2}{x^3}$$

2.7



$$F \geq Mg\sin\theta$$

2.8 $F = m\ddot{x} = m\dot{x}\frac{d\dot{x}}{dx}$

$$\dot{x} = bx^{-3}$$

$$\frac{d\dot{x}}{dx} = -3bx^{-4}$$

$$F = m\left(bx^{-3}\right)\left(-3bx^{-4}\right)$$

$$F = -3mb^2x^{-7}$$

2.9 (a) $V = mgx = (.145kg)\left(9.8\frac{m}{s^2}\right)(1250\,ft)\left(.3048\frac{m}{ft}\right) = 541J$

(b) $T = \frac{1}{2}mv^2 = \frac{1}{2}mv_t^2 = \frac{1}{2}m\left(\frac{mg}{c_2}\right) = \frac{1}{2}\frac{m^2 g}{.22D^2}$

$$T = \frac{(.145kg)^2\left(9.8\frac{m}{s^2}\right)}{(2)(.22)\left[(2)(.0366)\right]^2\frac{kg}{m}} = 87J$$

$$\int F dx = \int -cv^2 dx = -c\int v^3 dt = -c\int\left(-v_t \tanh\left(\frac{t}{\tau}\right)\right)^3 dt$$

$$= cv_t^3\tau\left[-\frac{1}{2}\tanh^2\left(\frac{t}{\tau}\right) + \int \tanh\left(\frac{t}{\tau}\right)d\left(\frac{t}{\tau}\right)\right]$$

$$= cv_t^3\tau\left[-\frac{1}{2}\tanh^2\left(\frac{t}{\tau}\right) + \ln\cosh\left(\frac{t}{\tau}\right)\right]$$

Now $\tanh^2\left(\frac{t}{\tau}\right) \cong 1$ for $t \ll \tau$

Meanwhile $x = \int v dt = \int\left(-v_t \tanh\left(\frac{t}{\tau}\right)\right)dt = v_t\tau \ln\cosh\left(\frac{t}{\tau}\right)$

$$\ln\cosh\left(\frac{t}{\tau}\right) = \frac{x}{v_t\tau}$$

$$x = (1250 ft)\left(.3048\frac{m}{ft}\right) = 381m$$

$$v_t = \left(\frac{mg}{c_2}\right)^{\frac{1}{2}} = \left[\frac{(.145kg)\left(9.8\frac{m}{s^2}\right)}{(.22)(.0732)^2\frac{kg}{m}}\right]^{\frac{1}{2}} = 34.72\frac{m}{s}$$

$$\tau = \left(\frac{m}{c_2 g}\right)^{\frac{1}{2}} = \left[\frac{(.145kg)}{(.22)(.0732)^2\frac{kg}{m}\left(9.8\frac{m}{s^2}\right)}\right]^{\frac{1}{2}} = 3.543s$$

$$\int F dx = (.22)(.0732)^2(34.72)^3(3.543)\left[-.5 + \frac{3.81}{(34.72)(3.54)}\right] = 454J$$

$$V - T = 541J - 87J = 454J$$

2.10 For $0 \le t \le t_1$: $v = \frac{F_\circ}{m}t$, $x = \frac{1}{2}\frac{F_\circ}{m}t^2$

For $t_1 \le t \le 2t_1$: $v_\circ = \frac{F_\circ}{m}t_1$, $x_\circ = \frac{F_\circ}{2m}t_1^2$, $t_\circ = t_1$

$$x = \frac{F_\circ}{2m}t_1^{\,2} + \frac{F_\circ}{m}t_1\left(t - t_1\right) + \frac{1}{2}\frac{2F_\circ}{m}\left(t - t_1\right)^2$$

At $\quad t = 2t_1: \quad x = \frac{F_\circ}{2m}t_1^{\,2} + \frac{F_\circ}{m}t_1^{\,2} + \frac{F_\circ}{m}t_1^{\,2} = \frac{5F_\circ}{2m}t_1^{\,2}$

2.11 $\quad a = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = v\cdot\frac{dv}{dx} = -\frac{c}{m}v^{\frac{3}{2}}$

$$v^{-\frac{1}{2}}dv = -\frac{c}{m}dx$$

$$\int_{v_\circ}^{v} v^{-\frac{1}{2}}dv = \int_0^{x_{max}} -\frac{c}{m}dx$$

$$-2v_\circ^{\frac{1}{2}} = -\frac{c}{m}x_{max}$$

$$x_{max} = \frac{2mv_\circ^{\frac{1}{2}}}{c}$$

2.12 $\quad$ Going up: $\; F_x = -mg\sin 30^\circ - \mu mg\cos 30^\circ$

$$\ddot{x} = -g\left(\sin 30^\circ + 0.1\cos 30^\circ\right) = -5.749\frac{m}{s^2}$$

$$v = v_\circ + at$$

at the highest point $v = 0$ so $t_{up} = -\frac{v_\circ}{a} = 0.174v_\circ s$

$$x_{up} = v_\circ t_{up} + \frac{1}{2}at_{up}^2 = 0.174v_\circ^{\,2} - .087v_\circ^{\,2} = 0.087v_\circ^{\,2}m$$

Going down: $\; x_\circ' = 0.087v_\circ^{\,2}, \;\; v_\circ' = 0, \;\; a' = -9.8\left(0.5 - 0.0866\right)$

$$x_{down} = 0 = 0.087v_\circ^{\,2} - \frac{1}{2}4.0513t_{down}^2$$

$$t_{down} = 0.207v_\circ s$$

$$t_{total} = t_{up} + t_{down} = 0.381v_\circ s$$

2.13 $\quad$ At the top $v = 0$ so $\; e^{-2kx_{max}} = \frac{\frac{g}{k}}{\frac{g}{k} + v_\circ^{\,2}}$

Coming down $x_\circ = x_{max}$ and at the bottom $x = 0$

$$v^2 = \frac{g}{k} - \left(\frac{g}{k}\right)^2 \frac{1}{\left(\frac{g}{k} + v_\circ^{\,2}\right)}(1) = \frac{\left(\frac{g}{k}\right)v_\circ^{\,2}}{\frac{g}{k} + v_\circ^{\,2}}$$

$$v = \frac{v_t v_\circ}{\left(v_t^2 + v_\circ^2\right)^{\frac{1}{2}}}, \quad v_t = \sqrt{\frac{g}{k}} = \sqrt{\frac{mg}{c_2}}$$

2.14 Going up: $F_x = -mg - c_2 v^2$

$$a = v\frac{dv}{dx} = -g - kv^2, \quad k = \frac{c^2}{m}$$

$$\int_{v_\circ}^{v} \frac{v\,dv}{-g - kv^2} = \int_0^x dx$$

$$-\frac{1}{2k}\ln\left(-g - kv^2\right)\Big|_{v_\circ}^{v} = x$$

$$\frac{g + kv^2}{g + kv_\circ^2} = e^{-2kx}$$

$$v^2 = \left(\frac{g}{k} + v_\circ^2\right)e^{-2kx} - \frac{g}{k}$$

Going down: $F_x = -mg + c_2 v^2$

$$v\frac{dv}{dx} = -g + kv^2$$

$$\int_0^{v} \frac{v\,dv}{-g + kv^2} = \int_0^x dx$$

$$\frac{1}{2k}\ln\left(-g + kv^2\right)\Big|_0^{v} = x - x_\circ$$

$$1 - \frac{k}{g}v^2 = e^{2kx}e^{-2kx}$$

$$v^2 = \frac{g}{k} - \left(\frac{g}{k}e^{-2kx}\right)e^{2kx}$$

2.15 $m\frac{dv}{dt} = mg - c_1 v - c_2 v^2$

$$\int_0^t \frac{dt}{m} = \int_0^v \frac{dv}{mg - c_1 v - c_2 v^2}$$

Using $\int \frac{dx}{a + bx + cx^2} = \frac{1}{\sqrt{b^2 - 4ac}}\ln\frac{2cx + b - \sqrt{b^2 - 4ac}}{2cx + b + \sqrt{b^2 - 4ac}}$,

$$\frac{t}{m}=\frac{1}{\sqrt{c_1^2+4mgc_2}}\ln\frac{-2c_2v-c_1-\sqrt{c_1^2+4mgc_2}}{-2c_2v-c_1+\sqrt{c_1^2+4mgc_2}}\Bigg|_0^v$$

$$\frac{t}{m}\left(c_1^2+4mgc_2\right)^{\frac{1}{2}}=\ln\frac{\left(2c_2v+c_1+\sqrt{c_1^2+4mgc_2}\right)\left(c_1-\sqrt{c_1^2+4mgc_2}\right)}{\left(2c_2v+c_1-\sqrt{c_1^2+4mgc_2}\right)\left(c_1+\sqrt{c_1^2+4mgc_2}\right)}$$

as $t\to\infty$, $2c_2v_t+c_1-\sqrt{c_1^2+4mgc_2}=0$

$$v_t=-\frac{c_1}{2c_2}+\left[\left(\frac{c_1}{2c_2}\right)^2+\frac{mg}{c_2}\right]^{\frac{1}{2}}$$

Alternatively, when $v=v_t$,

$$m\frac{dv}{dt}=0=mg-c_1v_t-c_2v_t^2$$

$$v_t=-\frac{c_1}{2c_2}+\left[\left(\frac{c_1}{2c_2}\right)^2+\frac{mg}{c_2}\right]^{\frac{1}{2}}$$

2.16 $a=v\dfrac{dv}{dx}=-\dfrac{k}{m}x^{-2}$

$$\int_0^v vdv=\int_b^x-\frac{kdx}{mx^2}$$

$$\frac{1}{2}v^2=\frac{k}{m}\left(\frac{1}{x}-\frac{1}{b}\right)$$

$$v=\frac{dx}{dt}=\left[\frac{2k}{m}\left(\frac{1}{x}-\frac{1}{b}\right)\right]^{\frac{1}{2}}=\left[\frac{2k}{mb}\left(\frac{b-x}{x}\right)\right]^{\frac{1}{2}}$$

$$\int_0^t dt=\int_b^0\left[\frac{mb}{2k}\left(\frac{x}{b-x}\right)\right]^{\frac{1}{2}}dx=\left(\frac{mb^3}{2k}\right)^{\frac{1}{2}}\int_1^0\left(\frac{\frac{x}{b}}{1-\frac{x}{b}}\right)^{\frac{1}{2}}d\left(\frac{x}{b}\right)$$

Since $x\le b$, say $\dfrac{x}{b}=\sin^2\theta$

$$t=\left(\frac{mb^3}{2k}\right)^{\frac{1}{2}}\int_{\frac{\pi}{2}}^0\frac{\sin\theta(2\sin\theta\cos\theta d\theta)}{\cos\theta}=\left(\frac{2mb^3}{k}\right)^{\frac{1}{2}}\int_{\frac{\pi}{2}}^0\sin^2\theta d\theta$$

$$t=\left(\frac{mb^3}{8k}\right)^{\frac{1}{2}}\pi$$

2.17 $m\frac{dv}{dt}=mv\frac{dv}{dx}=f(x)\cdot g(v)$

$$\frac{mvdv}{g(v)}=f(x)dx$$

By integration, get $v=v(x)=\frac{dx}{dt}$

If $F(x,t)=f(x)\cdot g(t)$:

$$m\frac{d^2x}{dt^2}=m\frac{d}{dt}\left(\frac{dx}{dt}\right)=f(x)\cdot g(t)$$

This cannot, in general, be solved by integration.

If $F(v,t)=f(v)\cdot g(t)$:

$$m\frac{dv}{dt}=f(v)\cdot g(t)$$

$$\frac{mdv}{f(v)}=g(t)dt$$

Integration gives $v=v(t)$

$$\frac{dx}{dt}=v(t)$$

$$dx=v(t)dt$$

A second integration gives $x=x(t)$

2.18

$$c_1=\left(1.55\times10^{-4}\right)\left(10^{-2}\right)=1.55\times10^{-6}\frac{kg}{s}$$

$$c_2=(0.22)\left(10^{-2}\right)^2=2.2\times10^{-5}\frac{kg}{s}$$

$$v_t=-\frac{1.55\times10^{-6}}{2\times2.2\times10^{-5}}+\left[\left(\frac{1.55\times10^{-6}}{2\times2.2\times10^{-5}}\right)^2+\frac{\left(10^{-7}\right)(9.8)}{2.2\times10^{-5}}\right]^{\frac{1}{2}}$$

$$v_t=0.179\frac{m}{s}$$

Using equation 2.29, $v_t=\sqrt{\frac{\left(10^{-7}\right)(9.8)}{2.2\times10^{-5}}}=0.211\frac{m}{s}$

2.19

$F(x)=-Ae^{\alpha\dot{x}}=m\ddot{x}$ or $F(v)=-Ae^{\alpha v}=m\dot{v}$ $\qquad \frac{dv}{e^{\alpha v}}=-\frac{A}{m}dt$

Let $u=e^{\alpha v}$ $\qquad du=\alpha e^{\alpha v}dv$ $\qquad dv=\frac{du}{\alpha e^{\alpha v}}=\frac{du}{\alpha u}$ $\qquad \therefore\frac{du}{u^2}=-\frac{\alpha A}{m}dt$

Integrating

$$\frac{1}{u}-\frac{1}{u_\circ}=\frac{A}{m}\alpha t \qquad \text{and substituting } e^{\alpha v}=u$$

(a) $v=v_\circ-\dfrac{1}{\alpha}\ln\left[1+\dfrac{A}{m}e^{\alpha v}\alpha t\right]$

(b) $t=T@v=0$

$$\alpha v_\circ=\ln\left[1+\frac{A}{m}e^{\alpha v}\alpha T\right]$$

$$e^{\alpha v}=1+\frac{A}{m}e^{\alpha v}\alpha T \qquad T=\frac{m}{\alpha A}\left[1-e^{-\alpha v}\right]$$

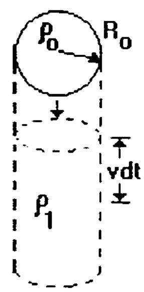
(c) $v\dfrac{dv}{dx}=v=-\dfrac{A}{m}e^{\alpha v} \qquad \dfrac{vdv}{e^{\alpha v}}=-\dfrac{A}{m}dx$

again, let $u=e^{\alpha v}$ $\quad du=\alpha u dv$ $\quad$ or $\quad dv=\dfrac{du}{\alpha u}$ $\quad v=\dfrac{1}{\alpha}\ln u$

$$\frac{\left[\frac{1}{\alpha}\ln u\right]\frac{du}{\alpha u}}{u}=-\frac{A}{m}dx \qquad \text{Integrating and solving}$$

$$x=\frac{m}{\alpha^2 A}\left[1-(1+\alpha v_\circ)e^{-\alpha v}\right]$$

2.20



$$F=\frac{d(mv)}{dt}=mv+vm=mg$$

but $\quad m=\rho_\circ\dfrac{4}{3}\pi r^3 \qquad m=\rho_1\pi r^2 v$

so (1) $\dfrac{4}{3}\pi\rho_\circ r^3 v+\pi\rho_1 r^2 v^2=\dfrac{4}{3}\pi^2=\dfrac{4}{3}\pi\rho_\circ r^3 g$

Now $\dfrac{\rho_1}{\rho_\circ}\approx 10^{-3}$ $\qquad$ so, second term is negligible-small

hence $\; v\approx g \;$ and $\boxed{v\approx gt}$ speed $\propto t$ but

$\dot m=\rho_\circ 4\pi r^2\dot r=\rho_1\pi r^2 v$ $\quad$ or $\quad \dot r\cong\dfrac{1}{4}\dfrac{\rho_1}{\rho_\circ}v$ $\qquad$ Hence $\boxed{r\approx\dfrac{1}{4}\dfrac{\rho_1}{\rho_\circ}gt}$ and rate of growth $\propto t$

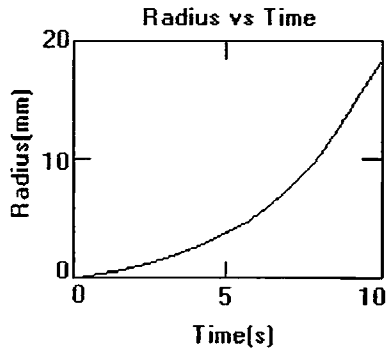
The exact differential equation from (1) above is:

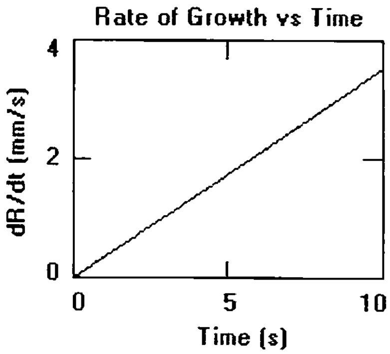
$$\frac{4}{3}\pi\rho_\circ r\left|\frac{4\rho_\circ}{\rho_1}\ddot r\right|+\pi\rho_1\left|\frac{4\rho_\circ\dot r}{\rho_1}\right|^2=\frac{4}{3}\pi\rho_\circ rg$$

which reduces to: $\ddot{r} + \frac{3\dot{r}^2}{r} = \frac{\rho_l}{4\rho_v} g$

Using Mathcad, solve the above non-linear d.e. letting $\frac{\rho_l}{\rho_v} \approx 10^{-3}$ and $R_o \approx 0.01mm$ (small raindrop). Graphs show that

$$v \propto \dot{r} \propto t \text{ and } r \propto t^2$$





CHAPTER 3
OSCILLATIONS

3.1 $x = 0.002\sin\left[2\pi\left(512\,s^{-1}\right)t\right]\,[m]$

$$\dot{x}_{max} = (0.002)(2\pi)(512)\left[\frac{m}{s}\right] = 6.43\left[\frac{m}{s}\right]$$

$$\ddot{x}_{max} = (0.002)(2\pi)^2(512)^2\left[\frac{m}{s^2}\right] = 2.07\times10^4\left[\frac{m}{s^2}\right]$$

3.2 $x = 0.1\sin\omega_o t\ \ [m]$ $\qquad \dot{x} = 0.1\,\omega_o\cos\omega_o t\left[\frac{m}{s}\right]$

When $t = 0$, $x = 0$ and $\dot{x} = 0.5\left[\frac{m}{s}\right] = 0.1\,\omega_o$

$\omega_o = 5\,s^{-1}$ $\qquad T = \dfrac{2\pi}{\omega_o} = 1.26\,s$

3.3 $x(t) = x_o\cos\omega_o t + \dfrac{\dot{x}_o}{\omega_o}\sin\omega_o t$ and $\omega_o = 2\pi f$

$$x = 0.25\cos(20\pi t) + 0.00159\sin(20\pi t)\,[m]$$

3.4 $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$x = A\cos(\omega_o t - \phi) = A\cos\phi\cos\omega_o t + A\sin\phi\sin\omega_o t$$

$$x = \mathrm{A}\cos\omega_o t + \mathrm{B}\sin\omega_o t\,,\ \mathrm{A} = A\cos\phi\,,\ \mathrm{B} = A\sin\phi$$

3.5 $\dfrac{1}{2}m\dot{x}_1^2 + \dfrac{1}{2}kx_1^2 = \dfrac{1}{2}m\dot{x}_2^2 + \dfrac{1}{2}kx_2^2$

$$k\left(x_1^2 - x_2^2\right) = m\left(\dot{x}_2^2 - \dot{x}_1^2\right)$$

$$\omega_o = \sqrt{\frac{k}{m}} = \left(\frac{\dot{x}_2^2 - \dot{x}_1^2}{x_1^2 - x_2^2}\right)^{\frac{1}{2}}$$

$$\frac{1}{2}kA^2 = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}kx_1^2$$

$$A^2 = \frac{m}{k}\dot{x}_1^2 + x_1^2 = \frac{x_1^2\dot{x}_1^2 - x_2^2\dot{x}_1^2}{\dot{x}_2^2 - \dot{x}_1^2} + x_1^2$$

$$A = \left(\frac{x_1^2\dot{x}_2^2 - x_2^2\dot{x}_1^2}{\dot{x}_2^2 - \dot{x}_1^2}\right)^{\frac{1}{2}}$$

3.6 $\frac{1}{2}T_o = \pi\sqrt{\frac{l}{g}} = \pi\sqrt{\frac{1}{\frac{9.8}{6}}}\,s \approx 2.5\,s$

3.7 For springs tied in parallel:

$F_s(x) = -k_1x - k_2x = -(k_1 + k_2)x$

$$\omega = \left[\frac{(k_1 + k_2)}{m}\right]^{\frac{1}{2}}$$

For springs tied in series:

The upward force m is $k_{eq}\,x$.

Therefore, the downward force on spring k_2 is $k_{eq}\,x$.

The upward force on the spring k_2 is k_1x' where x' is the displacement of P, the point at which the springs are tied.

Since the spring k_2 is in equilibrium, $k_1x' = k_{eq}\,x$.

Meanwhile,

The upward force at P is k_1x'.

The downward force at P is $k_2(x - x')$.

Therefore, $k_1x' = k_2(x - x')$

$$x' = \frac{k_2x}{k_1 + k_2}$$

And $k_{eq}\,x = k_1\left(\frac{k_2x}{k_1 + k_2}\right)$

$$\omega = \sqrt{\frac{k_{eq}}{m}} = \left[\frac{k_1k_2}{(k_1 + k_2)m}\right]^{\frac{1}{2}}$$

3.8 For the system $(M + m)$, $-kX = (M + m)\ddot{X}$

The position and acceleration of m are the same as for $(M + m)$:

$$\ddot{x}_m = -\frac{k}{M + m}x_m$$

$$x_m = A\cos\left(\sqrt{\frac{k}{M + m}}\,t + \delta\right) = d\cos\sqrt{\frac{k}{M + m}}\,t$$

The total force on m, $F_m = m\ddot{x}_m = mg - F_r$

$$F_r = mg + \frac{mk}{M+m}x_m = mg + \frac{mkd}{M+m}\cos\sqrt{\frac{k}{M+m}}t$$

For the block to just begin to leave the bottom of the box at the top of the vertical oscillations, $F_r = 0$ at $x_m = -d$:

$$0 = mg - \frac{mkd}{M+m}$$

$$d = \frac{g(M+m)}{k}$$

3.9 $x = e^{-\gamma t} A\cos(\omega_d t - \phi)$

$$\frac{dx}{dt} = -e^{-\gamma t} A\omega_d \sin(\omega_d t - \phi) - \gamma e^{-\gamma t} A\cos(\omega_d t - \phi)$$

maxima at $\frac{dx}{dt} = 0 = \omega_d \sin(\omega_d t - \phi) + \gamma\cos(\omega_d t - \phi)$

$$\tan(\omega_d t - \phi) = -\frac{\gamma}{\omega_d}$$

thus the condition of relative maximum occurs every time that t increases by $\frac{2\pi}{\omega_d}$:

$$t_{i+1} = t_i + \frac{2\pi}{\omega_d}$$

For the i th maximum: $x_i = e^{-\gamma t_i} A\cos(\omega_d t_i - \phi)$

$$x_{i+1} = e^{-\gamma t_{i+1}} A\cos(\omega_d t_{i+1} - \phi) = e^{-\gamma\frac{2\pi}{\omega_d}} x_i$$

$$\frac{x_i}{x_{i+1}} = e^{-\gamma\frac{2\pi}{\omega_d}} = e^{\gamma T_d}$$

3.10 (a) $\gamma = \frac{c}{2m} = 3\,s^{-1}$ $\quad\omega_o^2 = \frac{k}{m} = 25\,s^{-2}$

$\omega_d^2 = \omega_o^2 - \gamma^2 = 16\,s^{-2}$ $\quad\omega_r^2 = \omega_d^2 - \gamma^2 = 7\,s^{-2}$

$\therefore \omega_r = \sqrt{7}\,s^{-1}$

(b) $A_{max} = \frac{F_o}{C\omega_d} = \frac{48}{60.4}m = 0.2\ m$

(c) $\tan\phi = \frac{2\gamma\omega_r}{(\omega_o^2 - \omega_r^2)} = \frac{2\gamma\omega_r}{2\gamma^2} = \frac{\omega_r}{\gamma} = \frac{\sqrt{7}}{3}$ $\qquad\therefore \phi \approx 41.4^\circ$